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A note on uniform continuity of super-additive transformations of aggregation functions

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Abstract

Extending and completing earlier results on lifting certain continuity properties of aggregation functions by super- and sub-additive transformations (J. Mahani Math. Res. Center 8 (2019) 37–51, and Iranian J. Fuzzy Sets 17 (2020) 2, 165–168), we prove that uniform continuity of multi-dimensional aggregation functions is preserved under super-additive transformations.

Keywords: aggregation function, sub-additive and super-additive transformation, uniform continuity

1 Introduction

Let $\mathbb{R}_+$ denote the non-negative real half-axis $[0, \infty[$, including the origin. Following [2], we define an n-dimensional *aggregation function* to be an arbitrary non-constant mapping $A: \mathbb{R}_+^n \to \mathbb{R}_+$ that is monotone and has the property that $A(\mathbf{0}) = A(0, \ldots, 0) = 0$.

We remark that there is some ambiguity in the usage of the term 'aggregation function'; the one defined here (and coming from the influential paper [2]) emerged as a natural extension of the one originally introduced in [1], with domains $[0, 1]^n$ and range $[0, 1]$. In general, aggregation functions on both bounded and unbounded domains have been widely studied and as representative resources we mention here only the monograph [1], the paper [2] with a number of motivating situations, and the recent collections of mathematical reflections [5].

If an aggregation function A as above satisfies the inequality $A(\mathbf{u} + \mathbf{v}) \geq A(\mathbf{u}) + A(\mathbf{v})$ for every $\mathbf{u},\ \mathbf{v} \in \mathbb{R}_+^n$, then A is said to be *super-additive*; similarly, if $A(\mathbf{u} + \mathbf{v}) \leq A(\mathbf{u}) + A(\mathbf{v})$ for every $\mathbf{u},\ \mathbf{v} \in R_+^n$ then A is *sub-additive*. These concepts suggest looking at a given n-dimensional aggregation function A in terms of its super- and sub-additive 'envelope', that is, by considering the smallest and largest functions (in the ordering $f \leq g$ if $f(\mathbf{x}) \leq g(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}_+^n$) that are, respectively, super- additive and dominating A on the one hand, and sub-additive and dominated by A on the other hand. In the first case, the value of such a 'super-additive envelope' at every $\mathbf{x} \in \mathbb{R}_+^n$ must, at the very least, be never smaller than any finite sum of the form $\sum_j A(\mathbf{x}^{(j)})$ for $\sum_j \mathbf{x}^{(j)} = \mathbf{x}$; an analogous principle applies in the second case.

This way one arrives at the notion of a *super-* and *sub-additive transformation*, A^* and A_*, of an n-dimensional aggregation function A. Originally motivated by applications in economics, the transformations were first introduced in [2] by letting, for any $\mathbf{x} \in \mathbb{R}_+^n$,

$$A^*(\mathbf{x}) = \sup \left\{ \sum_{j=1}^{k} A(\mathbf{x}^{(j)}) \ ; \ \ \mathbf{x}^{(j)} \in \mathbb{R}_+^n,\ \sum_{j=1}^{k} \mathbf{x}^{(j)} \leq \mathbf{x} \right\}, \quad \text{and} \tag{1}$$

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$$A_*(\mathbf{x}) = \inf \left\{ \sum_{j=1}^{k} A(\mathbf{x}^{(j)}) \ ; \quad \mathbf{x}^{(j)} \in \mathbb{R}_+^n, \ \sum_{j=1}^{k} \mathbf{x}^{(j)} \geq \mathbf{x} \right\} . \tag{2}$$

Both A^* and A_* are aggregation functions, with A^* super-additive and A_* sub-additive, and our introductory discussion implies that they are exactly the super- and sub-additive envelopes of A, as expected. As an easy observation, by known properties of supremum and infimum of a set of real numbers it follows that one may equivalently replace the inequality signs in (1) and (2) by 'equals' signs.

Addressing a valid concern that the supreme in (1) may not be finite, in general, by a modification of a result of [3] it follows that if $A^*(\bar{\mathbf{x}}) = \infty$ for some $\bar{\mathbf{x}} \in \mathbb{R}_+^n$ with all entries positive, then $A^*(\mathbf{x}) = \infty$ for *all* $\mathbf{x} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$. Another view of this phenomenon is offered by an observation from [4] by which $A^*(\mathbf{x})$ is bounded below by the value of the dot product $\nabla^A \cdot \mathbf{x}$, where the i-th component of the n-dimensional vector ∇^A is equal to $\limsup_{t \to 0^+} A(t\mathbf{e}_i)/t$, with $\mathbf{e}_i$ being the i-th unit vector ($1 \leq i \leq n$). It follows that A^* has all its values finite if and only if all components of ∇^A are finite; in such a case we will say that A has a *non-escaping cover* and these will be the only aggregation functions considered here.

