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Diagonal conditions and uniformly continuous extension in ⊤-uniform limit spaces

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Abstract

We study suitable diagonal conditions for ⊤-uniform limit spaces. A dual diagonal condition is shown to be a suitable axiom for uniform regularity. We characterize this regularity concept by closures of L-sets. We apply all these diagonal axioms and prove an extension theorem for uniformly continuous mappings defined on a dense subspace.

Keywords: Topology, top-filter, uniform limit space, lattice-valued uniform convergence space, probabilistic uniform space, diagonal axiom, uniform regularity, extension of mappings.

1 Introduction

Recently, Fang and Yue [4] studied certain diagonal axioms for ⊤-convergence spaces and applied these to characterize strong L-topological spaces [22] by convergence. Also, in order to show that the diagonal axioms really work, they proved an extension theorem for a continuous mapping defined on a dense subspace.

In this paper, we study related diagonal axioms in the category$\top-\mathsf{ULim}$ of ⊤-uniform limit spaces that was introduced in [17] and [13]. This category has nice properties: it is topological and if the quantale is divisible or is a value quantale, it is Cartesian closed. This is an improvement over the supercategory $\mathsf{SL}-\mathsf{UCS}$ of stratified lattice-valued uniform convergence spaces studied earlier [3, 12] as here for the Cartesian closedness of the category a frame as quantale is needed.

We define Fischer-, Kowalsky- and Gähler-type diagonal axioms and generalize in this way diagonal conditions that were studied for classical uniform limit spaces by W.Gähler [6]. Dualizing the Fischer-type axiom results in a uniform regularity concept for ⊤-uniform limit spaces. We characterize this concept via closures of L-sets. Finally, as an application of the diagonal conditions that we introduce, we prove an extension theorem for uniformly continuous mappings defined on a dense subspace. Also here we are able to generalize a classical result for uniform limit spaces [6] to the lattice-valued case. This result again improves a related result obtained in $\mathsf{SL}-\mathsf{UCS}$, where again a frame as quantale was needed [11]. Our result works for commutative and integral quantales, a lattice background that is chosen in this paper.

The paper is organized as follows. In Section 2 we fix the notation and the lattice background of the paper. Section 3 is devoted to the study of ⊤-filters. Here, in particular, the monoidal product of two ⊤-filters is introduced. Using this product instead of the Cartesian product allows to relax assumptions on the quantale, e.g. distributivity or the frame law for the underlying lattice, for several results. Section 4 reviews ⊤-uniform limit spaces and ⊤-convergence spaces as far as is needed for this paper. Section 5 then studies the diagonal axioms and Section 6 the regularity concept. In Section 7 we tie all diagonal conditions together and prove an extension theorem for uniformly continuous mappings. Finally, in Section 8, we draw some conclusions.

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2 Preliminaries

We consider in this paper *commutative and integral quantales*, i.e. triples $\mathsf{L} = (L, \leq, *)$, where $(L, \leq)$ is a complete lattice with order relation $\leq$, and $(L, *)$ is a commutative semigroup for which the top element of L acts as the unit, i.e. $\alpha * \top = \alpha$ for all $\alpha \in L$, and $*$ is distributive over arbitrary joins, i.e. $(\bigvee_{i \in J} \alpha_i) * \beta = \bigvee_{i \in J} (\alpha_i * \beta)$, see e.g. [10]. For simplicity, we often speak of a quantale, always including that it is commutative and integral. Typical examples of such quantales are e.g. frames (if the quantale operation is the meet operation of the lattice) or the unit interval $[0, 1]$ with a left-continuous *t-norm* [19]. Another prominent example is *Lawvere's quantale*, the interval $[0, \infty]$ with the opposite order and addition as quantale operation $\alpha * \beta = \alpha + \beta$, extended by $\alpha + \infty = \infty + a = \infty$, see e.g. [15, 5]. A further noteworthy example is the quantale of *distance distribution functions*, i.e. of functions $\varphi : [0, \infty] \longrightarrow [0, 1]$, that satisfy $\varphi(x) = \sup\{\varphi(y) : y < x\}$ for all $x \in [0, \infty]$. The set of all distance distribution functions is denoted by Δ^+ and with the pointwise order Δ^+ becomes a completely distributive lattice [5]. A quantale operation $* : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$ is called a *sup-continuous triangle function* in [19].

In a commutative and integral quantale, we can define an *implication* by $\alpha \to \beta = \bigvee\{\delta \in L \ : \ \alpha * \delta \leq \beta\}$. Then $\delta \leq \alpha \to \beta \iff \delta * \alpha \leq \beta$.

We denote the set of L-subsets $a, b, c, \ldots$ on X by $L^X = \{a : X \longrightarrow L\}$. A constant L-subset with value $\alpha \in L$ is denoted by α_X. For a function $\varphi : X \longrightarrow Y$ and $a \in L^X$ and $b \in L^Y$ the *image of* a, $\varphi(a) \in L^Y$, is defined by $\varphi(a)(y) = \bigvee_{\varphi(x)=y} a(x)$ for $y \in Y$, and the *inverse image of* b is defined by $\varphi^{\leftarrow}(b) = b \circ \varphi$. The lattice operations are extended pointwisely from L to L^X. For $a \in L^X$ and $b \in L^Y$ we define the *Cartesian product* $a \times b \in L^{X \times Y}$ by $(a \times b)(x, y) = a(x) \wedge b(y)$ for all $(x, y) \in X \times Y$.

For $b, d \in L^X$ we denote the *fuzzy inclusion order* [2] by $[b, d] = \bigwedge_{x \in X} (b(x) \to d(x))$. If we want to emphasize the set X, then we will also write $[b, d]_X$ for $[b, d]$. We collect some of the properties that we will need later.

Lemma 2.1. *Let $a, a', b, b', c \in L^X$, $d \in L^Y$ and let $\varphi : X \longrightarrow Y$ be a mapping. Then*

(i) $a \leq b$ *if and only if* $[a, b] = \top$;

(ii) $a \leq a'$ *implies* $[a', b] \leq [a, b]$ *and* $b \leq b'$ *implies* $[a, b] \leq [a, b']$;

(iii) $[a, c] \wedge [b, c] = [a \vee b, c]$;

(iv) $[\varphi(a), d] = [a, \varphi^{\leftarrow}(d)]$.

If $a, c \in L^X$ *and* $b, d \in L^Y$ *then*

(v) $[a \times b, c \times d] \geq [a, c] \wedge [b, d]$.

For notions from category theory we refer to [1].

3 $\top$-filters

Definition 3.1. [8, 20] *A subset $\mathbb{F} \subseteq L^X$ is called a* $\top$-filter *if*
$(TF1)$ $\bigvee_{x \in X} b(x) = \top$ *for all* $b \in \mathbb{F}$;
$(TF2)$ $a, b \in \mathbb{F}$ *implies* $a \wedge b \in \mathbb{F}$;
$(TF3)$ $\bigvee_{b \in \mathbb{F}} [b, d] = \top$ *implies* $d \in \mathbb{F}$.

We denote the set of all $\top$-filters on X by $\mathsf{F}_{\mathsf{L}}^{\top}(X)$.

Example 3.2. *For an L-subset $a \in L^X$ with $a(x) = \top$, then $[a] = \{b \in L^X \ : \ a \leq b\}$ is a $\top$-filter. In particular, for $a = \top_{\{x\}}$, we denote the* point $\top$-filter *by* $[x] = \{a \in L^X \ : \ a(x) = \top\}$.

Definition 3.3. [8, 20] *A subset $\mathbb{B} \subseteq L^X$ is called a* $\top$-filter base *if*
$(TB1)$ $\bigvee_{x \in X} b(x) = \top$ *for all* $b \in \mathbb{B}$;
$(TB2)$ $a, b \in \mathbb{B}$ *implies* $\bigvee_{c \in \mathbb{B}} [c, a \wedge b] = \top$.

For a $\top$-filter base $\mathbb{B}$, $[\mathbb{B}] = \{a \in L^X \ : \ \bigvee_{b \in \mathbb{B}} [b, a] = \top\}$ is a $\top$-filter, the $\top$-*filter generated by* $\mathbb{B}$.

The set $\mathsf{F}_{\mathsf{L}}^{\top}(X)$ is ordered by set inclusion, i.e. $\mathbb{F} \leq \mathbb{G}$ if $\mathbb{F} \subseteq \mathbb{G}$. The meet of a non-empty family $(\mathbb{F}_j)_{j \in J}$ of $\top$-filters on X is given by $\bigwedge_{j \in J} \mathbb{F}_j = \bigcap_{j \in J} \mathbb{F}_j$ and a $\top$-filter base for $\mathbb{F} \wedge \mathbb{G}$ is given by $\{f \vee g \ : \ f \in \mathbb{F}, g \in \mathbb{G}\}$.

Proposition 3.4 (Join of two $\top$-filters, [20])**.** *Let $\mathbb{F}, \mathbb{G} \in \mathsf{F}_{\mathsf{L}}^{\top}(X)$. Then $\mathbb{B} = \{f \wedge g \ : \ f \in \mathbb{F}, g \in \mathbb{G}\}$ is a $\top$-filter base if and only if $\bigvee_{x \in X} f \wedge g(x) = \top$ for all $f \in \mathbb{F}, g \in \mathbb{G}$.*

If the condition of Proposition 3.4 is satisfied, then we denote the generated $\top$-filter by $\mathbb{F} \vee \mathbb{G}$ and say that $\mathbb{F} \vee \mathbb{G}$ exists and call it the *join of* $\mathbb{F}$ *and* $\mathbb{G}$.

It is well-known, that for a $\top$-filter $\mathbb{F} \in \mathsf{F}_L^\top(X)$ and a mapping $\varphi : X \longrightarrow Y$, the set $\mathbb{B} = \{\varphi(a) \ : \ a \in \mathbb{F}\}$ is a $\top$-filter base on Y and we denote $\varphi(\mathbb{F})$ the generated $\top$-filter on Y, the *image of* $\mathbb{F}$ *under* φ, see [8].

Lemma 3.5. *Let* $\mathbb{F} \in \mathsf{F}_\mathsf{L}^\top(X)$ *and* $\varphi : X \longrightarrow Y$ *be a mapping. Then* $b \in \varphi(\mathbb{F})$ *if and only if* $\varphi^\leftarrow(b) \in \mathbb{F}$.

We conclude that $\varphi([x]) = [\varphi(x)]$.

Proposition 3.6. *Let* $\mathbb{F}, \mathbb{G} \in \mathsf{F}_\mathsf{L}^\top(X)$ *and let* $\varphi : X \longrightarrow Y$ *be a mapping. Then* $\varphi(\mathbb{F}) \wedge \varphi(\mathbb{G}) = \varphi(\mathbb{F} \wedge \mathbb{G})$.

