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Original paper

Finite-time synchronization of fractional-order fuzzy Cohen-Grossberg neural networks with time delay

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Abstract

This paper deals with the issues of the finite-time synchronization (FTS) for a class of fractional-order fuzzy Cohen-Grossberg neural networks (FOFCGNNs) with time delay. Based on the finite-time stability theory, fractional-order Razumikhin theorem and applying fractional-order differential inequalities and other inequality techniques, a few new and effective criteria formulated by testable algebraic inequalities are derived to ensure the FTS for the concerned models via designing a discontinuous control strategy. Finally, two numerical simulations examples are furnished to demonstrate the feasibility and effectiveness of the derived theoretical results.

Keywords: Fuzzy Cohen-Grossberg neural network, fractional-order, finite-time synchronization, fractional-order inequality, fractional-order Razumikhin-type theorem.

1 Introduction

The fractional calculus has a long history of over 300 years, but at the beginning it did not attract the attention of researchers because of its unknown application background. Over the last couple of two decades, fractional-order calculus has been diffusely applied in physics and practical engineering [9, 16, 22]. Compared with integer-order derivatives, fractional-order derivatives are nonlocal and have weakly singular kernels, which is an exceedingly outstanding tool to depict the memory and genetic characteristics of dynamic processes [32]. A multitude of researchers have extensively researched the dynamical behaviours of fractional-order systems and quiet a few crucial results on stability theory were proposed for fractional-order systems [3, 35, 37]. Latterly, fractional calculus has been introduced into neural networks (NNs) to generate fractional-order NNs (FONNs) and some interesting dynamic results were obtained in [5, 14, 15, 33].

In 1983, Cohen and Grossberg [6] first introduced and studied Cohen-Grossberg NNs (CGNNs), which are the most paramount and intricate NNs and have been successfully applied to combinatorial optimization, pattern recognition, associative memory, signal processing, etc. As a kind of exceedingly general and typical NN models including Hopfield NNs and cellular NNs, various CGNNs have been concentrated by many researchers and rich fruits on their various dynamical behaviors have been obtained [2, 11, 13, 15, 25, 28, 41, 42, 43]. The authors [7] researched the global stability for neutral CGNNs with multiple delays. The authors [36] investigated the stability of CGNNs with saturated pulse input.

In addition, in mathematical modeling of practical problems, it is inevitable to encounter uncertainties, approximations and fuzziness. Fuzzy logic systems have been shown to approximate any nonlinear function. In 1985, Takagi and Sugeno [31] first introduced the T-S fuzzy system, which is a nonlinear system bewrote by a suit of IF-THEN rules. Now, IF-THEN rules have been applied to the research of CGNNs [30], complex-valued NNs [12] and quaternion NNs [14]. Later, Yang and Yang [39] originally proposed fuzzy cellular NNs, which are composed of AND operator and OR operator. Recently, the researchers [1, 17, 19, 20, 44] studied asymptotic properties for various fuzzy NNs with the fuzzy

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AND and fuzzy OR operator. He and Chu [8] analyzed the global exponential stability for fuzzy BAM-type CGNNs with mixed delays and impulses. Wan and Wu [34] discussed the Mittag-Leffler stability of FOFCGNNs with deviating arguments.

On the other hand, the synchronization between NNs has been diffusely applied to image processing, secure communication and other fields. In practice, systems are always expected to realize the synchronization as soon as possible. Compared to the asymptotic synchronization, the FTS is a kind of optimal synchronization, which has attracted extensive attention from many scholars [4, 24, 27, 29, 38]. In addition, the FTS can possess a pivotal issue that the settling time relies on the initial conditions, the maximum synchronization time can not only be calculated with the FTS, but also can speed up the convergence rate. The authors [1] researched the FTS of fuzzy cellular NNs. The FTS for some fractional-order dynamical networks were handled in [23, 40]. Kong et al. [17, 19, 20] studied the FTS and fixed-time synchronization of fuzzy CGNNs with uncertain external disturbances or discontinuous activation and parameter uncertainties, respectively. Peng et al. [26] discussed nonfragile synchronization in finite time for discontinuous FONNs with nonlinear growth activations. The authors [18] analyzed the fixed-time synchronization for a class of discontinuous fuzzy inertial NNs with the fuzzy AND and fuzzy OR operator. The authors [10] are concerned with the FTS of memristive FOCGNNs with time delays. But there is hardly any paper that considered the FTS for FOFCGNNs with time delays.

Inspired by the front statement, the goal of this paper is to research the FTS for FOFCGNNs. The contribution of this paper is primarily reflected in the following three aspects: (i) Fractional-order and fuzziness are all considered in the FTS analysis for the proposed NNs. (ii) A discontinuous control strategy is presented to acquire the FTS. (iii) By utilizing a few fractional-order differential inequalities, the finite-time stability theory and combining fractional-order Razumikhin-type theorem, some effective criteria according to algebraic inequalities are derived to accomplish the FTS of the concerned network, which cover some results about the corresponding integer-order system as special cases.

The rest of the paper is structured as follows. Section 2 introduces some needful definitions and lemmas. The FTS for FOFCGNNs is discussed and some new criteria are derived in Section 3. Section 4 provides two numerical simulations examples to clarify the validity of the obtained results. Eventually, some conclusions are presented in Section 5.

Notations: $\mathbb{R}^+, \mathbb{R}^*, \mathbb{R}$ mean, respectively, the set of positive numbers, non-negative numbers and real numbers. $\bigwedge$ and $\bigvee$ denote the fuzzy AND and fuzzy OR operations, respectively. $\mathbb{C}^n([t_0, +\infty), \mathbb{R})$ shows a suit of all differentiable functions of order n from $[t_0, +\infty)$ into $\mathbb{R}$. $\text{sign}(\cdot)$ denotes the sign function. $\Im = \{1, 2, \ldots, n\}$, $\mathbb{N}^+ = \{1, 2, \ldots\}$.

2 Preliminaries

Definition 2.1. [16] *The Caputo derivate of order κ for function $\zeta(t) \in \mathbb{C}^n([t_0, +\infty), \mathbb{R})$ is defined as*

$$ {}_{t_0}^{c}D_t^{\kappa}\zeta(t) = \frac{1}{\Gamma(n-\kappa)} \int_{t_0}^{t} \frac{\zeta^{(n)}(\tau)}{(t-\tau)^{\kappa-n+1}} d\tau, $$

where $t \geq t_0$ and $n \in \mathbb{N}^+$ such that $n-1 < \kappa < n$, $\Gamma(x) = \int_0^{+\infty} e^{-t}t^{x-1}dt$. Particularly,

$$ {}_{t_0}^{c}D_t^{\kappa}\zeta(t) = \frac{1}{\Gamma(1-\kappa)} \int_{t_0}^{t} \frac{\zeta'(\tau)}{(t-\tau)^{\kappa}} d\tau, \ 0 < \kappa < 1. $$

Definition 2.2. [16] *Mittag-Leffer function $E_{d,b}(\cdot)$ owning two parameters is defined as*

$$ E_{d,b}(z) = \sum_{m=0}^{+\infty} \frac{z^m}{\Gamma(dm+b)}, $$

where $d > 0, b > 0$ and z is a complex number. Evidently,

$$ E_{d,1}(z) = E_d(z) = \sum_{m=0}^{+\infty} \frac{z^m}{\Gamma(dm+1)}, \ E_{1,1}(z) = e^z = \sum_{m=0}^{+\infty} \frac{z^m}{m!}. $$

