Iranian Journal of Fuzzy Systems

Volume 19, Number 4, (2022), pp. 147-163

Original paper

Complex fuzzy sets with applications in decision-making

M. Zeeshan[1] and M. Khan[2]

[1]Department of Mathematics, COMSATS University Islamabad, Islamabad Campus, Pakistan
[2]Department of Mathematics, COMSATS University Islamabad, Abbottabad Campus, Pakistan

zeeshan.msc08@gmail.com, madadmath@yahoo.com

Abstract

In this paper, we discussed the conjunctive normal form, disjunctive normal form, duality principle, equality of two sets and a semi Boolean algebra of complex fuzzy sets (CFSs). We established some basic results and particular examples with respect to standard complex fuzzy intersection, standard complex fuzzy union and standard complex fuzzy complement functions with the same function for determining the phase term. We used CFSs in signals and systems because the behavior of CFSs is similar to Fourier transforms in certain cases. Moreover, we developed a new algorithm using a Cartesian product of complex fuzzy sets for applications in signals and systems by which we identified a reference signal out of the large number of signals detected by a digital receiver.

Keywords: Complex fuzzy sets, duality principle, semi-Boolean algebra, signals and systems.

1 Introduction

Models representing real life phenomenon with only choices of truth and falsehood are insufficient to represent the actual reality of the problems. The reason for this, is that there are several complexities exist in the models, that is why a system is needed to be develop for handling such complex situations of the models. There are now two ways of dealing with such cases, one is to find numerical solutions to the problems and the other is to construct a numerical model. In both cases, we get numerical solutions to the problems. The second one is about the fuzzy set theory, which includes the theory of fuzzy soft sets, theory of probability, theory of intuitive fuzzy sets, and theory of neutrosophic sets. The later approach to complex problems is more generalized.

Zadeh [30] gave the concept of a fuzzy set (FS) in 1965, which is similar to a probability function. A fuzzy set plays a crucial role in models of real-world problems in various branches of science. A fuzzy set has many applications in operational research, psychology, medicine, decision making, engineering design, thermodynamics, quantum physics, biological classification, image processing, economics, and mathematical chemistry. Torra [25] developed the concept of a hesitant fuzzy set. The concept of bipolar soft sets was developed by Mahmood [18]. Certain operational laws based on type-2 fuzzy sets were used by Karnik and Mendel [15].

Atanassov [3] proposed the concept of intuitionistic fuzzy sets (IFSs) which explicitly characterized the hesitancy. IFSs assign a membership grade and a non-membership grade to the objects which are taken from the set of attributes with a rule that is the sum duplet is limited to the unit interval. Atanassov [4] gave the concept of the interval-valued IFS and discussed their applications. Huang et, al., [14] discussed the complete ranking method for interval-valued IFSs. The interval-valued intuitionistic fuzzy parametrized interval-valued intuitionistic fuzzy soft sets were proposed by Aydin and Enginoglu [5]. Xue et, al., [27] utilized the measure-based belief function by using the IFSs.

The IFSs have been broadly applied in different areas of real-life phenomena but the principle of IFS has limited applications due to its structure. To overcome this deficiency, Yager proposed the concept of Pythagorean fuzzy set (PFS) in [28]. Garg [12] introduced the interval-valued PFS and discussed their applications. Similarity measures for

Corresponding Author: M. Zeeshan
Received: March 2021; Revised: April 2022; Accepted: May 2022.
https://doi.org/10.22111/IJFS.2022.7093

PFSs were studied by Pan et, al., in [20]. Rani et, al., [22] developed the weighted discrimination based approximation approach using PFSs. Chen (2021) developed the likelihood-based optimization based on PFSs.

Fuzzy sets, intuitionist fuzzy sets and Pythagorean fuzzy sets do not handle imprecise, inconsistent, and incomplete knowledge of periodic structure. These theories apply to different fields of research but there is one significant deficiency in both sets, that is, a lack of ability to solve two-dimensional phenomena. Ramot et, al., [21] gave the notion of a complex fuzzy set (CFS). A CFS is the extension of a fuzzy set, the range of which extends from a closed interval $[0, 1]$ to a circle of radius one in a complex plane. The membership function of CFS C is denoted as $\lambda_C(u)$ and the image value of $u \in U$ is in unit disc. Thus all values of $\lambda_C(u)$ exists inside a circle of radius one in complex plane and defined as $\lambda_C(u) = a_C(u)e^{ip_C(u)}$; where $i = \sqrt{-1}$. The term $p_C(u)$ is said to be phase term, $a_C(u)$ is said to be amplitude term and both of these are real-valued with $a_C(u) \in [0, 1]$. The CFS C is represented as $\{(u, \lambda_C(u))|u \in U\}$.

In defining the functionality of the complex fuzzy set model, the phase term for CFS plays a crucial role. This term identifies a CFS model from all other models in the literature available. The ability of a complex fuzzy set to represent two-dimensional phenomena make it superior for dealing ambiguous and intuitive information that is prevalent in time-periodic phenomena. Complex fuzzy sets, their classes, and logic play a significant role in applications such as periodic event prediction and advanced control systems. The complex fuzzy set is very similar to the Fourier transform, in fact, it is the particular form of the Fourier transformation by restricting the range of the Fourier transformation to a unit complex disc. In different fields such as signals and networks, communication, physics, geology, optics, etc, Fourier transform has a lot of applications. Dick suggested that using it to represent relatively periodic behavioral patterns is one of the beneficial applications of complex fuzzy sets [11]. Ma et, al., [17] suggested the model based on complex fuzzy sets to identify the reference signal out of large number of signals detected by a digital receiver..

Ngan et, al., [19] discussed the complex fuzzy set forms of t-norms and t-conorms and discussed their properties. They have applied the complex t-norm and t-conorm to multi-criteria decision-making in the context of medicine-related problems using medical data sets. Tuan et, al., [26] proposed a new mamdani complex fuzzy inference system with rule reduction using complex fuzzy measures in granular computing. The fast adaptive neuro-complex fuzzy inference system, which is designed for fast training of a compact, accurate forecasting model was proposed by Yazdanbakhsh and Dick in [29]. Liu [16] presented an adaptive neuro-complex-fuzzy-inferential modeling mechanism (ANCFIMM) for generating higher-order Takagi-Sugeno-Kang (TSK) models, and exploit its split complex-valued gradient descent algorithm (SCGDA). Selvachandran et, al., [24] discussed the Mamdani complex fuzzy inference system (Mamdani CFIS) to improve the performance of the classical FIS and complex FIS. Bi et, al., [6] defined two kinds of entropy measures for complex fuzzy sets, called type-A and type-B entropy measures, and analyzed their rotational invariance properties. Moreover, the distance measures between interval-valued complex fuzzy sets (IVCFSs) were discussed by Dai et, al., in [10].

Ramot initially introduced the set-theoretical operations on a CFS, such as intersection, union, complement, rotation, and reflection in [21]. De Morgan Laws for a CFS and CF relation were also introduced in [21]. Dai studied new types of rotational invariance for complex fuzzy operations in [8]. Moreover, he introduced the complex fuzzy ordered weighted distance measure [9].

The paper is organized as follows. A formal definition, set theoretic operations and laws of the complex fuzzy sets are provided in Section 2. Some particular examples of these novel concepts are also presented in this section. The basic results of complex fuzzy sets are discussed in Section 3. Section 4 is concerned with applications of complex fuzzy sets. In Section 5, the comparison analysis is provided. In Section 6, we discussed the conclusion of this manuscript.

2 Preliminaries

In this section, we will discuss some basic concepts of CFS and, also discuss particular examples.

Definition 2.1. [21] *A CFS S, defined on a universe of discourse U, is characterized by a grade value $Z_C(x)$ that assigns any element $x \in U$ a complex-valued grade of membership in S. The values $Z_C(x)$ all lie within the unit circle in the complex plane and thus all of the form $\epsilon_C(x)e^{i\vartheta_C(x)}$ where $\epsilon_C(x)$ and $\vartheta_C(x)$ are both real-valued and $\epsilon_C(x) \in [0, 1]$. Here, $\epsilon_C(x)$ is termed as amplitude term and $\vartheta_C(x)$ is termed as phase term. The complex fuzzy set may be represented in the set form as*

$$C = \{(x; Z_C(x)) : x \in U\}.$$

Definition 2.2. [31] *Let C_m, $m = 1, 2, 3, ..., M$ be M CFS defined on U and $Z_{C_m}(x) = \epsilon_{C_m}(x)e^{i\vartheta_{C_m}(x)}$ their membership functions, respectively. The complex fuzzy Cartesian product of C_m denoted $C_1 \times C_2 \times C_3 \times ... \times C_m$, is specified by a function*

$$\begin{aligned} Z_{C_1\times C_2\times C_3\times \ldots \times C_m}(x) &= \epsilon_{C_1\times C_2\times C_3\times \ldots \times C_m}(x)e^{i\vartheta_{C_1\times C_2\times C_3\times \ldots \times C_m}(x)} \\ &= \min(\epsilon_{C_1}(x_1), \epsilon_{C_2}(x_2), ..., \epsilon_{C_m}(x_m))e^{i\min(\vartheta_{C_1}(x_1),\vartheta_{C_2}(x_2),\ldots,\vartheta_{C_m}(x_m))}. \end{aligned}$$