Proposition 3.7. [8, 20] *Let* $\varphi : X \longrightarrow Y$ *and* $\mathbb{F} \in \mathsf{F}_\mathsf{L}^\top(Y)$. *Then* $\mathbb{B} = \{\varphi^\leftarrow(b) \ : \ b \in \mathbb{F}\}$ *is a* $\top$*-filter base if and only if* $\bigvee_{y\in\varphi(X)} b(y) = \top$ *for all* $b \in \mathbb{F}$.

A special case arises by considering, for $A \subseteq X$, the mapping $i_A : A \longrightarrow X$, $i_A(x) = x$ for all $x \in A$. For $\mathbb{F} \in \mathsf{F}_L^\top(X)$ then $i_A^\leftarrow(\mathbb{F}) = \mathbb{F}_A$ exists if $\bigvee_{a\in A} f(a) = \top$ for all $f \in \mathbb{F}$. We call $\mathbb{F}_A$ the *trace of* $\mathbb{F}$ *on* A in this case. Denoting for $\mathbb{G} \in \mathsf{F}_L^\top(A)$, $[\mathbb{G}] = i_A(\mathbb{G})$ we then have $[\mathbb{G}]_A = \mathbb{G}$ and $[\mathbb{F}_A] \geq \mathbb{F}$.

We now turn to products of $\top$-filters.

Proposition 3.8. [20] *Let* $\mathbb{F} \in \mathsf{F}_\mathsf{L}^\top(X)$ *and* $\mathbb{G} \in \mathsf{F}_\mathsf{L}^\top(Y)$. *Then* $\mathbb{B} = \{f \times g \ : \ f \in \mathbb{F}, g \in \mathbb{G}\}$ *is a* $\top$*-filter base on* $X \times Y$.

We denote the generated $\top$-filter by $\mathbb{F} \times \mathbb{G}$ and call it the *Cartesian product of* $\mathbb{F}$ *and* $\mathbb{G}$.

Many important properties of the Cartesian product of $\top$-filters need that the underlying lattice of the quantale is a frame, see e.g. [13]. For this reason another product will prove useful later.

Proposition 3.9. *Let* $\mathbb{F} \in \mathsf{F}_L^\top(X)$ *and* $\mathbb{G} \in \mathsf{F}_L^\top(Y)$. *Then* $\mathbb{B} = \{f \otimes g \ : \ f \in \mathbb{F}, g \in \mathbb{G}\}$ *is a* $\top$*-filter base on* $X \times Y$. *Here,* $f \otimes g(x, y) = f(x) * g(y)$ *for all* $x \in X, y \in Y$.

Proof. We have $\bigvee_{(x,y)\in X\times Y} f \otimes g(x, y) = \bigvee_{x\in X} f(x) * \bigvee_{y\in Y} g(y) = \top * \top = \top$ and we have (TB1). For (TB2), let $f_1, f_2 \in \mathbb{F}$ and $g_1, g_2 \in \mathbb{G}$. Noting that $(f_1 \wedge f_2)(u) * (g_1 \wedge g_2)(v) \leq (f_1(u) * g_1(v)) \wedge (f_2(u) * g_2(v))$ we conclude

$$\bigvee_{f_3\in\mathbb{F}, g_3\in\mathbb{G}} [f_3 \otimes g_3, (f_1 \otimes g_1) \wedge (f_2 \otimes g_2)] \geq [(f_1 \wedge f_2) \otimes (g_1 \wedge g_2), (f_1 \otimes g_1) \wedge (f_2 \otimes g_2)] = \top.$$

□

We denote the generated $\top$-filter by $\mathbb{F} \otimes \mathbb{G}$ and call it the *monoidal product of* $\mathbb{F}$ *and* $\mathbb{G}$. As $f(x) * g(y) \leq f(x) \wedge g(y)$ we see that $\mathbb{F} \times \mathbb{G} \leq \mathbb{F} \otimes \mathbb{G}$. If the quantale is a frame, i.e. if $* = \wedge$, then we have equality.

Proposition 3.10. *Let* $\mathbb{F}, \mathbb{G} \in \mathsf{F}_L^\top(X)$ *and* $\mathbb{H} \in \mathsf{F}_L^\top(Y)$. *Then* $(\mathbb{F} \wedge \mathbb{G}) \otimes \mathbb{H} = (\mathbb{F} \otimes \mathbb{H}) \wedge (\mathbb{G} \otimes \mathbb{H})$.

Proof. Clearly $(\mathbb{F} \wedge \mathbb{G}) \otimes \mathbb{H} \leq (\mathbb{F} \otimes \mathbb{H}) \wedge (\mathbb{G} \otimes \mathbb{H})$. So we prove the other inequality. Let $f, g \in L^X$ and $h, k \in L^Y$. For $x \in X$ and $y \in Y$ we then have

$$(f \otimes h) \vee (g \otimes k)(x, y) \geq (f(x) * (h \wedge k)(y)) \vee (g(x) * (h \wedge k)(y)) = (f \vee g)(x) * (h \wedge k)(y) = (f \vee g) \otimes (h \wedge k)(x, y).$$

If $a \in (\mathbb{F} \otimes \mathbb{H}) \wedge (\mathbb{G} \otimes \mathbb{H})$, then $\top = \bigvee_{f\in\mathbb{F},h\in\mathbb{H}}[f \otimes h, a]$ and $\top = \bigvee_{g\in\mathbb{G},k\in\mathbb{H}}[g \otimes k, a]$ and we conclude

$$\begin{aligned} \top \leq & \bigvee_{f\in\mathbb{F},g\in\mathbb{G},h,k\in\mathbb{H}} [f \otimes h, a] * [g \otimes k, a] \leq \bigvee_{f\in\mathbb{F},g\in\mathbb{G},h,k\in\mathbb{H}} [(f \otimes h) \vee (g \otimes k), a] \\ \leq & \bigvee_{f\in\mathbb{F},g\in\mathbb{G},h,k\in\mathbb{H}} [(f \vee g) \otimes (h \wedge k), a] \leq \bigvee_{f\in\mathbb{F},g\in\mathbb{G},m\in\mathbb{H}} [(f \vee g) \otimes m, a,], \end{aligned}$$

and hence $a \in (\mathbb{F} \wedge \mathbb{G}) \otimes \mathbb{H}$. □

Proposition 3.11. *Let* $\mathbb{F} \in \mathsf{F}_\mathsf{L}^\top(X)$ *and* $\mathbb{G} \in \mathsf{F}_\mathsf{L}^\top(Y)$ *and* $\varphi : X \longrightarrow U, \psi : Y \longrightarrow V$ *be mappings. Then* $(\varphi \times \psi)(\mathbb{F} \otimes \mathbb{G}) = \varphi(\mathbb{F}) \otimes \psi(\mathbb{G})$.

Proof. This follows from

$$\begin{aligned} (\varphi \times \psi)(a \otimes b)(u, v) &= \bigvee_{\varphi(x)=u,\psi(y)=v} a(x) * b(y) = \bigvee_{\varphi(x)=u} a(x) * \bigvee_{\psi(y)=v} b(y) \\ &= \varphi(a)(u) * \psi(b)(v) = \varphi(a) \otimes \psi(b)(u, v). \end{aligned}$$

□

Proposition 3.12. *Let $\mathbb{F} \in \mathsf{F}_L^\top(X), \mathbb{G} \in \mathsf{F}_L^\top(Y)$ and denote the projections by $p_1 : X \times Y \longrightarrow X$, $p_2 : X \times Y \longrightarrow Y$. Then $p_1(\mathbb{F} \otimes \mathbb{G}) = \mathbb{F}$ and $p_2(\mathbb{F} \otimes \mathbb{G}) = \mathbb{G}$.*

Proof. This follows, as e.g. for $f \in \mathbb{F}, g \in \mathbb{G}$ we have $p_1(f \otimes g)(x) = \bigvee_{y\in Y} f(x) * g(y) = f(x) * \bigvee_{y \in Y} g(y) = f(x) * \top = f(x)$, because $\mathbb{G}$ is a $\top$-filter. □

In [13] we defined for $\mathbb{F} \in \mathsf{F}_L^\top(X)$ and $x \in X$ the $\top$-filter $\mathbb{F}_x = \{d \in L^{X\times X} : d(\cdot, x) \in \mathbb{F}\}$ and we showed $\mathbb{F}_x = \mathbb{F} \times [x]$.

Lemma 3.13. *Let $\mathbb{F} \in \mathsf{F}_L^\top(X)$ and $x \in X$. Then $\mathbb{F}_x = \mathbb{F} \otimes [x]$.*

Proof. Let first $d \in \mathbb{F}_x$. Then $d(\cdot, x) \in \mathbb{F}$. Now we note $d(\cdot, x) \otimes \top_{\{x\}}(\cdot)(u, v) = d(u, x) * \top_{\{x\}}(v) \leq d(u, v)$ for all $u, v \in X$. Therefore, $\top = [d(\cdot, x) \otimes \top_{\{x\}}, d] \leq \bigvee_{f\in\mathbb{F}, b\in[x]}[f \otimes b, d]$ and hence $d \in \mathbb{F} \otimes [x]$.

Conversely, let $d \in \mathbb{F}\otimes[x]$. Then $\top = \bigvee_{f\in\mathbb{F},b\in[x]}[f\otimes b, d] \leq \bigvee_{f\in\mathbb{F},b\in[x]} \bigwedge_{u\in X}(f(u)*b(x) \to d(u, x)) = \bigvee_{f\in\mathbb{F}}[f, d(\cdot, x)]$, the latter because $b(x) = \top$ for $b \in [x]$. Hence $d(\cdot, x) \in \mathbb{F}$, i.e. $d \in \mathbb{F}_x$. □

For $\top$-filters $\Phi, \Psi \in \mathsf{F}_L^\top(X \times X)$ we define [18] $\Phi^{-1} = \{a^{-1} : a \in \Phi\}$ with the *inverse* a^{-1} of $a \in L^{X\times X}$ defined by $a^{-1}(x, y) = a(y, x)$ for $x, y \in X$. Then $(\Phi^{-1})^{-1} = \Phi$ and $\Phi \leq \Psi$ implies $\Phi^{-1} \leq \Psi^{-1}$.

Furthermore, $\mathbb{B} = \{\phi \circ \psi : \phi \in \Phi, \psi \in \Psi\}$ is a $\top$-filter base if and only if $\bigvee_{(x,y)\in X\times X} \phi \circ \psi(x, y) = \top$ for all $\phi \in \Phi, \psi \in \Psi$. Here, $\phi \circ \psi(x, y) = \bigvee_{z\in X} \phi(x, z) * \psi(z, y)$ for all $(x, y) \in X \times X$. In this case, we denote the generated $\top$-filter by $\Phi \circ \Psi$ and say that $\Phi \circ \Psi$ exists [18].