Definition 2.3. [35, 37] *Assume that $\omega(t) \in \mathbb{C}^1([t_0, +\infty), \mathbb{R})$, for $0 < \kappa < 1$, the generalized Caputo fractional derivative can be given by*

$$ {}_{t_0}^{c}D_t^{\kappa^+}\omega(t) = \frac{1}{\Gamma(1-\kappa)} \int_{t_0}^{t} \frac{D^+\omega(\tau)}{(t-\tau)^{\kappa}} d\tau, $$

where $D^+\omega(t)$ is the upper right-hand Dini derivative of $\omega(t)$ defined by

$$D^+\omega(t) = \overline{\lim_{s\to 0^+}} \frac{\omega(t+s)-\omega(s)}{s}.$$

Consider the following FOFCGNNs with time delay for $l \in \Im$

$$\begin{aligned} {}_{t_0}^{c}D_t^{\kappa}\alpha_l(t) = &- p_l(\alpha_l(t))\Bigg[c_l(\alpha_l(t)) - \sum_{r=1}^{n} a_{lr}f_r(\alpha_r(t)) - \sum_{r=1}^{n} b_{lr}f_r(\alpha_r(t-\tau)) - \sum_{r=1}^{n} d_{lr}v_r \\ &- \bigwedge_{r=1}^{n} T_{lr}v_r - \bigwedge_{r=1}^{n} \omega_{lr}f_r(\alpha_r(t-\tau)) - \bigvee_{r=1}^{n} \varpi_{lr}f_r(\alpha_r(t-\tau)) - \bigvee_{r=1}^{n} S_{lr}v_r - I_l\Bigg], \end{aligned} \tag{1}$$

the initial conditions are

$$\alpha_l(s) = \varphi_l(s), \quad s \in [t_0-\tau, t_0],$$

where ${}_{t_0}^{c}D_t^{\kappa}$ signifies Caputo fractional derivative of order κ with $0 < \kappa < 1$, $\alpha_l(t)$ represents the corresponding state of the lth neuron at time t. $p_l(\alpha_l(t))$ represents the amplification function, $c_l(\alpha_l(t))$ is suitable behaved function. a_{lr} and b_{lr} are connection weights, d_{lr} signifies fuzzy feed-forward template, the elements of fuzzy feedback MIN and MAX template are ω_{lr} and ϖ_{lr}, the elements of fuzzy feed-forward MIN and MAX template are T_{lr} and S_{lr}, respectively. $f_r(\cdot)$ is the activation function, τ is time delay. I_l and v_r denote external input and bias of the lth and rth neuron, respectively.

Let (1) be the drive system and the response system is

$$\begin{aligned} {}_{t_0}^{c}D_t^{\kappa}\beta_l(t) = &- p_l(\beta_l(t))\Bigg[c_l(\beta_l(t)) - \sum_{r=1}^{n} a_{lr}f_r(\beta_r(t)) - \sum_{r=1}^{n} b_{lr}f_r(\beta_r(t-\tau)) - \sum_{r=1}^{n} d_{lr}v_r \\ &- \bigwedge_{r=1}^{n} T_{lr}v_r - \bigwedge_{r=1}^{n} \omega_{lr}f_r(\beta_r(t-\tau)) - \bigvee_{r=1}^{n} \varpi_{lr}f_r(\beta_r(t-\tau)) - \bigvee_{r=1}^{n} S_{lr}v_r - I_l\Bigg] + u_l(t), \end{aligned} \tag{2}$$

where $u_l(t)$ is control input, the initial conditions are

$$\beta_l(s) = \psi_l(s), \quad s \in [t_0-\tau, t_0].$$

Let $e_l(t) = \beta_l(t) - \alpha_l(t)$ for $l \in \Im$, the error dynamical system can be derived as

$$\begin{aligned} {}_{t_0}^{c}D_t^{\kappa}e_l(t) = &- p_l(\beta_l(t))\Bigg[c_l(\beta_l(t)) - \sum_{r=1}^{n} a_{lr}f_r(\beta_r(t)) - \sum_{r=1}^{n} b_{lr}f_r(\beta_r(t-\tau)) - \sum_{r=1}^{n} d_{lr}v_r \\ &- \bigwedge_{r=1}^{n} T_{lr}v_r - \bigwedge_{r=1}^{n} \omega_{lr}f_r(\beta_r(t-\tau)) - \bigvee_{r=1}^{n} \varpi_{lr}f_r(\beta_r(t-\tau)) - \bigvee_{r=1}^{n} S_{lr}v_r - I_l\Bigg] + u_l(t) \\ &+ p_l(\alpha_l(t))\Bigg[c_l(\alpha_l(t)) - \sum_{r=1}^{n} a_{lr}f_r(\alpha_r(t)) - \sum_{r=1}^{n} b_{lr}f_r(\alpha_r(t-\tau)) - \sum_{r=1}^{n} d_{lr}v_r \\ &- \bigwedge_{r=1}^{n} T_{lr}v_r - \bigwedge_{r=1}^{n} \omega_{lr}f_r(\alpha_r(t-\tau)) - \bigvee_{r=1}^{n} \varpi_{lr}f_r(\alpha_r(t-\tau)) - \bigvee_{r=1}^{n} S_{lr}v_r - I_l\Bigg], \end{aligned} \tag{3}$$

the initial conditions are

$$e_l(s) = \psi_l(s) - \varphi_l(s), \quad s \in [t_0-\tau, t_0].$$

Definition 2.4. *For designed suitable controller $u_l(t)$, systems (1) and (2) are said to achieve the FTS, if there is a constant $T > 0$ such that*

$$\lim_{t\to T} |\beta_l(t) - \alpha_l(t)| = 0,$$

and $|\beta_l(t) - \alpha_l(t)| = 0$ for $t > T, l \in \Im$.

Assumption 2.5. *For function $p_l(\cdot)$, there are positive constants $\overline{p_l}$ and h_l such that*

$$0 < p_l(\nu) \le \overline{p_l} < \infty, \ |p_l(\mu) - p_l(\nu)| \le h_l|\mu - \nu|, \ \mu, \nu \in \mathbb{R}, \ l \in \Im.$$

Assumption 2.6. *For functions $c_l(\cdot)$ and $p_l(\cdot)$, there is a positive constant d_l such that*

$$\frac{c_l(\mu)p_l(\mu) - c_l(\nu)p_l(\nu)}{\mu - \nu} \geq d_l > 0,\ \ \mu, \nu \in \mathbb{R},\ \ \mu \neq \nu,\ \ l \in \Im.$$

Assumption 2.7. *There are positive constants M_r, m_r such that*

$$|f_r(\nu)| \leq M_r,\ \ \forall \nu \in \mathbb{R},\ \ |f_r(\mu) - f_r(\nu)| \leq m_r|\mu - \nu|,\ \ f_r(0) = 0,\ \ \forall \mu, \nu \in \mathbb{R},\ \ r \in \Im.$$

Lemma 2.8. [40] *Let $0 < \kappa < 1$, if $h(t) : ([t_0, +\infty), \mathbb{R})$ is a continuous differentiable function, then for $\forall \beta \geq 1$*

$${}^{c}_{t_0}D_t^{\kappa^+}|h(t)|^{\beta} \leq \beta \mathrm{sign}(h(t))|h(t)|^{\beta-1}\, {}^{c}_{t_0}D_t^{\kappa}h(t).$$

