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The Craig interpolation property for rational Gödel logic

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Abstract

In this article, the Craig interpolation property for rational Gödel logic is studied. Despite classical Gödel logic, this property can be proved in this new extension of Gödel logic. This new predicate version of Gödel logic is similar to continuous logic and also, its semantics is extended similar to metric model theory with some differences.

Keywords: Rational Gödel logic, the Craig interpolation property, standard Gödel logic.

1 Introduction

In this paper, some model-theoretic properties of first-order rational Gödel logic are introduced. This logic is an extension of Gödel logic. The relation of this extension to standard Gödel logic is the same as the rational Pavelka logic to Łukasiewicz logic. The approach which is used here is more similar to (metric) model theory [4] than the ones used in fuzzy logics.
The main subject of the model theory that is studied here is the Craig interpolation property. This property states for every two formulas φ and ψ in the languages $\mathcal{L}_1$ and $\mathcal{L}_2$ with the property $\varphi \models \psi$, there is a formula θ in $\mathcal{L}_1 \cap \mathcal{L}_2$ such that $\varphi \models \theta$ and $\theta \models \psi$. It is an open problem whether this property holds for standard Gödel logic, $\mathbb{G}_{[0,1]}$, see [2]. One of the obstacles to prove this property in the standard Gödel logic is that one can not determine the location of the value of a formula by another specific formula. But, in this new logic, by adding rational numbers to be as connectives and by extending the set of truth values, the existence of this property can be proved. Indeed, an approximate version of this property can be proved. By choosing $\mathbb{I} = [0,1]^2 \backslash \{(0,r) : r > 0\}$ as the set of truth values and showing a rational number r as $\hat{r} = (r,r)$, one can specify the location of the value of every formula accurately. The interpretation of $\rightarrow$ in the formula is the same as in $\mathbf{G}_{[0,1]}$. Also, for every $r \in [0,1]_{\mathbb{Q}}$, the nullary connective $\bar{r}$ is added to the language. The interpretation of $\bar{r}$ is $\hat{r}$, for every $r \in [0,1] \cap \mathbb{Q}$. The formulas used for the above purpose are in the forms of $\varphi \rightarrow \bar{r}$ or $\bar{r} \rightarrow \varphi$. So, the interpretations of the two above formulas in every structure $\mathcal{M}$ are as $\varphi^{\mathcal{M}} \geq \hat{r}$ or vice versa.
As a historical review, this property was proved in classical logic by Craig [6] and for intuitionistic predicate logic by Schütte [10]. After that, Maksimova studied completely this property for the intermediate propositional logic [8]. In 2013, Mints, Olkhovikov, and Urquhart in [9] showed that constant domain intuitionistic logic does not have the interpolation property. Also, in [3] it was shown that constant domain intermediate logics based on finite algebras of truth values as well as some fragments of Gödel logic do have the interpolation property. Finally, in 2018 Baaz, Gehrke, and Van Gool in a short preprint claimed that the counterexample in [9] for constant domain intuitionistic logic admits an interpolant for predicate Gödel logic, [2].

In the preliminaries, rational Gödel logic is introduced briefly. The syntax and semantics of this logic are explained. In the main section, an approximate version of Craig interpolation property for this logic is studied and proved. This approximation states that two sentences φ and ψ in $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, have approximate Craig interpolation property if $\varphi \models \psi$ implies that for every $s \in (0,1] \cap \mathbb{Q}$, there is a sentence θ in $\mathcal{L}_1 \cap \mathcal{L}_2$ such that $\varphi \models \theta$ and $\theta \models \bar{s} \rightarrow \psi$. The proof of the existence of this property is analogous to what is used in classical logic. Note that the semantics

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of rational Gödel logic is similar to the metric model theory. So, one may extend some proof in classical logic to the rational Gödel logic, which is a predicate extension of Gödel logic.

