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States on weak pseudo EMV-algebras. I. States and states morphisms

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Abstract

Recently in [17, 18], new algebras, called weak pseudo EMV-algebras, wPEMV-algebras for short, were introduced generalizing pseudo MV-algebras, generalized Boolean algebras and pseudo EMV-algebras. For these algebras a top element is not assumed a priori. For this class of algebras, we define a state as a finitely additive mapping from a wPEMV-algebra into the real interval $[0,1]$ which preserves a partial addition of two non-interactive elements and attaining the value 1 in some element. It can happen that some commutative wPEMV-algebras are stateless, e.g. cancellative ones.
The paper is divided into two parts. Part I deals with basic properties of states and state-morphisms which are wPEMV-homomorphisms from a wPEMV-algebra into the real interval $[0,1]$ endowed as a commutative wPEMV-algebra. We show that there is a one-to-one correspondence between the set of state-morphisms and the set of maximal and normal ideals having a special property.
In Part II, we present an analogue of the Krein-Mil'man theorem applied to the set of states. We characterize the space of the state-morphisms of a wPEMV-algebra without top element as a Hausdorff locally compact space in the weak topology of states and we present its Alexandroff's one-point compactification. Moreover, we give an integral representation of any (finitely additive) state by a unique regular Borel σ-additive probability measure.

Keywords: Pseudo MV-algebra, pseudo EMV-algebra, wPEMV-algebra, generalized Boolean algebra, state, state-morphism, extremal state, pre-state, maximal and normal ideal, weak convergence, simplex, integral representation.

1 Introduction

States on some structures connected either with the mathematical foundations of quantum mechanics (like orthomodular lattices or posets, quantum logics, effect algebras), [8, 11, 24, 34], or with algebraic structures connected with fuzzy reasonings in many-valued logics (like MV-algebras or pseudo MV-algebras) have been intensively studied during the last decades. The state is defined as a finitely additive mapping from an algebraic structure into the real interval $[0,1]$ preserving a special kind of addition and attaining the value 1. States on unital Abelian po-groups are deeply studied in [24]. It is interesting that states on MV-algebras started since appearing the paper [30], where states on MV-algebras were presented as averaging the truth-value in Łukasiewicz logic. States on pseudo MV-algebras were studied in [6].

Importance of algebraic structures corresponding to Łukasiewicz many-valued logic caused that there is a whole hierarchy of generalizations of MV-algebras. For example, pseudo MV-algebras, [22, 33], a non-commutative generalization of MV-algebras, BL-algebras, [25], hoops and their non-commutative generalizations, [3, 4, 23], are such algebras connected with many-valued reasoning or with fuzzy ideas. We note that Bosbach states on fuzzy structures were studied in [21]. And on all of these algebras, states were introduced and studied.

A standard approach to probability theory is due to Kolmogorov, [27], who has supposed that the probability world connected with some measuring process can be described by a triple $(\Omega, \mathcal{S}, P)$, where P is a σ-additive probability measure defined on a σ-algebra $\mathcal{S}$ of subsets of a universe $\Omega \neq \emptyset$. However, due to de Finetti, a probability measure

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has to be only a finitely additive measure. More precisely, de Finetti's Dutch book analysis showed that his coherence betting criterion is necessary and sufficient for existence of a finitely additive probability measure, see e.g. [2, p. 311–312].

An interpretation of de Finetti's coherence criterion in Łukasiewicz logic was applied in [31, p. 240]. In addition, finitely additive probability measures are intensively studied for example in books [1, 5]. Moreover, [28, 32] showed every finitely additive measure on an MV-algebra can be represented as an integral through a unique regular Borel σ-additive probability measure.

We note that in [19], there is an algebraizable logic whose equivalent algebraic semantics is the variety of state MV-algebras.

Recently, a new generalization of MV-algebras, EMV-algebras (EMV stands for extended MV), was presented, see [12], EMV-algebras do not assume a priori a top element and they resemble MV-algebras locally, i.e. every element is dominated by some idempotent element (a Boolean element) and every interval $[0,a]$, where a is an idempotent element, is an MV-algebras. They generalize MV-algebras and generalized Boolean algebras. They admit a representation saying that every EMV-algebra M without top element can be embedded into an EMV-algebra N with top element as a maximal ideal of N and every element either belongs to the image of M or is a complement of some element from the image of M. States on EMV-algebras were studied in [13]. EMV-algebras were generalized as (non-commutative) pseudo EMV-algebras in [14, 15] together with states in [15]. In these algebras we have described states as additive mappings with values in the interval $[0,1]$ attaining the value 1 in some element, and state-morphisms as homomorphisms from the algebra into the interval $[0,1]$ endowed as a kind of EMV-algebras attaining also the value 1. It was shown that state-morphisms are exactly extremal states, and their connection with maximal and normal ideals was underlined. The weak topology of states was exhibited and studied, when the state-space or the space of state-morphisms is compact or locally compact. We have presented an analogue of the Krein-Mil'man theorem for the state space and established an integral representation of states by regular Borel probability measures.

We note that these algebras do not form a variety, therefore, in [16] we have studied weak EMV-algebras as the least variety containing the class of EMV-algebras. Their non-commutative generalization was presented in [17, 18] as weak pseudo EMV-algebras, wPEMV-algebras in abbreviation. The states on wEMV-algebras were studied in [9].

The main aim of the present paper is to study states and state-morphisms on wPEMV-algebras and generalize some of their important properties known for states on MV-algebras.

We will exhibit the following goals which are divided into two parts.

Part I.

(1) Characterization of states and state-morphisms as extremal states on wPEMV-algebras. In contrast to MV-algebras, we show that even commutative wPEMV-algebras can be stateless. We describe a characterization of the state-morphisms as states whose kernel is a maximal and normal ideal with a special property.

(2) We define weak topologies of states and state-morphisms and we show that every state on a wPEMV-algebra M without top element can be uniquely extended to its representing wPEMV-algebra N with top element. We show that under the weak topology, the space of state-morphisms is a locally compact space whose Alexandroff's one-point compactification is exactly the space of state-morphisms on the representing wPEMV-algebra with top element. It contains a special two-valued state-morphism s_∞ which vanishes on M is one on the rest of M in N. Moreover, every state-morphism on N different from s_∞ is an extension of a state-morphism on M.

(3) Describe maximal and normal ideals I of M such that the quotient M/I is a bounded wPEMV-algebra and we show that the space of maximal and normal ideals with this property is homeomorphic under the hull-kernel topology with the set of state-morphisms on M under the weak topology.

Part II.

(4) The Krein–Mil'man theorem. We show that every state on a wPEMV-algebra M without top element is a weak limit of a net of convex combinations of state-morphisms on M. But not every limit of a net of convex combinations defines a state on M.

(5) We show that a one-point compactification of the set of state-morphisms of a wPEMV-algebra without top element is homeomorphic to the set of state-morphisms of the representing wPEMV-algebra with top element.

(6) Show that every state space of a wPEMV-algebra is either the empty set or a non-empty simplex. We present an integral representation of states showing that, for each state s on M, there is a unique regular Borel probability measure μ_s on the Borel σ-algebra generated by open sets of state-morphisms such that s is an integral through μ_s. This result generalizes the results by [28, 32] established for states on an MV-algebra.

The paper is organized as follows.

Part I. Section 2 gathers the basic facts and examples on wPEMV-algebras. States as finitely additive mappings with values in the interval $[0,1]$ attaining the value 1 and state-morphisms are defined and studied in Section 3. State-morphisms are described also as extremal states or equivalently, as states whose kernel is a maximal and normal ideal with a special property. The weak topology of states is exhibited in Section 4 and some its important topological properties are established.

Part II. In Section 5, we show an analogue of the Krein–Mil'man theorem for states. Moreover, if M has no top element the set of state-morphisms is homeomorphic to the special set of maximal and normal ideals endowed with the hull-kernel topology and it is locally compact whose one-point compactification is the set of state-morphisms on the representing wPEMV-algebra with top element. Finally, an integral representation of states by regular Borel σ-additive probability measures is proved in Section 6.

2 Basic facts on weak pseudo EMV-algebras

We present three families of not necessarily commutative algebras which will be in the center of our investigation and which are mutually connected: pseudo MV-algebras, pseudo EMV-algebras, and weak pseudo EMV-algebras. They are a different kind of non-commutative generalizations of MV-algebras.

Let $\oplus$ be a binary operation on M. An element $a \in M$ is said to be *idempotent* if $a \oplus a = a$. We denote by $\mathcal{I}(M)$ the set of idempotents of M.

2.1 Pseudo MV-algebras

A *pseudo MV-algebra* is an algebra $(M; \oplus, ^-, ^\sim, 0, 1)$ of type $(2, 1, 1, 0, 0)$ such that the following axioms hold for all $x, y, z \in M$ with an additional binary operation $\odot$ defined via

$$y \odot x = (x^- \oplus y^-)^\sim,$$

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(A2) $x \oplus 0 = 0 \oplus x = x$;

(A3) $x \oplus 1 = 1 \oplus x = 1$;

(A4) $1^\sim = 0$; $1^- = 0$;

(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;

(A6) $x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^-) \oplus y = (y \odot x^-) \oplus x$;

(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$;

(A8) $(x^-)^\sim = x$.

If $\oplus$ is commutative, then a pseudo MV-algebra is an MV-algebra.