Lemma 2.9. [39] *For $\alpha_r, \beta_r, \omega_{lr}, \varpi_{lr} \in \mathbb{R}$, $f_r : \mathbb{R} \to \mathbb{R}$ is continuous function for $l, r \in \Im$, then*

$$|\bigwedge_{r=1}^{n} \omega_{lr} f_r(\alpha_r) - \bigwedge_{r=1}^{n} \omega_{lr} f_r(\beta_r)| \leq \sum_{r=1}^{n} |\omega_{lr}||f_r(\alpha_r) - f_r(\beta_r)|,$$

$$|\bigvee_{r=1}^{n} \varpi_{lr} f_r(\alpha_r) - \bigvee_{r=1}^{n} \varpi_{lr} f_r(\beta_r)| \leq \sum_{r=1}^{n} |\varpi_{lr}||f_r(\alpha_r) - f_r(\beta_r)|.$$

Lemma 2.10. [21] *Let $\varrho_1 > 0, \varrho_2 > 0, \varrho_3 > 1, \varrho_4 > 1$ and $\frac{1}{\varrho_3} + \frac{1}{\varrho_4} = 1$, then for $\forall \varepsilon > 0$*

$$\varrho_1 \varrho_2 \leq \varepsilon^{\varrho_3} \frac{\varrho_1^{\varrho_3}}{\varrho_3} + \varepsilon^{-\varrho_4} \frac{\varrho_2^{\varrho_4}}{\varrho_4}.$$

Lemma 2.11. [1] *Suppose that $n \in \mathbb{N}^+$ and $a_l \in \mathbb{R}$ for $l \in \Im$ and $0 < \mu < \nu$, then*

$$|a_1|^{\mu} + |a_2|^{\mu} + \ldots + |a_n|^{\mu} \geq (|a_1|^{\nu} + |a_2|^{\nu} + \ldots + |a_n|^{\nu})^{\frac{\mu}{\nu}}.$$

Lemma 2.12. [40] *Let $\zeta(t) : ([t_0, +\infty), \mathbb{R}^*)$ be a positive definite and continuous function, if there are numbers $\eta \geq 0$ and $\chi > 0$ such that*

$${}^{c}_{t_0}D_t^{\kappa^+}\zeta(t) \leq -\eta\zeta(t) - \chi,\quad \zeta(t) \in \mathbb{R}^+,$$

where $0 < \kappa < 1$, then
(1) *when $\eta = 0$, $\zeta(t)$ trends to 0 inside the time t_1^*, which is expressed as*

$$t_1^* \leq \hat{t}_1^* = t_0 + \left(\frac{\Gamma(\kappa+1)\zeta(t_0)}{\chi}\right)^{\frac{1}{\kappa}}.$$

(2) *when $\eta > 0$, $\zeta(t)$ trends towards 0 inside the time t_2^* approximated by $t_2^* \leq \hat{t}_2^*$, where $\hat{t}_2^*$ is one and only positive root of the following equation*

$$E_{\kappa}(-\eta(t - t_0)^{\kappa}) = \frac{\chi}{\eta\zeta(t_0) + \chi}.$$

Lemma 2.13. [40] *If function $\zeta(t) : ([t_0, +\infty), \mathbb{R}^*)$ is positive definite and continuous, if there are numbers $\eta > 0$ and $\chi \geq 0$ such that*

$${}^{c}_{t_0}D_t^{\kappa^+}\zeta(t) \leq -\eta\zeta^{\theta}(t) - \chi,\quad \zeta(t) \in \mathbb{R}^+,$$

where $0 \leq \theta < \kappa < 1$, then
(1) *if $\theta = 0$, $\zeta(t)$ verges on 0 inside the time t_3^*, which is expressed as*

$$t_3^* \leq t_0 + \left(\frac{\Gamma(\kappa+1)\zeta(t_0)}{\eta + \chi}\right)^{\frac{1}{\kappa}};$$

(2) *if $0 < \theta < \kappa < 1$, $\zeta(t)$ tends towards 0 inside the time t_4^*, which is given by*

$$t_4^* \leq t_0 + \left(\frac{\kappa B(\kappa, 1-\theta)((\zeta(t_0) + (\frac{\chi}{\eta})^{\frac{1}{\theta}})^{\kappa-\theta} - (\frac{\chi}{\eta})^{\frac{\kappa-\theta}{\theta}})}{\eta}\right)^{\frac{1}{\kappa}},$$

where $B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$. Particularly, when $\chi = 0$, the time t_5^* is given by

$$t_5^* \leq t_0 + \left(\frac{\kappa B(\kappa, 1-\theta)\zeta(t_0)^{\kappa-\theta}}{\eta}\right)^{\frac{1}{\kappa}}.$$

3 Main results

The control input $u_l(t)$ in system (3) is designed as

$$u_l(t) = -\delta_l e_l(t) - \sigma_l \text{sign}(e_l(t)), \tag{4}$$

where $\delta_l > 0$, $\sigma_l > 0$.

For simplicity, denote

$$A_l = h_l|I_l| + h_l \sum_{r=1}^{n} (|d_{lr}| + |T_{lr}| + |S_{lr}|)|v_r| + m_l \sum_{r=1}^{n} \overline{p_r}|a_{rl}|,$$

$$B_l = h_l \sum_{r=1}^{n} (|a_{lr}| + |b_{lr}| + |\omega_{lr}| + |\varpi_{lr}|)M_r,\ C_{rl} = |b_{rl}| + |\omega_{rl}| + |\varpi_{rl}|.$$

Theorem 3.1. *Under Assumptions 2.5-2.6, systems (1) and (2) can be synchronized in a finite time T_1^* by the control law (4), if there are appropriate positive numbers δ_l, σ_l and ξ such that the following conditions hold*

$$\begin{cases} \xi_1 = \min\limits_{1\le l\le n} \Big\{ d_l + \delta_l - A_l - B_l \Big\} > 0, \\ \xi_2 = \max\limits_{1\le l\le n} \Big\{ m_l \sum\limits_{r=1}^{n} C_{rl}\overline{p_r} \Big\} > 0, \end{cases} \quad \xi_1 - \xi_2 \ge \xi, \tag{5}$$

where $T_1^* \le \hat{T}_1^*$, $\hat{T}_1^*$ denotes one and only positive root of $E_\kappa(-\xi(t-t_0)^\kappa) = \frac{\lambda}{\xi \bar{V}(t_0)+\lambda}$ and $\bar{V}(t_0) = \sum\limits_{l=1}^{n} |e_l(t_0)|$, $\lambda = \sum\limits_{l=1}^{n} \sigma_l$.

Proof. Consider a Lyapunov function as

$$\bar{V}(t) = \sum_{l=1}^{n} |e_l(t)|. \tag{6}$$

Employing Lemma 2.8 and computing the fractional derivative of $\bar{V}(t)$ along system (3), one can obtain

$$\begin{aligned} {}_{t_0}^{c}D_t^{\kappa^+} \bar{V}(t) =& \sum_{l=1}^{n} {}_{t_0}^{c}D_t^{\kappa^+} |e_l(t)| \\ \le& \sum_{l=1}^{n} \text{sign}(e_l(t)) {}_{t_0}^{c}D_t^{\kappa} e_l(t) \\ =& \sum_{l=1}^{n} \text{sign}(e_l(t)) \Bigg\{ - p_l(\beta_l(t)) \Big[c_l(\beta_l(t)) - \sum_{r=1}^{n} a_{lr} f_r(\beta_r(t)) \\ & - \sum_{r=1}^{n} b_{lr} f_r(\beta_r(t-\tau)) - \sum_{r=1}^{n} d_{lr} v_r - \bigwedge_{r=1}^{n} T_{lr} v_r - \bigvee_{r=1}^{n} S_{lr} v_r \\ & - \bigwedge_{r=1}^{n} \omega_{lr} f_r(\beta_r(t-\tau)) - \bigvee_{r=1}^{n} \varpi_{lr} f_r(\beta_r(t-\tau)) - I_l \Big] - \delta_l e_l(t) - \sigma_l \text{sign}(e_l(t)) \\ & + p_l(\alpha_l(t)) \Big[c_l(\alpha_l(t)) - \sum_{r=1}^{n} a_{lr} f_r(\alpha_r(t)) - \sum_{r=1}^{n} b_{lr} f_r(\alpha_r(t-\tau)) - \sum_{r=1}^{n} d_{lr} v_r \\ & - \bigwedge_{r=1}^{n} T_{lr} v_r - \bigvee_{r=1}^{n} S_{lr} v_r - \bigwedge_{r=1}^{n} \omega_{lr} f_r(\alpha_r(t-\tau)) - \bigvee_{r=1}^{n} \varpi_{lr} f_r(\alpha_r(t-\tau)) - I_l \Big] \Bigg\}. \end{aligned} \tag{7}$$