Example 2.3. *Let*

$$C_1 = \frac{0.8e^{i\pi}}{a} + \frac{0.1e^{i\frac{\pi}{2}}}{b} + \frac{0.6e^{i\frac{3\pi}{2}}}{c},$$
$$C_2 = \frac{0.3e^{i\frac{\pi}{4}}}{a} + \frac{0.7e^{i2\pi}}{b} + \frac{0.9e^{i\frac{\pi}{5}}}{c},$$

then

$$C_1\times C_2 = \frac{0.3e^{i\frac{\pi}{4}}}{(a,a)} + \frac{0.7e^{i\pi}}{(a,b)} + \frac{0.8e^{i\frac{\pi}{5}}}{(a,c)} + \frac{0.1e^{i\frac{\pi}{4}}}{(b,a)} + \frac{0.1e^{i\frac{\pi}{2}}}{(b,b)} + \frac{0.1e^{i\frac{\pi}{5}}}{(b,c)} + \frac{0.3e^{i\frac{\pi}{4}}}{(c,a)} + \frac{0.6e^{i\frac{3\pi}{2}}}{(c,b)} + \frac{0.6e^{i\frac{\pi}{5}}}{(c,c)}.$$

Definition 2.4. [21] *Let C_1 and C_2 be two complex fuzzy sets on U, and $Z_{C_1}(x) = \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)}$ and $Z_{C_2}(x) = \epsilon_{C_2}(x)e^{i\vartheta_{C_2}(x)}$ their grade values, respectively. The complex fuzzy intersection of C_1 and C_2, denoted by $C_1\cap C_2$, is specified by a function*

$$C_1\cap C_2 = \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} \cap \epsilon_{C_2}(x)e^{i\vartheta_{C_2}(x)} = \min\left[\epsilon_{C_1}(x), \epsilon_{C_2}(x)\right] e^{i\min\left[\vartheta_{C_1}(x),\vartheta_{C_2}(x)\right]}.$$

Definition 2.5. [21] *Let C_1 and C_2 be two complex fuzzy sets on U, and $Z_{C_1}(x) = \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)}$ and $Z_{C_2}(x) = \epsilon_{C_2}(x)e^{i\vartheta_{C_2}(x)}$ their grade values, respectively. The complex fuzzy union of C_1 and C_2, denoted by $C_1\cup C_2$, is specified by a function*

$$C_1\cup C_2 = \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} \cup \epsilon_{C_2}(x)e^{i\vartheta_{C_2}(x)} = \max\left[\epsilon_{C_1}(x), \epsilon_{C_2}(x)\right] e^{i\max\left[\vartheta_{C_1}(x),\vartheta_{C_2}(x)\right]}.$$

Definition 2.6. *A compound statement for a complex fuzzy set is said to be in conjunctive normal form if it is obtained by operating an intersection '$\cap$' among complex fuzzy sets connected with the union '$\cup$'.*

Example 2.7. *Let C_1, C_2, and C_3 be three complex fuzzy sets. The conjunctive normal form of these complex fuzzy sets is obtained as:*

$$(C_1\cup C_2)\cap(C_1\cup C_3)\cap(C_2\cup C_3).$$

Also, $(C_1\cup C_2)\cap(C_2\cup C_3)$.

Definition 2.8. *A compound statement for a complex fuzzy set is in disjunctive normal form if it is obtained by operating union '$\cup$' among complex fuzzy sets connected with the intersection '$\cap$'.*

Example 2.9. *Let C_1, C_2, and C_3 be three complex fuzzy sets. The disjunctive normal form of these complex fuzzy sets is obtained as:*

$$(C_1\cap C_2)\cup(C_1\cap C_3)\cup(C_2\cap C_3).$$

Also, $(C_1\cap C_2)\cup(C_2\cap C_3)$.

Definition 2.10. *Let C_1 and C_2 be two complex fuzzy sets on U, and $Z_{C_1}(x) = \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)}$ and $Z_{C_2}(x) = \epsilon_{C_2}(x)e^{i\vartheta_{C_2}(x)}$ their grade values, respectively. Then, C_1 is equal to C_2 if $\epsilon_{C_1}(x) = \epsilon_{C_2}(x)$ and $\vartheta_{C_1}(x) = \vartheta_{C_2}(x)$.*

Definition 2.11. *If we interchange the complex fuzzy union by complex fuzzy intersection and complex fuzzy intersection by complex fuzzy union in any result for a complex fuzzy set, we obtain the true result. Therefore, any algebraic equality derived from these axioms will still be valid whenever the complex fuzzy intersection and complex fuzzy union have been interchanged. i.e. changing every $\cap$ into $\cup$ and vice versa, This is called the duality principle.*

Example 2.12. *Let*

$$C_1 = \frac{0.5e^{i0}}{a} + \frac{1e^{i\pi}}{b} + \frac{0.4e^{i\frac{\pi}{2}}}{c},$$
$$C_2 = \frac{0.6e^{i\frac{3\pi}{4}}}{a} + \frac{0.8e^{i2\pi}}{b} + \frac{0.2e^{i\frac{\pi}{6}}}{c},$$

and

$$C_3 = \frac{0.9e^{i\frac{3\pi}{2}}}{a} + \frac{1e^{i\frac{\pi}{4}}}{b} + \frac{0.2e^{i\frac{\pi}{5}}}{c}.$$

be three CFSs. Now, the distributive law of union over intersection is:

$$\begin{aligned} C_1 \cup (C_2 \cap C_3) &= [\frac{0.5e^{i0}}{a} + \frac{1e^{i\pi}}{b} + \frac{0.4e^{i\frac{\pi}{2}}}{c}] \cup ([\frac{0.6e^{i\frac{3\pi}{4}}}{a} + \frac{0.8e^{i2\pi}}{b} + \frac{0.2e^{i\frac{\pi}{6}}}{c}] \cap [\frac{0.9e^{i\frac{3\pi}{2}}}{a} + \frac{1e^{i\frac{\pi}{4}}}{b} + \frac{0.2e^{i\frac{\pi}{5}}}{c}]) \\ &= [\frac{0.5e^{i0}}{a} + \frac{1e^{i\pi}}{b} + \frac{0.4e^{i\frac{\pi}{2}}}{c}] \cup [\frac{0.6e^{i\frac{3\pi}{2}}}{a} + \frac{0.8e^{i2\pi}}{b} + \frac{0.2e^{i\frac{\pi}{5}}}{c}] \\ &= \frac{0.6e^{i\frac{3\pi}{2}}}{a} + \frac{1e^{i2\pi}}{b} + \frac{0.4e^{i\frac{\pi}{2}}}{c}. \end{aligned} \tag{1}$$

$$\begin{aligned} (C_1 \cup C_2) \cap (C_1 \cup C_3) &= ([\frac{0.5e^{i0}}{a} + \frac{1e^{i\pi}}{b} + \frac{0.4e^{i\frac{\pi}{2}}}{c}] \cup [\frac{0.6e^{i\frac{3\pi}{4}}}{a} + \frac{0.8e^{i2\pi}}{b} + \frac{0.2e^{i\frac{\pi}{6}}}{c}]) \\ &\quad \cap ([\frac{0.5e^{i0}}{a} + \frac{1e^{i\pi}}{b} + \frac{0.4e^{i\frac{\pi}{2}}}{c}] \cup [\frac{0.9e^{i\frac{3\pi}{2}}}{a} + \frac{1e^{i\frac{\pi}{4}}}{b} + \frac{0.2e^{i\frac{\pi}{5}}}{c}]) \\ &= [\frac{0.6e^{i\frac{3\pi}{4}}}{a} + \frac{1e^{i2\pi}}{b} + \frac{0.4e^{i\frac{\pi}{2}}}{c}] \cap [\frac{0.9e^{i\frac{3\pi}{2}}}{a} + \frac{1e^{i\pi}}{b} + \frac{0.4e^{i\frac{\pi}{2}}}{c}] \\ &= \frac{0.6e^{i\frac{3\pi}{2}}}{a} + \frac{1e^{i2\pi}}{b} + \frac{0.4e^{i\frac{\pi}{2}}}{c}. \end{aligned} \tag{2}$$

From (1) and (2), we have

$$C_1 \cup (C_2 \cap C_3) = (C_1 \cup C_2) \cap (C_1 \cup C_3).$$

We interchange union by intersection and intersection by union we get the true result, that is:

$$\begin{aligned} C_1 \cap (C_2 \cup C_3) &= [\frac{0.5e^{i0}}{a} + \frac{1e^{i\pi}}{b} + \frac{0.4e^{i\frac{\pi}{2}}}{c}] \cap ([\frac{0.6e^{i\frac{3\pi}{4}}}{a} + \frac{0.8e^{i2\pi}}{b} + \frac{0.2e^{i\frac{\pi}{6}}}{c}] \cup [\frac{0.9e^{i\frac{3\pi}{2}}}{a} + \frac{1e^{i\frac{\pi}{4}}}{b} + \frac{0.2e^{i\frac{\pi}{5}}}{c}]) \\ &= [\frac{0.5e^{i0}}{a} + \frac{1e^{i\pi}}{b} + \frac{0.4e^{i\frac{\pi}{2}}}{c}] \cap [\frac{0.9e^{i\frac{3\pi}{2}}}{a} + \frac{1e^{i2\pi}}{b} + \frac{0.2e^{i\frac{\pi}{5}}}{c}] \\ &= \frac{0.5e^{i\frac{3\pi}{2}}}{a} + \frac{1e^{i2\pi}}{b} + \frac{0.4e^{i\frac{\pi}{2}}}{c}. \end{aligned} \tag{3}$$

$$\begin{aligned} (C_1 \cap C_2) \cup (C_1 \cap C_3) &= ([\frac{0.5e^{i0}}{a} + \frac{1e^{i\pi}}{b} + \frac{0.4e^{i\frac{\pi}{2}}}{c}] \cap [\frac{0.6e^{i\frac{3\pi}{4}}}{a} + \frac{0.8e^{i2\pi}}{b} + \frac{0.2e^{i\frac{\pi}{6}}}{c}]) \\ &\quad \cup ([\frac{0.5e^{i0}}{a} + \frac{1e^{i\pi}}{b} + \frac{0.4e^{i\frac{\pi}{2}}}{c}] \cap [\frac{0.9e^{i\frac{3\pi}{2}}}{a} + \frac{1e^{i\frac{\pi}{4}}}{b} + \frac{0.2e^{i\frac{\pi}{5}}}{c}]) \\ &= [\frac{0.5e^{i\frac{3\pi}{4}}}{a} + \frac{1e^{i2\pi}}{b} + \frac{0.4e^{i\frac{\pi}{2}}}{c}] \cup [\frac{0.5e^{i\frac{3\pi}{2}}}{a} + \frac{1e^{i\pi}}{b} + \frac{0.2e^{i\frac{\pi}{2}}}{c}] \\ &= \frac{0.5e^{i\frac{3\pi}{2}}}{a} + \frac{1e^{i2\pi}}{b} + \frac{0.4e^{i\frac{\pi}{2}}}{c}. \end{aligned} \tag{4}$$