Pseudo MV-algebras are intimately connected with unital ℓ-groups. We note that an element u from the positive cone G^+ of an ℓ-group G is said to be a *strong unit* if given $x \in G$, there is an integer $n \geq 0$ such that $x \leq nu$. A couple (G, u), where u is a fixed strong unit of G, is said to be a *unital ℓ-group*.

Whence, if (G, u) is a unital ℓ-group (not necessarily Abelian), we define

$$\Gamma(G, u) := [0, u],$$

and

$$x \oplus y := (x + y) \wedge u, \quad x^- := u - x, \quad x^\sim := -x + u, \quad x \odot y := (x - u + y) \vee 0,$$

then $\Gamma(G, u) := ([0, u]; \oplus, ^-, ^\sim, 0, u)$ is a pseudo MV-algebra [22]. Moreover, according to [7], these interval pseudo MV-algebras are of crucial importance because they are prototypical examples of pseudo MV-algebras, that is, every pseudo MV-algebra is isomorphic to some $\Gamma(G, u)$. Moreover, the category of unital ℓ-groups is categorically equivalent to the category of pseudo MV-algebras. This result generalizes a famous one of Mundici [29] for MV-algebras.

2.2 Pseudo EMV-algebras

Pseudo MV-algebras and MV-algebras have been generalized in different ways. One of them are pseudo EMV-algebras introduced in [14, 15]:

An algebra $(M;\vee,\wedge,\oplus,0)$ of type $(2,2,2,0)$ is called a *pseudo EMV-algebra* (EMV stands for extended MV-algebras) if it satisfies the following conditions:

(E1) $(M;\vee,\wedge,0)$ is a distributive lattice with the least element 0;

(E2) $(M;\oplus,0)$ is an ordered monoid with a neutral element 0;

(E3) for each idempotent $a\in\mathcal{I}(M)$,

$$\lambda_a(x):=\min\{z\in[0,a]\colon z\oplus x=a\},\quad \rho_a(x):=\min\{z\in[0,a]\colon x\oplus z=a\},$$

exist in M for all $x\in[0,a]$, and the algebra $([0,a];\oplus,\lambda_a,\rho_a,0,a)$ is a pseudo MV-algebra;

(E4) for each $x\in M$, there is $a\in\mathcal{I}(M)$ such that $x\le a$.

If $\oplus$ is commutative, then $\lambda_a=\rho_a$ and $(M;\vee,\wedge,\oplus,0)$ is an *EMV-algebra* originally introduced in [12].

In the same way as for pseudo MV-algebras, we can introduce a total binary operation $\odot$ in the following way: For all $x,y\in M$, we define

$$x\odot y=\rho_a(\lambda_a(y)\oplus\lambda_a(x)),$$

where $a\in\mathcal{I}(M)$ and $x,y\in[0,a]$. Then $x\odot y$ is correctly defined and it does not depend on $a\in\mathcal{I}(M)$. It is clear that if $(M;\oplus,^-,^\sim,0,1)$ is a pseudo MV-algebra, then $(M;\vee,\wedge,\oplus,0)$ is a pseudo EMV-algebra with top element. Conversely, if $(M;\vee,\wedge,\oplus,0)$ is a pseudo EMV-algebra with top element 1, then $(M;\oplus,\lambda_1,\rho_1,0,1)$ is a pseudo MV-algebra. This shows that there is a one-to-one correspondence between pseudo MV-algebras and pseudo EMV-algebras with top element.

We note that a top element is not assumed a priori in pseudo EMV-algebras, so a pseudo EMV-algebra is not necessarily a pseudo MV-algebra, but "locally" it has features of pseudo MV-algebras. They generalize generalized Boolean algebras. On the other hand, every pseudo EMV-algebra M without top element can be embedded into a pseudo EMV-algebra N with top element as a maximal and normal ideal of N and every element of N belongs either to the ideal or is a complement of some element from the ideal, [18, Tm 6.4]. This result is generalized in Theorem 2.2 below. We note that not every maximal and normal ideal of a pseudo MV-algebra can serve as an example of a pseudo EMV-algebra, the problem is that it can happen that the set of idempotents is a singleton $\{0\}$. It is necessary to emphasize that the class of pseudo EMV-algebras is not a variety.

2.3 Weak Pseudo EMV-algebras

According to [17], an algebra $(M;\vee,\wedge,\oplus,\ominus,\oslash,0)$ of type (2,2,2,2,2,0) is said to be a *weak pseudo EMV-algebra* (wPEMV-algebra for short) if the following conditions axioms hold:

(W1) $(M,\vee,\wedge,0)$ is a distributive lattice with the least element 0;

(W2) $(M;\oplus,0)$ is a monoid;

(W3) $(y\oplus x)\ominus x\le y$ and $x\oslash(x\oplus y)\le y$;

(W4) $(y\ominus x)\oplus x=x\vee y=x\oplus(x\oslash y)$;

(W5) $x\ominus(x\wedge y)=x\ominus y$ and $(x\wedge y)\oslash y=x\oslash y$;

(W6) $y\ominus(x\oslash y)=x\wedge y=(y\ominus x)\oslash y$;

(W7) $z\ominus(x\vee y)=(z\ominus x)\wedge(z\ominus y)$ and $(x\vee y)\oslash z=(x\oslash y)\wedge(y\oslash z)$;

(W8) $(x\wedge y)\ominus z=(x\ominus z)\wedge(y\ominus z)$ and $z\oslash(x\wedge y)=(z\oslash x)\wedge(z\oslash y)$;

(W9) $x\ominus(y\oplus z)=(x\ominus z)\ominus y$ and $(y\oplus z)\oslash x=z\oslash(y\oslash x)$;

(W10) $x\oplus(y\vee z)=(x\oplus y)\vee(x\oplus z)$ and $(y\vee z)\oplus x=(y\oplus x)\vee(z\oplus x)$.

If the binary operation $\oplus$ is commutative, then M is said to be a *wEMV*-algebra introduced in [16], it is a commutative version of wPEMV-algebra. An equivalent condition is $x \ominus y = y \tilde{\ominus} x$ for all $x, y \in M$.

If a wPEMV-algebra $(M; \vee, \wedge, \oplus, \ominus, \tilde{\ominus}, 0)$ admits a top element 1, then $(M; \oplus, \lambda_1, \rho_1, 0, 1)$ is a pseudo MV-algebra. Conversely, if $(M; \oplus, ^-, ^\sim, 0, 1)$ is a pseudo MV-algebra, then $(M; \vee, \wedge, \oplus, \ominus, \tilde{\ominus}, 0)$, where $x \ominus y = x \odot y^-$, $x \tilde{\ominus} y = x^\sim \odot y$, is a wPEMV-algebra.

We note that wPEMV-algebras can be studied also as integral GMV-algebras, see e.g. [20].

Important classes of wPEMV-algebras are: (1) Every pseudo EMV-algebra generalizes pseudo MV-algebras, and every wPEMV-algebra generalizes pseudo EMV-algebras.

(2) If $(M; \vee, \wedge, \oplus, \ominus, \tilde{\ominus}, 0)$ is a wPEMV-algebra such that its reduct $(M; \vee, \wedge, \oplus, 0)$ is a pseudo EMV-algebra, then the wPEMV-algebra M is said to be an *associated wPEMV-algebra*. For example, let (G, u) be a unital ℓ-group with the pseudo MV-algebra $\Gamma(G, u)$. We denote by $\Gamma_a(G, u)$ the associated wPEMV-algebra corresponding to $\Gamma(G, u)$. It is possible to show that a wPEMV-algebra M is associated iff every element $x \in M$ is dominated by some idempotent $a \in M$, see [17, Thm 4.3].

(3) Let G^+ be the positive cone of an ℓ-group G. Set $x \oplus y = x + y$, $x \ominus y = (x - y) \vee 0$, and $x \tilde{\ominus} y = (-x + y) \vee 0$ for all $x, y \in G^+$. Then $(G^+; \vee, \wedge, \oplus, \ominus, \tilde{\ominus}, 0)$ is a wPEMV-algebra called a *wPEMV-algebra of a positive cone* or a *conic algebra*. It is equivalent to be cancellative and the class of cancellative wPEMV-algebras forms a proper subvariety of the variety of wPEMV-algebras, [18, Thm 8.1]. We note that a wPEMV-algebra M is *cancellative* if $a \oplus b_1 = a \oplus b_2$ and $a_1 \oplus b = a_2 \oplus b$ iff $b_1 = b_1$ and $a_1 = a_2$

The basic properties of wPEMV-algebras are as follows, see [17, Prop 3.2].