From the point of view of theory, the existence of super- and sub-additive envelopes A^* and A_* of an aggregation function $A : \mathbb{R}_+^n \to \mathbb{R}_+$ with $A_*(\mathbf{x}) \leq A(\mathbf{x}) \leq A^*(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}_+^n$ pose the following intriguing question: For which pairs of functions $f, g : \mathbb{R}_+^n \to \mathbb{R}_+$, with f sub-additive g super-additive and $f \leq g$, does there exist an aggregation function $A : \mathbb{R}_+^n \to \mathbb{R}_+$ with the property that $A_* = f$ and $A^* = g$? The question is far from resolved, and since some necessary conditions proved, e.g., in [4] involve continuity, this prompted research into continuity properties inherited from an aggregation function by its super- and sub-additive transformations. A number of results in this direction were obtained in [7], most notably for Lipschitz and Hölder continuity; inheritance of uniform continuity was established there for sub-additive transformations in any dimension and for super-additive transformations in dimension one. Also, for a variant of aggregation functions defined on compact domains, inheritance of continuity of both transformations and in any dimension was studied in [8].

Returning to [7], the only basic question left open there was the one of 'lifting' uniform continuity to super-additive transformations in dimension greater than one, which we answer in the affirmative in this note. In the exposition we also include extensions of relevant results from [9], [7] and [8].

By a standard definition, a function $f : \mathbb{R}_+^n \to \mathbb{R}_+$ is uniformly continuous (on its domain) if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^n$ for which $||\mathbf{x} - \mathbf{y}|| < \delta$ one has $|f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon$; here $||\cdot||$ stands for the usual Euclidean norm. In the case of an aggregation function, however, one may use an equivalent description that follows from [8] (cf. also [7]) by which an n-dimensional aggregation function $A : \mathbb{R}_+^n \to \mathbb{R}_+$ is uniformly continuous if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^n$ satisfying $\mathbf{x} \geq \mathbf{y}$ (with inequality to be understood coordinate-wise) and such that $||\mathbf{x} - \mathbf{y}|| < \delta$ one has $A(\mathbf{x}) - A(\mathbf{y}) < \varepsilon$; note that the last difference is automatically non-negative. In what follows we will be working with this way of handling uniform continuity, which has the advantage of assuming the partial order $\mathbf{x} \geq \mathbf{y}$ of the points referred to.

2 Results

We begin with re-proving, in a much shorter way, an inheritance result for sub-additive transformations that can be found in [7] as Theorem 3.

Proposition 2.1. *If $A : \mathbb{R}_+^n \to \mathbb{R}_+$ is an aggregation function continuous at the origin $\mathbf{0}$, then its sub-additive transformation A_* is uniformly continuous. In particular, uniform continuity of A is inherited by A_*.*

Proof. For any given $\varepsilon > 0$, by continuity of A at $\mathbf{0}$ there exists a $\delta > 0$ such that for any $\mathbf{z} \in \mathbb{R}_+^n$ with $||\mathbf{z}|| < \delta$ one has $A(\mathbf{z}) < \varepsilon$. Let $\mathbf{x} \geq \mathbf{y}$ be an arbitrary pair of points in $\mathbb{R}_+^n$ such that $||\mathbf{x} - \mathbf{y}|| < \delta$ for the same δ as before. Letting $\mathbf{z} = \mathbf{x} - \mathbf{y}$, by sub-additivity of A_* one has $A_*(\mathbf{y}) + A_*(\mathbf{z}) \geq A_*(\mathbf{x})$, which is equivalent to $A_*(\mathbf{x}) - A_*(\mathbf{y}) \leq A_*(\mathbf{z}) = A_*(\mathbf{x} - \mathbf{y})$. By the obvious dominance of A_* by A we then have $A_*(\mathbf{x}) - A_*(\mathbf{y}) \leq A_*(\mathbf{x} - \mathbf{y}) \leq A(\mathbf{x} - \mathbf{y}) = A(\mathbf{z}) < \varepsilon$, implying uniform continuity of A_*. □

We are now ready to turn to our main goal, which is proving inheritance of uniform continuity by super-additive transformations. We will do this by converting the problem to Proposition 2.1, but prior to this we need to establish two estimates that may be of interest on their own. The first one builds on bounds on components of the vector ∇^A from the Introduction which were proved in [9] and elaborated on later in [4].