Lemma 3.14. *Let $\mathbb{F} \in \mathsf{F}_L^\top(X), \mathbb{G} \in \mathsf{F}_L^\top(Y)$. Then $(\mathbb{F} \otimes \mathbb{G})^{-1} = \mathbb{G} \otimes \mathbb{F}$.*

Lemma 3.15. *Let $\mathbb{F}, \mathbb{H}, \mathbb{G}, \mathbb{K} \in \mathsf{F}_L^\top(X)$. If $\bigvee_{z\in X} g(z) * h(z) = \top$ for all $g \in \mathbb{G}, h \in \mathbb{H}$, then $\mathbb{F} \otimes \mathbb{K} = (\mathbb{F} \otimes \mathbb{G}) \circ (\mathbb{H} \otimes \mathbb{K})$.*

Proof. This follows, as for $f \in \mathbb{F}, g \in \mathbb{G}, h \in \mathbb{H}$ and $k \in \mathbb{K}$ we have

$$(f \otimes g) \circ (h \otimes k)(x, y) = \bigvee_{z\in X} f(x) * g(z) * h(z) * k(y) = f(x) * k(y) * \bigvee_{z\in X} g(z) * h(z) = f \otimes k(x, y).$$

□

In particular, we have $\mathbb{F}_x \circ (\mathbb{F}_x)^{-1} = \mathbb{F} \otimes \mathbb{F}$.

4 $\top$-uniform limit spaces

Definition 4.1. [18] *A pair (X, Λ) is called a* $\top$-uniform limit space *if $\Lambda \subseteq \mathsf{F}_\mathsf{L}^\top(X \times X)$ satisfies*

$(TUC1)$ $[(x, x)] \in \Lambda$ for all $x \in X$;
$(TUC2)$ If $\Phi \leq \Psi$ and $\Phi \in \Lambda$, then $\Psi \in \Lambda$;
$(TUC3)$ $\Phi \wedge \Psi \in \Lambda$ whenever $\Phi, \Psi \in \Lambda$;
$(TUC4)$ $\Phi \circ \Psi \in \Lambda$ whenever $\Phi, \Psi \in \Lambda$ and $\Phi \circ \Psi$ exists;
$(TUC5)$ $\Phi^{-1} \in \Lambda$ whenever $\Phi \in \Lambda$.

A mapping $\varphi : (X, \Lambda) \longrightarrow (X', \Lambda')$ is called uniformly continuous *if $(\varphi \times \varphi)(\Phi) \in \Lambda'$ whenever $\Phi \in \Lambda$. The category which has as objects the $\top$-uniform limit spaces and as morphisms the uniformly continuous mappings is denoted by* $\top$-ULim.

We showed in [13] that the category $\top$-ULim is well-fibred and topological over Set and that if $\mathsf{L} = (L, \leq, *)$ is a commutative and integral quantale which is divisible [9] or is a value quantale [5], then the category $\top$-ULim is Cartesian closed.

Initial constructions are done as follows [13]. For a source $(\varphi_\gamma : X \longrightarrow (X_\gamma, \Lambda_\gamma))_{\gamma\in\Gamma}$, the initial structure $\Lambda = init(\Lambda_\gamma)$ on X is defined by

$$\Phi \in \Lambda \iff (\varphi_\gamma \times \varphi_\gamma)(\Phi) \in \Lambda_\gamma \text{ for all } \gamma \in \Gamma.$$

In particular, *subspaces* are defined as follows. For $A \subseteq X$ and $(X, \Lambda) \in |\top{-}\mathsf{ULim}|$ we define the $\top$-uniform convergence on A, $\Lambda|_A$, by $\Phi \in \Lambda|_A$ iff $(i_A \times i_A)(\Phi) = [\Phi] \in \Lambda$.

An important example for $\top$-uniform limit spaces is given by *probabilistic uniform spaces* in the definitions of [8, 21], i.e. spaces $(X,\mathcal{U})$ with a $\top$-filter $\mathcal{U} \in \mathsf{F}_\mathsf{L}^\top(X \times X)$ satisfying (TU1) $\mathcal{U} \leq [(x, x)]$ for all $x \in X$; (TU2) $\mathcal{U} \leq \mathcal{U} \circ \mathcal{U}$ and (TU3) $\mathcal{U} \leq \mathcal{U}^{-1}$. Morphisms are the uniformly continuous mappings $\varphi : (X,\mathcal{U}) \longrightarrow (X',\mathcal{U}')$ with $(\varphi \times \varphi)(\mathcal{U}) \geq \mathcal{U}'$.

For a probabilistic uniform space $(X,\mathcal{U})$ we define $\Lambda^{\mathcal{U}} = \{\Phi \in \mathsf{F}_{\mathsf{L}}^{\top}(X \times X) \ : \ \mathcal{U} \le \Phi\}$. Then $(X,\Lambda^{\mathcal{U}}) \in |\top\text{-}\mathsf{ULim}|$. Furthermore, for a uniformly continuous mapping $\varphi : (X,\mathcal{U}) \longrightarrow (X',\mathcal{U}')$ then $\varphi : (X,\Lambda^{\mathcal{U}}) \longrightarrow (X',\Lambda^{\mathcal{U}'})$ is uniformly continuous.

Probabilistic uniform spaces can be identified with *principal* $\top$*-uniform limit spaces*, i.e. $\top$-uniform limit spaces (X,Λ) for which $\mathcal{U}^{\Lambda} \in \Lambda$ [13], where $\mathcal{U}^{\Lambda} = \bigwedge_{\Phi\in\Lambda} \Phi$.

For $(X,\Lambda) \in |\top\text{-}\mathsf{ULim}|$, we define [13] $q^{\Lambda} : \mathsf{F}_{\mathsf{L}}^{\top}(X) \longrightarrow \mathsf{P}(X)$ by $x \in q^{\Lambda}(\mathbb{F}) \iff \mathbb{F} \times [x] \in \Lambda$.

Then (X, q^{Λ}) is a $\top$*-limit space*, i.e. satisfies the axioms

$(TC1)$ $x \in q^{\Lambda}([x])$ for all $x \in X$;

$(TC2)$ $q^{\Lambda}(\mathbb{F}) \subseteq q^{\Lambda}(\mathbb{G})$ whenever $\mathbb{F} \le \mathbb{G}$;

$(TC3)$ $q^{\Lambda}(\mathbb{F}) \cap q^{\Lambda}(\mathbb{G}) \subseteq q^{\Lambda}(\mathbb{F} \wedge \mathbb{G})$.

In the absence of (TC3) we speak of a $\top$-convergence space [4].

Furthermore, for a uniformly continuous mapping $\varphi(X,\Lambda) \longrightarrow (X',\Lambda')$, $\varphi : (X,q^{\Lambda}) \longrightarrow (X',q^{\Lambda'})$ is continuous, i.e. $\varphi(x) \in q^{\Lambda'}(\varphi(\mathbb{F}))$ whenever $x \in q^{\Lambda}(\mathbb{F})$.

Remark 4.2. *For a probabilistic uniform space $(X,\mathcal{U})$ we define convergence via $q^{\mathcal{U}} = q^{\Lambda^{\mathcal{U}}}$. We showed in [13] that $x \in q^{\mathcal{U}}(\mathbb{F})$ if and only if $\mathbb{F} \ge \mathcal{U}(x) = \{d(\cdot,x) \ : \ d \in \mathcal{U}\}$.*

Denoting the category of $\top$-limit spaces with continuous mappings as morphisms by $\top\text{-}\mathsf{Lim}$, we have a functor

$$\mathsf{H} : \begin{cases} \top\text{-}\mathsf{ULim} & \longrightarrow & \top\text{-}\mathsf{Lim} \\ (X,\Lambda) & \longmapsto & (X,q^{\Lambda}) \\ \varphi & \longmapsto & \varphi \end{cases}$$

The category $\top\text{-}\mathsf{Lim}$ is topological and initial constructions are done as follows [4]. For a source $(\varphi_\gamma : X \longrightarrow (X_\gamma, q_\gamma))_{\gamma\in\Gamma}$, the initial structure $q = init(q_\gamma)$ on X is defined by

$$x \in q(\mathbb{F}) \iff \varphi_\gamma(x) \in q_\gamma(\varphi_\gamma(\mathbb{F})) \text{ for all } \gamma \in \Gamma.$$

The functor H preserves initial constructions in the sense that for a source $(\varphi_\gamma : X \longrightarrow (X_\gamma,\Lambda_\gamma))_{\gamma\in\Gamma}$ in $\top\text{-}\mathsf{ULim}$ and if Λ is the initial structure on X, then q^{Λ} is the initial structure in $\top\text{-}\mathsf{Lim}$ for the source $(\varphi_\gamma : X \longrightarrow (X_\gamma, q^{\Lambda_\gamma}))_{\gamma\in\Gamma}$, [13].

In particular, for $A \subseteq X$ we have $q^{\Lambda|_A} = (q^{\Lambda})|_A$ with the subspace structure $x \in q^{\Lambda}|_A(\mathbb{F})$ iff $x \in q^{\Lambda}(i_A(\mathbb{F}))$.

The product space $(X \times X, q \times q)$ of a $\top$-limit space (X,q) is defined as initial construction for the projection mappings $p_1, p_2 : X \times X \longrightarrow X$ and results in $(x,y) \in q \times q(\Phi)$ if and only if $x \in q(p_1(\Phi))$ and $y \in q(p_2(\Phi))$.

We will later need the following result.

Proposition 4.3. *Let $(X,q_X),(Y,q_Y)$ be $\top$-convergence spaces and let $\varphi : X \longrightarrow Y$ be continuous. Then $\varphi \times \varphi : (X \times X, q_X \times q_X) \longrightarrow (Y \times Y, q_Y \times q_Y)$ is continuous.*

Proof. Let $(x_1,x_2) \in q_X \times q_X(\Phi)$. Then $x_1 \in q_X(p_1(\Phi))$ and $x_2 \in q_X(p_2(\Phi))$. The continuity yields $\varphi(x_1) \in q_Y(\varphi(p_1(\Phi)))$ and $\varphi(x_2) \in q_Y(\varphi(p_2(\Phi)))$ and hence $(\varphi\times\varphi)(x_1,x_2) \in q_Y \times y_Y(\varphi(p_1(\Phi)) \times \varphi(p_2(\Phi)))$. Noting that $\varphi(p_1(\Phi)) \times \varphi(p_2(\Phi)) \le (\varphi\times\varphi)(p_1(\Phi) \times p_2(\Phi))) \le (\varphi\times\varphi)(\Phi)$ establishes $(\varphi\times\varphi)(x_1,x_2) \in q_Y \times q_Y((\varphi\times\varphi)(\Phi))$. □

5 Uniform $\top$-diagonal axioms

5.1 A uniform Fischer $\top$-diagonal condition

Let J be a set, $\Psi \in \mathsf{F}_L^{\top}(J)$ and consider a *selection function* $\sigma : J \longrightarrow \mathsf{F}_L^{\top}(X \times X)$. We define, for $a \in L^{X\times X}$ the L-set $\widehat{\sigma}(a) \in L^J$ by

$$\widehat{\sigma}(a)(j) = \bigvee_{d\in\sigma(j)} [d,a] = \bigvee_{d\in\sigma(j)} \bigwedge_{(x_1,x_2)\in X\times X} (d(x_1,x_2) \to a(x_1,x_2)).$$