Proposition 2.1. *Let $(M; \vee, \wedge, \oplus, \ominus, \tilde{\ominus}, 0)$ be a wPEMV-algebra and $x, y, z \in M$. Then the following hold:*

(i) *$(M; \oplus, 0)$ is an ordered monoid which is right and left naturally ordered (that is, $x \le y$ if and only if there is $u \in M$ such that $x \oplus u = y$, equivalently, there is $v \in M$ such that $v \oplus x = y$).*

(ii) *$(a \ominus x) \tilde{\ominus} a = x = a \ominus (x \tilde{\ominus} a)$ if $x \le a$.*

(iii) *$x \wedge y = ((a \ominus x) \vee (a \ominus y)) \tilde{\ominus} a$ and $x \wedge y = a \ominus ((x \tilde{\ominus} a) \vee (y \tilde{\ominus} a))$ if $x, y \le a$.*

(iv) *$x \le y$ implies that $x \ominus z \le y \ominus z$ and $z \tilde{\ominus} x \le z \tilde{\ominus} y$. Also, $z \ominus y \le z \ominus x$ and $y \tilde{\ominus} z \le x \tilde{\ominus} z$.*

(v) *$z \le x \oplus y$ if and only if $z \ominus y \le x$ if and only if $x \tilde{\ominus} z \le y$.*

(vi) *$(x \wedge y) \oplus z = (x \oplus z) \wedge (y \oplus z)$ and $z \oplus (x \wedge y) = (z \oplus x) \wedge (z \oplus y)$.*

(vii) *$z \ominus (x \wedge y) = (z \ominus x) \vee (z \ominus y)$ and $(x \wedge y) \tilde{\ominus} z = (x \tilde{\ominus} z) \vee (y \tilde{\ominus} z)$.*

(viii) *$x \ominus x = 0 = x \tilde{\ominus} x$ and $x \ominus 0 = x = 0 \tilde{\ominus} x$.*

(ix) *$x \le y$ if and only if $x \ominus y = 0$ if and only if $y \tilde{\ominus} x = 0$.*

(x) *$(x \vee y) \ominus z = (x \ominus z) \vee (y \ominus z)$ and $z \tilde{\ominus} (x \vee y) = (z \tilde{\ominus} x) \vee (z \tilde{\ominus} y)$.*

(xi) *$x \ominus y \le x$ and $x \tilde{\ominus} y \le y$.*

(xii) *$(x \ominus y) \wedge (y \ominus x) = 0 = (x \tilde{\ominus} y) \wedge (y \tilde{\ominus} x)$.*

(xiii) *If $a \oplus a = a$, then $a \oplus x = a \vee x = x \oplus a$.*

(xiv) *The binary operation $\oplus$ is commutative if and only if $x \ominus y = y \tilde{\ominus} x$.*

A non-empty set I of a wPEMV-algebra M is called an *ideal* if I is closed under $\oplus$ and for each $x, y \in M$, $x \le y \in I$ implies $x \in I$. It is possible to show that each ideal I is closed under $\tilde{\ominus}$, $\ominus$ and $\odot$, too. An ideal I of M is (i) *maximal* if I is a proper subset of M and it cannot be a proper subset of any proper ideal of M, and (ii) *normal* if, for each $x, y \in M$, $y \ominus x \in I$ if and only if $x \tilde{\ominus} y \in I$; equivalently, $x \oplus I = I \oplus x$ for each $x \in M$, where $x \oplus I = \{x \oplus i \colon i \in I\}$ and $I \oplus x = \{i \oplus x \colon i \in I\}$. We know that congruences on M are in a one-to-one relationship with normal ideals. So for any normal ideal, we can define the quotient wPEMV-algebra M/I. For more details on wPEMV-algebras, we recommend to consult with [17, 18].

A basic representation of wPEMV-algebras by wPEMV-algebras with top element was established in [18, Thm 7.9]:

Theorem 2.2. [Basic Representation Theorem for wPEMV-algebras]
Every wPEMV-algebra M either has a top element and so it is an associated wPEMV-algebra or it can be embedded into an associated wPEMV-algebra N with top element as a maximal and normal ideal of N. Moreover, every element of N is either in the image of $x \in M$ or is a right complement of the image of some element $x \in M$.

Any wPEMV-algebra N with top element from the latter theorem is said to be *representing* M. We note that all representing wPEMV-algebras of M are isomorphic.

Moreover, we say that a wPEMV-algebra M is *proper* if it does not have a top element. The class of proper wPEMV-algebras is not a variety. By Basic Representation Theorem, every proper wPEMV-algebra M can be embedded into a wPEMV-algebra N as a maximal and normal ideal of N and every element of N is either in the image of some element from M or is a complement of some element from the image of M.

Given an element $x \in M$, we set $0.x = 0$ and $(n+1).x = (n.x) \oplus x$, $n \geq 0$.

Due to the latter theorem, on N we have the associative binary operation $\odot$ that is defined by $x \odot y = (y^- \oplus x^-)^\sim$, $x, y \in N$, where $^-$ and $^\sim$ denotes the left and right complements taken in N. If $x, y \in M$, then $x \odot y \in M$. Hence, for each $x \in N$, and any integer $n \geq 0$, we set

$$x^0 := 1, \quad x^n := x^{n-1} \odot x, \quad \text{if } n \geq 1.$$

3 States and state-morphisms on weak pseudo EMV-algebras

In the section, we define the basic notions of the paper - states and state-morphisms. States are presented as a generalization of finitely additive measures, and state-morphisms are wPEMV-homomorphisms from M into the real interval $[0,1]$ understood as a commutative wPEMV-algebra. It will be shown that the latter ones are exactly the extremal points of the state space of M. In particular, we show that the state space can be empty even in a commutative wPEMV-algebra which in the case of the state space of an MV-algebra is not possible. State-morphisms are also states whose kernel is a special maximal and normal ideal of M.

Let M be a wPEMV-algebra. Given an element $c \in M$, we define two mappings λ_c and ρ_c from M into M such that $\lambda_c(x) = c \ominus x$ and $\rho_c(x) = x \circleddash c$, $x \in M$. We define a partial operation $+$ in M as follows: Given $x, y \in M$, $x + y$ is defined in M iff $(x \oplus y) \ominus y = x$, and in such a case, we define $x + y = x \oplus y$. We note that for example, in G^+, this partial operation coincides with the total operation $\oplus$. The basic properties of this partial addition $+$ are as follows, see [17, Lem 4.4]:

(i) The elements $x + 0$ and $0 + x$ are defined in M and $x + 0 = x = 0 + x$.

(ii) The element $x + y$ is defined in M if and only if $x \circleddash (x \oplus y) = y$, equivalently if $(x \oplus y) \ominus y = x$.

(iii) If $x + y$ is defined and $x_1 \leq x$ and $y_1 \leq y$, then $x_1 + y_1$ is also defined in M.

If M is with top element, then $x + y$ is defined iff $x \leq 1 \ominus y$, equivalently, $y \leq x \circleddash 1$.

For every integer $n \geq 0$, we define $0x := 0$, and for $n \geq 0$, we put $(n+1)x := nx + x$ assuming that the right-hand side exists.

A bounded wPEMV-algebra M is said to be *Archimedean* if whenever nx exists for each integer $n \geq 1$, then $x = 0$.

A mapping $s : M \to [0,1]$ is said to be a *state* if (i) $s(x+y) = s(x) + s(y)$ whenever $x + y$ is defined, (ii) there is an element $z \in M$ such that $s(z) = 1$. Then (i) $s(0) = 0$, (ii) $s(x \ominus y) = s(x) - s(y) = s(y \circleddash x)$ whenever $y \leq x$, so that s is monotone. We note that the property (i) is equivalent to $s(x \ominus y) = s(x) - s(y)$ whenever $y \ominus x = 0$, equivalently, $s(y \circleddash x) = s(x) - s(y)$ whenever $y \circleddash x = 0$. A mapping $s : M \to [0,1]$ is said to be a *pre-state* if $s(x+y) = s(x) + s(y)$ whenever $x + y$ is defined in M. We denote by $\mathcal{S}(M)$ and $\mathcal{S}_p(M)$ the set of states and pre-states, respectively, on M. We note that $\mathcal{S}(M)$ can be empty. Indeed, if $M = G^+$ is any conic algebra of some (even commutative) ℓ-group G, then M has no states and it possesses only a pre-state $s = 0$, i.e. $s(x) = 0$, $x \in M$. On the other hand, if M is with top element $1 \neq 0$, then $\mathcal{S}(M)$ is non-empty whenever $\oplus$ is commutative because in such a case, M is equivalent to an MV-algebra, see [24, Cor 4.4]. In a non-commutative case $\mathcal{I}(M)$ can be empty, see [6, Cor 7.4]. In general, the state space $\mathcal{S}(M)$ can be only one of the following three forms (i) empty, (ii) a singleton, and (iii) uncountable.

If s_1 and s_2 are states on M and λ is a real number from $[0,1]$, then $\lambda s_1 + (1-\lambda)s_2$ is a *convex combination* of s_1 and s_2, and it is evident that it is also a state. A state s is *extremal* if from $s = \lambda s_1 + (1-\lambda)s_2$, where $\lambda \in (0,1)$, we conclude $s = s_1 = s_2$. We denote by $\partial\mathcal{S}(M)$ the set of extremal states on M.

Proposition 3.1. *Let s be a state on a wPEMV-algebra M. For all $x, y \in M$, we have:*

(i) $s(x \vee y) + s(x \wedge y) = s(x) + s(y)$.

(ii) $s(x \oplus y) + s(x \odot y) = s(x) + s(y)$.

(iii) $\mathrm{Ker}(s)$ *is a proper and normal ideal of* M, *where* $\mathrm{Ker}(s) = \{x \in M : s(x) = 0\}$.

(iv) *If we define a mapping* $\hat{s}$ *on the quotient wPEMV-algebra* $M/\mathrm{Ker}(s)$ *by* $\hat{s}(x/\mathrm{Ker}(s)) := s(x)$, $(x \in M)$, *then* $\hat{s}$ *is a state on* $M/\mathrm{Ker}(s)$, *and* $M/\mathrm{Ker}(s)$ *is with top element* $[z]$, *where* z *is any element with* $s(z) = 1$.