Proposition 2.2. *Let $A:\ \mathbb{R}_+^n \to \mathbb{R}_+$ be a continuous aggregation function with a non-escaping cover. Then, for every $c>0$ there is a positive α_c such that $A(\mathbf{x}) \le \alpha_c||\mathbf{x}||$ for every $\mathbf{x}\in\mathbb{R}_+^n$ with $||\mathbf{x}||\le c$.*

Proof. We first show that $\limsup_{\mathbf{x}\to\mathbf{0}} A(\mathbf{x})/||\mathbf{x}|| < \infty$. Suppose the contrary and let the limit superior be equal to $+\infty$ for an aggregation function A with a non-escaping cover. This means that there exists a sequence $(\mathbf{z}^{(j)})_{j=1}^{\infty}$ as above with limit $\mathbf{0}$ such that $A(\mathbf{z}^{(j)}) > b_j||\mathbf{z}^{(j)}||$ for every $j\ge 1$, with $b_j\to\infty$ as $j\to\infty$. For $j\ge 1$ let $a_j = ||\mathbf{z}^{(j)}||$ and let $n_j = \lceil a_j^{-1}\rceil$, that is, n_j is the unique positive integer satisfying $n_j - 1 < a_j^{-1} \le n_j$. We have the following chain of inequalities:

$$1 \le n_j a_j = n_j||\mathbf{z}^{(j)}|| = ||n_j\mathbf{z}^{(j)}|| = (n_j-1)a_j + a_j < 1 + a_j\ ;$$

as we may without loss of generality assume that $a_j \le 1$ for any $j\ge 1$ it follows that we may also assume that $1 \le ||n_j\mathbf{z}^{(j)}|| \le 2$. The definition of A^*, the assumed inequality $A(\mathbf{z}^{(j)}) > b_j||\mathbf{z}^{(j)}||$ the just established estimate $n_j||\mathbf{z}^{(j)}||\ge 1$ then imply

$$A^*(n_j\mathbf{z}^{(j)}) \ge n_jA(\mathbf{z}^{(j)}) > n_jb_j||\mathbf{z}^{(j)}|| \ge b_j\ .$$

This way we have established the existence of an infinite sequence of points $n_j\mathbf{z}^{(j)}$, all lying in the compact domain $D=\{\mathbf{x}\in\mathbb{R}_+^n;\ 1\le||\mathbf{x}||\le 2\}$, with an unbounded sequence of values $A^*(n_j\mathbf{z}^{(j)}) > b_j$. But by [8] and [3], continuity of A (with a non-escaping cover) on a compact set D implies continuity of A^* on D as well, and hence A^* is bounded on D. This contradiction shows that $\limsup_{\mathbf{x}\to\mathbf{0}} A(\mathbf{x})/||\mathbf{x}|| < \infty$, as claimed. The function $A(\mathbf{x})/||\mathbf{x}||$ is thus bounded on a set of the form $\{\mathbf{x}\in\mathbb{R}_+^n\setminus\{\mathbf{0}\};\ ||\mathbf{x}||\le\eta\}$ for some (arbitrarily small) $\eta>0$. By the same argument as above, however, continuity of A implies that the function $A(\mathbf{x})/||\mathbf{x}||$ is bounded in any compact set of the form $\{\mathbf{x}\in\mathbb{R}_+^n;\ \eta\le||\mathbf{x}||\le c\}$ for any $c>\eta$. Our statement now follows by merging the last two facts. □

Observe that essentially the same arguments imply also an upper bound on $A^*(\mathbf{x})$ by a scalar multiple of $||\mathbf{x}||$ on some neighbourhood of $\mathbf{0}$ within $\mathbb{R}_+^n$.

Our second ingredient will be, in a sense, complementary to Proposition 2.2. We prove a bound on super-additive transformations of certain aggregation functions (which include those that are *uniformly* continuous) on unbounded regions avoiding a neighbourhood of the origin. The statement has been inspired by Theorem 3.1 of [6] in dimension 1.