Based on Assumption 2.5, then

$$\begin{aligned}&p_l(\beta_l(t))\sum_{r=1}^{n}d_{lr}v_r-p_l(\alpha_l(t))\sum_{r=1}^{n}d_{lr}v_r\\=&\sum_{r=1}^{n}d_{lr}v_r(p_l(\beta_l(t))-p_l(\alpha_l(t)))\\\leq&\sum_{r=1}^{n}h_l|d_{lr}||v_r||e_l(t)|.\end{aligned}\tag{8}$$

From Assumption 2.6, one can gain

$$\operatorname{sign}(e_l(t))[-(p_l(\beta_l(t))c_l(\beta_l(t))-p_l(\alpha_l(t))c_l(\alpha_l(t)))]\leq -d_l|e_l(t)|.\tag{9}$$

According to Assumptions 2.6-2.7, then

$$\begin{aligned}&p_l(\beta_l(t))\sum_{r=1}^{n}a_{lr}f_r(\beta_r(t))-p_l(\alpha_l(t))\sum_{r=1}^{n}a_{lr}f_r(\alpha_r(t))\\=&p_l(\beta_l(t))\sum_{r=1}^{n}a_{lr}(f_r(\beta_r(t))-f_r(\alpha_r(t)))+(p_l(\beta_l(t))-p_l(\alpha_l(t)))\sum_{r=1}^{n}a_{lr}f_r(\alpha_r(t))\\\leq&\sum_{r=1}^{n}|a_{lr}|(\overline{p_l}m_r|e_r(t)|+h_lM_r|e_l(t)|).\end{aligned}\tag{10}$$

$$\begin{aligned}&p_l(\beta_l(t))\sum_{r=1}^{n}b_{lr}f_r(\beta_r(t-\tau))-p_l(\alpha_l(t))\sum_{r=1}^{n}b_{lr}f_r(\alpha_r(t-\tau))\\=&p_l(\beta_l(t))\sum_{r=1}^{n}b_{lr}(f_r(\beta_r(t-\tau))-f_r(\alpha_r(t-\tau)))+(p_l(\beta_l(t))-p_l(\alpha_l(t)))\sum_{r=1}^{n}b_{lr}f_r(\alpha_r(t-\tau))\\\leq&\sum_{r=1}^{n}|b_{lr}|(\overline{p_l}m_r|e_r(t-\tau)|+h_lM_r|e_l(t)|).\end{aligned}\tag{11}$$

From Assumption 2.5 and Lemma 2.9, then

$$\begin{aligned}&p_l(\beta_l(t))\bigwedge_{r=1}^{n}T_{lr}v_r-p_l(\alpha_l(t))\bigwedge_{r=1}^{n}T_{lr}v_r+p_l(\beta_l(t))\bigvee_{r=1}^{n}S_{lr}v_r-p_l(\alpha_l(t))\bigvee_{r=1}^{n}S_{lr}v_r\\=&(p_l(\beta_l(t))-p_l(\alpha_l(t)))\bigwedge_{r=1}^{n}T_{lr}v_r+(p_l(\beta_l(t))-p_l(\alpha_l(t)))\bigvee_{r=1}^{n}S_{lr}v_r\\\leq&\sum_{r=1}^{n}h_l|T_{lr}||v_r||e_l(t)|+\sum_{r=1}^{n}h_l|S_{lr}||v_r||e_l(t)|\\=&\sum_{r=1}^{n}h_l|v_r|(|T_{lr}|+|S_{lr}|)|e_l(t)|.\end{aligned}\tag{12}$$

Using Assumption 2.7 and Lemma 2.9, one has

$$\begin{aligned}&p_l(\beta_l(t))\bigwedge_{r=1}^{n}\omega_{lr}f_r(\beta_r(t-\tau))-p_l(\alpha_l(t))\bigwedge_{r=1}^{n}\omega_{lr}f_r(\alpha_r(t-\tau))\\=&p_l(\beta_l(t))\bigwedge_{r=1}^{n}\omega_{lr}f_r(\beta_r(t-\tau))-p_l(\beta_l(t))\bigwedge_{r=1}^{n}\omega_{lr}f_r(\alpha_r(t-\tau))\\&+p_l(\beta_l(t))\bigwedge_{r=1}^{n}\omega_{lr}f_r(\alpha_r(t-\tau))-p_l(\alpha_l(t))\bigwedge_{r=1}^{n}\omega_{lr}f_r(\alpha_r(t-\tau))\\\leq&\sum_{r=1}^{n}|\omega_{lr}|(\overline{p_l}m_r|e_r(t-\tau)|+h_lM_r|e_l(t)|).\end{aligned}\tag{13}$$

$$
\begin{aligned}
&p_l(\beta_l(t))\bigvee_{r=1}^{n}\varpi_{lr}f_r(\beta_r(t-\tau))-p_l(\alpha_l(t))\bigvee_{r=1}^{n}\varpi_{lr}f_r(\alpha_r(t-\tau))\\
=&p_l(\beta_l(t))\bigvee_{r=1}^{n}\varpi_{lr}f_r(\beta_r(t-\tau))-p_l(\beta_l(t))\bigvee_{r=1}^{n}\varpi_{lr}f_r(\alpha_r(t-\tau))\\
&+p_l(\beta_l(t))\bigvee_{r=1}^{n}\varpi_{lr}f_r(\alpha_r(t-\tau))-p_l(\alpha_l(t))\bigvee_{r=1}^{n}\varpi_{lr}f_r(\alpha_r(t-\tau))\\
\leq&\sum_{r=1}^{n}|\varpi_{lr}|(\overline{p_l}m_r|e_r(t-\tau)|+h_lM_r|e_l(t)|).
\end{aligned}\tag{14}
$$

Combining with (7)-(14), one can get

$$
\begin{aligned}
{}_{t_0}^{c}D_t^{\kappa^+}\bar{V}(t)\leq&-\sum_{l=1}^{n}\Big[d_l+\delta_l-h_l|I_l|-h_l\sum_{r=1}^{n}(|d_{lr}|+|T_{lr}|+|S_{lr}|)|v_r|-m_l\sum_{r=1}^{n}\overline{p_r}|a_{rl}|\\
&-h_lM_r\sum_{r=1}^{n}(|a_{lr}|+|b_{lr}|+|\omega_{lr}|+|\varpi_{lr}|)\Big]|e_l(t)|\\
&+\sum_{l=1}^{n}\Big[m_l\sum_{r=1}^{n}\overline{p_r}(|b_{rl}|+|\omega_{rl}|+|\varpi_{rl}|)\Big]|e_l(t-\tau)|-\sum_{l=1}^{n}\sigma_l\\
\leq&-\xi_1\bar{V}(t)+\xi_2\bar{V}(t-\tau)-\lambda.
\end{aligned}\tag{15}
$$