(v) $s(x \oplus y) = s(y \oplus x)$, $s(x \ominus y) = s(y \oslash x)$, $x, y \in M$, *and* $M/\mathrm{Ker}(s)$ *is a commutative bounded wPEMV-algebra.*

Proof. (i), (ii). As it was shown just after the Basic Representation Theorem 2.2, for the wPEMV-algebra M, there are a binary operation $\odot$, and a wPEMV-algebra N with top element such that either $M = N$ if M is with top element or M is a maximal and normal ideal of N. In either case, $(N; \oplus, ^-, ^\sim, 0, 1)$ is a pseudo MV-algebra, where $x^- = 1 \ominus x$ and $x^\sim = x \oslash 1$, $x \in N$. In every wPEMV-algebra, we have $(x \vee y) \ominus x = y \ominus x = y \ominus (x \wedge y)$ and $(x \oplus y) \ominus y = x \ominus (x \odot y)$ which yields (i) and (ii).

(iii) To show that $\mathrm{Ker}(s)$ is an ideal it is enough to assume if $x, y \in \mathrm{Ker}(s)$, then $x \oplus y \in \mathrm{Ker}(s)$: Using (W3), we have $s(x \oplus y) = s((x \oplus y) \ominus y) + s(y) \le s(x) + s(y) = 0$. Normality: Let $x \in M$ and $y \in \mathrm{Ker}(s)$. Then $s(x) \le s(x \oplus y) = s(x) + s(y) - s(x \odot y) = s(x)$. Whence $x \oplus y = ((x \oplus y) \ominus x) \oplus x$ and $s((x \oplus y) \ominus x) = s(y) - s(x \odot y) = s(y) = 0$, that is $x \oplus \mathrm{Ker}(s) \subseteq \mathrm{Ker}(s) \oplus x$. In a dual way, we show the converse inclusion.

(iv) Given $x, y \in M$, we have $x/\mathrm{Ker}(s) = y/\mathrm{Ker}(s)$ iff $s(x) = s(x \wedge y) = s(y)$. Therefore, $\hat{s}$ is correctly defined. For each $x \in M$, let $[x] = x/\mathrm{Ker}(s)$. Then $[x] \le [y]$ iff $s(x) = s(x \wedge y)$, so that for $[x] \le [y]$, we have

$$\hat{s}([y] \ominus [x]) = \hat{s}([y \ominus x]) = s(y \ominus (x \wedge y)) = s(y) - s(x \wedge y) = s(y) - s(x) = \hat{s}([y]) - \hat{s}([x]),$$

proving $\hat{s}$ is a pre-state.

Since s is a state, there is an element $z \in M$ such that $s(z) = 1$, and $s(x) \le 1$. Property (i) yields $s(x \wedge z) + s(x \vee z) = s(z) + s(x)$ giving $s(x \wedge z) = s(x)$, so that the element $[z]$ is the top element for $M/\mathrm{Ker}(s)$.

(v) According to (iv), the quotient $M/\mathrm{Ker}(s)$ is a bounded wPEMV-algebra, so it can be viewed equivalently also as a pseudo MV-algebra, and the state $\hat{s}$ can be viewed as a state on the pseudo MV-algebra. Due to (iv), we have $\hat{s}([x]) = 0$ iff $s(x) = 0$. We show that $M/\mathrm{Ker}(s)$ is Archimedean: Suppose that $n[x]$ exists in $M/\mathrm{Ker}(s)$ for each $n \ge 1$. Then $\hat{s}(n[x]) = n\hat{s}([x]) = ns(x) \le 1$, so that $s(x) = \hat{s}(x) = 0$. The Archimedeanicity of the quotient pseudo MV-algebra $M/\mathrm{Ker}(s)$ entails that it is commutative, see [7, Thm 4.2]. Whence, the wPEMV-algebra $M/\mathrm{Ker}(s)$ is commutative, so that $s(x \oplus y) = \hat{s}([x] \oplus [y]) = \hat{s}([y] \oplus [x]) = s(y \oplus x)$. The property $s(x \ominus y) = s(y \oslash x)$ entails the commutativity of the quotient. ☐

Properties (i)–(ii) hold also for pre-states.

Now, we introduce an important notion. Take the real interval $[0, 1]$ and convert it into a (commutative) wPEMV-algebra with the top element 1 putting $\vee = \max$, $\wedge = \min$, $s \oplus t = \min\{s + t, 1\}$ and $s \ominus t = t \oslash s = \max\{s - t, 0\}$. Any wPEMV-homomorphism $s : M \to [0, 1]$ is said to be a *state-morphism* if there is an element $z \in M$ such that $s(z) = 1$. We denote by $\mathcal{SM}(M)$ the set of state-morphisms on M. Then $\mathcal{SM}(M) \subseteq \mathcal{S}(M)$. Any wPEMV-homomorphism $s : M \to [0, 1]$ is said to be a *pre-state-morphism* (we do not require existence of an element $z \in M$ with $s(z) = 1$), and let $\mathcal{SM}_p(M)$ be the set of pre-state-morphisms on M. In Lemma 4.4, we show that $\mathcal{SM}_p(M) = \{o\} \cup \mathcal{SM}(M)$, where o is the zero mapping on M, it is a pre-state-morphism.

Proposition 3.2. *Let* M *be a wPEMV-algebra and let* s *be a state on* M. *The following statements are equivalent:*

(i) *s is a state-morphism.*

(ii) $s(x \wedge y) = \min\{s(x), s(y)\}$, $x, y \in M$.

(iii) $s(x \vee y) = \max\{s(x), s(y)\}$, $x, y \in M$.

(iv) $s(x \oplus y) = \min\{s(x) + s(y), 1\}$, $x, y \in M$.

(v) $s(x \ominus y) = \max\{s(x) - s(y), 0\}$, $x, y \in M$.

(vi) $s(x \oslash y) = \max\{s(y) - s(x), 0\}$, $x, y \in M$.

Proof. Let $z \in M$ with $s(z) = 1$ be given. Without loss of generality, we can assume that $x, y, x \oplus y \le z$. Trivially, (i) implies (ii)–(v).

(ii) $\Leftrightarrow$ (iii). By (W7), we have $z \ominus (x \vee y) = (z \ominus x) \wedge (z \ominus y)$ which entails the equivalence of (ii) and (iii).

The rest of implications (ii)–(v) can be verified as follows. Due to (iv) of Proposition 3.1, we know that $\hat{s}$ defined on the quotient $M/\mathrm{Ker}(s)$ is a state on the commutative wPEMV-algebra $M/\mathrm{Ker}(s)$ with top element $[a] := a/\mathrm{Ker}(s)$.

It is clear that s is a state-morphism iff so is $\hat{s}$ on the quotient $M/\mathrm{Ker}(s)$. On the other hand, $(M/\mathrm{Ker}(s);\oplus,', [0],[a])$ is in fact an MV-algebra with a state $\hat{s}$. For MV-algebras, the conditions (i)–(v) are equivalent. The equivalence of (v) and (vi) follows from (v) of Proposition 3.1. ☐

As a direct corollary of (iv) of the latter proposition we have the following lemma:

Lemma 3.3. *If a is an idempotent element and s is a state-morphism on a wPEMV-algebra M, then $s(a)\in\{0,1\}$.*

Proof. We have $s(a)=s(a\oplus a)=\min\{2s(a),1\}\in\{0,1\}$. Then $s(a)\in\{0,1\}$. ☐

The proofs of the following propositions follow ideas of the analogous results from [9] improved for our non-commutative case of M.

Proposition 3.4. *A state s on a wPEMV-algebra M is a state-morphism if and only if the ideal* $\mathrm{Ker}(s)$ *is maximal.*

Proof. Due to Proposition 3.2, a state s on M is a state-morphism on M iff so is $\hat{s}$ on $M/\mathrm{Ker}(s)$, in addition, $\mathrm{Ker}(s)$ is always a normal ideal of M, see Proposition 3.1(iv).

Assume s is a state-morphism on M. Then $\hat{s}$ is a state-morphism on $M/\mathrm{Ker}(s)$, and since $M/\mathrm{Ker}(s)$ is a commutative wPEMV-algebras with top element, it can be viewed as an MV-algebra and $\hat{s}$ is an MV-state-morphism on $M/\mathrm{Ker}(s)$. Due to [6, Prop 4.3], $\mathrm{Ker}(\hat{s})=\{[0]\}$ is a maximal ideal of $M/\mathrm{Ker}(s)$. Now, let I be any proper ideal of M containing $\mathrm{Ker}(s)$. Then $\phi(I)=\{[x]\colon x\in I\}$ is a proper ideal of $M/\mathrm{Ker}(s)$, so that $\mathrm{Ker}(\hat{s})=\phi(I)$ and hence, $s(x)=0$ for each $x\in I$, giving $I=\mathrm{Ker}(s)$. Hence, $\mathrm{Ker}(s)$ is a maximal ideal of M.