Proposition 2.3. *Let $A:\ \mathbb{R}_+^n \to \mathbb{R}_+$ be an aggregation function. Assume that there exists a constant $d>0$ such that for any $\mathbf{x},\mathbf{y}\in\mathbb{R}_+^n$ with $\mathbf{x}\ge\mathbf{y}$ and $||\mathbf{x}-\mathbf{y}||\le d$ one has $A(\mathbf{x})-A(\mathbf{y})\le 1$ (which is satisfied if A is uniformly continuous). Then, for each $\mathbf{x}\in\mathbb{R}_+^n$, one has $A(\mathbf{x})\le 1+d^{-1}||\mathbf{x}||$.*

Proof. Let d be as in the statement and let $\mathbf{x}\in\mathbb{R}_+^n$ be arbitrary but $\mathbf{x}\ne\mathbf{0}$. Let t be the unique positive integer such that $d(t-1) < ||\mathbf{x}|| \le dt$. For $j=1,\ldots,t$ define $\mathbf{y}^{(j)} = b_j\mathbf{x}$ for $b_j=(j-1)d/||\mathbf{x}||$; note that $||\mathbf{y}^{(j)}|| = (j-1)d$ and, in particular, $\mathbf{y}^{(1)}=\mathbf{0}$. Since $||\mathbf{y}^{(j+1)}-\mathbf{y}^{(j)}|| = d$ for $1\le j\le t-1$, we have $A(\mathbf{y}^{(j+1)})-A(\mathbf{y}^{(j)})\le 1$ by our assumption on A. One may check that the inequality $||\mathbf{x}||\le dt$ implies $||\mathbf{x}-\mathbf{y}^{(t)}||\le d$, so that by the same assumption we also have $A(\mathbf{x})-A(\mathbf{y}^{(t)})\le 1$.

Adding the $t-1$ inequalities $A(\mathbf{y}^{(j+1)})-A(\mathbf{y}^{(j)})\le 1$ for $1\le j\le t-1$ together with $A(\mathbf{x})-A(\mathbf{y}^{(t)})\le 1$ give $A(\mathbf{x})\le t$. Now, $d(t-1)<||\mathbf{x}||$ implies that $t<1+d^{-1}||\mathbf{x}||$, which together with $A(\mathbf{x})\le t$ gives $A(\mathbf{x})<1+d^{-1}||\mathbf{x}||$, an inequality valid for *every* $\mathbf{x}\in\mathbb{R}_+^n$. □

As an aside, the function A defined on $[0,\infty[=\cup_{j\ge1}[j-1,j]$ for every positive integer j by

$$A(x) = \begin{cases} j-1 & \text{if } x\in[j-1, j-\frac{1}{j}] \\ jx + j(1-j) & \text{if } x\in[j-\frac{1}{j}, j]\ . \end{cases}$$

is an example of a (one-dimensional) aggregation satisfying the assumptions of Proposition 2.3 for $d=1/2$ which is *not* uniformly continuous because of unboundedly increasing slopes on the non-horizontal straight line segments in the intervals $[j-1/j, j]$.

We note that in the statement of Proposition 2.3 we could have assumed the existence of some positive d, d' such that $A(\mathbf{x})-A(\mathbf{y})\le d'$ whenever $\mathbf{x}\ge\mathbf{y}$ and $||\mathbf{x}-\mathbf{y}||\le d$; we have chosen to work with $d'=1$ just for simplicity.

It now remains to put the pieces together. We begin with a fundamental and universal estimate on values of the super-additive transformation of a uniformly continuous aggregation function, which we state in a much more general form.

Theorem 2.4. *Let $A:\ \mathbb{R}_+^n \to \mathbb{R}_+$ be a continuous aggregation function with a non-escaping cover, for which there is a constant $d>0$ such that for any $\mathbf{x},\mathbf{y}\in\mathbb{R}_+^n$ with $\mathbf{x}\ge\mathbf{y}$ and $||\mathbf{x}-\mathbf{y}||\le d$ it holds that $A(\mathbf{x})-A(\mathbf{y})\le 1$ (existence of such d is automatic if A is uniformly continuous). Then there is an $\alpha_A>0$ such that $A(\mathbf{x})\le\alpha_A||\mathbf{x}||$ for every $\mathbf{x}\in\mathbb{R}_+^n$.*