According to the fractional-order Razumikhin theorem [3], one has

$$
\bar{V}(s)\leq\bar{V}(t),t-\tau\leq s\leq t.\tag{16}
$$

Combining (15) with (16), then

$$
\begin{aligned}
{}_{t_0}^{c}D_t^{\kappa^+}\bar{V}(t)\leq&-(\xi_1-\xi_2)\bar{V}(t)-\lambda\\
\leq&-\xi\bar{V}(t)-\lambda.
\end{aligned}\tag{17}
$$

Hence, from Lemma 2.12, formula (17) implies that $\bar{V}(t)$ trends towards 0 and the settling time is approximated by $T_1^*\leq\hat{T}_1^*$, $\hat{T}_1^*$ denotes one and only positive root of the following equation

$$
E_\kappa(-\xi(t-t_0)^\kappa)=\frac{\lambda}{\xi\bar{V}(t_0)+\lambda}.
$$

According to Definition 2.4, systems (1) and (2) can achieve the FTS under controller (4). The proof is completed. □

Corollary 3.2. *Under the conditions of Theorem 3.1, if $\xi_1=\xi_2$, then systems (1) and (2) can be synchronized by the control law (4) in a finite time T_1^*, which is evaluated by*

$$
T_1^*\leq t_0+\left(\frac{\Gamma(\kappa+1)\sum\limits_{l=1}^{n}|e_l(t_0)|}{\sigma_1+\sigma_2+\ldots+\sigma_n}\right)^{\frac{1}{\kappa}}.
$$

Remark 3.3. *Under the conditions of Theorem 3.1, systems (1) and (2) with $\kappa=1$ can achieve the FTS and the settling time is approximated by $T_2^*\leq\hat{T}_2^*$, $\hat{T}_2^*$ is one and only root of $E_{1,1}(-\xi(t-t_0))=e^{-\xi(t-t_0)}=\frac{\lambda}{\xi\bar{V}(t_0)+\lambda}$ with $\hat{T}_2^*=t_0+\frac{\ln(\xi\bar{V}(t_0)+\lambda)-\ln\lambda}{\xi}$.*

Theorem 3.4. *Under Assumptions 2.5-2.7, for $\rho>1$ and $\frac{\rho-1}{\rho}<\kappa<1$, systems (1) and (2) can be synchronized in a finite time T_3^* by the control law (4), if there are appropriate positive numbers δ_l,σ_l and ε such that the following conditions hold*

$$
\begin{cases}
\eta_1=\rho\min\limits_{1\leq l\leq n}\left\{d_l+\delta_l-A_l-B_l-\dfrac{\rho-1}{\rho}\varepsilon^{\frac{\rho}{\rho-1}}\overline{p_l}\sum\limits_{r=1}^{n}C_{lr}m_r\right\}>0,\\
\eta_2=\max\limits_{1\leq l\leq n}\left\{\dfrac{1}{\varepsilon^\rho}m_l\sum\limits_{r=1}^{n}C_{lr}\overline{p_r}\right\},
\end{cases}\quad\eta_1=\eta_2,\tag{18}
$$

where $T_3^* \le \hat{T}_3^* = t_0 + \left(\frac{\kappa B(\kappa,\frac{1}{\rho})\tilde{V}^{\kappa-\frac{\rho-1}{\rho}}(t_0)}{\mu}\right)^{\frac{1}{\kappa}}$ *with* $\mu = \min_{1\le l\le n}\{\rho\sigma_l\}$ *and* $\tilde{V}(t_0) = \sum_{l=1}^{n}|e_l(t_0)|^\rho$.

Proof. Construct a Lyapunov function as:

$$\tilde{V}(t) = \sum_{l=1}^{n}|e_l(t)|^\rho. \tag{19}$$

Employing Lemma 2.8 and computing the fractional derivative of $\tilde{V}(t)$ along system (3), then

$$\begin{aligned}
{}_{t_0}^{c}D_t^{\kappa^+}\tilde{V}(t) =& \sum_{l=1}^{n} {}_{t_0}^{c}D_t^{\kappa^+}|e_l(t)|^\rho \\
\le& \sum_{l=1}^{n}\rho\mathrm{sign}(e_l(t))|e_l(t)|^{\rho-1}{}_{t_0}^{c}D_t^{\kappa}e_l(t) \\
=& \sum_{l=1}^{n}\rho\mathrm{sign}(e_l(t))|e_l(t)|^{\rho-1}\Bigg\{-p_l(\beta_l(t))\Big[c_l(\beta_l(t)) - \sum_{r=1}^{n}a_{lr}f_r(\beta_r(t)) \\
&- \sum_{r=1}^{n}b_{lr}f_r(\beta_r(t-\tau)) - \sum_{r=1}^{n}d_{lr}v_r - \bigwedge_{r=1}^{n}T_{lr}v_r - \bigvee_{r=1}^{n}S_{lr}v_r \\
&- \bigwedge_{r=1}^{n}\omega_{lr}f_r(\beta_r(t-\tau)) - \bigvee_{r=1}^{n}\varpi_{lr}f_r(\beta_r(t-\tau)) - I_l\Big] - \delta_l e_l(t) - \sigma_l\mathrm{sign}(e_l(t)) \\
&+ p_l(\alpha_l(t))\Big[c_l(\alpha_l(t)) - \sum_{r=1}^{n}a_{lr}f_r(\alpha_r(t)) - \sum_{r=1}^{n}b_{lr}f_r(\alpha_r(t-\tau)) - \sum_{r=1}^{n}d_{lr}v_r \\
&- \bigwedge_{r=1}^{n}T_{lr}v_r - \bigvee_{r=1}^{n}S_{lr}v_r - \bigwedge_{r=1}^{n}\omega_{lr}f_r(\alpha_r(t-\tau)) - \bigvee_{r=1}^{n}\varpi_{lr}f_r(\alpha_r(t-\tau)) - I_l\Big]\Bigg\}.
\end{aligned} \tag{20}$$

Substituting (8)-(14) into (20), one can get

$$\begin{aligned}
{}_{t_0}^{c}D_t^{\kappa^+}\tilde{V}(t) \le& -\rho\sum_{l=1}^{n}\Bigg[d_l + \delta_l - h_l|I_l| - h_l\sum_{r=1}^{n}(|d_{lr}| + |T_{lr}| + |S_{lr}|)|v_r| - m_l\sum_{r=1}^{n}\overline{p_r}|a_{rl}| \\
&- h_l\sum_{r=1}^{n}(|a_{lr}| + |b_{lr}| + |\omega_{lr}| + |\varpi_{lr}|)M_r\Bigg]\sum_{l=1}^{n}|e_l(t)|^\rho - \rho\sum_{l=1}^{n}\sigma_l|e_l(t)|^{\rho-1} \\
&+ \rho\sum_{l=1}^{n}\Bigg[\overline{p_l}\sum_{r=1}^{n}(|b_{lr}| + |\omega_{lr}| + |\varpi_{lr}|)m_r\Bigg]|e_l(t)|^{\rho-1}|e_r(t-\tau)|.
\end{aligned} \tag{21}$$

Utilizing Lemma 2.10, one can gain

$$|e_l(t)|^{\rho-1}|e_r(t-\tau)| \le \frac{\rho-1}{\rho}\varepsilon^{\frac{\rho}{\rho-1}}|e_l(t)|^\rho + \frac{1}{\rho\varepsilon^\rho}|e_r(t-\tau)|^\rho. \tag{22}$$