2 Preliminaries

Let $\mathcal{L}$ be a first-order language consisting of countably many predicate, function, and constant symbols. As usual, $\mathcal{L}$ has a countable set of variables, the quantifiers $\{\forall, \exists\}$, a set of Boolean connectives $\{\wedge, \vee, \rightarrow, \neg\}$ and a set of nullary connectives $\{\bar{r} : \ r \in [0,1]_{\mathbb{Q}}\}$, where $[0,1]_{\mathbb{Q}} = [0,1] \cap \mathbb{Q}$. Adding the extra nullary connectives relative to each rational number is the reason of calling this new logic, the rational Gödel logic.
Unlike the standard semantics of Gödel logic, $\mathbb{I} = [0,1]^2 \backslash \{(0,r) : r > 0\}$ with lexicographical ordering is the set of truth values; Otherwise, the compactness theorem does not hold for rational Gödel logic, RGL*, [7].
The concepts of an $\mathcal{L}$-structure, an assignment, $\mathcal{L}$-terms, their interpretations, and $\mathcal{L}$-formulas are defined as usual, see Definitions 2.1 and 2.2 in [7]. The interpretation of $\mathcal{L}$-formulas is defined as follows.

Definition 2.1. [7] Let $\mathcal{M}$ be an $\mathcal{L}$-structure and σ be an $\mathcal{M}$-assignment.

1. for every $\bar{r} \in [0,1]_{\mathbb{Q}}$, $\bar{r}^{\mathcal{M},\sigma} = (r,r) = \hat{r}$. Particularly, $\bar{1}^{\mathcal{M},\sigma} = \hat{1}$ and $\bar{0}^{\mathcal{M},\sigma} = \hat{0}$.
2. $P^{\mathcal{M},\sigma}(t_1(\bar{x}), ..., t_n(\bar{x})) = P^{\mathcal{M}}(t_1^{\mathcal{M},\sigma}(\bar{x}), ..., t_n^{\mathcal{M},\sigma}(\bar{x}))$.
3. $(\varphi \wedge \psi)^{\mathcal{M},\sigma}(\bar{x}) = \max\{\varphi^{\mathcal{M},\sigma}(\bar{x}), \psi^{\mathcal{M},\sigma}(\bar{x})\}$.
4. $(\varphi \rightarrow \psi)^{\mathcal{M},\sigma}(\bar{x}) = \begin{cases} 0 & \varphi^{\mathcal{M},\sigma}(\bar{x}) \geq \psi^{\mathcal{M},\sigma}(\bar{x}), \\ \psi^{\mathcal{M},\sigma}(\bar{x}) & \varphi^{\mathcal{M},\sigma}(\bar{x}) < \psi^{\mathcal{M},\sigma}(\bar{x}). \end{cases}$
5. $(\forall x \ \varphi(x))^{\mathcal{M},\sigma} = \sup\{\varphi^{\mathcal{M},\sigma'}(x) : \sigma(x) = \sigma'(x)\}$.
6. $(\exists x \ \varphi(x))^{\mathcal{M},\sigma} = \inf\{\varphi^{\mathcal{M},\sigma'}(x) : \sigma(x) = \sigma'(x)\}$.

The connectives $\vee, \neg, \leftrightarrow$ are defined by $\wedge$, $\rightarrow$, $\bar{1}$:

$$\neg\varphi := \varphi \rightarrow \bar{1}.$$
$$\varphi \vee \psi := ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi).$$
$$\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi).$$

Also, their interpretations are defined as

$$(\varphi \vee \psi)^{\mathcal{M},\sigma}(\bar{x}) = \min\{\varphi^{\mathcal{M},\sigma}(\bar{x}), \psi^{\mathcal{M},\sigma}(\bar{x})\},$$
$$(\varphi \leftrightarrow \psi)^{\mathcal{M},\sigma}(\bar{x}) = d_{max}(\varphi^{\mathcal{M},\sigma}(\bar{x}), \psi^{\mathcal{M},\sigma}(\bar{x})),$$

where

$$d_{max}(x,y) = \begin{cases} \max\{x,y\} & x \neq y, \\ \hat{0} & x = y. \end{cases}$$