From (3) and (4), we have

$$C_1 \cap (C_2 \cup C_3) = (C_1 \cap C_2) \cup (C_1 \cap C_3).$$

Thus, we get the true results by interchanging intersection by union and union by intersection.

Definition 2.13. *A semi Boolean algebra $(\tau, \cap, \cup)$ is a set τ together with two binary operations $\cap$ and $\cup$ on τ satisfying the following axioms: for any C_1, C_2, $C_3 \in \tau$ (collection of complex fuzzy sets)*

O_1). Associative law holds with respect to standard complex fuzzy intersection and complex fuzzy union, that is;

$$C_1 \cap (C_2 \cap C_3) = (C_1 \cap C_2) \cap C_3,$$

and

$$C_1 \cup (C_2 \cup C_3) = (C_1 \cup C_2) \cup C_3.$$

O_2). Commutative law holds with respect to standard complex fuzzy union and complex fuzzy intersection, that is; $C_1 \cup C_2 = C_2 \cup C_1$, and $C_1 \cap C_2 = C_2 \cap C_1$.

O_3). Distributive law of union over intersection and intersection over union holds with respect to standard complex fuzzy union and complex fuzzy intersection, that is;

$$C_1 \cup (C_2 \cap C_3) = (C_1 \cup C_2) \cap (C_1 \cup C_3),$$

and

$$C_1 \cap (C_2 \cup C_3) = (C_1 \cap C_2) \cup (C_1 \cap C_3).$$

Example 2.14. *Let $\tau = \{C_1, C_2, C_3\}$ be a collection of complex fuzzy sets together with two binary operations $\cap$ and $\cup$. We have to show that $(\tau, \cap, \cup)$ is a semi Boolean algebra. Let*

$$C_1 = \frac{1e^{i2\pi}}{a} + \frac{0.1e^{i\frac{\pi}{4}}}{b} + \frac{0.8e^{i\pi}}{c},$$

$$C_2 = \frac{0.5e^{i\frac{3\pi}{5}}}{a} + \frac{0.9e^{i\pi}}{b} + \frac{0.4e^{i\frac{\pi}{2}}}{c},$$

and

$$C_3 = \frac{0.8e^{i\frac{3\pi}{4}}}{a} + \frac{0e^{i\frac{\pi}{2}}}{b} + \frac{0.6e^{i\frac{\pi}{3}}}{c},$$

be three CFSs. Now we have to check O_1.

O_1).

$$\begin{aligned} C_1 \cap (C_2 \cap C_3) &= [\frac{1e^{i2\pi}}{a} + \frac{0.1e^{i\frac{\pi}{4}}}{b} + \frac{0.8e^{i\pi}}{c}] \cap ([\frac{0.5e^{i\frac{3\pi}{5}}}{a} + \frac{0.9e^{i\pi}}{b} + \frac{0.4e^{i\frac{\pi}{2}}}{c}] \cap [\frac{0.8e^{i\frac{3\pi}{4}}}{a} + \frac{0e^{i\frac{\pi}{2}}}{b} + \frac{0.6e^{i\frac{\pi}{3}}}{c}]) \\ &= [\frac{1e^{i2\pi}}{a} + \frac{0.1e^{i\frac{\pi}{4}}}{b} + \frac{0.8e^{i\pi}}{c}] \cap [\frac{0.5e^{i\frac{3\pi}{4}}}{a} + \frac{0e^{i\pi}}{b} + \frac{0.4e^{i\frac{\pi}{2}}}{c}] \\ &= [\frac{0.5e^{i2\pi}}{a} + \frac{0e^{i\pi}}{b} + \frac{0.4e^{i\pi}}{c}]. \end{aligned} \tag{1}$$

$$\begin{aligned} (C_1 \cap C_2) \cap C_3 &= ([\frac{1e^{i2\pi}}{a} + \frac{0.1e^{i\frac{\pi}{4}}}{b} + \frac{0.8e^{i\pi}}{c}] \cap [\frac{0.5e^{i\frac{3\pi}{5}}}{a} + \frac{0.9e^{i\pi}}{b} + \frac{0.4e^{i\frac{\pi}{2}}}{c}]) \cap [\frac{0.8e^{i\frac{3\pi}{4}}}{a} + \frac{0e^{i\frac{\pi}{2}}}{b} + \frac{0.6e^{i\frac{\pi}{3}}}{c}] \\ &= [\frac{0.5e^{i2\pi}}{a} + \frac{0.1e^{i\pi}}{b} + \frac{0.4e^{i\pi}}{c}] \cap [\frac{0.8e^{i\frac{3\pi}{4}}}{a} + \frac{0e^{i\frac{\pi}{2}}}{b} + \frac{0.6e^{i\frac{\pi}{3}}}{c}] \\ &= [\frac{0.5e^{i2\pi}}{a} + \frac{0e^{i\pi}}{b} + \frac{0.4e^{i\pi}}{c}]. \end{aligned} \tag{2}$$

From (1) and (2), we have

$$C_1 \cap (C_2 \cap C_3) = (C_1 \cap C_2) \cap C_3.$$

Thus associative law of intersection holds. Therefore by duality principle, the associative law of union also holds. Now we have to check O_2.

$$\begin{aligned} C_1 \cap C_2 &= [\frac{1e^{i2\pi}}{a} + \frac{0.1e^{i\frac{\pi}{4}}}{b} + \frac{0.8e^{i\pi}}{c}] \cap [\frac{0.5e^{i\frac{3\pi}{5}}}{a} + \frac{0.9e^{i\pi}}{b} + \frac{0.4e^{i\frac{\pi}{2}}}{c}] \\ &= [\frac{0.5e^{i2\pi}}{a} + \frac{0.1e^{i\pi}}{b} + \frac{0.4e^{i\pi}}{c}]. \end{aligned} \tag{3}$$

$$\begin{aligned} C_2 \cap C_1 &= [\frac{0.5e^{i\frac{3\pi}{5}}}{a} + \frac{0.9e^{i\pi}}{b} + \frac{0.4e^{i\frac{\pi}{2}}}{c}] \cap [\frac{1e^{i2\pi}}{a} + \frac{0.1e^{i\frac{\pi}{4}}}{b} + \frac{0.8e^{i\pi}}{c}] \\ &= [\frac{0.5e^{i2\pi}}{a} + \frac{0.1e^{i\pi}}{b} + \frac{0.4e^{i\pi}}{c}]. \end{aligned} \tag{4}$$

From (3) and (4), the commutative law of intersection holds. Therefore by duality principle, the commutative law of union also holds.

Now we have to check O_3.

$$\begin{aligned} C_1 \cup (C_2 \cap C_3) &= [\frac{1e^{i2\pi}}{a} + \frac{0.1e^{i\frac{\pi}{4}}}{b} + \frac{0.8e^{i\pi}}{c}] \cup ([\frac{0.5e^{i\frac{3\pi}{5}}}{a} + \frac{0.9e^{i\pi}}{b} + \frac{0.4e^{i\frac{\pi}{2}}}{c}] \cap [\frac{0.8e^{i\frac{3\pi}{4}}}{a} + \frac{0e^{i\frac{\pi}{2}}}{b} + \frac{0.6e^{i\frac{\pi}{3}}}{c}]) \\ &= [\frac{1e^{i2\pi}}{a} + \frac{0.1e^{i\frac{\pi}{4}}}{b} + \frac{0.8e^{i\pi}}{c}] \cup [\frac{0.5e^{i\frac{3\pi}{4}}}{a} + \frac{0e^{i\pi}}{b} + \frac{0.4e^{i\frac{\pi}{2}}}{c}] \\ &= [\frac{1e^{i2\pi}}{a} + \frac{0.1e^{i\pi}}{b} + \frac{0.8e^{i\pi}}{c}]. \end{aligned} \tag{5}$$

$$\begin{aligned} (C_1 \cup C_2) \cap (C_1 \cup C_3) &= ([\frac{1e^{i2\pi}}{a} + \frac{0.1e^{i\frac{\pi}{4}}}{b} + \frac{0.8e^{i\pi}}{c}] \cup [\frac{0.5e^{i\frac{3\pi}{5}}}{a} + \frac{0.9e^{i\pi}}{b} + \frac{0.4e^{i\frac{\pi}{2}}}{c}]) \\ &\cap ([\frac{1e^{i2\pi}}{a} + \frac{0.1e^{i\frac{\pi}{4}}}{b} + \frac{0.8e^{i\pi}}{c}] \cup [\frac{0.8e^{i\frac{3\pi}{4}}}{a} + \frac{0e^{i\frac{\pi}{2}}}{b} + \frac{0.6e^{i\frac{\pi}{3}}}{c}]) \\ &= [\frac{1e^{i2\pi}}{a} + \frac{0.9e^{i\pi}}{b} + \frac{0.8e^{i\pi}}{c}] \cap [\frac{1e^{i2\pi}}{a} + \frac{0.1e^{i\frac{\pi}{2}}}{b} + \frac{0.8e^{i\pi}}{c}] \\ &= [\frac{1e^{i2\pi}}{a} + \frac{0.1e^{i\pi}}{b} + \frac{0.8e^{i\pi}}{c}]. \end{aligned} \tag{6}$$

From (5) and (6), the distributive law of union over intersection holds. Therefore by duality principle, the intersection over union holds.