In view of Lemma 2.11, one can obtain

$$-\rho\sum_{l=1}^{n}\sigma_l|e_l(t)|^{\rho-1} \le -\mu\Bigg[\sum_{l=1}^{n}|e_l(t)|^\rho\Bigg]^{\frac{\rho-1}{\rho}}. \tag{23}$$

Integrating (16) and (21)-(23), one has

$$
\begin{aligned}
{}_{t_0}^{c}D_t^{\kappa^+}\tilde{V}(t) \leq & -\rho\sum_{l=1}^{n}\Bigg[d_l+\delta_l-h_l|I_l|-h_l\sum_{r=1}^{n}(|a_{lr}|+|b_{lr}|+|\omega_{lr}|+|\varpi_{lr}|)M_r-m_l\sum_{r=1}^{n}\overline{p_r}|a_{rl}| \\
& -h_l\sum_{r=1}^{n}(|d_{lr}|+|T_{lr}|+|S_{lr}|)|v_r|-\frac{\rho-1}{\rho}\varepsilon^{\frac{\rho}{\rho-1}}\overline{p_l}\sum_{r=1}^{n}(|b_{lr}|+|\omega_{lr}|+|\varpi_{lr}|)m_r\Bigg]|e_l(t)|^{\rho} \\
& -\mu\Bigg[\sum_{l=1}^{n}|e_l(t)|^{\rho}\Bigg]^{\frac{\rho-1}{\rho}}+\sum_{l=1}^{n}\Bigg[\frac{1}{\varepsilon^{\rho}}m_l\sum_{r=1}^{n}(|b_{rl}|+|\omega_{rl}|+|\varpi_{rl}|)\overline{p_r}\Bigg]|e_l(t-\tau)|^{\rho} \\
\leq & -(\eta_1-\eta_2)\tilde{V}(t)-\mu\tilde{V}^{\frac{\rho-1}{\rho}}(t) \\
= & -\mu\tilde{V}^{\frac{\rho-1}{\rho}}(t).
\end{aligned} \tag{24}
$$

Hence, from Lemma 2.13, formula (24) implies that $\tilde{V}(t)$ trends towards 0 and the settling time T_3^* is given by

$$
T_3^* \leq t_0+\left(\frac{\kappa B(\kappa,\frac{1}{\rho})\tilde{V}^{\kappa-\frac{\rho-1}{\rho}}(t_0)}{\mu}\right)^{\frac{1}{\kappa}}.
$$

According to Definition 2.4, systems (1) and (2) can achieve the FTS under controller (4). This achieves the proof. □

Remark 3.5. *As $\kappa = 1$ and the conditions of Theorem 3.4 hold, then systems (1) and (2) can still accomplish the FTS and the settling time T_4^* is expressed as*

$$
T_4^* \leq t_0+\frac{\rho\tilde{V}^{\frac{1}{\rho}}(t_0)}{\mu}.
$$

Remark 3.6. *It should be noted that the settling time can be theoretically determined according to T_1^* in Theorem 3.1 or T_3^* in Theorem 3.4. One can see that the settling time T_1^* or T_3^* depends on not only the relevant initial state $\bar{V}(t_0)$ or $\tilde{V}(t_0)$, but also the fractional-order κ and the control parameter σ_l.*

Remark 3.7. *Using different analytical methods, we can get Theorem 3.1 and Theorem 3.4, respectively. Because Theorem 3.1 is based on Lemma 2.12, and Theorem 3.4 with $\rho > 1$ is based on Lemma 2.10 and Lemma 2.13, but Lemma 2.10 is invalid for $\rho = 1$, Theorem 3.4 is invalid for $\rho = 1$. Meanwhile, the settling time T_3^* in Theorem 3.4 also relies on ρ. So, Theorem 3.1 and Theorem 3.4 will complement and enrich both.*

Remark 3.8. *So far, there have been some results on the stability [28, 41] and the FTS [10, 17, 20] of CGNNs. As far as we know, no previous research has investigated the FTS for FOFCGNNs. Thus, the results here have supplemented some previous works, which means that the results here are new.*

4 Numerical simulations

Example 4.1. *Consider models (1) and (2) with $n = 2$ and the following parameters*
$\kappa = 0.95$, $f_1(\upsilon) = f_2(\upsilon) = \tanh(\upsilon)$, $p_1(\upsilon) = p_2(\upsilon) = 1 - 1/(1 + \upsilon^2)$, $c_1(\upsilon) = c_2(\upsilon) = 1.6\upsilon + 0.1\sin(\upsilon)$, $d_{lr} = T_{lr} = S_{lr} = v_r = 1$, $a_{11} = a_{22} = 0.2$, $a_{12} = a_{21} = 0.1$, $b_{11} = b_{12} = b_{21} = b_{22} = 0.1$, $\omega_{11} = \omega_{22} = 0.1$, $\omega_{12} = \omega_{21} = -0.2$, $\varpi_{11} = \varpi_{22} = 0.2$, $\varpi_{12} = \varpi_{21} = -0.2$, $\tau = 1$, $I_1 = I_2 = 0$.

By simple calculation, one can get

$$
\overline{p_1}=\overline{p_2}=1, h_1=h_2=\frac{3\sqrt{3}}{8}\approx 0.65, d_1=d_2=1.75, M_1=M_2=m_1=m_2=1.
$$

So, Assumptions 2.5-2.7 hold. For the initial conditions

$$
(\alpha_1(s),\alpha_2(s))^T=(-0.5,0.5)^T,\ (\beta_1(s),\beta_2(s))^T=(0.5,-0.5)^T,\ \forall s\in[-1,0].
$$

Let $t_0 = 0$ and choosing $\delta_1 = \delta_2 = 5.23$, $\sigma_1 = \sigma_2 = 0.01$ in formula (4), then

$$
A_1=A_2=4.2,\ B_1=B_2=0.78,\ C_{11}=C_{22}=0.4,\ C_{12}=C_{21}=0.5,
$$

$$\xi_1 = 2,\ \xi_2 = 0.9,\ \xi_1 > \xi_2,\ \lambda = 0.02,\ \bar{V}(t_0) = \sum_{l=1}^{2} |e_l(t_0)| = 2.$$

Thus, all the conditions of Theorem 3.1 hold. Accordingly, systems (1) and (2) can be synchronized under the controller (4) in a finite time $T_1^* \leq 2.3902$. *Fig.1 describes the phase trajectories of states* $\alpha_1(t), \alpha_2(t)$ *of (1) in 2-D. Fig.2 shows time responses of states* $\alpha_1(t), \alpha_2(t)$ *of (1) in 3-D. Fig.3 manifests time responses of states of systems (1) and (2) without controller. Fig.4 shows synchronization curves of* $\alpha_l(t), \beta_l(t)$ *of systems (1) and (2) with controller (4). Fig.5 demonstrates time responses of synchronization errors* $e_1(t), e_2(t)$ *between models (1) and (2) with (4).*

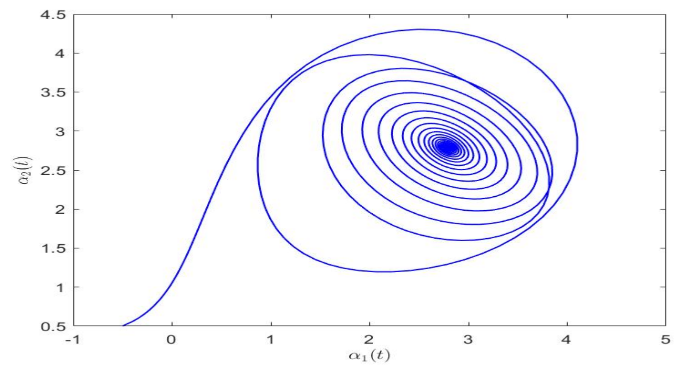

Fig. 1: Phase trajectories of states $\alpha_1(t), \alpha_2(t)$ for Example 4.1 in $\mathbb{R}^2$.