Definition 2.2. [7] Let $\mathcal{M}$ be an $\mathcal{L}$-structure. For an $\mathcal{L}$-formula $\varphi(\bar{x})$ and an $\mathcal{L}$-theory T,

1. $\varphi(\bar{x})$ is *satisfied* by $\bar{a} \in M$ if $\varphi^{\mathcal{M}}(\bar{a}) = \hat{0}$. In this situation, we write $\mathcal{M} \models \varphi(\bar{a})$. T is satisfiable in $\mathcal{M}$, if $\mathcal{M} \models \psi$ for every $\psi \in T$. This is denoted by $\mathcal{M} \models T$.
2. For an $\mathcal{L}$-sentence φ, we say that T *entails* φ, $T \models \varphi$, if for any $\mathcal{L}$-structure $\mathcal{M} \models T$ we have $\mathcal{M} \models \varphi$.

Remark 2.3. Let $u : [0,1]^2 \rightarrow [0,1]^2$ defined by $u(x,y) = (1-x, 1-y)$ and $\mathbb{I}^* = [0,1]^2 \backslash \{(1,r) : r < 1\}$. By using the function u, $\mathbb{I}^*$ can be considered as the set of truth values. Moreover, one may translate all semantical issues of RLG* , e.g, satisfiability and (strong) entailment to the fuzzy first-order rational Gödel logic. Hence all the results given in this section remain valid for the fuzzy first-order rational Gödel logic.

Satisfiability and finitely satisfiability are defined as usual.

The proof system for RGL* can be defined as Section 2.1 in [7]. This proof system has some axioms for propositional Gödel logic and quantifiers. Also, there are some book keeping axioms:

- (RGL1) $\bar{r} \wedge \bar{s} \leftrightarrow \overline{\max\{r,s\}}$,

- (RGL2) $\begin{cases} \bar{r} \to \bar{s} & \text{if } r \geq s, \\ (\bar{r} \to \bar{s}) \leftrightarrow \bar{s} & \text{if } r < s, \end{cases}$
- (RGL3) $\neg\neg\bar{r}$, for all $r < 1$.

The inference rules are *modus ponens* and *generalization*. The notion of proof is defined as usual. When an $\mathcal{L}$-theory T *proves* an $\mathcal{L}$-sentence φ, we denote it by $T \vdash \varphi$. T is called *consistent* if $T \nvdash \bar{1}$. Otherwise, it is *inconsistent*.

Definition 2.4. [7] An $\mathcal{L}$-theory T is called strongly consistent if $T \nvdash \bar{r}$ for every $r > 0$.

Note that for any $0 < r < 1$, $T = \{\bar{r}\}$ is consistent but it is not strongly consistent.

The Soundness Theorem holds in this logic and can be proved in the usual way.

Remark 2.5. If $T \vdash \varphi$ then $T \models \varphi$ for every $\mathcal{L}$-theory T and $\mathcal{L}$-sentence φ.

Therefore, if a theory T is satisfiable in an $\mathcal{L}$-structure $\mathcal{M}$ then T is strongly consistent [7].

The following lemma shows some properties of the proof in RGL*.

Lemma 2.6. [7] Suppose that T is an $\mathcal{L}$-theory and $\varphi, \psi, \chi \in Sent(\mathcal{L})$. Then,

(i) $\vdash \varphi \to (\psi \to \varphi)$.

(ii) $\vdash \varphi \to (\psi \to (\varphi \wedge \psi))$.

(iii) $\vdash (\bar{0} \to \varphi) \to \varphi$.

(iv) If $T \vdash \varphi$ and $T \vdash \psi$ then $T \vdash \varphi \wedge \psi$.

(v) If $T \vdash \varphi \to \psi$ and $T \vdash \psi \to \chi$ then $T \vdash \varphi \to \chi$.

(vi) $\vdash \big((\varphi \to \psi) \to (\psi \to \varphi)\big) \to (\psi \to \varphi)$.

(vii) $\vdash \varphi \to (\varphi \vee \psi)$.

(viii) $\vdash \big((\varphi \to \psi) \to \chi\big) \to \big((\psi \to \varphi) \vee \chi\big)$.