The $\top$*-diagonal filter* $\kappa\sigma\Psi \in \mathsf{F}_L^{\top}(X \times X)$ is then defined by [4]

$$a \in \kappa\sigma\Psi \iff \widehat{\sigma}(a) \in \Psi.$$

Let now $(X,\Lambda) \in |\top{-}\mathsf{ULim}|$. We say that (X,Λ) satisfies the *uniform Fischer* $\top$*-diagonal axiom* (TUF) if

$$\text{for all } J, \sigma : J \longrightarrow \mathsf{F}_L^{\top}(X \times X), \Psi \in \mathsf{F}_L^{\top}(J), \psi : J \longrightarrow X \times X \text{ we have:}$$

$$\text{if } \psi(\Psi) \in \Lambda \text{ and } \psi(j) \in q^\Lambda \times q^\Lambda(\sigma(j)) \text{ for all } j \in J, \text{ then } \kappa\sigma\Psi \in \Lambda.$$

We first show that if (X, Λ) satisfies (TUF), then (X, q^Λ) satisfies the axiom (TF) [4]

$$\text{for all } J, \sigma : J \longrightarrow \mathsf{F}_L^\top(X), \mathbb{G} \in \mathsf{F}_L^\top(J), \varphi : J \longrightarrow X \text{ we have:}$$

$$\text{if } x \in q^\Lambda(\varphi(\mathbb{G})) \text{ and } \varphi(j) \in q^\Lambda(\sigma(j)) \text{ for all } j \in J, \text{ then } x \in q^\Lambda(\kappa\sigma\mathbb{G}).$$

We need the following results.

Lemma 5.1. *Let $(X, \Lambda) \in |\top\!-\!\mathsf{ULim}|$ and let $\varphi : J \longrightarrow X$. Then $(\varphi(j), x) \in q^\Lambda \times q^\Lambda(\mathbb{F} \times [x])$ if and only if $\varphi(j) \in q^\Lambda(\mathbb{F})$.*

Proof. We have $(\varphi(j), x) \in q^\Lambda \times q^\Lambda(\mathbb{F} \times [x])$ if and only if $\varphi(j) \in q^\Lambda(p_1(\mathbb{F} \times [x]))$ and $x \in q^\Lambda(p_2(\mathbb{F} \times [x]))$ if and only if $\varphi(j) \in q^\Lambda(\mathbb{F})$ and $x \in q^\Lambda([x])$ if and only if $\varphi(j) \in q^\Lambda(\mathbb{F})$, as $x \in q^\Lambda([x])$ is always true. □

Lemma 5.2. *We have $\varphi \times id_X(\mathbb{G} \times [x]) = \varphi(\mathbb{G}) \times [x]$.*

Proof. From [13], Proposition 3.14, we get $\varphi \times id_X(\mathbb{G} \times [x]) = (\varphi(\mathbb{G}))_{id_X(x)} = (\varphi(\mathbb{G}))_x = \varphi(\mathbb{G}) \times [x]$. □

Proposition 5.3. *Let $(X, \Lambda) \in |\top\!-\!\mathsf{ULim}|$ satisfy (TUF). Then (X, q^Λ) satisfies (TF).*

Proof. Let J be a set, $\mathbb{G} \in \mathsf{F}_L^\top(J), \sigma : J \longrightarrow \mathsf{F}_L^\top(X), \varphi : J \longrightarrow X$ and $x \in X$. We define $\tilde{J} = J \times \{x\}$, $\psi = \phi \times id_{\{x\}}$, $\Psi = \mathbb{G}_x = \mathbb{G} \times [x]$ and $\tilde{\sigma} : \tilde{J} \longrightarrow \mathsf{F}_L^\top(X \times X)$ by $\tilde{\sigma}((j, x)) = \sigma(j) \times [x] = \sigma(j)_x$. If $x \in q^\Lambda(\varphi(\mathbb{G}))$, then $\psi(\mathbb{G}_x) = \varphi(\mathbb{G})_x \in \Lambda$. Also, if $\varphi(j) \in q^\Lambda(\sigma(j))$, then $\psi(j, x) \in (q^\Lambda \times q^\Lambda)(\sigma(j) \times [x]) = (q^\Lambda \times q^\Lambda)(\tilde{\sigma}(j))$. By the axiom (TUF) we conclude $\kappa\tilde{\sigma}\mathbb{G}_x \in \Lambda$. We show that $\kappa\tilde{\sigma}\mathbb{G}_x = (\kappa\sigma\mathbb{G})_x$. To this end, we first note that for $g \in L^J$ and $f \in L^{J \times X}$ we have

$$[g \times \top_{\{x\}}, f] = \bigwedge_{(j,y) \in J \times X} (g \times \top_{\{x\}}(j, y) \to f(j, y)) = \bigwedge_{j \in J} (g(j) \to f(j, x)) = [g, f(\cdot, x)].$$

This implies $\widehat{\tilde{\sigma}}(f)(j, x) = \bigvee_{f_j \in \sigma(j)} [f_j \times \top_{\{x\}}, f] = \bigvee_{f_j \in \sigma(j)} [f_j, f(\cdot, x)] = \widehat{\sigma}(f(\cdot, x))(j)$, i.e. we have $\widehat{\tilde{\sigma}}(f)(\cdot, x) = \widehat{\sigma}(f(\cdot, x))$. Hence we have $f \in \kappa\tilde{\sigma}\mathbb{G}_x$ if and only if $\widehat{\tilde{\sigma}}(f) \in \mathbb{G}_x$, which is equivalent to $\widehat{\tilde{\sigma}}(f)(\cdot, x) = \widehat{\sigma}(f(\cdot, x)) \in \mathbb{G}$, i.e. to $f(\cdot, x) \in \kappa\sigma\mathbb{G}$, which is equivalent to $f \in (\kappa\sigma\mathbb{G})_x$. Therefore we know $(\kappa\sigma\mathbb{G})_x \in \Lambda$, which means $x \in q^\Lambda(\kappa\sigma\mathbb{G})$ and (TF) is true. □

We will now show that the axiom (TUF) is preserved under initial constructions.

Lemma 5.4. *Let $(\varphi_\lambda : X \longrightarrow (X_\lambda, \Lambda_\gamma))_{\gamma \in \Gamma}$ be a source and denote $init(\Lambda_\gamma)$ the initial $\top$-uniform limit structure on X. Then $(u, v) \in q^{init(\Lambda_\gamma)} \times q^{init(\Lambda_\gamma)}(\Phi)$ implies $(\varphi_\gamma(u), \varphi_\gamma(v)) \in q^{\Lambda_\gamma} \times q^{\Lambda_\gamma}((\varphi_\gamma \times \varphi_\gamma)(\Phi))$ for all $\gamma \in \Gamma$.*

Proof. Let $(u, v) \in q^{init(\Lambda_\gamma)} \times q^{init(\Lambda_\gamma)}(\Phi)$. Then $u \in q^{init(\Lambda_\gamma)}(p_1(\Phi))$ and $v \in q^{init(\Lambda_\gamma)}(p_2(\Phi))$. As $q^{init(\Lambda_\gamma)} = init(q^{\Lambda_\gamma})$ this is equivalent to $\varphi_\gamma(u) \in q^{\Lambda_\gamma}(\varphi_\gamma(p_1(\Phi)))$ and $\varphi_\gamma(v) \in q^{\Lambda_\gamma}(\varphi_\gamma(p_2(\Phi)))$ for all $\gamma \in \Gamma$. Hence $(\varphi_\gamma(u), \varphi_\gamma(v)) \in (q^{\Lambda_\gamma} \times q^{\Lambda_\gamma})(\varphi_\gamma(p_1(\Phi)) \times \varphi_\gamma(p_2(\Phi)))$ for all $\gamma \in \Gamma$. As $\varphi_\gamma(p_1(\Phi)) \times \varphi_\gamma(p_2(\Phi)) \leq (\varphi_\gamma \times \varphi_\gamma)(p_1(\Phi) \times p_2(\Phi)) \leq (\varphi_\gamma \times \varphi_\gamma)(\Phi)$ for all $\gamma \in \Gamma$, the result follows. □

Proposition 5.5. *Let $(X_\gamma, \Lambda_\gamma)$ satisfy the axiom (TUF) for all $\gamma \in \Gamma$ and let $\varphi_\gamma : X \longrightarrow X_\gamma$ be mappings for all $\gamma \in \Gamma$. Then $(X, init(\Lambda_\gamma))$ satisfied (TUF).*

Proof. Let J be a set, $\Psi \in \mathsf{F}_L^\top(J), \sigma : J \longrightarrow \mathsf{F}_L^\top(X \times X)$, $\psi : J \longrightarrow X \times X$ and let $\psi(\Psi) \in init(\Lambda_\gamma)$ and $\psi(j) \in q^{init(\Lambda_\gamma)} \times q^{init(\Lambda_\gamma)}(\sigma(j))$ for all $j \in J$.

We define $\psi_\gamma = (\varphi_\gamma \times \varphi_\gamma) \circ \psi : J \longrightarrow X_\gamma \times X_\gamma$ and $\sigma_\gamma = (\varphi_\gamma \times \varphi_\gamma) \circ \sigma : J \longrightarrow \mathsf{F}_L^\top(X_\gamma \times X_\gamma)$.

As $\psi(\Psi) \in init(\Lambda_\gamma)$ we have $\psi_\gamma(\Psi) = (\varphi_\gamma \times \varphi_\gamma)(\psi(\Psi)) \in \Lambda_\gamma$ for all $\gamma \in \Gamma$ and likewise $\psi(j) \in q^{init(\Lambda_\gamma)} \times q^{init(\Lambda_\gamma)}(\sigma(j))$ implies $\psi_\gamma(j) = (\varphi_\gamma \times \varphi_\gamma)(\psi(j)) \in q^{\Lambda_\gamma} \times q^{\Lambda_\gamma}((\varphi_\gamma \times \varphi_\gamma)(\sigma(j))) = q^{\Lambda_\gamma} \times q^{\Lambda_\gamma}(\sigma_\gamma(j))$ for all $j \in J$. By the axiom (TUF) we conclude $\kappa\sigma_\gamma(\Psi) \in \Lambda_\gamma$ for all $\gamma \in \Gamma$.