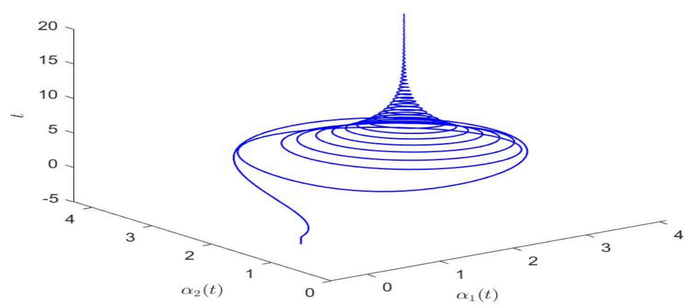

Fig. 2: Time responses of states $\alpha_1(t), \alpha_2(t)$ for Example 4.1 in $\mathbb{R}^3$.

Example 4.2. *Consider models (1) and (2) with* $n = 2$ *and the following parameters*
$\kappa = 0.95,\ f_1(\upsilon) = f_2(\upsilon) = 0.5(|\upsilon + 1| - |\upsilon - 1|),\ p_1(\upsilon) = p_2(\upsilon) = 1 - 1/(1 + \upsilon^2),\ c_1(\upsilon) = c_2(\upsilon) = 1.4\upsilon + 0.1\sin(\upsilon),$ $d_{lr} = 0.5,\ T_{lr} = S_{lr} = v_r = 1,\ a_{11} = a_{22} = 0.2,\ a_{12} = a_{21} = -0.1,\ b_{11} = b_{12} = b_{21} = b_{22} = 0.2,\ \omega_{11} = \omega_{22} = 0.15,$ $\omega_{12} = \omega_{21} = -0.1,\ \varpi_{11} = \varpi_{22} = 0.1,\ \varpi_{12} = \varpi_{21} = -0.2,\ \tau = 1,\ I_1 = I_2 = 1.$

By simple calculation, one can obtain

$$\overline{p_1} = \overline{p_2} = 1, h_1 = h_2 = \frac{3\sqrt{3}}{8} \approx 0.65, d_1 = d_2 = 1.5, M_1 = M_2 = m_1 = m_2 = 1.$$

So, Assumptions 2.5-2.7 hold. For the initial conditions

$$(\alpha_1(s), \alpha_2(s))^T = (0.2, -0.3)^T,\ (\beta_1(s), \beta_2(s))^T = (-0.3, 0.2)^T,\ \forall s \in [-1, 0].$$

Let $\rho = 2, \varepsilon = 1, t_0 = 0$ *and choosing* $\delta_1 = \delta_2 = 4.4625, \sigma_1 = \sigma_2 = 0.4$, *one can get*

$$A_1 = A_2 = 4.2,\ B_1 = B_2 = 0.8125,\ C_{11} = C_{22} = 0.45,\ C_{12} = C_{21} = 0.5,$$

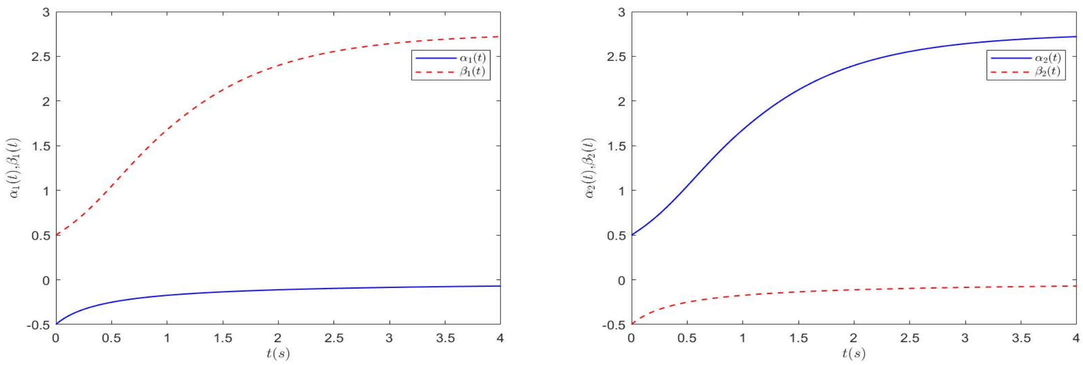

Fig. 3: Evaluations of states $\alpha_l(t), \beta_l(t)$ of Example 4.1 with $u_1(t) = u_2(t) = 0$.

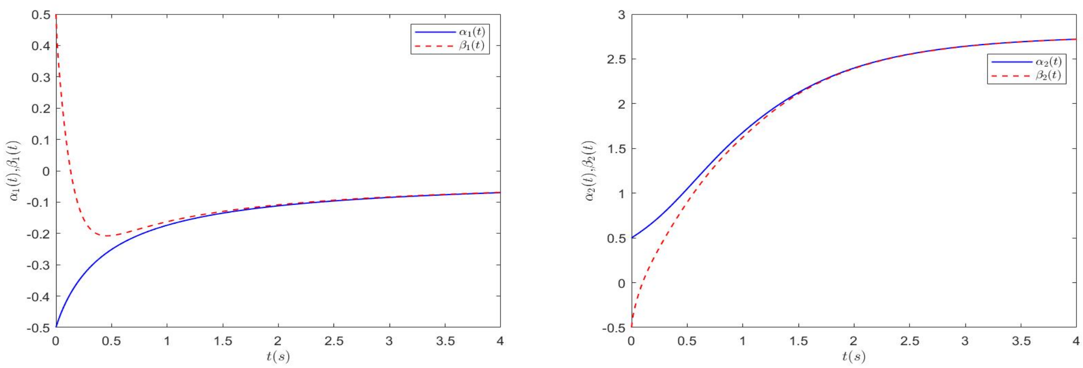

Fig. 4: Synchronization curves of $\alpha_l(t), \beta_l(t)$ of Example 4.1 with controller (4).

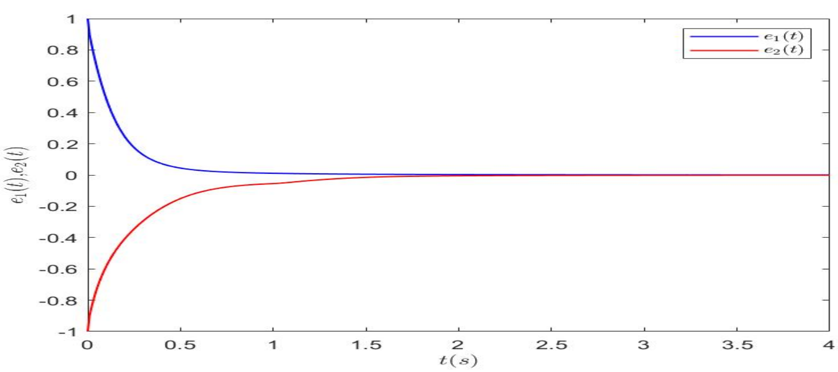

Fig. 5: Evaluations of errors $e_1(t), e_2(t)$ of Example 4.1 with controller (4).

$$\eta_1 = \eta_2 = 0.95,\ \mu = 0.8,\ \tilde{V}(t_0) = \sum_{l=1}^{2} |e_l(t_0)|^2 = 0.5.$$