We will also use the followings in the main proof which are from [7]. So, next, the concepts of a linear complete and a maximally consistent theory are expressed. Also, a characterization of a maximally consistent theory is mentioned. Finally, a weak version of the completeness theorem is presented.

Lemma 2.7. Let T be a strongly consistent $\mathcal{L}$-theory in RGL* and $\varphi, \psi \in Sent(\mathcal{L})$. Then, either $T \cup \{\varphi \to \psi\}$ or $T \cup \{\psi \to \varphi\}$ are strongly consistent.

A theory T is *linear complete* if for every two sentences φ, ψ, either $T \vdash \varphi \to \psi$ or $T \vdash \psi \to \varphi$.

Definition 2.8. A strongly consistent theory T is called maximally strongly consistent if it can not be properly included in any strongly consistent theory, i.e., for any strongly consistent theory Σ, if $T \subseteq \Sigma$ then $T = \Sigma$.

Lemma 2.9. Let T be a strongly consistent $\mathcal{L}$-theory. Then, T is maximally strongly consistent if and only if

1. for all $\varphi, \psi \in Sent(\mathcal{L})$, either $\varphi \to \psi \in T$ or $\psi \to \varphi \in T$,
2. if $\varphi \in Sent(\mathcal{L})$ and $T \vdash \bar{r} \to \varphi$ for all $r > 0$, then $\varphi \in T$.

Definition 2.10. An $\mathcal{L}$-theory T is Henkin complete if $T \nvdash \forall x\, \varphi(x)$ then there is a constant symbol $c \in \mathcal{L}$ such that $T \nvdash \varphi(c)$.

Theorem 2.11. In RGL*, any strongly consistent theory is satisfiable.

The followings are the deduction theorem and weak completeness theorem.

Theorem 2.12. (Deduction Theorem)For every theory T and $\mathcal{L}$-sentence φ, $T \cup \{\varphi\} \vdash \psi$ if and only if $T \vdash \varphi \to \psi$.

Theorem 2.13. *(Weak completeness theorem) For every sentence φ, if $\models \varphi$, then $\vdash \varphi$.*

3 The Craig interpolation property

In this section, first, the notions of separability and approximate Craig interpolation property are introduced.

Definition 3.1. Two sentences φ and ψ in $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, have *approximate Craig interpolation property* whenever $\varphi \models \psi$ implies that for every $s \in (0,1]_{\mathbb{Q}}$ there is an $\mathcal{L}_1 \cap \mathcal{L}_2$-sentence θ such that $\varphi \models \theta$ and $\theta \models \bar{s} \to \psi$.

In the above definition, $\varphi \models \psi$ implies for every $s \in [0,1]_{\mathbb{Q}}$, one can find some θ in the common part of the both related languages, such that for every structure $\mathcal{M}$, $\varphi^{\mathcal{M}} = \hat{0}$ deduces $\theta^{\mathcal{M}} = \hat{0}$. But, for every structure $\mathcal{M}$, $\theta^{\mathcal{M}} = \hat{0}$ concludes $\hat{s} \geq \psi^{\mathcal{M}}$. The last part is different to the one in the classical logic. Since in the classical model theory, it should be $\psi^{\mathcal{M}} = 0$. Therefore, this new version of Craig interpolation property is called approximate one. Since we can not find some θ in the above framework such that $\theta^{\mathcal{M}} = \hat{0}$ implies that $\psi^{\mathcal{M}} = \hat{0}$ and $\psi^{\mathcal{M}}$ has a distance to $\hat{0}$ but it is less than $\hat{s}$. Note that θ is related to s and changes by varying s.

To begin the proof, first, the following definition is needed. One of the methods to prove the Craig interpolation property in the classical logic is based on using the concept of separable theories, [5]. Due to the approximate nature of the new framework, the notion of separability is defined as follows.

Definition 3.2. $\mathcal{L}_1$-theory T and $\mathcal{L}_2$-theory U are called *separable* if there are an $\mathcal{L}_1 \cap \mathcal{L}_2$-sentence θ and $r \in (0,1]_{\mathbb{Q}}$ such that $T \models \theta$ and $U \models \theta \to \bar{r}$.