We show that $\kappa\sigma_\gamma(\Psi) = (\varphi_\gamma \times \varphi_\gamma)(\kappa\sigma\Psi)$. To this end, we note that

$$\begin{aligned}
\widehat{\sigma_\gamma}(f)(j) &= \bigvee_{d \in (\varphi_\gamma \times \varphi_\gamma)(\sigma(j))} [d, f] \\
&= \bigvee_{h \in \sigma(j)} [(\varphi_\gamma \times \varphi_\gamma)(h), f] \\
&= \bigvee_{h \in \sigma(j)} [h, (\varphi_\gamma \times \varphi_\gamma)^{\leftarrow}(f)] \\
&= \widehat{\sigma}((\varphi_\gamma \times \varphi_\gamma)^{\leftarrow}(f))(j).
\end{aligned}$$

Hence $f \in \kappa\sigma_\gamma\Psi$ is equivalent to $\widehat{\sigma}((\varphi_\gamma \times \varphi_\gamma)^{\leftarrow}(f)) \in \Psi$, i.e. to $(\varphi_\gamma \times \varphi_\gamma)^{\leftarrow}(f) \in \kappa\sigma\Psi$, which is equivalent to $f \in (\varphi_\gamma \times \varphi_\gamma)(\kappa\sigma\Psi)$. Therefore we conclude that $(\varphi_\gamma \times \varphi_\gamma)(\kappa\sigma\Psi) \in \Lambda_\gamma$ for all $\gamma \in \Gamma$ and we have $\kappa\sigma\Psi \in init(\Lambda_\gamma)$. □

5.2 A uniform Kowalsky ⊤-diagonal condition

If we specialize in the condition (TUF) the set J to be $X \times X$, then we obtain a diagonal condition that was introduced and studied in the case $L = \{0,1\}$ by Gähler [6]. This condition resembles the diagonal condition that Kowalsky introduced for convergence spaces [14], why we call it the *uniform Kowalsky ⊤-diagonal condition* (TUK):

$$\text{For all } \sigma : X \times X \longrightarrow \mathsf{F}_L^\top(X \times X), \Psi \in \mathsf{F}_L^\top(X \times X) \text{ we have:}$$

$$\text{if } \Psi \in \Lambda \text{ and } (u,v) \in q^\Lambda \times q^\Lambda(\sigma(u,v)) \text{ for all } (u,v) \in X \times X, \text{ then } \kappa\sigma\Psi \in \Lambda.$$

Clearly (TUF) implies (TUK).

Similarly as in Proposition 5.3, we can show that if $(X,\Lambda) \in |\top{-}\mathsf{ULim}|$ satisfies the axiom (TUK), then the induced ⊤-limit space (X, q^Λ) satisfies the axiom (TK)

$$\text{for all } \sigma : X \longrightarrow \mathsf{F}_L^\top(X), \mathbb{G} \in \mathsf{F}_L^\top(X) \text{ we have:}$$

$$\text{if } x \in q^\Lambda(\mathbb{G}) \text{ and } y \in q^\Lambda(\sigma(y)) \text{ for all } y \in X, \text{ then } x \in q^\Lambda(\kappa\sigma\mathbb{G}).$$

This axiom was introduced in [4]. We state this as a proposition but do not provide a proof as it is similar to the proof of Proposition 5.3.

Proposition 5.6. *Let $(X,\Lambda) \in |\top{-}\mathsf{ULim}|$ satisfy (TUK). Then (X, q^Λ) satisfies (TK).*

Unlike the axiom (TUF), the axiom (TUK) is not always preserved by initial constructions. However, we have the following result.

Proposition 5.7. *Let $(X_\gamma, \Lambda_\gamma)$ satisfy the axiom (TUK) for all $\gamma \in \Gamma$ and let $\varphi_\gamma : X \longrightarrow X_\gamma$ be* injective *for all $\gamma \in \Gamma$. Then $(X, init(\Lambda_\gamma))$ satisfied (TUK).*

Proof. Let $\sigma : X \times X \longrightarrow \mathsf{F}_L^\top(X \times X), \Psi \in \mathsf{F}_L^\top(X \times X)$ and let $\Psi \in q^{init(\Lambda_\gamma)}$ and $(u,v) \in q^{init(\Lambda_\gamma)} \times q^{init(\Lambda_\gamma)}(\sigma(u,v)) \forall (u,v) \in X \times X$. We then have for all $\gamma \in \Gamma$ that $(\varphi_\gamma \times \varphi_\gamma)(\Psi) \in \Lambda_\gamma$ and $(\varphi_\gamma(u), \varphi_\gamma(v)) \in q^{\Lambda_\gamma} \times q^{\Lambda_\gamma}((\varphi_\gamma \times \varphi_\gamma)(\sigma(u,v)))$.

We define $\sigma_\gamma : X_\gamma \times X_\gamma \longrightarrow \mathsf{F}_L^\top(X_\gamma \times X_\gamma)$ as follows. If $(u_\gamma, v_\gamma) \in (\varphi_\gamma \times \varphi_\gamma)(X \times X)$, then $\sigma_\gamma(u_\gamma, v_\gamma) = (\varphi_\gamma \times \varphi_\gamma)(\sigma(u,v))$ with the unique $(u,v) \in X \times X$ such that $(\varphi_\gamma \times \varphi_\gamma)(u,v) = (u_\gamma, v_\gamma)$. If $(u_\gamma, v_\gamma) \notin (\varphi_\gamma \times \varphi_\gamma)(X \times X)$, then we define $\sigma_\gamma(u_\gamma, v_\gamma) = [(u_\gamma, v_\gamma)]$. In the first case then $(u_\gamma, v_\gamma) \in q^{\Lambda_\gamma} \times q^{\Lambda_\gamma}((\varphi_\gamma \times \varphi_\gamma)(\sigma(u,v))) = q^{\Lambda_\gamma} \times q^{\Lambda_\gamma}(\sigma_\gamma(u_\gamma, v_\gamma))$ and in the second case trivially $(u_\gamma, v_\gamma) \in q^{\Lambda_\gamma} \times q^{\Lambda_\gamma}([(u_\gamma, v_\gamma)])$. The axiom (TUK) implies $\kappa\sigma_\gamma(\varphi_\gamma \times \varphi_\gamma)(\Psi) \in \Lambda_\gamma$ for all $\gamma \in \Gamma$ and we need to show that $\kappa\sigma_\gamma(\varphi_\gamma \times \varphi_\gamma)(\Psi) = (\varphi_\gamma \times \varphi_\gamma)(\kappa\sigma\Psi)$.

Let $d \in L^{X_\gamma \times X_\gamma}$. Then $(\varphi_\gamma \times \varphi_\gamma)(\widehat{\sigma_\gamma}(d))(u,v) = \bigvee_{d_\gamma \in \sigma_\gamma((\varphi_\gamma \times \varphi_\gamma)(u,v))} [d_\gamma, d] = \bigvee_{e \in \sigma(u,v))} [(\varphi_\gamma \times \varphi_\gamma)(e), d] = \bigvee_{e \in \sigma(u,v)} [e, (\varphi_\gamma \times \varphi_\gamma)^{\leftarrow}(d)] = \widehat{\sigma}((\varphi_\gamma \times \varphi_\gamma)^{\leftarrow}(d))(u,v)$. Hence we conclude $d \in \kappa\sigma_\gamma((\varphi_\gamma \times \varphi_\gamma)(\Psi))$ if and only if $\widehat{\sigma_\gamma}(d) \in (\varphi_\gamma \times \varphi_\gamma)(\Psi)$, which is equivalent to $\widehat{\sigma}((\varphi_\gamma \times \varphi_\gamma)^{\leftarrow}(d)) = (\varphi_\gamma \times \varphi_\gamma)^{\leftarrow}(\widehat{\sigma_\gamma}(d)) \in \Psi$, i.e. to $(\varphi_\gamma \times \varphi_\gamma)^{\leftarrow}(d) \in \kappa\sigma\Psi$. This, however, is equivalent to $d \in (\varphi_\gamma \times \varphi_\gamma)(\kappa\sigma\Psi)$, as desired.

Hence, we have $(\varphi_\gamma \times \varphi_\gamma)(\kappa\sigma\Psi) \in \Lambda_\gamma$ for all $\gamma \in \Gamma$, which means $\kappa\sigma\Psi \in init(\Lambda_\gamma)$. □

Corollary 5.8. *Let $(X,\Lambda) \in |\top{-}\mathsf{ULim}|$ satisfy the axiom (TUK) and let $A \subseteq X$. Then the subspace $(A, \Lambda|_A)$ satisfies (TUK).*

5.3 A uniform Gähler ⊤-neighbourhood condition

Let $(X,\Lambda) \in |\top{-}\mathsf{ULim}|$. We define the ⊤-neighbourhood filter with respect to $q^\Lambda \times q^\Lambda$ by

$$\mathbb{U}^{(x,y)}_{q^\Lambda \times q^\Lambda} = \mathbb{U}^{(x,y)} = \bigwedge_{(x,y) \in q^\Lambda \times q^\Lambda(\Phi)} \Phi,$$

and the selection function $\sigma_{\mathbb{U}} : X \times X \longrightarrow \mathsf{F}_L^\top(X \times X)$ by $\sigma_{\mathbb{U}}(x,y) = \mathbb{U}^{(x,y)}$ for all $(x,y) \in X \times X$. The diagonal ⊤-filter $\mathbb{U}(\Phi) = \kappa\sigma_{\mathbb{U}}\Phi$ is called the *uniform ⊤-neighbourhood filter of* Φ.

From $(x,y) \in q^\Lambda \times q^\Lambda([(x,y)])$ and $d \in \mathbb{U}(\Phi)$ iff $\widehat{\sigma_{\mathbb{U}}}(d) \in \Phi$ it follows with $\widehat{\sigma_{\mathbb{U}}}(d)(x,y) \le \bigvee_{e \in \mathbb{U}^{(x,y)}} (e(x,y) \to d(x,y)) = \top \to d(x,y) = d(x,y)$ that $\mathbb{U}(\Phi) \le \Phi$.

We say that $(X,\Lambda) \in |\top{-}\mathsf{ULim}|$ satisfies the *uniform Gähler ⊤-neighbourhood condition* (TUG) if $\Phi \in \Lambda$ implies $\mathbb{U}(\Phi) \in \Lambda$ for all $\Phi \in \mathsf{F}_L^\top(X \times X)$. This condition goes back to the work of W. Gähler [6].

Proposition 5.9. *Let $(X,\Lambda) \in |\top{-}\mathsf{ULim}|$. Then (X,Λ) satisfies (TUF) whenever (X,Λ) satisfies (TUG). If the underlying lattice of the quantale is continuous [7], then also the converse is true, i.e. (TUF) and (TUG) are equivalent.*

Proof. Let first (TUG) be satisfied and let J be a set, $\psi : J \longrightarrow X \times X$, $\Psi \in \mathsf{F}_L^\top(J)$ and $\sigma : J \longrightarrow \mathsf{F}_L^\top(X \times X)$ such that $\psi(\Psi) \in \Lambda$ and $\psi(j) \in q^\Lambda \times q^\Lambda(\sigma(j))$ for all $j \in J$. Then $\sigma(j) \geq \mathbb{U}^{\psi(j)}$ for all $j \in J$ and (TUG) implies $\mathbb{U}(\psi(\Psi)) \in \Lambda$. Hence it is sufficient to show $\mathbb{U}(\psi(\Psi)) \leq \kappa\sigma\Psi$. If $a \in \mathbb{U}(\psi(\Psi))$, then $\widehat{\sigma_{\mathbb{U}}}(a) \in \psi(\Psi)$, i.e. $\psi^{\leftarrow}(\widehat{\sigma_{\mathbb{U}}}(a)) \in \Psi$. For $j \in J$ we have $\psi^{\leftarrow}(\widehat{\sigma_{\mathbb{U}}}(a))(j) = \bigvee_{d \in \mathbb{U}^{\psi(j)}}[d, a] \leq \bigvee_{d \in \sigma(j)}[d, a] = \widehat{\sigma}(a)(j)$. Hence also $\widehat{\sigma}(a) \in \Psi$, which is equivalent to $a \in \kappa\sigma\Psi$.