Thus, all the conditions of Theorem 3.4 hold. Accordingly, systems (1) and (2) can be synchronized under the controller (4) in a finite time $T_3^* \leq 1.8505$. *Fig.6 describes the phase trajectories of states* $\alpha_1(t), \alpha_2(t)$ *of (1) in 2-D. Fig.7 shows time responses of states* $\alpha_1(t), \alpha_2(t)$ *of (1) in 3-D. Fig.8 manifests time responses of states of systems (1) and (2) without controller. Fig.9 shows synchronization curves of* $\alpha_l(t), \beta_l(t)$ *of systems (1) and (2) with controller (4). Fig.10 demonstrates the trajectories of synchronization errors* $e_1(t), e_2(t)$ *between models (1) and (2) with (4).*

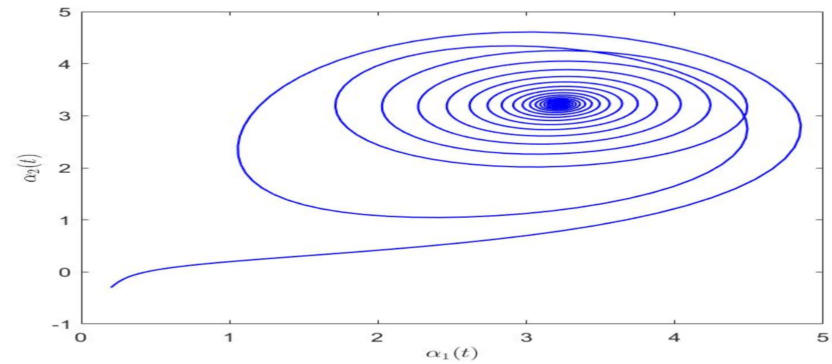

Fig. 6: Phase trajectories of states $\alpha_1(t), \alpha_2(t)$ for Example 4.2 in $\mathbb{R}^2$.

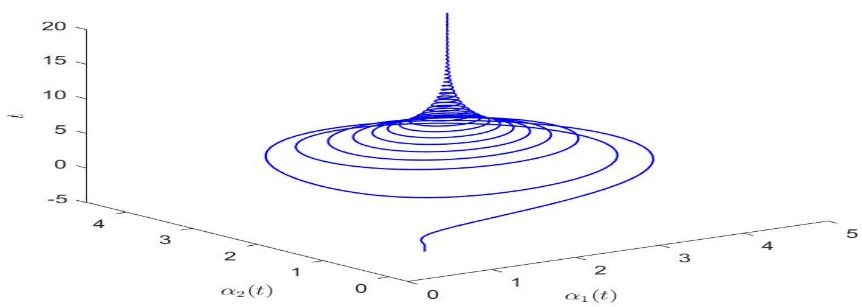

Fig. 7: Time responses of states $\alpha_1(t), \alpha_2(t)$ for Example 4.2 in $\mathbb{R}^3$.

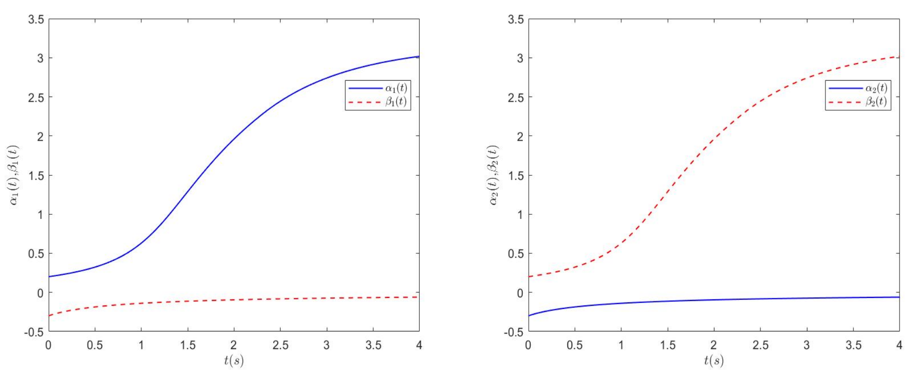

Fig. 8: Evaluations of states $\alpha_l(t), \beta_l(t)$ of Example 4.2 with $u_1(t) = u_2(t) = 0$.

5 Conclusions

By introducing fractional calculus and fuzziness with fuzzy AND and fuzzy OR operator into CGNNs, this paper has studied the FTS for a class of FOFCGNNs. On the basis of the finite-time stability theory, fractional-order Razumikhin theorem and applying fractional differential inequalities, several new and useful criteria are established for the FTS by engineering a discontinuous controller. In addition, the settling time is also presented. Moreover, the results here improve and enrich the case of the integer-order. Finally, two numerical simulations examples are provided to demonstrate the availability of the obtained results. The future work will focus on the research of the finite-time

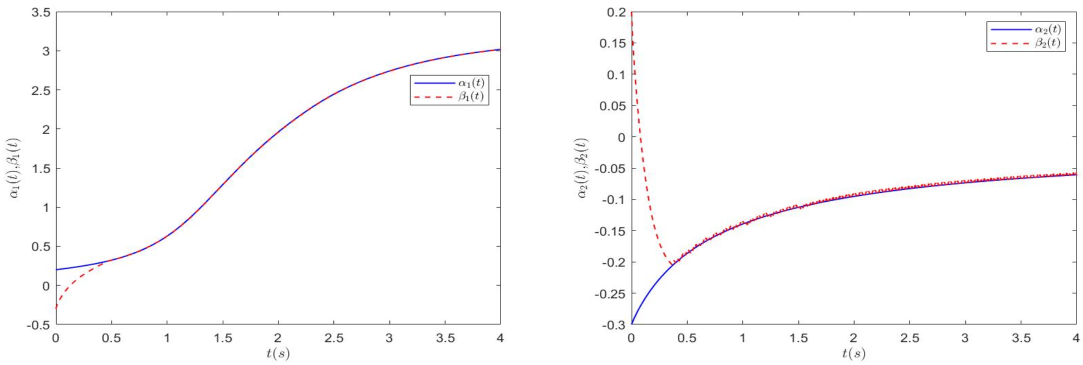

Fig. 9: Synchronization curves of $\alpha_l(t), \beta_l(t)$ of Example 4.2 with controller (4).

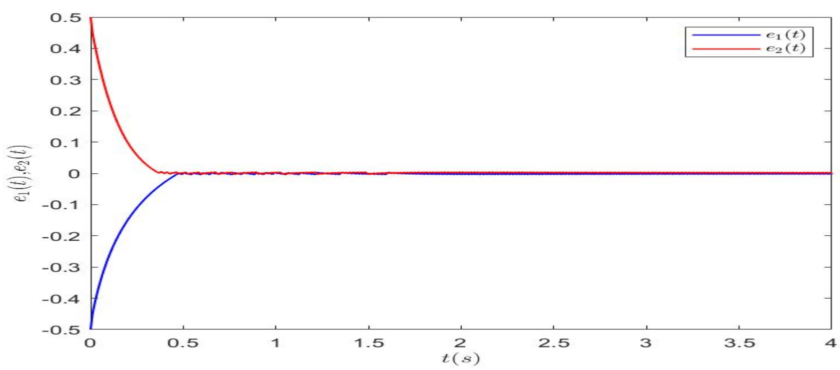

Fig. 10: Evaluations of errors $e_1(t), e_2(t)$ of Example 4.2 with controller (4).

stability and the FTS of memristive CGNNs with parameters uncertainties or impulse disturbance.

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