Conversely, let $\mathrm{Ker}(s)$ be a maximal ideal of M. We assert that $\mathrm{Ker}(\hat{s})=\{[0]\}$ is a maximal ideal of $M/\mathrm{Ker}(s)$. Let J be any proper ideal of $M/\mathrm{Ker}(s)$. If we set $\phi^{-1}(J)=\{x\in M\colon [x]\in J\}$, $\phi^{-1}(J)$ is a proper ideal of M containing $\mathrm{Ker}(s)$. Since $\mathrm{Ker}(s)$ is maximal, $\mathrm{Ker}(s)=\phi^{-1}(J)$, so that $J=\mathrm{Ker}(\hat{s})$ and $\mathrm{Ker}(\hat{s})$ is a maximal ideal of $M/\mathrm{Ker}(s)$. Since $M/\mathrm{Ker}(s)$ can be viewed also as an MV-algebra with a state $\hat{s}$, due to [6, Prop 4.3], $\hat{s}$ is a state-morphism, consequently, s is a state-morphism on M. ☐

Proposition 3.5. *If s_1 and s_2 are state-morphisms on a wPEMV-algebra M, then $s_1=s_2$ if and only if* $\mathrm{Ker}(s_1)=\mathrm{Ker}(s_2)$.

Proof. One direction is trivial. Conversely, let $\mathrm{Ker}(s_1)=\mathrm{Ker}(s_2)$. Then $M/\mathrm{Ker}(s_1)=M/\mathrm{Ker}(s_2)$ and let us determine $\hat{s}_1$ and $\hat{s}_2$ on $M/\mathrm{Ker}(s_1)=M/\mathrm{Ker}(s_2)$ by (iv) of Proposition 3.1. Since s_1 and s_2 are state-morphisms, by Proposition 3.4, $\mathrm{Ker}(s_1)$ and $\mathrm{Ker}(s_2)$ are maximal ideals of M. We note that there is $z\in M$ such that $s_1(z)=s_2(z)$. Since $\mathrm{Ker}(\hat{s}_1)=\{[0]\}=\mathrm{Ker}(\hat{s}_2)$ and $\hat{s}_1$ and $\hat{s}_2$ can be viewed also as state-morphisms on the equivalent MV-algebra $M/\mathrm{Ker}(s_1)$, so that by [6, Prop 4.5], we have $\hat{s}_1=\hat{s}_2$ and finally, $s_1=s_2$. ☐

Given a pseudo MV-algebra M we know that, for each maximal and normal ideal I of M, there is a unique state-morphism s on M such that $\mathrm{Ker}(s)=I$, see e.g. [6, Prop 4.6], and the mapping $x\mapsto x/I$ is in fact a state-morphism on M. Such a statement is not true in general for wPEMV-algebras. Indeed, take the conic wPEMV-algebra corresponding to $M:=\mathbb{Z}^+$, then $\{0\}$ is a unique maximal (and normal) ideal of M, but M is stateless.

On the other hand, if s is a state-morphism on a wPEMV-algebra M, then $I=\mathrm{Ker}(s)$ is a maximal and normal ideal of M such that there is an element $a\in M$ with $s(a)=1$ so that $a\notin I$ and a/I is a top element of M/I. Inspired by this property, we establish the following one-to-one correspondence between the state-morphisms s and such a kind of maximal and normal ideals I. Then the mapping $x\mapsto x/I$ can be viewed as a state-morphism s on M such that $\mathrm{Ker}(s)=I$.

Theorem 3.6. *Let I be a maximal and normal ideal of a wPEMV-algebra M such that there is $a\in M\setminus I$ and a/I is a top element of M/I, then there is a unique state-morphism s on M such that* $\mathrm{Ker}(s)=I$.

Moreover, there is a one-to-one correspondence between the set $\mathcal{SM}(M)$ of state-morphisms on M and the set $\mathrm{MaxN}_0(M)$ *of maximal and normal ideals I of M such that there is an element $a\in M\setminus I$ and a/I is a top element of M/I.*

Proof. (1) If a/I is a top element of M/I, then the quotient wPEMV-algebra M/I can be viewed as a pseudo MV-algebra. As in the proof of Proposition 3.4, we can show that $\{0/I\}$ is a maximal and normal ideal of M/I. Using [6, Prop 4.6], we see that the mapping $\mu(x/I)=x/I$, $x\in M$, is in fact a pseudo MV-state-morphism in the equivalent pseudo MV-algebra M/I. Define $s(x):=\mu(x/I)$, $x\in M$. Then s is a state on M such that $\hat{s}(x)=\mu(x/I)=x/I$, that is, s is a state-morphism on M such that $\mathrm{Ker}(s)=I$.

(2) Let I be a maximal and normal ideal of M such that there are $a,b\in M\setminus I$ and a/I and b/I are top elements of M/I. Then clearly $a/I=b/I$. Due to [17, Lem 5.12], M/I is a linearly ordered wPEMV-algebra.

First we show that M/I is simple, i.e. it has only two ideals, $\{0/I\}$ and M/I. Indeed, take a proper ideal J of M/I and define $\{x\colon a/I \in J\}$. It is an ideal of M containing I, and maximality of I yields $J = \{0/I\}$.

Second, we show that M/I is Archimedean. Thus let $n(x/I)$ be defined in M/I for each integer $n \geq 1$. Let $J(x)$ be the ideal of M/I generated by x/I. Then $J(x) = \{z/I\colon z/I \leq n.(x/I), \exists n \geq 1\}$. Since M/I is simple and $a/I \notin J(x)$, we have $x/I = 0/I$.

In other words, the equivalent pseudo MV-algebra M/I is Archimedean, and by [7, Thm 4.2], M/I is an Archimedean MV-algebra and it is a subalgebra of the MV-algebra of the real interval $[0,1]$. Whence, the mapping $s(x) = x/I$, $x \in M$, is a state-morphism not depending on a and b. The mapping $\kappa : \mathcal{SM}(M) \to \mathrm{MaxN}_0(M)$ defined by $\kappa(s) = \mathrm{Ker}(s)$ is the bijective mapping in question. ☐

We note that according to [17], a wPEMV-algebra M is said to be *strict* if $\mathcal{I}(M) = \{0\}$. Due to [17, Thm 5.8], every linearly ordered wPEMV-algebra is either strict or is with top element. Moreover, a linearly ordered M is strict iff M is cancellative. In such a case, M has no state. If it has a top element, then $\mathcal{S}(M)$ is a singleton and $\mathcal{S}(M) = \mathcal{SM}(M)$.

Theorem 3.7. *Let M be a wPEMV-algebra. Then $\mathcal{SM}(M) = \partial\mathcal{S}(M)$.*

Proof. Let s be a state-morphism and let $s = \lambda s_1 + (1-\lambda)s_2$, where $\lambda \in (0,1)$ and s_1 and s_2 are states on M. Then $\mathrm{Ker}(s) = \mathrm{Ker}(s_1) \cap \mathrm{Ker}(s_2)$. Since by Proposition 3.4, $\mathrm{Ker}(s)$ is a maximal and normal ideal, and $\mathrm{Ker}(s_1)$ and $\mathrm{Ker}(s_2)$ are proper ideals of M, from maximality of $\mathrm{Ker}(s)$ we have $\mathrm{Ker}(s) = \mathrm{Ker}(s_1) = \mathrm{Ker}(s_2)$. By Proposition 3.4, we conclude that s_1 and s_2 are state-morphisms which by Proposition 3.5 yields $s_1 = s_2 = s$, that is, s is an extremal state.

Conversely, let s be an extremal state on M. We define $\hat{s}$ on the quotient wPEMV-algebra $M/\mathrm{Ker}(s)$ by $\hat{s}([x]) = s(x)$. We assert that $\hat{s}$ is an extremal state on $M/\mathrm{Ker}(s)$. Indeed, let $\hat{s} = \lambda t_1 + (1-\lambda)t_2$, where $\lambda \in (0,1)$ and $t_1, t_2 \in \mathcal{S}(M/\mathrm{Ker}(s))$. Then mappings $s_i(x) := t_i([x])$ are states on M and $s = \lambda s_1 + (1-\lambda)s_2$ which entails $s_1 = s_2$ and $t_1 = t_2$. We can choose an element $z \in M$ such that $s(z) = s_1(z) = s_2(z) = 1$. Because on the equivalent MV-algebra $(M/\mathrm{Ker}(s); \oplus, ', [0], [z])$ extremal states are state-morphisms and vice versa, see [6, Prop 4.7], we conclude that $\hat{s}$ is a state-morphism on $M/\mathrm{Ker}(s)$, and finally, s is a state-morphism on M. ☐

We show that between the set of states and the set of state-morphisms there is a close connection. The following result gives one of such relations. More will be said in Theorem 5.6 in [10].

Proposition 3.8. *The set $\mathcal{S}(M)$ on a wPEMV-algebra M is non-empty if and only if $\mathcal{SM}(M)$ is non-empty.*

Proof. Since $\mathcal{SM}(M) \subseteq \mathcal{S}(M)$, one direction is clear. Now, suppose M admits at least one state, say s. According to Proposition 3.1(iv), $\hat{s}$ defined on the quotient wPEMV-algebra $M/\mathrm{Ker}(s)$ by $\hat{s}([x]) = s(x)$ $(x \in M)$ is in fact a state on the equivalent MV-algebra $M/\mathrm{Ker}(s)$ and every non-trivial MV-algebra admits a state-morphism, say μ, because contains a maximal ideal and each its maximal ideal does not contain the top element (then apply Theorem 3.6). Then m defined by $m(x) = \mu([x])$, $x \in M$, is a state-morphism on M. ☐

Proposition 3.9. *Every cancellative wPEMV-algebra and every linearly ordered strict wPEMV-algebra is stateless.*

Proof. (1) Due to [17, Thm 5.8](2), every cancellative wPEMV-algebra M is isomorphic to a conic wPEMV-algebra of the positive cone of some ℓ-group G. Therefore, M is stateless and it contains only the zero pre-state.