Thus all the conditions of semi Boolean algebra hold. Hence $(\tau, \cap, \cup)$ is a semi Boolean algebra.

3 Main results

Proposition 3.1. *Let C_1 and C_2 be two complex fuzzy sets. Then $(C_1 \cap C_2) \cup (C_1 \cap C_2') = C_1$ and $(C_1 \cup C_2) \cap (C_1 \cup C_2') = C_1$.*

Proof. Let $Z_{C_1}(x) = \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)}$ and $Z_{C_2} = \epsilon_{C_2}(x)e^{i\vartheta_{C_2}(x)}$ be the membership functions of complex fuzzy sets C_1 and C_2, respectively. Assume that $\epsilon_{C_1}(x) \geq \epsilon_{C_2}(x)$ and $\vartheta_{C_1}(x) \geq \vartheta_{C_2}(x)$, then $\epsilon_{C_1}(x) \leq \epsilon_{C_2'}(x)$. Using standard complex fuzzy union, standard complex fuzzy intersection, and standard complex fuzzy complement function with the same function for determining the phase term, we have

$$\begin{aligned} (C_1 \cap C_2) \cup (C_1 \cap C_2') &= [\epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} \cap \epsilon_{C_2}(x)e^{i\vartheta_{C_2}(x)}] \cup [\epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} \cap \epsilon_{C_2'}(x)e^{i\vartheta_{C_2'}(x)}] \\ &= \epsilon_{C_2}(x)e^{i\vartheta_{C_1}(x)} \cup \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} \\ &= \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} \\ &= C_1. \end{aligned}$$

$$\begin{aligned} (C_1 \cup C_2) \cap (C_1 \cup C_2') &= [\epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} \cup \epsilon_{C_2}(x)e^{i\vartheta_{C_2}(x)}] \cap [\epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} \cup \epsilon_{C_2'}(x)e^{i\vartheta_{C_2'}(x)}] \\ &= \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} \cap \epsilon_{C_2'}(x)e^{i\vartheta_{C_1}(x)} \\ &= \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} \\ &= C_1. \end{aligned}$$

Note: For the second result, we may apply duality principle, that is, changing every complex fuzzy union by complex fuzzy intersection and every complex fuzzy intersection by complex fuzzy union, we obtain the true result. □

Proposition 3.2. *Let C_1 and C_2 be two complex fuzzy sets. Then $C_1 \cup (C_2 \cap C_1') = C_1 \cup C_2$ and $C_1 \cap (C_2 \cup C_1') = C_1 \cap C_2$.*

Proof. Let $Z_{C_1}(x) = \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)}$ and $Z_{C_2}(x) = \epsilon_{C_2}(x)e^{i\vartheta_{C_2}(x)}$ be the membership functions of complex fuzzy sets C_1 and C_2, respectively. Assume that $\epsilon_{C_1}(x) \geq \epsilon_{C_2}(x)$ and $\vartheta_{C_1}(x) \geq \vartheta_{C_2}(x)$, then $\epsilon_{C_1'}(x) \leq \epsilon_{C_2}(x)$. Using standard complex fuzzy union, standard complex fuzzy intersection, and standard complex fuzzy complement function with the same function for determining the phase term, we have

$$\begin{aligned} C_1 \cup (C_2 \cap C_1') &= \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} \cup [\epsilon_{C_2}(x)e^{i\vartheta_{C_2}(x)} \cap \epsilon_{C_1'}(x)e^{i\vartheta_{C_1'}(x)}] \\ &= \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} \cup \epsilon_{C_1'}(x)e^{i\vartheta_{C_1'}(x)} \\ &= \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)}. \end{aligned} \tag{1}$$

$$C_1 \cup C_2 = \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} \cup \epsilon_{C_2}(x)e^{i\vartheta_{C_2}(x)} = \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)}. \tag{2}$$

From (1) and (2), we have

$$C_1 \cup (C_2 \cap C_1') = C_1 \cup C_2.$$

Now by duality principle the statement $C_1 \cap (C_2 \cup C_1') = C_1 \cap C_2$ also holds. □

Proposition 3.3. *Show that* $(\tau = \{C_1, C_2, C_3\}, \cap, \cup)$ *is a semi Boolean algebra with* $\epsilon_{C_1}(x) \geq \epsilon_{C_2}(x) \geq \epsilon_{C_3}(x)$ *and*

$\vartheta_{C_1}(x) \geq \vartheta_{C_2}(x) \geq \vartheta_{C_3}(x)$, where $\epsilon_{C_1}(x), \epsilon_{C_2}(x), \epsilon_{C_3}(x)$ and $\vartheta_{C_1}(x), \vartheta_{C_2}(x), \vartheta_{C_3}(x)$ represent the amplitude and phase terms of complex fuzzy sets C_1, C_2, and C_3, respectively.

Proof. To prove that $(\tau = \{C_1, C_2, C_3\}, \cap, \cup)$ is a semi Boolean algebra, we have to check the three conditions for a semi Boolean algebra using standard complex fuzzy union, standard complex fuzzy intersection and standard complex fuzzy complement function. □

O_1).

$$\begin{aligned} C_1 \cap (C_2 \cap C_3) &= \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} \cap [\epsilon_{C_2}(x)e^{i\vartheta_{C_2}(x)} \cap \epsilon_{C_3}(x)e^{i\vartheta_{C_3}(x)}] \\ &= \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} \cap \epsilon_{C_3}(x)e^{i\vartheta_{C_2}(x)} \\ &= \epsilon_{C_3}(x)e^{i\vartheta_{C_1}(x)}. \end{aligned} \tag{1}$$

$$\begin{aligned} (C_1 \cap C_2) \cap C_3 &= [\epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} \cap \epsilon_{C_2}(x)e^{i\vartheta_{C_2}(x)}] \cap \epsilon_{C_3}(x)e^{i\vartheta_{C_3}(x)} \\ &= \epsilon_{C_2}(x)e^{i\vartheta_{C_1}(x)} \cap \epsilon_{C_3}(x)e^{i\vartheta_{C_3}(x)} \\ &= \epsilon_{C_3}(x)e^{i\vartheta_{C_1}(x)}. \end{aligned} \tag{2}$$

From (1) and (2), we have

$$C_1 \cap (C_2 \cap C_3) = (C_1 \cap C_2) \cap C_3.$$

By duality principle, we have

$$C_1 \cup (C_2 \cup C_3) = (C_1 \cup C_2) \cup C_3.$$

Thus O_1 holds.
O_2).

$$C_1 \cup C_2 = \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} \cup \epsilon_{C_2}(x)e^{i\vartheta_{C_2}(x)} = \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)}. \tag{3}$$

$$C_2 \cup C_1 = \epsilon_{C_2}(x)e^{i\vartheta_{C_2}(x)} \cup \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} = \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)}. \tag{4}$$

From (3) and (4), we have

$$C_1 \cup C_2 = C_2 \cup C_1.$$

Thus by duality principle, we have

$$C_1 \cap C_2 = C_2 \cap C_1.$$

Hence O_2 holds.
O_3).

$$
\begin{aligned}
C_1 \cup (C_2 \cap C_3) &= \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} \cup [\epsilon_{C_2}(x)e^{i\vartheta_{C_2}(x)} \cap \epsilon_{C_3}(x)e^{i\vartheta_{C_3}(x)}] \\
&= \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} \cup \epsilon_{C_3}(x)e^{i\vartheta_{C_2}(x)} \\
&= \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)}. \qquad (5)
\end{aligned}
$$

$$
\begin{aligned}
(C_1 \cup C_2) \cap (C_1 \cup C_3) &= [\epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} \cup \epsilon_{C_2}(x)e^{i\vartheta_{C_2}(x)}] \cap [\epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} \cup \epsilon_{C_3}(x)e^{i\vartheta_{C_3}(x)}] \\
&= \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} \cap \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} \\
&= \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)}. \qquad (6)
\end{aligned}
$$

From (5) and (6), we have

$$C_1 \cup (C_2 \cap C_3) = (C_1 \cup C_2) \cap (C_1 \cup C_3).$$

By duality principle, we have

$$C_1 \cap (C_2 \cup C_3) = (C_1 \cap C_2) \cup (C_1 \cap C_3).$$

Thus O_3 holds and hence $(\tau = \{C_1, C_2, C_3\}, \cap, \cup)$ is a semi Boolean algebra.