Assume T and U are separable as above. Let $\theta' = \theta \to \bar{r}$. Then, $T \models \theta' \to \bar{r}$ and $U \models \theta'$ are also obtained.

In the rest of the article, approximate Craig interpolation property will be proved for RGL* by contradiction. So, let $\varphi \models \psi$ and φ and ψ have no approximate Craig interpolant. Let $\mathcal{L}_1$ be the language of all symbols of φ and $\mathcal{L}_2$ be the language of all symbols of ψ. Also, assume $\mathcal{L}_0 = \mathcal{L}_1 \cap \mathcal{L}_2$.

Lemma 3.3. By the above assumptions, there is some $s \in (0,1]_{\mathbb{Q}}$ such that $\{\varphi\}$ and $\{\psi \to \bar{s}\}$ are inseparable.

Proof. Let $\{\varphi\}$ and $\{\psi \to \bar{s}\}$ be separable. So, for each $s \in (0,1]_{\mathbb{Q}}$, there are an $\mathcal{L}_0$-sentence θ and $r \in (0,1]_{\mathbb{Q}}$ such that $\varphi \models \theta$ and $\psi \to \bar{s} \models \theta \to \bar{r}$. Therefore, for every structure $\mathcal{M} \models \psi \to \bar{s}$ we have $\mathcal{M} \models \theta \to \bar{r}$. It means if $\psi^{\mathcal{M}} \geq \hat{s} > \hat{0}$ then $\theta^{\mathcal{M}} \geq \hat{r} > \hat{0}$. Thus, for every structure $\mathcal{M}$, $\theta^{\mathcal{M}} = \hat{0}$ implies $\psi^{\mathcal{M}} < \hat{s}$, i.e. $(\bar{s} \to \psi)^{\mathcal{M}} = 0$ and so, $\theta \models \bar{s} \to \psi$. Therefore, φ and ψ have an approximate Craig interpolant for every $s \in (0,1]_{\mathbb{Q}}$, which is a contradiction. □

Also, one can conclude the following propositions for separability and inseparability.

Proposition 3.4. If T and U are separable with some θ and $r \in (0,1]_{\mathbb{Q}}$, then, they are separable with θ and every $s \leq r$.

Proof. Let $T \models \theta$ and $U \models \theta \to \bar{r}$. Assume $U \models \theta \to \bar{s}$ is not correct. So, for a model $\mathcal{M} \models U$, $(\theta \to \bar{s})^{\mathcal{M}} \neq \hat{0}$. Thus, $\theta^{\mathcal{M}} < \hat{s}$. Since $s \leq r$ we have $\hat{s} \leq \hat{r}$. Then, $\theta^{\mathcal{M}} < \hat{r}$ and this is a contradiction by hypothesis. □

Proposition 3.5. Let $T \cup \{\varphi \to \bar{r}\}$ and U be inseparable for some sentence φ and $r \in (0,1]_{\mathbb{Q}}$. Then $T \cup \{\varphi \to \bar{s}\}$ and U are inseparable for every $s \leq r$.

Proof. Let $T \cup \{\varphi \to \bar{s}\}$ and U be separable. Then, there exist a sentence θ and $t \in (0,1]_{\mathbb{Q}}$ such that $T \cup \{\varphi \to \bar{s}\} \models \theta$ and $U \models \theta \to \bar{t}$. Thus, for every model $\mathcal{M} \models T$, if $\varphi^{\mathcal{M}} \geq \hat{s}$ then $\theta^{\mathcal{M}} = 0$. If $\varphi^{\mathcal{M}} \geq \hat{r}$, since $r \geq s$, then $\hat{r} \geq \hat{s}$ and $\theta^{\mathcal{M}} = 0$. Therefore, $T \cup \{\varphi \to \bar{r}\}$ and U are separable which is a contradiction. □

The next lemma is used in the main proof.