(2) According to [17, Thm 5.8](3), a linearly ordered wPEMV-algebra M is strict iff it is cancellative. By (1), M is stateless. ☐

In the following result, we show that if M is an associated wPEMV-algebra, Theorem 3.6 can be weakened as follows.

Theorem 3.10. *Let M be an associated wPEMV-algebra. Then the set $\mathcal{SM}(M)$ of state-morphisms on M is in a one-to-one relationship with the set of maximal and normal ideals of M.*

Proof. If M is an associated wPEMV-algebra, then it is in fact equivalent to a pseudo EMV-algebra. Due to [15, Prop 8.6], for each maximal and normal ideal I of M, there is a unique state-morphism s on M such that $\mathrm{Ker}(s) = I$. The final conclusion, that the mapping $s \mapsto \mathrm{Ker}(s)$ from $\mathcal{SM}(M)$ onto the set of maximal and normal ideals of M is bijective, follows from Proposition 3.5. ☐

A topological characterization of the set of state-morphisms and the class $\mathrm{MaxN}_0(M)$ will be given in Theorem 5.6 in [10].

We note that an associated wPEMV-algebra is (i) *representable* if it is a subdirect product of associated linearly ordered wPEMV-algebras, and (ii) *normal-valued* if each its value is normal in its cover (for more details about this notion see [17, Sec 5]). Then every normal-valued associated wPEMV-algebra has the property that each its maximal ideal is normal, see [18, Thm 6.2]. Consequently each linearly ordered associated wPEMV-algebra, every associated representable wPEMV-algebra, and every normal-valued associated wPEMV-algebra admits at least one state. We note that there are linearly ordered wPEMV-algebras which are not associated and that are stateless, e.g. the conic wPEMV-algebras $\mathbb{Z}^+$ or $\mathbb{R}^+$. On the other hand, there are associated wPEMV-algebras that are stateless, e.g. those corresponding to stateless pseudo MV-algebras, see [6, Cor 7.4].

Finally, we characterize maximal and normal ideals of any wPEMV-algebra that are kernel of some state-morphisms.

Proposition 3.11. *A maximal and normal ideal I of a wPEMV-algebra M is a kernel of some state-morphism if and only if M/I is a bounded wPEMV-algebra.*

Proof. Let s be a state-morphism on M. Then $I := \mathrm{Ker}(s)$ is a maximal and normal ideal of M such that M/I is a bounded wPEMV-algebra, see Proposition 3.1(iv).

Conversely, the quotient wPEMV-algebra M/I is linearly ordered, so by [17, Thm 5.8], M/I is either cancellative, i.e. equivalent to a conic wPEMV-algebra or is bounded. In the first case I cannot be a kernel of some state-morphism. In the second case, due to the proof of Theorem 3.6, M/I is Archimedean, so that it is a subalgebra of the real interval $[0, 1]$, and the mapping $s : x \mapsto x/I$, $x \in I$, is the unique state-morphism such that $\mathrm{Ker}(s) = I$. ☐

4 States and state-morphisms on representing wPEMV-algebras

In the section, we show that every state and every state-morphism on a proper wPEMV-algebra M can be extended to ones on the representing wPEMV-algebra N with top element. In addition, N admits a special two-valued state-morphism s_∞ such that on M it vanishes and on the complement of M in N it takes the only value 1. We show that every proper wPEMV-algebra M induces the restriction property, i.e. the restriction of every state-morphism $s \in \mathcal{SM}(N)$ onto M, $s \neq s_\infty$, is a state-morphism on M. We introduce the weak topology of states on M and we show that every state s on M is a weak limit of a net of convex combinations of state-morphisms. We study some topological properties of the weak topology, like local compactness of the space of state-morphisms. In addition, we show that the space of state-morphisms is homeomorphic to a special set of maximal and normal ideals of M endowed with the hull-kernel topology.

For the next motivation, we start with two examples of the sets of state-morphisms.

Example 4.1. *Let M be the set of all finite subsets of the set of positive integers $\mathbb{N}$. It can be converted into a generalized Boolean algebra, where $\oplus = \vee = \cup$ and $\wedge = \cap$. Its representing wPEMV-algebra is the set of subsets of $\mathbb{N}$ that are either finite or co-finite. For each integer $n \geq 1$, let $s_n(A) = \chi_A(n)$, $A \in M$; it is a state-morphism and $\mathcal{SM}(M) = \{s_n \colon n \geq 1\}$. Every state-morphism s on N is of the form $\tilde{s}_n(A) = \chi_A(n)$, $A \in M$, $n \geq 1$, or $s = s_\infty$, where $s_\infty(A) = 0$ if A is finite, otherwise $s_\infty(A) = 1$.*

Example 4.2. *Let M be a conic wPEMV-algebra and N its representing one, then $\mathcal{SM}(M) = \emptyset$ and $\mathcal{SM}(N) = \{s_\infty\}$, where $s_\infty(a) = 0$ if $a \in M$ and $s_\infty(a) = 1$ if $a \in N \setminus M$. Indeed, we have $M \cong G^+$ and $N \cong \Gamma_a(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$.*

We say that a net of states $(s_\alpha)_\alpha$ *converges weakly* to a state s if $s(x) = \lim_\alpha s_\alpha(x)$, $x \in M$. Due to Proposition 3.2, we have that if a net $(s_\alpha)_\alpha$ is a net of state-morphisms, then the weak limit is a pre-state-morphism. The set $\mathcal{S}(M)$ is a convex and Hausdorff topological space that is compact whenever M is with top element. For each $x \in M$ and all u, v real numbers with $u < v$, we define sets $S(x)_{u,v} = \{s \in \mathcal{SM}(M) \colon u < s(x) < v\}$. They are open sets and they form a subbase of the weak topology.

We note that if $s(x) = \lim_\alpha s_\alpha(x)$ for each $x \in M$, then s is always a pre-state but not necessarily a state and $\mathcal{S}(M)$ is not necessarily compact. Indeed, take Example 4.1, then $s(A) = \lim_n s_n(A) = 0$ for each $A \in M$, but s is not a state, so that $\mathcal{S}(M)$ is not compact. In what follows, we will assume that M is a subset of its representing wPEMV-algebra N, otherwise, we have to use the embedding of M into N.

Motivated by the above two examples, we show that every state (state-morphism) s on a wPEMV-algebra M can be extended to a unique state (state-morphism) on the representing wPEMV-algebra N. For simplicity, we assume $M \subseteq N$.

Proposition 4.3. *Let M be a wPEMV-algebra. Given a state (state-morphism) s on M, the mapping $\tilde{s} : N \to [0, 1]$, defined by*

$$\tilde{s}(x) = \begin{cases} s(x) & \text{if } x \in M, \\ 1 - s(x_0) & \text{if } x = 1 \ominus x_0,\ x_0 \in M, \end{cases} \qquad x \in N, \tag{1}$$

where 1 is a top element of N, is a state (state-morphism) on N, and the mapping $s_\infty : N \to [0,1]$ defined by

$$s_\infty(x) = \begin{cases} 0 & \text{if } x \in M, \\ 1 & \text{if } x \in N \setminus M, \end{cases} \quad x \in N, \tag{2}$$

is a state-morphism on N.

Let s be a state on M. A net $(s_\alpha)_\alpha$ of states on M converges weakly to s on M if and only if $(\tilde{s}_\alpha)_\alpha$ converges weakly to the state $\tilde{s}$ on N, and the mapping $\phi : \mathcal{S}(M) \to \mathcal{S}(N)$ defined by $\phi(s) = \tilde{s}$, $s \in \mathcal{S}(M)$, is injective, continuous and affine.

In addition, if M has no top element and M is an associated wPEMV-algebra, then

$$\mathcal{SM}(N) = \{\tilde{s} \colon s \in \mathcal{S}(M)\} \cup \{s_\infty\}. \tag{3}$$

Proof. If M is with top element, the statement is trivial. So assume that M has no top element. Then for each $z \in N$, there is a unique $x_0 \in M$ such that $x = 1 \ominus x_0$. Similarly, there is a unique $x_1 \in M$ such that $x = x_1 \circleddash 1$, where 1 is the top element of N. Given $x \in M$, we write $x^- = 1 \ominus x$ and $x^\sim = x \circleddash 1$. Let $x_0, y_0 \in M$.

Clearly, $\tilde{s}(1) = 0$. Suppose $x + y$ is defined in N, $x, y \in N$. There are four cases.

(1) $x, y \in M$. Then $\tilde{s}(x+y) = s(x) + s(y) = \tilde{s}(x) + \tilde{s}(y)$.