Proposition 3.4. *Two complex fuzzy sets C_1 and C_2 are equal if $C_1 \cap C_2 = C_1$ and $C_1 \cup C_2 = C_1$ using the* min *function and* max *function for determining the phase term.*

Proof. Assume that for any two complex fuzzy sets C_1 and C_2, $Z_{C_1}(x) = \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)}$ and $Z_{C_2}(x) = \epsilon_{C_2}(x)e^{i\vartheta_{C_2}(x)}$ their membership functions, respectively. Also given that $C_1 \cap C_2 = C_1$ and $C_1 \cup C_2 = C_1$. We have to show that the two complex fuzzy sets are equal, that is, $C_1 = C_2$. Since $C_1 \cap C_2 = C_1$ implies that

$$\epsilon_{C_1}(x) \leq \epsilon_{C_2}(x) \text{ and } \vartheta_{C_1}(x) \leq \vartheta_{C_2}(x). \qquad (1)$$

Also $C_1 \cup C_2 = C_1$ implies that

$$\epsilon_{C_2}(x) \leq \epsilon_{C_1}(x) \text{ and } \vartheta_{C_2}(x) \leq \vartheta_{C_1}(x). \qquad (2)$$

From (1) and (2), we have $\epsilon_{C_2}(x) = \epsilon_{C_1}(x)$ and $\vartheta_{C_2}(x) = \vartheta_{C_1}(x)$. Thus the two complex fuzzy sets C_1 and C_2 are equal. □

Proposition 3.5. *Two complex fuzzy sets C_1 and C_2 are equal if $C_1 \cap C_2 = C_2$ and $C_1 \cup C_2 = C_2$ using* min *function and* max *function for determining the phase term.*

Proof. It is easy to prove. □

Proposition 3.6. *For any three complex fuzzy sets C_1, C_2, and C_3, the following identity holds*

$$C_1 \cap (C_2 \cap C_3)' = (C_1 \cap C_2') \cup (C_1 \cap C_3').$$

Proof. Let C_1, C_2, and C_3 be three complex fuzzy sets, $Z_{C_1}(x) = \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)}$, $Z_{C_2}(x) = \epsilon_{C_2}(x)e^{i\vartheta_{C_2}(x)}$ and $Z_{C_3}(x) = \epsilon_{C_3}(x)e^{i\vartheta_{C_3}(x)}$ their membership functions, respectively. Assume that $\epsilon_{C_1}(x) \leq \epsilon_{C_2}(x) \leq \epsilon_{C_3}(x)$ and $\vartheta_{C_1}(x) \leq \vartheta_{C_2}(x) \leq \vartheta_{C_3}(x)$. Note that $\epsilon_{C_1'}(x) \geq \epsilon_{C_2'}(x) \geq \epsilon_{C_3'}(x)$ and $\vartheta_{C_1'}(x) \leq \vartheta_{C_2'}(x) \leq \vartheta_{C_3'}(x)$. Now using standard complex fuzzy intersection, complex fuzzy union and complex fuzzy complement function with the same function for determining the phase term, we have

$$
\begin{aligned}
C_1 \cap (C_2 \cap C_3)' &= C_1 \cap (C_2' \cup C_3') \\
&= \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} \cap (\epsilon_{C_2'}(x)e^{i\vartheta_{C_2'}(x)} \cup \epsilon_{C_3'}(x)e^{i\vartheta_{C_3'}(x)}) \\
&= \epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} \cap \epsilon_{C_2'}(x)e^{i\vartheta_{C_3'}(x)} \\
&= \epsilon_{C_2'}(x)e^{i\vartheta_{C_3'}(x)}. \qquad (1)
\end{aligned}
$$

Now we have to find $(C_1 \cap C_2') \cup (C_1 \cap C_3')$ as:

$$
\begin{aligned}
(C_1 \cap C_2') \cup (C_1 \cap C_3') &= (\epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} \cap \epsilon_{C_2'}(x)e^{i\vartheta_{C_2'}(x)}) \cup (\epsilon_{C_1}(x)e^{i\vartheta_{C_1}(x)} \cap \epsilon_{C_3'}(x)e^{i\vartheta_{C_3'}(x)}) \\
&= \epsilon_{C_2'}(x)e^{i\vartheta_{C_2'}(x)} \cup \epsilon_{C_3'}(x)e^{i\vartheta_{C_3'}(x)} \\
&= \epsilon_{C_2'}(x)e^{i\vartheta_{C_3'}(x)}. \qquad (2)
\end{aligned}
$$

From (1) and (2), we have

$$C_1 \cap (C_2 \cap C_3)' = (C_1 \cap C_2') \cup (C_1 \cap C_3').$$

□

4 Applications

In this section, we will discuss the applications of CFSs in signals and systems.

Definition 4.1. *The discrete Fourier transform transforms a sequence of N complex numbers $\{x_n(N)\}$ into another sequence of complex numbers $X(k)$, which is defined by*

$$X(k) = \sum_{n=0}^{N-1} x_i(n) e^{i(-\frac{2\pi}{N}kn)} \; ; k, n \in \{0, 1, 2, ..., N-1\}. \tag{1}$$

The inverse discrete Fourier transform of (1) is defined as:

$$x_i(n) = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{i\frac{2\pi}{N}kn} \; ; k, n \in \{0, 1, 2, ..., N-1\}. \tag{2}$$

which is also N-periodic and X_k has different values [23].

We take a particular case, that is, $X(k)$ is restricted to a closed interval $[0, 1]$ because, in CFS, the amplitude term has all the values in the closed interval $[0, 1]$.

In the following, we develop an algorithm using CFSs in signals and systems for the identification of a reference signal received by a particular receiver.

Let n be different electromagnetic signals, $x_1(n), x_2(n), x_3(n), ..., x_n(n)$ have been received by a particular receiver. Each of these signals is noted N different *times*. Let $x_i(n)$ be the $i - th$ $(1 \leq n \leq N)$ signal. The inverse discrete Fourier transform of this $i - th$ signal is

$$x_i(n) = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{i\frac{2\pi}{N}kn} \qquad ; k, n = 0, 1, 2, ..., N-1. \tag{3}$$

We restrict the range of $X[k]$ as $0 \leq X[k] \leq 1$ $(0 \leq k \leq N-1)$. Here, $X[k]$ is known as amplitude term and $\frac{2\pi}{N}kn = \omega_s(q)$ is known as phase term and the first one having the range as real numbers.

Thus a general signal representing by equations (3) is model for signal representation using a CFS.

We use the CFS in signals and systems utilizing a new kind of matrix to identifies a reference signal out of large signals detected by a digital receiver. For this, we have a reference signal r. This reference signal r is noted N times. The IDFT of this reference signal r is

$$r(n) = \frac{1}{N} \sum_{k=0}^{N-1} X'[k] e^{i\frac{2\pi}{N}kn} \qquad ; k, n = 0, 1, 2, ..., N-1. \tag{4}$$

where $X'[k] \in [0, 1]$; $(0 \leq k \leq N-1)$. To compare the similarity between two signals, we apply the following method.

Algorithm

Step 1.

Expand $x_i(n) = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{i\frac{2\pi}{N}kn}$ for $k = 0, 1, 2, ..., N-1$, we get

$$\begin{aligned} x_i(n) &= \frac{1}{N}[X[0]e^{i\frac{2\pi}{N}n(0)} + X[1]e^{i\frac{2\pi}{N}n(1)} + X[2]e^{i\frac{2\pi}{N}n(2)} + ...X[N-1]e^{i\frac{2\pi}{N}n(N-1)}] \\ &= \frac{1}{N}[X[0].1 + X[1]e^{i\frac{2\pi}{N}n(1)} + X[2]e^{i\frac{2\pi}{N}n(2)} + ... + X[N-1]e^{i\frac{2\pi}{N}n(N-1)}]. \end{aligned} \tag{5}$$

From equation (5) we get $N-samples$ by putting $n=0,1,2,3,...,N-1$.
For $n=0$, we have

$$\begin{aligned} x_i(0) &= \frac{1}{N}[X[0].1 + X[1]e^{i\frac{2\pi}{N}(0)(1)} + X[2]e^{i\frac{2\pi}{N}(0)(2)} + ... + X[N-1]e^{i\frac{2\pi}{N}(0)(N-1)}] \\ &= \frac{1}{N}[X[0].1 + X[1].1 + X[2].1 + ... + X[N-1].1]. \end{aligned} \tag{6}$$

For $n=1$, we have

$$\begin{aligned} x_i(1) &= \frac{1}{N}[X[0].1 + X[1]e^{i\frac{2\pi}{N}(1)(1)} + X[2]e^{i\frac{2\pi}{N}(1)(2)} + ... + X[N-1]e^{i\frac{2\pi}{N}(1)(N-1)}] \\ &= \frac{1}{N}[X[0].1 + X[1]e^{i\frac{2\pi}{N}(1)} + X[2]e^{i\frac{2\pi}{N}(2)} + ... + X[N-1]e^{i\frac{2\pi}{N}(N-1)}]. \end{aligned} \tag{7}$$

For $n=2$, we have

$$\begin{aligned} x_i(2) &= \frac{1}{N}[X[0].1 + X[1]e^{i\frac{2\pi}{N}(2)(1)} + X[2]e^{i\frac{2\pi}{N}(2)(2)} + ... + X[N-1]e^{i\frac{2\pi}{N}(2)(N-1)}] \\ &= \frac{1}{N}[X[0].1 + X[1]e^{i\frac{2\pi}{N}(2)} + X[2]e^{i\frac{2\pi}{N}(4)} + ... + X[N-1]e^{i\frac{2\pi}{N}2(N-1)}]. \end{aligned} \tag{8}$$

Continuing this process, for $n=N-1$, we have

$$\begin{aligned} x_i(N-1) &= \frac{1}{N}[X[0].1 + X[1]e^{i\frac{2\pi}{N}(N-1)(1)} + X[2]e^{i\frac{2\pi}{N}(N-1)(2)} + ... + X[N-1]e^{i\frac{2\pi}{N}(N-1)(N-1)}] \\ &= \frac{1}{N}[X[0].1 + X[1]e^{i\frac{2\pi}{N}(N-1)} + X[2]e^{i\frac{2\pi}{N}2(N-1)} + ... + X[N-1]e^{i\frac{2\pi}{N}(N-1)^2}]. \end{aligned} \tag{9}$$

A similar argument repeats for the reference signal $r(n)$, we get the $N-samples$ of the reference signal r putting $n=0,1,2,3,...,N-1$.