For the converse, let $\Phi \in \Lambda$. We define $J = \{(\Psi, (x, y)) : (x, y) \in q^\Lambda \times q^\Lambda(\Psi)\}$ and define $\psi : J \longrightarrow X \times X$ by $\psi((\Psi, (x, y))) = (x, y)$ and $\sigma : J \longrightarrow \mathsf{F}_L^\top(X \times X)$ by $\sigma((\Psi, (x, y))) = \Psi$. As $(x, y) \in q^\Lambda \times q^\Lambda([(x, y)])$, the mapping ψ is surjective and hence $\mathcal{K} = \psi^{\leftarrow}(\Phi) \in \mathsf{F}_L^\top(J)$ and $\psi(\mathcal{K}) = \Phi$. Moreover, $\psi((\Psi, (x, y))) = (x, y) \in q^\Lambda \times q^\Lambda(\Psi) = q^\Lambda \times q^\Lambda(\sigma(\Psi, (x, y)))$ for all $(\Psi, (x, y)) \in J$. The axiom (TUF) then implies $\kappa\sigma\mathcal{K} \in \Lambda$ and it is sufficient to show that $\kappa\sigma\mathcal{K} = \mathbb{U}(\Phi)$.

In [16] it is shown that for a continuous lattice, with the notation $P(\mathbb{F})(a) = \bigvee_{f \in \mathbb{F}}[f, a]$ for a $\top$-filter $\mathbb{F}$ and $a \in L^X$, we have $\bigwedge_{\alpha \in A} P(\mathbb{F}_\alpha) \leq P(\bigwedge_{\alpha \in A} \mathbb{F}_\alpha)$. It follows for $a \in L^{X \times X}$ and $d \in \Phi$ with the notation $J(x, y) = \{\Psi \in \mathsf{F}_L^\top(X \times X) : (x, y) \in q^\Lambda \times q^\Lambda(\Psi)\}$,

$$\begin{aligned}
[\psi^{\leftarrow}(d), \widehat{\sigma}(a)] &= \bigwedge_{(\Psi,(x,y)) \in J} (\psi^{\leftarrow}(d)((\Psi, (x, y))) \to \widehat{\sigma}(a)((\Psi, (x, y)))) \\
&= \bigwedge_{(x,y) \in X \times X} \bigwedge_{\Psi \in J(x,y)} (d(x, y) \to P(\sigma((\Psi, (x, y)))(a))) \\
&= \bigwedge_{(x,y) \in X \times X} (d(x, y) \to \bigwedge_{\Psi \in J(x,y)} P(\sigma((\Psi, (x, y)))(a))) \\
&= \bigwedge_{(x,y) \in X \times X} (d(x, y) \to \bigwedge_{\Psi \in J(x,y)} P(\Psi)(a))) \\
&= \bigwedge_{(x,y) \in X \times X} (d(x, y) \to P(\mathbb{U}^{(x,y)})(a))) \\
&= \bigwedge_{(x,y) \in X \times X} (d(x, y) \to \widehat{\sigma_{\mathbb{U}}}(a)) \\
&= [d, \widehat{\sigma_{\mathbb{U}}}(a)].
\end{aligned}$$

We conclude

$$\begin{aligned}
a \in \kappa\sigma\mathcal{K} &\iff \widehat{\sigma}(a) \in \mathcal{K} = \psi^{\leftarrow}(\Phi) \\
&\iff \bigvee_{d \in \Phi} [\psi^{\leftarrow}(d), \widehat{\sigma}(a)] = \top \\
&\iff \bigvee_{d \in \Phi} [d, \widehat{\sigma_{\mathbb{U}}}(a)] = \top \\
&\iff \widehat{\sigma_{\mathbb{U}}}(a) \in \Phi \\
&\iff a \in \kappa\sigma_{\mathbb{U}}\Phi = \mathbb{U}(\Phi).
\end{aligned}$$

□

In order to show that a probabilistic uniform space $(X, \mathcal{U})$ satisfies the axiom (TUG), we digress here and introduce the concept of *interior of an L-subset* in a $\top$-convergence space. Let (X, q) be a $\top$-convergence space. For $x \in X$ the $\top$-*neighbourhood filter of* x is defined by $\mathbb{U}_q^x = \mathbb{U}^x = \bigwedge_{x \in q(\mathbb{F})} \mathbb{F}$, [4, 20]. Let now $a \in L^X$. We define the *interior of* a, $\underline{a} \in L^X$, by

$$\underline{a}(x) = \underline{a}_q(x) = \bigvee_{u \in \mathbb{U}^x} [u, a].$$

The value $\underline{a}(x)$ can be interpreted as the grade to which a is a neighbourhood of x. Clearly, we have $u \in \mathbb{U}^x$ if and only if $\underline{u}(x) = \top$.

Lemma 5.10 (Properties of interiors)**.** *Let* $(X, q) \in |\top\text{−}\mathsf{Conv}|$ *and let* $a, b \in L^X$.

1. $\underline{\top_X} = \top_X$.
2. $a \leq b$ *implies* $\underline{a} \leq \underline{b}$.

3. $\underline{a} \leq a$.
4. *If the lattice* $(L, \leq)$ *is a frame, then* $\underline{a} \wedge \underline{b} \leq \underline{a \wedge b}$.

Lemma 5.11. *Let* (X, q) *be a* ⊤*-convergence space and let* $a, b \in L^X$. *Then* $[a, b] \leq [\underline{a}, \underline{b}]$.

Proof. We have

$$\begin{aligned}[\underline{a}, \underline{b}] &= \bigwedge_{x \in X} \left(\left(\bigvee_{u \in \mathbb{U}^x} [u, a] \right) \to \left(\bigvee_{u \in \mathbb{U}^x} [u, b] \right) \right) \\ &\geq \bigwedge_{x \in X} \bigwedge_{u \in \mathbb{U}^x} ([u, a] \to [u, b]) \\ &\geq \bigwedge_{x \in X} \bigwedge_{u \in \mathbb{U}^x} [a, b] \;=\; [a, b].\end{aligned}$$

□

Fang and Yue [4] introduced the "topological axiom"

$$\text{(TT)} \qquad \kappa\sigma_{\mathbb{U}}\mathbb{U}^x = \mathbb{U}^x \text{ for all } x \in X$$

for a ⊤-convergence space (X, q). Noting that $\widehat{\sigma_{\mathbb{U}}}(a) = \underline{a}$, we can rephrase this using interiors as

$$\text{(TT)} \qquad \mathbb{U}^x = \{u \in L^X \,:\, \underline{u} \in \mathbb{U}^x\} \text{ for all } x \in X.$$

Proposition 5.12. *Let* (X, q) *be a* ⊤*-convergence space. Then (TT) is equivalent to* $\underline{a} \leq \underline{\underline{a}}$ *for all* $a \in L^X$.

Proof. If (TT) is valid, then $\underline{\underline{a}}(x) = \bigvee_{u \in \mathbb{U}^x} [u, \underline{a}] = \bigvee_{u \in \mathbb{U}^x} [\underline{u}, \underline{a}] \geq \bigvee_{u \in \mathbb{U}^x} [u, a] = \underline{a}(x)$.

For the converse, let $u \in \mathbb{U}^x$. Then $\underline{u}(x) = \top$ and hence also $\underline{\underline{u}}(x) = \top$. This implies $\widehat{\sigma_{\mathbb{U}}}(u) = \underline{u} \in \mathbb{U}^x$, i.e. $u \in \kappa\sigma_{\mathbb{U}}\mathbb{U}^x$. The other inequality being trivial, this completes the proof. □

Lemma 5.13. *Let* (X, q) *be a* ⊤*-convergence space. Then for all* $x, y \in X$ *we have* $\mathbb{U}_q^x \times \mathbb{U}_q^y \leq \mathbb{U}_{q \times q}^{(x,y)}$. *If* $x \in q(\mathbb{U}_q^x)$ *for all* $x \in X$ *we have equality.*

Proof. Let first $u \in \mathbb{U}_q^x$ and $v \in \mathbb{U}_q^y$. Then $u \in \mathbb{F}$ for all $\mathbb{F}$ with $x \in q(\mathbb{F})$ and $v \in \mathbb{G}$ for all $\mathbb{G}$ with $y \in q(\mathbb{G})$. If $(x, y) \in q \times q(\Phi)$, then $x \in q(p_1(\Phi))$ and $y \in q(p_2(\Phi))$ and hence $u \in p_1(\Phi)$ and $v \in p_2(\Phi)$ which implies $u \times v \in p_1(\Phi) \times p_2(\Phi) \leq \Phi$. Hence $u \times v \in \mathbb{U}_{q \times q}^{(x,y)}$.

Let now $w \in \mathbb{U}_q^x \times \mathbb{U}_q^y$. Then $\top = \bigvee_{u \in \mathbb{U}_q^x, v \in \mathbb{U}_q^y} [u \times v, w] \leq \bigvee_{u \times v \in \mathbb{U}_{q \times q}^{(x,y)}} [u \times v, w]$ which implies $w \in \mathbb{U}_{q \times q}^{(x,y)}$.

The converse follows, as $x \in q(\mathbb{U}_q^x), y \in q(\mathbb{U}_q^y)$ implies $(x, y) \in q \times q(\mathbb{U}_q^x \times \mathbb{U}_q^y)$ and hence $\mathbb{U}_{q \times q}^{(x,y)} \leq \mathbb{U}_q^x \times \mathbb{U}_q^y$. □

If $(X, \mathcal{U})$ is a probabilistic uniform space, then we conclude $\mathbb{U}_{q^{\mathcal{U}} \times q^{\mathcal{U}}}^{(x,y)} = \mathcal{U}(x) \times \mathcal{U}(y)$, with the ⊤-filters $\mathcal{U}(x) = \{u(\cdot, x) \,:\, u \in \mathcal{U}\}$.

Proposition 5.14. *Let the quantale* L *be a frame, i.e.* $* = \wedge$, *and let* $(X, \mathcal{U})$ *be a probabilistic uniform space. Then for* $u \in \mathcal{U}$ *we have* $u \leq \underline{u \circ u \circ u}_{q^{\mathcal{U}} \times q^{\mathcal{U}}}$.