Proof. Let A and d be as in the statement. By the result of Proposition 2.2 applied to the value of $c = d$ there exists an $\alpha_c = \alpha_d > 0$ such that $A(\mathbf{x}) \le \alpha_d ||\mathbf{x}||$ for every $\mathbf{x} \in \mathbb{R}^n_+$ with $||\mathbf{x}|| \le d$. Since the term $\alpha_d ||\mathbf{x}||$ is an upper bound on $A(\mathbf{x})$ for $||\mathbf{x}|| \le d$, we may (and will) assume that $\alpha_d \ge 2d^{-1}$. Defining now $\alpha_A = \alpha_d$, from $\alpha_A \ge 2d^{-1}$ we have $\alpha_A - d^{-1} \ge d^{-1}$ and so $d \ge (\alpha_A - d^{-1})^{-1}$. It follows that for every $\mathbf{x} \in \mathbb{R}^n_+$ such that $||\mathbf{x}|| \ge d$ we also have $||\mathbf{x}|| \ge (\alpha_A - d^{-1})^{-1}$. Multiplication of both sides of this inequality by the inverse of the right-hand side gives $(\alpha_A - d^{-1})||\mathbf{x}|| \ge 1$, which after multiplying through simplifies to $\alpha_A ||\mathbf{x}|| \ge 1 + d^{-1}||\mathbf{x}||$. But the right-hand side of the last inequality is, by Proposition 2.3, an upper bound for $A(\mathbf{x})$. This yields $A(\mathbf{x}) \le \alpha_A ||\mathbf{x}||$ for every $\mathbf{x} \in \mathbb{R}^n_+$ such that $||\mathbf{x}|| \ge d$. By the first part of the proof, however, we have $A(\mathbf{x}) \le \alpha_A ||\mathbf{x}||$ for every $\mathbf{x} \in \mathbb{R}^n_+$ such that $||\mathbf{x}|| \le d$. Summing up, we have shown that $A(\mathbf{x}) \le \alpha_A ||\mathbf{x}||$ for *every* $\mathbf{x} \in \mathbb{R}^n_+$. □

Our main result on inheritance of uniform continuity of aggregation functions by super-additive transformations will now be a relatively easy consequence of the previously established facts.

Theorem 2.5. *Let $A : \mathbb{R}^n_+ \to \mathbb{R}_+$ be an aggregation function with a non-escaping cover. If A is uniformly continuous, then so is its super-additive transformation A^*.*

Proof. An aggregation function A as in the statement satisfies the assumptions of Theorem 2.4 and so there is an $\alpha_A > 0$ such that $A(\mathbf{x}) \le \alpha_A ||\mathbf{x}||$ for every $\mathbf{x} \in \mathbb{R}^n_+$. For $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$ let $\sigma(\mathbf{x}) = \sum_{i=1}^n x_i$. Since obviously $||\mathbf{x}|| \le \sigma(\mathbf{x})$ for our points $\mathbf{x}$, it follows that $A(\mathbf{x}) \le \alpha_A \sigma(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n_+$. This inequality shows that the function $B : \mathbb{R}^n_+ \to \mathbb{R}_+$ given by $B(\mathbf{x}) = \alpha_A \sigma(\mathbf{x}) - A(\mathbf{x})$ is also an aggregation function. Moreover, since $\sigma(\mathbf{x})$ is a linear function, a straightforward calculation employing the fact that $\inf(-M) = -\sup(M)$ for any non-empty set M of real numbers shows that the sub-additive transformation B_* of B is related to the super-additive transformation A^* of A by $B_*(\mathbf{x}) = \alpha_A \sigma(\mathbf{x}) - A^*(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^n_+$. The function B is obviously continuous at the origin, and so B_* is uniformly continuous by Proposition 2.1. But $\sigma : \mathbf{x} \mapsto \alpha_A \sigma(\mathbf{x})$ is clearly uniformly continuous on $\mathbb{R}^n_+$, and hence also $A^*(\mathbf{x}) = \alpha_A \sigma(\mathbf{x}) - B_*(\mathbf{x})$ is uniformly continuous. □

We remark that the assumption of uniform continuity of A in Theorem 2.5 can be replaced by a much weaker assumption of Proposition 2.3. However, the assumption of A having a non-escaping cover cannot be omitted even in simplest cases, as shown e.g. in dimension 1 by the function $A(x) = \sqrt{x}$, which is uniformly continuous on $[0, \infty[$ but its super-additive transformation A^* is equal to infinity on the open interval $]0, \infty[$.

3 Summary

In this note we have completed a study of inheritance of continuity properties of aggregation functions to their super- and sub-additive transformations, initiated in [7] for Lipschitz and Hölder continuity and for uniform continuity with respect to sub-additive transformations, and in [8] for pointwise continuity on compact domains, by proving that uniform continuity of aggregation functions is inherited also by their super-additive transformations.

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