Step 2.

Now we find the cross product of these $N-samples$ of the signal $x_i(n)$ and the reference signal $r(n)$, where $n=0,1,2,...,N-1$.

$$\begin{aligned} x_i(0)\times r(0) = \frac{1}{N^2}[&\frac{\min\{X[0]\times X'[0]\}e^{i\min\{\frac{2\pi}{N}N(0),\frac{2\pi}{N}N(0)\}}}{(0,0)}+ \\ &\frac{\min\{X[0]\times X'[1]\}e^{i\min\{\frac{2\pi}{N}N(0),\frac{2\pi}{N}(0)(1)\}}}{(0,1)}+...+ \\ &\frac{\min\{X[0]\times X'[N-1]\}e^{i\min\{\frac{2\pi}{N}N(0),\frac{2\pi}{N}(0)(N-1)\}}}{(0,N-1)}+ \\ &\frac{\min\{X[1]\times X'[0]\}e^{i\min\{\frac{2\pi}{N}(0)(1),\frac{2\pi}{N}(0)(0)\}}}{(1,0)}+ \\ &\frac{\min\{X[1]\times X'[1]\}e^{i\min\{\frac{2\pi}{N}(0)(1),\frac{2\pi}{N}(0)(1)\}}}{(1,1)}+...+ \\ &\frac{\min\{X[1]\times X'[N-1]\}e^{i\min\{\frac{2\pi}{N}(0)(1),\frac{2\pi}{N}(0)(N-1)\}}}{(1,N-1)}+...+ \\ &\frac{\min\{X[N-1]\times X'[0]\}e^{i\min\{\frac{2\pi}{N}(0)(N-1),\frac{2\pi}{N}(0)(0)\}}}{(N-1,0)}+...+ \\ &\frac{\min\{X[N-1]\times X'[N-1]\}e^{i\min\{\frac{2\pi}{N}(0)(N-1),\frac{2\pi}{N}(0)(N-1)\}}}{(0,1)}. \end{aligned}$$

Compute the absolute value of every term of $x_i(0)\times r(0)$ and divided by N^2. Now, we take the maximum value from the absolute values of $x_i(0)\times r(0)$.

A similar process repeats for $x_i(1)\times r(1)$, $x_i(2)\times r(2),...,x_i(N-1)\times r(N-1)$. Now we develop the column matrix from all these max values of $x_i(0)\times r(0), x_i(1)\times r(1)$, $x_i(2)\times r(2),...,x_i(N-1)\times r(N-1)$; that is,

$$\begin{bmatrix} \max \mid x_i(0) \times r(0) \mid \\ \max \mid x_i(1) \times r(1) \mid \\ \max \mid x_i(2) \times r(2) \mid \\ . \\ . \\ . \\ \max \mid x_i(N-1) \times r(N-1) \mid \end{bmatrix}. \tag{10}$$

Similarly, we develop the column matrix from the cross product of the $N - samples$ of the signal $x_j(n)$ and the reference signal $r(n)$, that is, we have

$$\begin{bmatrix} \max \mid x_j(0) \times r(0) \mid \\ \max \mid x_j(1) \times r(1) \mid \\ \max \mid x_j(2) \times r(2) \mid \\ . \\ . \\ . \\ \max \mid x_j(N-1) \times r(N-1) \mid \end{bmatrix}. \tag{11}$$

Step 3.

For identification of the reference signal of $x_i(n)$ and $x_j(n)$; $n = 0, 1, 2, ..., N-1$, we again take the maximum value of the column matrix (10) and (11). If the max value of column matrix (10) is greater than the max value of the column matrix (11), then $x_i(n)$ shows the reference signal and if the max value of the column matrix (11) is greater than the max value of the column matrix (10), then $x_j(n)$ is the reference signal.

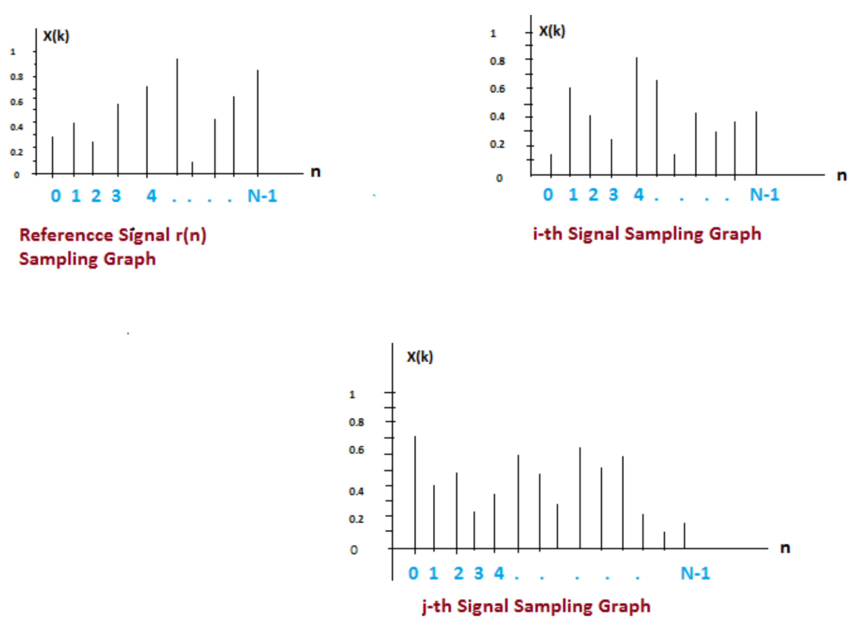
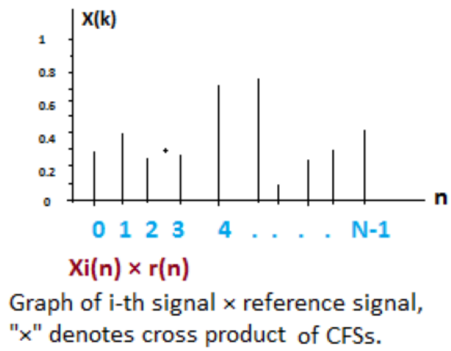
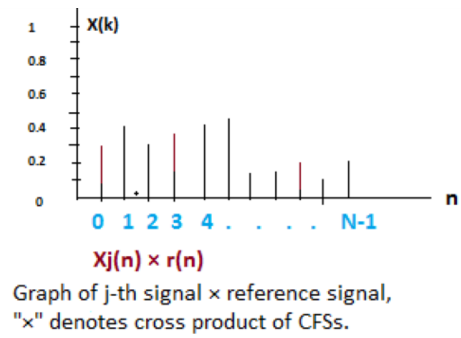


Figure 1: Fig. a

Figure (a) shows the graphs of the $N-$sampled reference signal, i-th signal, and j-th signal. Note that these are discrete signal graphs.



(a) Fig. b



(b) Fig. c

From Figure (b) and Figure (c), it is clear that the i-th signal shows a high degree of resemblance to the reference signal. Thus the i-th signal is the reference signal.

Note that our proposed model discussed the resemblance of signals from different sources. If the electromagnetic signals are from one source then, there is no need to check the resemblance.

We may apply different techniques and operations to solve noise signal problems. If the Fourier transform of noise signals is in discrete form, then we can apply our proposed algorithm. If the Fourier transform of noise signals is in continuous form, then we may apply continuous Fourier transform instead of discrete Fourier transform.