Lemma 3.6. Let $r_1, r_2, \ldots, r_n \in (0,1]_{\mathbb{Q}}$ and φ be a sentence. Then,

$$\overline{\min\{r_1, r_2, \ldots, r_n\}} \to \varphi \vdash \bar{r}_i \to \varphi,$$

for every $1 \leq i \leq n$.

Proof. Set $t = \min\{r_1, \ldots, r_n\}$. First, we prove $\models (\bar{t} \to \varphi) \to (\bar{r}_i \to \varphi)$, for every $1 \leq i \leq n$. For every structure $\mathcal{M}$, we have two cases:

1. If $\hat{t} \geq \varphi^{\mathcal{M}}$ then $\hat{r_i} \geq \varphi^{\mathcal{M}}$ and $(\bar{t} \to \varphi) \to (\bar{r}_i \to \varphi)^{\mathcal{M}} = \hat{0}$, for every $1 \leq i \leq n$.

2. If $\hat{t} < \varphi^{\mathcal{M}}$ then $(\bar{t} \to \varphi)^{\mathcal{M}} = \varphi^{\mathcal{M}}$. For every $1 \leq i \leq n$, if $\hat{r_i} < \varphi^{\mathcal{M}}$ then $(\bar{r_i} \to \varphi)^{\mathcal{M}} = \varphi^{\mathcal{M}}$. Else, $(\bar{r_i} \to \varphi)^{\mathcal{M}} = \hat{0}$. In both cases, again, we have $(\bar{t} \to \varphi) \to (\bar{r_i} \to \varphi)^{\mathcal{M}} = \hat{0}$.

So, by Theorems 2.12 and 2.13 , the result is obvious. □

To resumption of the proof of existence of an approximate interpolant for RGL*, let s_0 be the rational number in $(0,1]$ for which $\{\varphi\}$ and $\{\psi \to \bar{s}_0\}$ are inseparable (Lemma 3.3). Assume $\mathcal{C}$ is a new set of constants and for $i = 0,1,2$,

$$\mathcal{L}'_i = \mathcal{L}_i \cup \mathcal{C}.$$

Let $\varphi_1, \varphi_2, \ldots$ be an enumeration of sentences in $\mathcal{L}'_1$ and $\psi_1, \psi_2, \ldots$ be an enumeration of sentences in $\mathcal{L}'_2$. For every $i \in \mathbb{N}$, $\mathcal{L}'_1$-theory T_i and $\mathcal{L}'_2$-theory U_i are constructed in the following way;

$$\{\varphi\} = T_1 \subseteq T_2 \subseteq T_3 \subseteq \ldots$$

and

$$\{\psi \to \bar{s}_0\} = U_1 \subseteq U_2 \subseteq U_3 \subseteq \ldots$$

such that

1. T_i and U_i are inseparable.
2. (a) If there exists $r \in (0,1]_{\mathbb{Q}}$ such that $T_n \cup \{\varphi_n \to \bar{r}\}$ and U_n are inseparable then set $T \cup \{\varphi_n \to \bar{r}\} \subseteq T_{n+1}$ and $r_n = r$. Otherwise, $\{\bar{r} \to \varphi_n\}_{r \in (0,1]_{\mathbb{Q}}} \cup \{\varphi_n\} \subseteq T_{n+1}$.
   (b) If there exists $s \in (0,1]_{\mathbb{Q}}$ such that $U_n \cup \{\psi_n \to \bar{s}\}$ and T_{n+1} are inseparable then set $U_n \cup \{\psi_n \to \bar{s}\} \subseteq U_{n+1}$ and $s_n = s$. Otherwise, $\{\bar{s} \to \psi_n\}_{s \in (0,1]_{\mathbb{Q}}} \cup \{\psi_n\} \subseteq U_{n+1}$.
3. (a) If $\varphi_n = \forall x\ \delta(x)$ and $\varphi_n \notin T_{n+1}$ then there exist $c \in \mathcal{C}$ and $r \in (0,1]_{\mathbb{Q}}$ such that $\delta(c) \to \bar{r} \in T_{n+1}$.
   (b) If $\psi_n = \forall x\ \sigma(x)$ and $\psi_n \notin U_{n+1}$ then there exist $d \in \mathcal{C}$ and $s \in (0,1]_{\mathbb{Q}}$ such that $\sigma(d) \to \bar{s} \in U_{n+1}$.

   c and d are not used before.