(2) $x = 1 \ominus x_0$, $y = y_0$. Then $y_0 \le x_0$, so that

$$\tilde{s}(x+y) = s((1 \ominus x_0) \oplus y_0) = 1 - s(y_0^\sim \odot x_0) = 1 - s(y_0 \circleddash x_0) = 1 + s(y_0) - s(x_0) = \tilde{s}(x) + \tilde{s}(y).$$

(3) $x = x_0$, $y = 1 \ominus y_0$. Then there is a unique $y_1 \in M$ such that $y = y_1 \circleddash 1 = y_1^\sim$. Therefore, $x \le y_1$. Since N is equivalent to a pseudo MV-algebra, there is a unital ℓ-group (G, u) such that $N \cong \Gamma_a(G, u)$. Without loss of generality, we can assume $M \subset N = [0, u]$ and $1 = u$ and therefore, we can count as in the ℓ-group G. Whence,

$$\begin{aligned} \tilde{s}(x+y) &= \tilde{s}(x_0 + y_1^\sim) = \tilde{s}(x_0 - y_1 + 1) = \tilde{s}(-(y_1 - x_0) + 1) = \tilde{s}((y_1 - x_0)^\sim) \\ &= 1 - s(y_1) + s(x_0) = \tilde{s}(x) + \tilde{s}(y). \end{aligned}$$

(4) $x = 1 \ominus x_0$ and $y = 1 \ominus y_0$. This case cannot happen.

Summarizing (1)–(4), we see that $\tilde{s}$ is a state on N.

It is easy to see that s_∞ is a two-valued state-morphism on N. Now, assume that s is a state-morphism on M, we show that $\tilde{s}$ is a state-morphism on N. Therefore, we verify condition (iii) of Proposition 3.2. There are three cases:

(1) $x = x_0$ and $y = y_0$. Then trivially the condition holds.

(2) $x = x_0$ and $y = 1 \ominus y_0$. There is an element $z \in M$ such that $s(z) = 1$ and $x_0, y_0 \le z$. There is $y_1 \in M$ such that $y = y_1^\sim$. Then $x \vee y = x_0 \vee y_1^\sim = (x_0^- \wedge y_0)^\sim$. Since N can be viewed as a pseudo MV-algebra, applying [22, Prop 1.17], we have

$$\begin{aligned} (z \ominus x_0) \wedge y_1 &\le (1 \ominus x_0) \wedge y_1 = ((1 \ominus z) \oplus (z \ominus x_0)) \wedge y_1 \le ((1 \ominus z) \wedge y_1) \oplus ((z \ominus x_0) \wedge y_1 \\ &= ((1 \ominus z) \wedge y_1) + ((z \ominus x_0) \wedge y_1) \in M. \end{aligned}$$

Hence, $\tilde{s}((1 \ominus x_0) \wedge y_1) = \tilde{s}((z \ominus x_0) \wedge y_1) = s((z \ominus x_0) \wedge y_1) = \min\{s(z \ominus x_0), s(y_1)\} = \min\{\tilde{s}(1 \ominus x_0), \tilde{s}(y_1)\}$ which implies easily $\tilde{s}(x \vee y) = \max\{\tilde{s}(x), \tilde{s}(y)\}$.

(3) $x = 1 \ominus x_0$ and $y = 1 \ominus y_0$. This gives

$$\begin{aligned} \tilde{s}(x \vee y) &= \tilde{s}((1 \ominus x_0) \vee (1 \ominus y_0)) = \tilde{s}(1 - (x_0 \wedge y_0)) \\ &= 1 - s(x_0 \wedge y_0) = 1 - \min\{s(x_0), s(y_0)\} \\ &= \max\{1 - s(x_0), 1 - s(y_0)\} = \max\{\tilde{s}(x), \tilde{s}(y)\}. \end{aligned}$$

The statements on the weak convergence are straightforward.

Now, let M be an associated wPEMV-algebra without top element. If s is a state-morphism on N such that $s \ne s_\infty$, we assert that the restriction s_0 of s onto M is a state-morphism on M. Of course, s_0 is additive, i.e. $s_0(x+y) = s_0(x) + s_0(y)$ and $s_0(x \wedge y) = \min\{s_0(x), s(y)\}$, $x, y \in M$. Since M is associated, for each $x \in M$ there is $a \in \mathcal{I}(M)$ such that $x \le a$. According to Lemma 3.3, $s_0(a) = s(a) \in \{0,1\}$. If $s(a) = 0$ for each idempotent $a \in M$, then $s(x) = 0$ for each $x \in M$ and $s = s_\infty$ which is a contradiction. Therefore, there is $a \in M$ such that $s_0(a) = 1$. It is clear that $\tilde{s}_0 = s$ on N which proves (3). ☐

We underline that the weak topology can be defined in the same way also for pre-states, and in such a case, $\mathcal{S}_p(M)$ and $\mathcal{SM}_p(M)$ are always compact sets. It is possible to show that every pre-state (pre-state-morphism) s on M is the restriction of a unique state (state-morphism) on the representing wPEMV-algebra N. Moreover, the spaces $\mathcal{S}_p(M)$ and $\mathcal{S}(N)$ are affinely homeomorphic and $\mathcal{SM}_p(M)$ and $\mathcal{SM}(N)$ are also homeomorphic. Homeomorphisms are given by $s \mapsto \tilde{s}$, $s \in \mathcal{S}_p(M)$, and $s \mapsto \tilde{s}$, $s \in \mathcal{SM}_p(M)$, respectively.

We note that if s is a state-morphism on N, then we cannot guarantee automatically that its restriction onto M is also a state-morphism. What we know, in general, is that it is a pre-state-morphism. Therefore, motivated by [9], we say that a wPEMV-algebra M induces the *restriction property* if the restriction of each state-morphism s of the representing wPEMV-algebra N, $s \neq s_\infty$ (s_∞ defined by (2)), onto M is a state-morphism on M. According to (3), if M is an associated wPEMV-algebra, then it induces the restriction property. The same is true if M is a conic wPEMV-algebra described by the positive cone G^+ of any ℓ-group. This follows from the fact for its representing wPEMV-algebra N, we have $\mathcal{SM}(N) = \{s_\infty\}$. If M is with top element, then it trivially induces the restriction property. We note that M induces the restriction property if and only if $\mathcal{SM}(N) = \{\tilde{s} \colon s \in \mathcal{S}(M)\} \cup \{s_\infty\}$.

If M induces the restriction property, then it does not mean that the restriction of any state on N is a state on M. Indeed, let M be a wPEMV-algebra without top element and let it induce the restriction property. Assume that $\mathcal{S}(M)$ is non-void. Take a state-morphism $s \neq s_\infty$ on N and let $\lambda \in (0,1)$ be any real number. Then the restriction of each state $s_\lambda = \lambda s + (1-\lambda)s_\infty$, $0 < \lambda < 1$, onto M is not a state on M because $s_\lambda(x) = \lambda s(x) < s(x) \leq 1$. In addition, using this trick, we can show that the restriction of any state from $\mathcal{S}(N)$ onto M is a state on M iff M possesses a top element.

In the following simple but very useful lemma, we show that every proper wPEMV-algebra induces the restriction property.

Lemma 4.4. *Every proper wPEMV-algebra induces the restriction property. In addition,* (3) *holds.*

Moreover, if M is any wPEMV-algebra, then

$$\mathcal{SM}_p(M) = \{o\} \cup \mathcal{SM}(M), \tag{4}$$

where o is the zero mapping on M

Proof. Thus let M be a proper wEMV-algebra. Then s_∞ is a two-valued state-morphism on N. Take a state-morphism s on N different from s_∞. If, for each $x \in M$, $s(x) = 0$, then $s(1 \ominus x) = 1$, so that $s = s_\infty$, a contradiction. Therefore, there is an element $x_0 \in M$ such that $s(x_0) > 0$. There is also an integer $n \geq 1$ such that $1 = n.s(x_0) = s(n.x_0)$ proving the restriction of s onto M is a state-morphism on M, alias, M induces the restriction property.

In particular, (3) holds.

Now, let M be an arbitrary wPEMV-algebra and let s be any non-zero pre-state-morphism. Therefore, there is an element $x_0 \in M$ such that $s(x_0) > 0$. We can find an integer $n \geq 1$ such that $1 = n.s(x_0) = s(n.x_0)$, so that s is a state-morphism on M. Whence, we have (4). ☐

The following notion will be used in the next theorem. We remind that a topological space Ω is *locally compact* if every point of Ω has a compact neighborhood.

Theorem 4.5. *Let M be a wPEMV-algebra. Given $x \in M$, we define $S(x) = \{s \in \mathcal{SM}(M) \colon s(x) > 0\}$. If $a \in \mathcal{I}(M)$, then $S(a)$ is both an open and closed set in the weak topology of state-morphisms.*

Moreover, if a wPEMV-algebra M is associated, then $\mathcal{SM}(M)$ is either an empty set or is a locally compact non-empty Hausdorff space in the weak topology of state-morphisms such that $S(a)$ is a compact set for each $a \in \mathcal{I}(M)$.

Proof. If M is with top element, it is equivalent to a pseudo MV-algebra so that the statements hold. Thus, let M be without top element. It is clear that every $S(x)$ is an open set. If $x = a \in \mathcal{I}(M)$, then by Lemma 3.3, $s(a) \in \{0,1\}$ for each state-morphism s and $S(a) = \{s \in \mathcal{SM}(M) \colon s(a) > 0\} = \{s \in \mathcal{SM}(M) \colon s(a) = 1\}$, so $S(a)$ is both an open and closed set.