Proof. We have

$$\underline{u \circ u \circ u}_{q^{\mathcal{U}} \times q^{\mathcal{U}}}(x, y) = \bigvee_{v \in \mathcal{U}(x) \times \mathcal{U}(y)} [v, u \circ u \circ u] \geq [u(\cdot, x) \times u^{-1}(\cdot, y), u \circ u \circ u].$$

Now we note that for $z_1, z_2 \in X$ we have

$$\begin{aligned}u(x, y) \wedge u(z_1, x) \wedge u^{-1}(z_2, y) &= u(x, y) \wedge u(z_1, x) \wedge u(y, z_2) \\ &\leq u \circ u(z_1, y) \wedge u(y, z_2) \;\leq\; u \circ u \circ u(z_1, z_2),\end{aligned}$$

and hence

$$[u(\cdot, x) \times u^{-1}(\cdot, y), u \circ u \circ u] = \bigwedge_{z_1, z_2 \in X} \left(u(z_1, x) \wedge u^{-1}(z_2, y) \to u \circ u \circ u(z_1, z_2)\right) \geq u(x, y).$$

□

Theorem 5.15. *Let the quantale* L *be a frame, i.e.* $* = \wedge$, *and let* $(X, \mathcal{U})$ *be a probabilistic uniform space. Then* $\mathcal{U} \leq \mathbb{U}(\mathcal{U})$, *i.e. the axiom (TUG) is valid for* $(X, \Lambda^{\mathcal{U}})$.

Proof. Let $u \in \mathcal{U}$. It was shown in [21] that $\bigvee_{w \in \mathcal{U}} [w \circ w \circ w, u] = \top$ and hence $\top = \bigvee_{w \in \mathcal{U}} [w \circ w \circ w, u] \leq \bigvee_{w \in \mathcal{U}} [\underline{w \circ w \circ w}, \underline{u}] \leq \bigvee_{w \in \mathcal{U}} [w, \underline{u}]$. Therefore, $\underline{u} \in \mathcal{U}$ and $u \in \mathbb{U}(\mathcal{U})$. □

6 Uniform regularity

Let $(X,\Lambda)\in|\top-\mathsf{ULim}|$. We say that (X,Λ) is *uniformly* $\top$*-regular* if the axiom (TUR)

$$\text{for all } J, \sigma: J\longrightarrow \mathsf{F}_L^{\top}(X\times X), \Psi\in \mathsf{F}_L^{\top}(J), \psi: J\longrightarrow X\times X \text{ we have:}$$

$$\text{if } \kappa\sigma\Psi\in\Lambda \text{ and } \psi(j)\in q^{\Lambda}\times q^{\Lambda}(\sigma(j))\forall j\in J \text{ then } \psi(\Psi)\in\Lambda$$

is satisfied. This axiom can be considered as a "dual axiom" of the axiom (TUF). In the case $L=\{0,1\}$ the axiom was introduced and studied in [6].

In a similar way, [4] define the $\top$-regularity of a $\top$-convergence space (X,q), if the axiom (TR)

$$\text{for all } J, \sigma: J\longrightarrow \mathsf{F}_L^{\top}(X), \mathbb{G}\in \mathsf{F}_L^{\top}(J), \varphi: J\longrightarrow X \text{ we have:}$$

$$\text{if } x\in q^{\Lambda}(\kappa\sigma\mathbb{G}) \text{ and } \varphi(j)\in q^{\Lambda}(\sigma(j))\forall j\in J, \text{ then } x\in q^{\Lambda}(\varphi(\mathbb{G})),$$

is valid.

The following two results have proofs that are similar to the proofs of Propositions 5.3 and 5.5 and are therefore not presented.

Proposition 6.1. *Let $(X,\Lambda)\in|\top-\mathsf{ULim}|$ satisfy (TUR). Then (X,q^{Λ}) satisfies (TR).*

Proposition 6.2. *Let $(X_\gamma,\Lambda_\gamma)$ satisfy the axiom (TUR) for all $\gamma\in\Gamma$ and let $\varphi_\gamma: X\longrightarrow X_\gamma$ be mappings for all $\gamma\in\Gamma$. Then $(X, init(\Lambda_\gamma))$ satisfies (TUR).*

In [17] the axiom (TR) in the category of $\top$-convergence spaces was characterized using *closures of L-sets*. We are now going to characterize the axiom (TUR) in $|\top-\mathsf{ULim}|$ in a similar way. For $L=\{0,1\}$ the corresponding results are due to Gähler [6].

Let (X,q) be a $\top$-convergence space and let $a\in L^X$. Then we define [17] the *closure of a* by

$$\overline{a}(x)=\bigvee_{x\in q(\mathbb{G})}\bigvee_{g\in\mathbb{G}}[g,a], \qquad (x\in X).$$

The closure has then the usual properties [17]: $\overline{\top_X}=\top_X$, $a\le\overline{a}$, $a\le b$ implies $\overline{a}\le\overline{b}$ and, in case that L is a complete Boolean algebra, $\overline{a\vee b}\le\overline{a}\vee\overline{b}$, for all $a,b\in L^X$.

Proposition 6.3. *Let $(X,q_X),(Y,q_Y)$ be $\top$-convergence spaces and let $\varphi: X\longrightarrow Y$ be continuous and $a\in L^X$. Then $\varphi(\overline{a})\le\overline{\varphi(a)}$.*

Proof. We have

$$\begin{aligned}\varphi(\overline{a})(y) &= \bigvee_{\varphi(x)=y}\bigvee_{x\in q_X(\mathbb{G})}\bigvee_{g\in\mathbb{G}}[g,a]\le\bigvee_{\varphi(x)=y}\bigvee_{x\in q_X(\mathbb{G})}\bigvee_{g\in\mathbb{G}}[\varphi(g),\varphi(a)]\\ &\le\bigvee_{y\in q_Y(\varphi(\mathbb{G}))}\bigvee_{h\in\varphi(\mathbb{G})}[h,\varphi(a)]\le\bigvee_{y\in q_Y(\mathbb{H})}\bigvee_{h\in\mathbb{H}}[h,\varphi(a)]=\overline{\varphi(a)}(y).\end{aligned}$$

□

Important for us is the following result.

Lemma 6.4. [17] *Let (X,q) be a $\top$-convergence space and let $a,b\in L^X$. Then $[a,b]\le[\overline{a},\overline{b}]$.*

We deduce that for a $\top$-filter $\mathbb{F}\in\mathsf{F}_L^{\top}(X)$, $\mathbb{B}=\{\overline{f}: f\in\mathbb{F}\}$ is a $\top$-filter base and we denote the generated $\top$-filter by $\overline{\mathbb{F}}$, [17].

Proposition 6.5. *Let $(X,\Lambda)\in|\top-\mathsf{ULim}|$. Then (X,Λ) satisfies (TUR) if and only if $\Phi\in\Lambda$ implies $\overline{\Phi}\in\Lambda$ for all $\Phi\in\mathsf{F}_L^{\top}(X\times X)$. Here, the closure is taken with respect to $(X\times X, q^{\Lambda}\times q^{\Lambda})$.*

Proof. Let first (X, Λ) satisfy (TUR) and let $\Phi \in \Lambda$. We define $J = \{(\Psi, (x, y)) : (x, y) \in q^\Lambda \times q^\Lambda(\Psi)\}$ and we define $\psi : J \longrightarrow X \times X$ by $\psi((\Psi, (x, y))) = (x, y)$ and $\sigma : J \longrightarrow \mathsf{F}_L^\top(X \times X)$ by $\sigma((\Psi, (x, y))) = \Psi$. For $(\Psi, (x, y)) \in J$ we then have $\psi((\Psi, (x, y))) \in q^\Lambda \times q^\Lambda(\sigma((\Psi, (x, y))))$ and for $a \in L^{X \times X}$ we get $\widehat{\sigma}(a)((\Psi, (x, y))) = \bigvee_{d \in \Psi}[d, a]$. It is not difficult to show that $\mathcal{B} = \{\widehat{\sigma}(a) : a \in \Phi\}$ is a ⊤-filter base and we denote the generated ⊤-filter on $X \times X$ by $[\mathcal{B}]$. For $d \in \Phi$ we have $\widehat{\sigma}(d) \in [\mathcal{B}]$ and by definition of the ⊤-diagonal filter then $d \in \kappa\sigma[\mathcal{B}]$. We show that $\psi([\mathcal{B}]) \leq \overline{\Phi}$. We have for $a \in \Phi$ that

$$\psi(\widehat{\sigma}(a))(x, y) = \bigvee_{\psi((\Psi,(u,v)))=(x,y)} \widehat{\sigma}(a)((\Psi, (u, v))) = \bigvee_{(\Psi,(x,y))\in J} \bigvee_{e\in\Psi} [e, a] = \overline{a}(x, y).$$

Hence, if $d \in \psi([\mathcal{B}])$, then $\top = \bigvee_{a\in\Phi}[\psi(\widehat{\sigma}(a)), d] = \bigvee_{a\in\Phi}[\overline{a}, d]$ which implies $d \in \overline{\Phi}$. Therefore, $\Phi \in \Lambda$ implies $\kappa\sigma[\mathcal{B}] \in \Lambda$ and, by (TUR), then $\psi([\mathcal{B}]) \in \Lambda$, leading to $\overline{\Phi} \in \Lambda$.

For the converese, let J be a set, $\psi : J \longrightarrow X \times X$, $\sigma : J \longrightarrow \mathsf{F}_L^\top(X \times X)$ and $\Psi \in \mathsf{F}_L^\top(J)$ such that $\psi(j) \in q^\Lambda \times q^\Lambda(\sigma(j))$ and $\kappa\sigma\Psi \in \Lambda$. Then $\overline{\kappa\sigma\Psi} \in \Lambda$ and we need to show that $\overline{\kappa\sigma\Psi} \leq \psi(\Psi)$. To this end, we note that for $d \in L^{X \times X}$ we have

$$\psi(\widehat{\sigma}(d))(x, y) = \vee_{\psi(j)=(x,y)} \bigvee_{e\in\sigma(j)} [e, d] \leq \bigvee_{(x,y)\in q^\Lambda\times q^\Lambda(\Theta)} \bigvee_{e\in\Theta} [e, d] = \overline{d}(x, y).$$

Hence, $\widehat{\sigma}(d) \in \Psi$ implies $\overline{d} \in \psi(\Psi)$ and we conclude

$$a \in \overline{\kappa\sigma\Psi} \iff \top = \bigvee_{d\in\kappa\sigma\Psi} [\overline{d}, a] \iff \bigvee_{\widehat{\sigma}(d)\in\Psi} [\overline{d}, a] \leq \bigvee_{\overline{d}\in\psi(\Psi)} [\overline{d}, a],$$

i.e. $a \in \psi(\Psi)$. This completes the proof. □

We are finally showing that if the quantale is a frame, i.e. if $* = \wedge$, then a probabilistic uniform space is uniformly ⊤-regular.