Example 4.2. *Assume that two different electromagnetic signals, $x_1(n)$, $x_2(n)$ have been received by a receiver. Each of these signals is sampled two times. Let $r(n)$ be the reference signal. The discrete Fourier transform of the signal $x_i(n); n = 0, 1$ and reference signal $r(n)$ for $N = 2$ is*

$$x_i(n) = \frac{1}{2}\sum_{k=0}^{1} X[k]e^{i\frac{2\pi}{2}kn} \;; k, n = 0, 1, \tag{1}$$

where $X[k] \in [0, 1]$. Also

$$r(n) = \frac{1}{2}\sum_{k=0}^{1} X'[k]e^{i\frac{2\pi}{2}nk} \;; n, k = 0, 1, \tag{2}$$

where $X'[k] \in [0, 1]$. For $k = 0, 1$, equation (1) becomes

$$x_i(n) = \frac{1}{2}[X[0].e^{i\frac{2\pi}{2}n(0)} + X[1].e^{i\frac{2\pi}{2}n(1)}] = \frac{1}{2}[X[0].1 + X[1].e^{i\pi n}]. \tag{3}$$

Now put $n = 0$ and $i = 1$ in (3), we have

$$x_1(0) = \frac{1}{2}[X[0].e^{i\pi(0)(0)} + X[1].e^{i\pi(0)(1)}] = \frac{1}{2}[X[0].e^{i(0)} + X[1].e^{i(0)}] \tag{3.1}$$

Put $n = 1$, we have

$$x_1(1) = \frac{1}{2}[X[0].e^{i\pi(0)(1)} + X[1].e^{i\pi(1)}] = \frac{1}{2}[X[0].e^{i\pi(0)} + X[1].e^{i\pi(1)}]. \tag{3.2}$$

A similar process for the reference signal, we have

$$r(0) = \frac{1}{2}[X'[0].e^{i\pi(0)(0)} + X'[1].e^{i\pi(0)(1)}] = \frac{1}{2}[X'[0].e^{i\pi(0)} + X'[1].e^{i\pi(0)}]. \tag{2.1}$$

$$r(1) = \frac{1}{2}[X'[0].e^{i\pi(0)(1)} + X'[1].e^{i\pi(1)(1)}] = \frac{1}{2}[X'[0].e^{i\pi(0)} + X'[1].e^{i\pi(1)}]. \tag{2.2}$$

We take the amplitude term of the signal $x_1(0)$, $x_1(1)$, $r(0)$ and $r(1)$ respectively as

$$X[k] = \begin{cases} 1 & ; k = 0 \\ 0.5 & ; k = 1 \end{cases},$$

$$X[k] = \begin{cases} 0.8 & ; k = 0 \\ 0.2 & ; k = 1 \end{cases},$$

$$X'[k] = \begin{cases} 0 & ; k = 0 \\ 1 & ; k = 1 \end{cases},$$

$$X'[k] = \begin{cases} 0.8 & ; k = 0 \\ 0.4 & ; k = 1 \end{cases}.$$

Therefore $(2.1), (2.2), (3.1)$ and (3.2) becomes respectively

$$r(0) = \frac{1}{2}[0.e^{i\pi(0)(0)} + 1.e^{i\pi(0)(1)}] \tag{2.3}$$

$$r(1) = \frac{1}{2}[0.8.e^{i\pi(0)(1)} + 0.4.e^{i\pi(1)(1)}]. \tag{2.4}$$

$$x_1(0) = \frac{1}{2}[1.e^{i\pi(0)(0)} + 0.5e^{i\pi(0)(1)}]. \tag{3.3}$$

$$x_1(1) = \frac{1}{2}[0.8e^{i\pi(0)(1)} + 0.2e^{i\pi(1)}]. \tag{3.4}$$

Now, we find the cross product of $x_1(n)$ and the reference signal $r(N)$; that is

$$x_1(0) \times r(0) = \frac{1}{4}[\frac{\min\{0,1\}^{i\min\{0,0\}}e}{(0,0)} + \frac{\min\{0,0.5\}e^{i\min\{0,0\}}}{(0,1)} + \frac{\min\{1,1\}e^{i\min\{0,0\}}}{(1,0)} + \frac{\min\{1,0.5\}e^{i\min\{0,0\}}}{(1,1)}].$$

$$x_1(0) \times r(0) = \frac{1}{4}[0(1) + 0(1) + 1(1) + 0.5(1)] = \frac{0}{4} + \frac{0}{4} + \frac{1}{4} + \frac{1.5}{4}.$$

Now,

$$\max \mid x_1(0) \times r(0) \mid = \max[\mid \frac{0}{4} \mid + \mid \frac{0}{4} \mid + \mid \frac{1}{4} \mid + \mid \frac{0.5}{4} \mid] = \max[0 + 0 + 0.25 + 0.125].$$

Thus,

$$\max \mid x_1(0) \times r(0) \mid = 0.25.$$

Also

$$x_1(1) \times r(1) = \frac{1}{4}[\frac{\min\{0.8,0.8\}^{i\min\{0,0\}}e}{(0,0)} + \frac{\min\{0.8,0.4\}e^{i\min\{0,\pi\}}}{(0,1)} + \frac{\min\{0.2,0.8\}e^{i\min\{\pi,0\}}}{(1,0)} + \frac{\min\{0.2,0.4\}e^{i\min\{\pi,\pi\}}}{(1,1)}],$$

$$x_1(1) \times r(1) = \frac{1}{4}[0.8(1) + 0.4(1) + 1.2(1) + 0.2(-1)] = \frac{0.8}{4} + \frac{0.4}{4} + \frac{1.2}{4} + \frac{(-0.2)}{4}.$$

Now,

$$\max \mid x_1(1) \times r(1) \mid = \max[\mid \frac{0.8}{4} \mid + \mid \frac{0.4}{4} \mid + \mid \frac{1.2}{4} \mid + \mid \frac{-0.2}{4} \mid] = \max[0.2 + 0.1 + 0.3 + 0.05].$$

Thus,

$$\max \mid x_1(1) \times r(1) \mid = 0.3.$$

Now, we develop the column matrix from all these max values of $x_1(0) \times r(0)$ and $x_1(1) \times r(1)$, $x_n(2) \times r(2)$; that is,

$$\begin{bmatrix} \max \mid x_1(0) \times r(0) \mid \\ \max \mid x_1(1) \times r(1) \mid \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.3 \end{bmatrix}. \tag{A}$$

Now consider the signal

$$x_2(n) = \frac{1}{2}\sum_{k=0}^{1} X[k]e^{i\frac{2\pi}{2}kn} \;; k, n = 0, 1, \tag{4}$$

where $X[k] \in [0, 1]$. For $k = 0, 1$, equation (1) becomes

$$x_2(n) = \frac{1}{2}[X[0].e^{i\frac{2\pi}{2}n(0)} + X[1].e^{i\frac{2\pi}{2}n(1)}] = \frac{1}{2}[X[0].1 + X[1].e^{i\pi n}]. \tag{4.1}$$

Now put $n = 0$ & 1 in (4.1), we get

$$x_2(0) = \frac{1}{2}[X[0].e^{i\pi(0)(0)} + X[1].e^{i\pi(0)(1)}], \tag{4.2}$$

$$x_2(1) = \frac{1}{2}[X[0].e^{i\pi(0)(1)} + X[1].e^{i\pi(1)}]. \tag{4.3}$$

We take the amplitude term of the signal $x_2(0)$, $x_2(1)$, respectively as:

$$X[k] = \begin{cases} 0.6 & ; k = 0 \\ 1 & ; k = 1 \end{cases}$$

$$X[k] = \begin{cases} 0.9 & ; k = 0 \\ 0.5 & ; k = 1 \end{cases}.$$

Therefore (4.2) and (4.3) becomes as:

$$x_2(0) = \frac{1}{2}[0.6e^{i\pi(0)(0)} + 1e^{i\pi(0)(1)}], \tag{4.4}$$

$$x_2(1) = \frac{1}{2}[0.9e^{i\pi(0)(1)} + 0.5e^{i\pi(1)}]. \tag{4.5}$$

Now we find the cross product of $u_2(n)$ and $r(n)$, that is,

$$x_2(0) \times r(0) = \frac{1}{4}[\frac{\min\{0, 0.6\}^{i\min\{0,0\}}e}{(0, 0)} + \frac{\min\{0, 1\}e^{i\min\{0,0\}}}{(0, 1)} + \frac{\min\{1, 0.6\}e^{i\min\{0,0\}}}{(1, 0)} + \frac{\min\{1, 1\}e^{i\min\{0,0\}}}{(1, 1)}],$$

$$x_2(0) \times r(0) = \frac{1}{4}[0(1) + 0(1) + 0.6(1) + 1(1)] = \frac{0}{4} + \frac{0}{4} + \frac{0.6}{4} + \frac{1}{4}.$$

Now,

$$\max \mid x_2(0) \times r(0) \mid = \max[\mid \frac{0}{4} \mid + \mid \frac{0}{4} \mid + \mid \frac{0.6}{4} \mid + \mid \frac{1}{4} \mid] = \max[0 + 0 + 0.15 + 0.25].$$

Thus,

$$\max \mid x_2(0) \times r(0) \mid = 0.25.$$

Now,

$$x_2(1) \times r(1) = \frac{1}{4}[\frac{\min\{0.8, 0.9\}^{i\min\{0,0\}}e}{(0, 0)} + \frac{\min\{0.8, 0.5\}e^{i\min\{0,\pi\}}}{(0, 1)} + \frac{\min\{0.2, 0.9\}e^{i\min\{\pi,0\}}}{(1, 0)} + \frac{\min\{0.2, 0.5\}e^{i\min\{\pi,\pi\}}}{(1, 1)}],$$

$$x_2(1) \times r(1) = \frac{1}{4}[0.8(1) + 0.5(1) + 0.2(1) + 0.2(-1)] = \frac{0.8}{4} + \frac{0.5}{4} + \frac{0.2}{4} + \frac{(-0.2)}{4}.$$

Now,

$$\max \mid x_2(1) \times r(1) \mid = \max[\mid \frac{0.8}{4} \mid + \mid \frac{0.5}{4} \mid + \mid \frac{0.2}{4} \mid + \mid \frac{-0.2}{4} \mid] = \max[0.2 + 0.125 + 0.05 + 0.05].$$

Thus,

$$\max \mid x_2(1) \times r(1) \mid = 0.2.$$

Now, we develop the column matrix from all these max values of $x_2(0) \times r(0)$ and $x_2(1) \times r(1)$, that is,

$$\begin{bmatrix} \max \mid x_2(0) \times r(0) \mid \\ \max \mid x_2(1) \times r(1) \mid \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.2 \end{bmatrix}. \tag{B}$$

From (A) and (B), we have

$$\max \begin{bmatrix} \max \mid x_1(0) \times r(0) \mid \\ \max \mid x_1(1) \times r(1) \mid \end{bmatrix} > \max \begin{bmatrix} \max \mid x_2(0) \times r(0) \mid \\ \max \mid x_2(1) \times r(1) \mid \end{bmatrix}.$$

Thus, the signal $x_1(n)$ identifies the reference signal.