Now, let $T_\omega = \bigcup_{i \in \mathbb{N}} T_i$ and $U_\omega = \bigcup_{i \in \mathbb{N}} U_i$. Then, T_ω and U_ω have the following properties:

1. T_ω and U_ω are inseparable.

   Assume they are separable. So, there are $r \in (0,1]_{\mathbb{Q}}$ and an $\mathcal{L}'_0$-sentence θ such that $T_\omega \models \theta$ and $U_\omega \models \theta \to \bar{r}$. So, by the construction, for some $n \in \mathbb{N}$, $T_n \models \theta$ and $U_n \models \theta \to \bar{r}$. This is a contradiction.

2. T_ω and U_ω are strongly consistent.

   Claim 3.7. For each $n \in \mathbb{N}$, if T_n is strongly consistent then so is T_{n+1}. Similar property holds for U_n.

   We prove it for T_n, for every $n \in \mathbb{N}$. Let T_n be strongly consistent. There are two cases:

   (a) Let $\varphi_n \to \bar{r}_n \in T_{n+1}$ and T_{n+1} be not strongly consistent. So, there is $k \in (0,1]_{\mathbb{Q}}$ such that $T_{n+1} \vdash \bar{k}$. If $T_n \cup \{\varphi_n \to \bar{r}_n\} \vdash \bar{k}$ then $T_n \vdash (\varphi_n \to \bar{r}_n) \to \bar{k}$. Also, by Lemma 2.6(i), $T_n \vdash \bar{r}_n \to (\varphi_n \to \bar{r}_n)$. By Lemma 2.6(v), $T_n \vdash \bar{r}_n \to \bar{k}$. By the procedure of constructing T_{n+1} and Proposition 3.5, $T_n \vdash \bar{r} \to \bar{k}$ for some $r < k$. So, by RGL2, $T_n \vdash \bar{k}$, which is a contradiction. So, $S_n = T_n \cup \{\varphi_n \to \bar{r}_n\}$ is strongly consistent.
   Now, let $\varphi_n = \forall x\ \delta(x)$ and $S_n \cup \{\delta(c) \to \bar{s}\} \vdash \bar{k}$ for some $s < k$. Then, $S_n \vdash (\delta(c) \to \bar{s}) \to \bar{k}$. Again by Lemma 2.6(i), $S_n \vdash \bar{s} \to (\delta(c) \to \bar{s})$. Therefore, $S_n \vdash \bar{s} \to \bar{k}$. Since $s < k$, we have $S_n \vdash \bar{k}$, which is a contradiction.

   (b) Assume that $T_{n+1} = T_n \cup \{\bar{r} \to \varphi_n\}_{r \in (0,1]_{\mathbb{Q}}} \cup \{\varphi_n\}$ and T_{n+1} is not strongly consistent. So, there is $k \in (0,1]_{\mathbb{Q}}$ such that $T_{n+1} \vdash \bar{k}$. Thus, $T_n \cup \{\bar{r} \to \varphi_n\}_{r \in \mathbb{Q}} \vdash \varphi_n \to \bar{k}$. Therefore, there are $r_1, \ldots, r_m \in (0,1]_{\mathbb{Q}}$ such that

   $$T_n \cup \{\bar{r}_1 \to \varphi_n, \ldots, \bar{r}_m \to \varphi_n\} \vdash \varphi_n \to \bar{k}.$$

   By Lemma 3.6,

   $$T_n \cup \{\overline{\min\{r_1, r_2, \ldots, r_m\}} \to \varphi_n\} \vdash \varphi_n \to \bar{k}.$$

   Set $t = \min\{r_1, r_2, \ldots, r_m\}$. So,

   $$T_n \cup \{\bar{t} \to \varphi_n\} \vdash \varphi_n \to \bar{k}.$$

   And,

   $$T_n \vdash (\bar{t} \to \varphi_n) \to (\varphi_n \to \bar{k}).$$

   Therefore, $T_n \vdash \bar{t} \to \bar{k}$. By choosing in a suitable way, t can be less than k. So, $T_n \vdash \bar{k}$, by RGL2. This is a contradiction.