Now, let M be an associated wPEMV-algebra. Assume that N is a wPEMV-algebra with top element representing M. Given $y \in N$, let $S_N(y) = \{s \in \mathcal{SM}(N) \colon s(y) > 0\}$, it is an open set in the weak topology on $\mathcal{SM}(N)$.

Define a mapping $\phi : \mathcal{SM}(M) \to \mathcal{SM}(N)$ by $\phi(s) = \tilde{s}$, $s \in \mathcal{SM}(M)$, where $\tilde{s}$ is defined by (1). Then ϕ is an injective mapping and the restriction property yields $\phi(S(x)) = S_N(x)$ for each $x \in M$. Take an idempotent element $a \in \mathcal{I}(N)$. Then $S_N(a) = \{s \in \mathcal{SM}(N) \colon s(a) > 0\}$ is open and it is compact because $\mathcal{SM}(N)$ is compact.

For each $x \in M$, $y \in N$, and u, v real numbers with $u < v$, the sets $S(x)_{u,v} = \{s \in \mathcal{SM}(M) \colon u < s(x) < v\}$ and $S_N(y)_{u,v} = \{s \in \mathcal{SM}(N) \colon u < s(y) < v\}$ are open and they form a subbase of the weak topologies. Then $\phi(S(x)_{u,v}) = S_N(x)_{u,v}$ if $u \geq 0$, $\phi(S(x)_{u,v}) = S_N(x)_{u,v} \setminus \{s_\infty\}$ if $u < 0$, and $\phi(S(x)) = S_N(x)$ whenever $x \in M$.

To show that $S(a)$ is a compact set in $\mathcal{SM}(M)$ for each $a \in \mathcal{I}(M)$, we choose an open cover of $S(a)$ in the form $\{S(x_\alpha)_{u_\alpha,v_\alpha} \colon \alpha \in A\}$, where $x_\alpha \in M$ and u_α, v_α are real numbers such that $u_\alpha < v_\alpha$ for each $\alpha \in A$. Then

$$\begin{aligned} S(a) &\subseteq \bigcup_\alpha S(x_\alpha)_{u_\alpha,v_\alpha}, \\ \phi(S(a)) &\subseteq \bigcup_\alpha \phi(S(x_\alpha)_{u_\alpha,v_\alpha}), \\ S_N(a) &\subseteq \bigcup_\alpha \phi(S(x_\alpha)_{u_\alpha,v_\alpha}). \end{aligned}$$

Due to compactness of $S_N(a)$, there is a finite subset F of A such that $S_N(a) \subseteq \bigcup\{\phi(S(x_\alpha)_{u_\alpha,v_\alpha}) \colon \alpha \in F\}$, whence, $S(a) \subseteq \bigcup\{S(x_\alpha)_{u_\alpha,v_\alpha} \colon \alpha \in F\}$. Since the system of all open sets $S(x)_{u,v}$ forms a subbase of the weak topology of $\mathcal{SM}(M)$, by [26, Thm 5.6], $S(a)$ is compact and clopen as well. In addition, given a state-morphism $s \in \mathcal{SM}(M)$, there is an element $x \in M$ with $s(x) = 1$. Because M is associated, there is an idempotent element $a \in M$ such that $x \le a$ which entails $s \in S(x) \subseteq S(a)$. Whence, $\mathcal{SM}(M)$ is locally compact. □

The latter theorem will be generalized in Theorem 5.7 and Theorem 5.8 in [10].

We note that we do not know whether $\mathcal{SM}(M)$ is also locally compact for all wPEMV-algebras or even for every wPEMV-algebra without top element.

We know that if a topological space is locally compact, there is a one-point compactification in the sense of the Alexandroff theorem, see [26, Thm 4.21]. Repeating ideas from [15, Thm 8.17], we can establish the following result.

Theorem 4.6. *Let M be a proper associated wPEMV-algebra and N be its representing wPEMV-algebra. If $\mathcal{SM}(M)$ is non-empty, then $\mathcal{SM}(N)$ is the one-point compactification of the space $\mathcal{SM}(M)$.*

Now we exhibit the compactness of the weak topology of states and state-morphisms, respectively.

Theorem 4.7. *Let M be an associated wPEMV-algebra such that every maximal ideal is normal. The following statements are equivalent.*

(i) *M has a top element.*

(ii) *The space $\mathcal{S}(M)$ is compact.*

(iii) *The space $\mathcal{SM}(M)$ is compact.*

Proof. If M is a singleton, then 0 is the top element and $\mathcal{S}(M) = \emptyset = \mathcal{SM}(M)$ and the theorem holds.

Now, let $M \ne \{0\}$. Since M is associated, by [14, Thm 4.17], M has at least one maximal ideal, and hypothesis on M entails M has at least one maximal and normal ideal. By Theorem 3.10, M has at least one state-morphism.

It is simple to verify that (i) ⇒ (ii) and (ii) ⇒ (iii).

(iii) ⇒ (i). Given $x \in M$, let $S(x) = \{s \in \mathcal{SM}(M) \colon s(x) > 0\}$. Then each $S(x)$ is an open set of $\mathcal{SM}(M)$. Given $s \in \mathcal{SM}(M)$, there is an idempotent element $a \in M$ such that $s(a) = 1$, so that $s \in S(a)$ which means that $\{S(a) \colon a \in \mathcal{I}(M)\}$ is an open cover of $\mathcal{SM}(M)$. The compactness of $\mathcal{SM}(M)$ entails there are elements $a_1, \ldots, a_n \in M$ such that $\mathcal{SM}(M) = \bigcup_{i=1}^n S(a_i) = S(a_0)$, where $a_0 = a_1 \vee \cdots \vee a_n > 0$.

Define the ideal $I_0(a_0)$ generated by a_0, then $I_0(a_0) = [0, a_0]$ and it is a normal ideal. We assert a_0 is a top element of M. Suppose the converse. Then $[0, a_0]$ is a proper ideal of M, so it can be embedded into a maximal ideal I of M, see [14, Prop 5.4-5.5]. Due to the assumptions, I is normal, and in view of Theorem 3.10, there is a unique state-morphism s on M such that $I = \mathrm{Ker}(s) \supseteq [0, a_0]$. Then we obtain a contradiction $s(a_0) = 1$ and $s(a_0) = 0$. Whence, a_0 is a top element of M. □

We note that if M is not associated, then compactness of $\mathcal{SM}(M)$ or of $\mathcal{S}(M)$ does not necessary entails that M is with top element. This follows e.g. from Example 5.2 from the second part [10]

From the latter theorem and Theorem 4.5, we conclude:

Corollary 4.8. *If M is an associated wPEMV-algebra M without top element such that every its maximal ideal is normal, then $\mathcal{SM}(M)$ is locally compact but not compact.*

Now, we present a result connected with Theorem 4.7.

Theorem 4.9. *Let M be a wPEMV-algebra such that each maximal ideal I of M is normal, M/I is bounded, and every proper ideal is a subset of some maximal ideal. If $\mathcal{SM}(M)$ is compact, then there is an element $x \in M \setminus \{0\}$ such that, given $s \in \mathcal{SM}(M)$, there is an integer $n \ge 1$ such that $s(n.x) = 1$.*

Proof. Given $x \in M \setminus \{0\}$, let $S(x) = \{s \in \mathcal{SM}(M) \colon s(x) > 0\}$. Then $S(x)$ is an open set in the weak topology of state-morphisms. Then $\{S(x) \colon x \in M\}$ is an open cover of $\mathcal{SM}(M)$ and compactness of $\mathcal{SM}(M)$ yields that there are finitely many elements $x_1, \ldots, x_n > 0$ such that $\mathcal{SM}(M) = \bigcup_{i=1}^n S(x_i)$. If we set $x_0 = x_1 \vee \cdots \vee x_n$, then $\mathcal{SM}(M) = S(x_0)$. Let $I_0(x_0)$ be the ideal of M generated by the element x_0. We assert that $I_0(x_0)$ is a proper ideal of M. Suppose the converse, then there is a maximal ideal I of M containing $I_0(x_0)$, and moreover, I is a normal ideal of M. By Theorem 3.6, there is a unique state-morphism s_0 such that $\mathrm{Ker}(s_0) = I$. Then we have $s_0(x_0) > 0$ as well as $s_0(x_0) = 0$, a contradiction.

Therefore, $I_0(x_0) = M$, $I_0(x_0) = \{z \in M \colon z \le n.x_0, \exists n \ge 1\}$. Then, for each $s \in \mathcal{SM}(M)$, there is an integer $n \ge 1$ such that $s(n.x_0) = 1$. The element $x = x_0$ is that in question. $\square$

We have to underline, that from the latter theorem it does not follow that M has necessarily top element, see e.g. Example 5.2 in the second part [10].

Conclusion

The paper is divided into two parts.

Part I: States are defined as finitely additive mappings with values in the real interval $[0, 1]$ attaining the value 1. We have shown that even non-trivial commutative wPEMV-algebras can be stateless which in the case of MV-algebras or EMV-algebras is impossible. Characterization of state-morphisms as extremal states was established in Theorem 3.7 and they are in a one-to-one correspondence with maximal and normal ideals with a special property, see Theorem 3.6.

Part II. It will continue in [10]. We will study the weak topology of states, an analogue of the Krein–Mil'man theorem for states will be present, and an integral representation of states will be studied.

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