5 Comparison

There are many applications of the complex fuzzy set, particularly in signal processing and image restoration, as the behaviour of CFSs is similar to the inverse discrete Fourier transform. Here, we discussed the applications of CFSs in signals and systems. One of the key problems with this practical application is that how to select a suitable model. We explored this concept in detail and used CFSs in signals and systems by adding the concept of matrices. In this application, we presented an algorithm for the identification of a reference signal out of large interest signals detected by a digital receiver. Ramot et al., in [21] proposed an algorithm to classify the unknown signal received by the digital receiver with reference signal R and in [31] the authors modified the method introduced in [21]. Xueling [17] et al . proposed the model for identifying the reference signal out of large interested signals by using new kind of matrices. Further, Smarandache and Ali [1] worked on the same algorithm for complex neutrosophic sets. The authors actually attempted to find the greatest similarity to the known signal R in all these algorithms, while the method we proposed gives a high degree of resemblance of the unknown signals received by the digital receiver. We utilized the cross product of the known signal and unknown signals and take the maximum value. We compared it with all the cross products of the known and unknown signals. We again take the maximum value among all of these values and observed that the unknown signal shows a high degree of resemblance to the reference signal. Thus we identified one unknown signal as a reference signal R among the several signals detected by the receiver. The model for identifying a reference signal provided in this paper is more efficient than the methods previously developed. Here we used the IDFT to develop an algorithm for further use in signal processing. Moreover, we compared the value of each signal separately in our proposed algorithm. Our designed model, however, is not a perfect one, but it is stuck with a lack of theoretical support. For applications, the definition of a matrix for CFSs can be useful. Therefore it will be significant for future work.

6 Conclusion

In this paper, we discussed the conjunctive normal form, disjunctive normal form, duality principle, equality of two sets and a semi Boolean algebra of complex fuzzy sets (CFSs). We established some basic results and particular examples with respect to standard complex fuzzy intersection, complex fuzzy union and complex fuzzy complement functions with the same function for determining the phase term. We utilized CFSs in signals and systems because the behavior of CFSs is similar to Fourier transforms in certain cases. Moreover, we proposed a new algorithm using a Cartesian product of complex fuzzy sets for applications in signals and systems by which we identified a reference signal out of the large number of signals detected by a digital receiver.

In future, the same technique can be used for continuous signal data using continuous Fourier transformation. This approach may also be used in geology for the detection of signals. In addition, this work and further analysis of complex fuzzy sets will provide a new path for applications in various science and engineering fields.

Conflict of Interest
The authors declare that they have no conflict of interests.

Acknowledgments: This work is financially supported by the Higher Education Commission of Pakistan (Grant No: 7750/Federal/NRPU/R&D/HEC/2017).

References

[1] M. Ali, F. Smarandache, *Complex neutrosophic set*, Neural Computing and Applications, **28**(7) (2017), 1817-1834.

[2] A. Alkouri, A. R. Salleh, *Complex intuitionistic fuzzy sets*, In: International Conference on Fundamental and Applied Sciences, AIP Conference Proceedings, **1482** (2012), 464-470.

[3] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20**(1) (1986), 87-96.

[4] K. T. Atanassov, *Interval valued intuitionistic fuzzy sets*, In Intuitionistic Fuzzy Sets, Physica, Heidelberg, (1999), 139-177.

[5] T. Aydın, S. Enginoğlu, *Interval-valued intuitionistic fuzzy parameterized interval-valued intuitionistic fuzzy soft sets and their application in decision-making*, Journal of Ambient Intelligence and Humanized Computing, **12**(1) (2021), 1541-1558.

[6] L. Bi, Z. Zeng, B. Hu, S. Dai, *Two classes of entropy measures for complex fuzzy sets*, Mathematics, **7**(1) (2019), 96.

[7] T. Y. Chen, *The likelihood-based optimization ordering model for multiple criteria group decision making with Pythagorean fuzzy uncertainty*, Neural Computing and Applications, **33**(10) (2021), 4865-4900.

[8] S. Dai, *A generalization of rotational invariance for complex fuzzy operations*, IEEE Transactions on Fuzzy Systems, **29**(5) (2020), 1152-1159.

[9] S. Dai, *Complex fuzzy ordered weighted distance measures*, Iranian Journal of Fuzzy Systems, **17**(6) (2020), 107-114.

[10] S. Dai, L. Bi, B. Hu, *Distance measures between the interval-valued complex fuzzy sets*, Mathematics, **7**(6) (2019), 549.

[11] S. Dick, *Toward complex fuzzy logic*, IEEE Transactions on Fuzzy Systems, **13**(3) (2005), 405-414.

[12] H. Garg, *A novel accuracy function under interval-valued Pythagorean fuzzy environment for solving multicriteria decision making problem*, Journal of Intelligent and Fuzzy Systems, **31**(1) (2016), 529-540.

[13] A. Garrido, *Fuzzy mathematical analysis*, In Conference Paper, September 2007.

[14] W. Huang, F. Zhang, S. Xu, *A complete ranking method for interval-valued intuitionistic fuzzy numbers and its applications to multicriteria decision making*, Soft Computing, **25**(3) (2021), 2513-2520.

[15] N. N. Karnik, J. M. Mendel, *Operations on type-2 fuzzy sets*, Fuzzy Sets and Systems, **122**(2) (2001), 327-348.

[16] Y. Liu, F. Liu, *An adaptive neuro-complex-fuzzy-inferential modeling mechanism for generating higher-order TSK models*, Neurocomputing, **365** (2019), 94-101.

[17] X. Ma, J. Zhan, M. Khan, M. Zeeshan, S. Anis, A. S. Awan, *Complex fuzzy sets with applications in signals*, Computational and Applied Mathematics, **38**(4) (2019), 1-34.

[18] T. Mahmood, *A novel approach towards bipolar soft sets and their applications*, Journal of Mathematics, (2020). DOI: 10.1155/2020/4690808.

[19] T. T. Ngan, L. T. H. Lan, M. Ali, D. Tamir, L. H. Son, T. M. Tuan, N. Rishe, A. Kandel, *Logic connectives of complex fuzzy sets*, Romanian Journal of Information Science and Technology, **21**(4) (2018), 344-358.

[20] L. Pan, X. Gao, Y. Deng, K. H. Cheong, *The constrained Pythagorean fuzzy sets and its similarity measure*, IEEE Transactions on Fuzzy Systems, **30**(4) (2021), 1102-1113.

[21] D. Ramot, R. Milo, M. Friedman, A. Kandel, *Complex fuzzy sets*, IEEE Transactions on Fuzzy Systems, **10**(2) (2002), 171-186.

[22] P. Rani, A. R. Mishra, A. Saha, D. Pamucar, *Pythagorean fuzzy weighted discrimination-based approximation approach to the assessment of sustainable bioenergy technologies for agricultural residues*, International Journal of Intelligent Systems, **36**(6) (2021), 2964-2990.

[23] I. W. Selesnick, G. Schuller, *The discrete fourier transform*, 2nd chapter of the book The transform and data compression Handbook, editted by K. R. Rao and P. C. Yip, CRC Press, Boca Raton, 2001.

[24] G. Selvachandran, S. G. Quek, L. T. H Lan, N. L. Giang, W. Ding, M. Abdel-Basset, V. H. C. Albuquerque, *A new design of Mamdani complex fuzzy inference system for multi-attribute decision making problems*, IEEE Transactions on Fuzzy Systems, **29**(4) (2019), 716-730.

[25] V. Torra, *Hesitant fuzzy sets*, International Journal of Intelligent Systems, **25**(6) (2010), 529-539.

[26] T. M. Tuan, L. T. H. Lan, S. Y. Chou, T. T. Ngan, L. H. Son, N. L. Giang, M. Ali, *M-CFIS-R: Mamdani complex fuzzy inference system with rule reduction using complex fuzzy measures in granular computing*, Mathematics, **8**(5) (2020), 707.

[27] Y. Xue, Y. Deng, H. Garg, *Uncertain database retrieval with measure–Based belief function attribute values under intuitionistic fuzzy set*, Information Sciences, **546** (2021), 436-447.

[28] R. R. Yager, *Pythagorean fuzzy subsets*, In 2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), IEEE, 57-61. DOI: 10.1109/IFSA-NAFIPS.2013.6608375.

[29] O. Yazdanbakhsh, S. Dick, *FANCFIS: Fast adaptive neuro-complex fuzzy inference system*, International Journal of Approximate Reasoning, **105** (2019), 417-430.

[30] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338-353.

[31] G. Zhang, T. S. Dillon, K. Y. Cai, J. Ma, J. Lu, *Operation properties and δ-equalities of complex fuzzy sets*, International Journal of Approximate Reasoning, **50**(8) (2009), 1227-1249.

[32] Y. Zhou, W. Cao, L. Liu, S. Agaian, C. P. Chen, *Fast fourier transform using matrix decomposition*, Information Sciences, **291** (2015), 172-183.