3. T_ω and U_ω are maximally strongly consistent.

   By Lemma 2.9.2, it is sufficient to prove that for every θ and χ in $\mathcal{L}_1$, $\theta \to \chi$ or $\chi \to \theta$ is in T_ω. Assume $\theta \to \chi$ and $\chi \to \theta$ are not in T_ω. So, $\theta \to \chi = \varphi_m$ and $\chi \to \theta = \varphi_n$. Therefore, φ_m and φ_n are not in T_ω and $\varphi_m \to \bar{r}_m$ and $\varphi_n \to \bar{r}_n$ are in T_ω. By Theorem 2.11, since T_ω is strongly consistent, there is a model $\mathcal{M} \models T_\omega$. So, $\varphi_m^{\mathcal{M}} \geq \hat{r}_m$ and $\varphi_n^{\mathcal{M}} \geq \hat{r}_n$. Therefore, $\theta^{\mathcal{M}} < \chi^{\mathcal{M}}$ and $\chi^{\mathcal{M}} \geq \hat{r}_m$. Also, $\chi^{\mathcal{M}} < \theta^{\mathcal{M}}$ and $\theta^{\mathcal{M}} \geq \hat{r}_n$. This is impossible. In a similar way, the proof is done for U_ω.

4. $T_\omega \cap U_\omega$ is maximally strongly consistent.
   Let $T_\omega \cap U_\omega$ is not maximally strongly consistent. Then, there are φ and ψ in $\mathcal{L}'_0$ such that $\varphi \to \psi$ and $\psi \to \varphi$ are not in $T_\omega \cap U_\omega$. Since T_ω and U_ω are maximally strongly consistent, $\varphi \to \psi \in T_\omega$ and $\psi \to \varphi \in U_\omega$ or vice versa. let $\varphi \to \psi \in T_\omega$ and not in U_ω. So, there is $r \in (0,1]_{\mathbb{Q}}$ such that $T_\omega \models \varphi \to \psi$ and $U_\omega \models (\varphi \to \psi) \to \bar{r}$ which contradicts with inseparability of T_ω and U_ω.

5. Now, let $\mathcal{M}' = (M', b_0, b_1, \dots)$ be a model of T_ω. By the above properties, the submodel $\mathcal{M} = (M, b_0, b_1, \dots)$ with universe $M_1 = \{b_0, b_1, \dots\}$ is also a model of T_ω. Similarly, U_ω has a model $\mathcal{N} = (N_2, d_0, d_1, \dots)$ with universe $N = \{d_0, d_1, \dots\}$. By 4, $\mathcal{L}'_0$-reducts of $\mathcal{M}$ and $\mathcal{N}$ are isomorphic with b_n corresponding to d_n. So, we can let for every $n \in \mathbb{N}$, $b_n = d_n$. Then, $\mathcal{M}$ and $\mathcal{N}$ have the same $\mathcal{L}'_0$-reduct. Assume $\mathcal{A}$ is a structure in $\mathcal{L}_1 \cup \mathcal{L}_2$ with $\mathcal{L}_1$-reduct M and $\mathcal{L}_2$-reduct N. So, it is a model of $T_\omega \cup U_\omega$. Since $\varphi \in T_\omega$ and $\psi \to \bar{s}_0 \in U_\omega$, $\mathcal{A}$ is a model of φ and $\psi \to \bar{s}_0$. So, $\varphi \models \psi$ can not be correct. This is a contradiction.

4 Conclusions

In this article, a version of Craig interpolation property is proved for an extension of predicate Gödel logic, RGL*. In this new framework, one can study the model-theoretic and also, fuzzy logic properties. In the presented article, an approximate version of the above property is presented. Also, it is proved that this property holds in RGL*.

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