Lemma 6.6. *Let the quantale* L *be a frame and let* $(X, \mathcal{U})$ *be a probabilistic uniform space and let* $u \in \mathcal{U}$. *Then* $\overline{u} \leq u \circ u \circ u$.

Proof. We first note that $(x, y) \in q^\mathcal{U} \times q^\mathcal{U}(\Phi)$ if and only if $\Phi \geq \mathcal{U}(x) \times \mathcal{U}(y)$, and $\Phi \geq \mathcal{U}(x) \times \mathcal{U}(y)$ if and only if $\top = \bigvee_{f\in\Phi}[f, u^{-1}(\cdot, x) \times u(\cdot, y)]$ for all $u \in \mathcal{U}$. We conclude, for $u \in \mathcal{U}$,

$$\begin{aligned}
\overline{u}(x, y) &= \bigvee_{\Phi\geq\mathcal{U}(x)\times\mathcal{U}(y)} \bigvee_{f\in\Phi} [f, u] \\
&= \bigvee_{\Phi\geq\mathcal{U}(x)\times\mathcal{U}(y)} \left(\bigvee_{g\in\Phi} [g, u^{-1}(\cdot, x) \times u(\cdot, y)] \right) \wedge \bigvee_{f\in\Phi} [f, u] \\
&\leq \bigvee_{\Phi\geq\mathcal{U}(x)\times\mathcal{U}(y)} \left(\bigvee_{h\in\Phi} [h, u^{-1}(\cdot, x) \times u(\cdot, y)] \wedge [h, u] \right) \\
&= \bigvee_{\Phi\geq\mathcal{U}(x)\times\mathcal{U}(y)} \bigvee_{h\in\Phi} [h, (u^{-1}(\cdot, x) \times u(\cdot, y)) \wedge u].
\end{aligned}$$

Now we have, for $z_1, z_2 \in X$

$$(u^{-1}(\cdot, x) \times u(\cdot, y)) \wedge u(z_1, z_2) = u(x, z_1) \wedge u(z_2, y) \wedge u(z_1, z_2) \leq u(x, z_1) \wedge u \circ u(z_1, y) \leq u \circ u \circ u(x, y).$$

Consequently, for $h \in \Phi$, we conclude

$$\begin{aligned}
& [h, (u^{-1}(\cdot, x) \times u(\cdot, y)) \wedge u] \\
\leq\ & \bigwedge_{z_1,z_2\in X} (h(z_1, z_2) \to u \circ u \circ u(x, y)) \\
=\ & (\bigvee_{z_1,z_2\in X} h(z_1, z_2)) \to u \circ u \circ u(x, y) \\
=\ & \top \to u \circ u \circ u(x, y) \ = \ u \circ u \circ u(x, y).
\end{aligned}$$

Hence we obtain

$$\overline{u}(x,y) \leq \bigvee_{\Phi \geq \mathcal{U}(x)\times\mathcal{U}(y)} \bigvee_{h\in\Phi} u\circ u\circ u(x,y) = u\circ u\circ u(x,y).$$

□

Proposition 6.7. *Let the quantale* L *be a frame and let* $(X,\mathcal{U})$ *be a probabilistic uniform space. Then* $\mathcal{U} \leq \overline{\mathcal{U}}$, *i.e.* $(X,\Lambda^{\mathcal{U}})$ *is uniformly* $\top$*-regular.*

Proof. Let $u \in \mathcal{U}$. Then $\bigvee_{w\in\mathcal{U}}[\overline{w},u] \geq \bigvee_{w\in\mathcal{U}}[w\circ w\circ w,u] = \top$, and hence $u \in \overline{\mathcal{U}}$. From this the claim immediately follows. □

We would like to point out here, that Gähler [6], page 303, for the case $L = \{0,1\}$ gave an example that shows that uniform limit spaces that satisfy (TUG) and (TUR) are in general not uniform spaces. So unlike for $\top$-convergence spaces, where the axiom (TF) characterizes the strong L-topological spaces among the $\top$-convergence spaces [4], the axiom (TUF) cannot characterize the probabilistic uniform spaces among the $\top$-uniform limit spaces.

7 An extension theorem for uniformly continuous mappings

In [4] an extension theorem for a continous mapping from a dense subspace was given. We shall use this result and derive from it a related extension theorem in $\top-\mathsf{ULim}$. First we need some concepts. Let $(X,q_X),(Y,q_Y)$ be $\top$-convergence spaces, let $A \subseteq X$ be non-empty and let $\varphi : A \longrightarrow Y$ be a mapping. We denote

$$H_A(x) = \{\mathbb{F} \in \mathsf{F}_L^{\top}(X) \,:\, \mathbb{F}_A \in \mathsf{F}_L^{\top}(A), x \in q_X(\mathbb{F})\},$$

$$F_A(x) = \{y \in Y \,:\, H_A(x) \neq \emptyset, y \in q_Y(\varphi(\mathbb{F}_A))\,\forall \mathbb{F} \in H_A(x)\}.$$

We call $A \subseteq X$ *dense in* (X,q_X) if $H_A(x) \neq \emptyset$ for all $x \in X$. The space (X,q_X) is called a *T2-space* if $x = y$ whenever $x,y \in q_X(\mathbb{F})$.

Fang and Yue [4] proved the following result.

Theorem 7.1. [4] *Let* (X,q_X) *satisfy the axiom (TK) and let* (Y,q_Y) *be a regular T2-space. Let further* $A \subseteq X$ *be dense in* (X,q_X) *and let* $\varphi : (A,q_X|_A) \longrightarrow (Y,q_Y)$ *be continous. Then* φ *has a continuous extension,* $\psi : (X,q_X) \longrightarrow (Y,q_Y)$, *if and only if* $F_A(x) \neq \emptyset$ *for all* $x \in X$.

If we denote the embedding mapping $i_A : A \longrightarrow X$, then ψ being an extension of φ means that $\psi\circ i_A = \varphi$.

Following [17] we call $(X,\Lambda) \in |\top-\mathsf{ULim}|$ *complete* if $\mathbb{F}\otimes\mathbb{F} \in \Lambda$ implies $\mathbb{F}\otimes[x] \in \Lambda$ for some $x \in X$. Note that [17] only consider frames as quantales and then $\otimes = \times$. We call (X,Λ) a T2-space if (X,q^{Λ}) is a T2-space.

Theorem 7.2. *Let* $(X,\Lambda_X),(Y,\Lambda_Y) \in |\top-\mathsf{ULim}|$ *and let* (X,Λ_X) *satisfy the axiom (TUG) and let* (Y,Λ_Y) *be a complete, uniformly* $\top$*-regular T2-space. If* $A \subseteq X$ *is dense in* (X,q^{Λ_X}) *and* $\varphi : (A,\Lambda_X|_A) \longrightarrow (Y,\Lambda_Y)$ *is uniformly continuous, then* φ *has a uniformly continuous extension* $\psi : (X,\Lambda_X) \longrightarrow (Y,\Lambda_Y)$.

Proof. The space (X,Λ_X) satisfies (TUF) and hence (X,q^{Λ_X}) satisfies (TF) and therefore also (TK). Also, (Y,q^{Λ_Y}) satisfies (TR) and is a T2-space. We show that for each $x \in X$, the set $F_A(x)$ is not empty. As $A \subseteq X$ is dense in (X,q^{Λ_X}) the set $H_A(x) \neq \emptyset$ for all $x \in X$. For $\mathbb{F} \in H_A(x)$ we have $x \in q^{\Lambda_X}(\mathbb{F})$, i.e. $\mathbb{F}\times[x] \in \Lambda_X$. Then $\mathbb{F}\otimes\mathbb{F} = (\mathbb{F}\otimes[x])\circ([x]\otimes\mathbb{F}) \in \Lambda_X$. As $i_A^{\leftarrow}(\mathbb{F}) = \mathbb{F}_A$ exists and $i_A(\mathbb{F}_A) \geq \mathbb{F}$ we conclude $(i_A\times i_A)(\mathbb{F}_A\otimes\mathbb{F}_A) = i_A(\mathbb{F}_A)\otimes i_A(\mathbb{F}_A) \in \Lambda_X$ and therefore $\mathbb{F}_A\otimes\mathbb{F}_A \in \Lambda_X|_A$. From the uniform continuity of φ we conclude that $(\varphi\times\varphi)(\mathbb{F}_A\otimes\mathbb{F}_A) = \varphi(\mathbb{F}_A)\otimes\varphi(\mathbb{F}_A) \in \Lambda_Y$. As (Y,Λ_Y) is complete, there is $y_0 \in Y$ such that $\varphi(\mathbb{F}_A)\otimes[y_0] \in \Lambda_Y$. We claim that $y_0 \in F_A(x)$.

Let $\mathbb{G} \in H_A(x)$. Then $\mathbb{G}_A \in \mathsf{F}_L^{\top}(A)$ and $x \in q^{\Lambda_X}(\mathbb{G})$. Then also $x \in q^{\Lambda_X}(\mathbb{F}\wedge\mathbb{G})$ and hence $(\mathbb{F}\wedge\mathbb{G})\otimes(\mathbb{F}\wedge\mathbb{G}) \in \Lambda_X$. We conclude that $(\mathbb{F}\wedge\mathbb{G})_A \in \mathsf{F}_L^{\top}(A)$ and $(\mathbb{F}\wedge\mathbb{G})_A\otimes(\mathbb{F}\wedge\mathbb{G})_A \in \Lambda_X|_A$. The uniform continuity of φ then yields $(\varphi\times\varphi)((\mathbb{F}\wedge\mathbb{G})_A\otimes(\mathbb{F}\wedge\mathbb{G})_A) \in \Lambda_Y$ and again the completeness of (Y,Λ_Y) implies $y_1 \in q^{\Lambda_Y}(\varphi((\mathbb{F}\wedge\mathbb{G})_A)) \subseteq q^{\Lambda_Y}(\varphi(\mathbb{G}_A))$. The T2-property implies $y_0 = y_1$ and as $\mathbb{G} \in H_A(x)$ was arbitrarily chosen, we conclude $y_0 \in F_A(x)$.

This implies that φ has a continuous extension $\psi : (X,q^{\Lambda_X}) \longrightarrow (Y,q^{\Lambda_Y})$ and we are going to show that $\psi : (X,\Lambda_X) \longrightarrow (Y,\Lambda_Y)$ is uniformly continuous.

Let $\Phi \in \Lambda_X$. From the axiom (TUG) we conclude $\mathbb{U}(\Phi) \in \Lambda_X$. In a first step, we show that $\mathbb{U}(\Phi)_{A\times A} \in \mathsf{F}_L^{\top}(A\times A)$.

First we note that because $A \subseteq X$ is dense in (X,q^{Λ_X}) for $(x,y) \in X\times X$ we find $\top$-filters $\mathbb{F},\mathbb{G}$ such that $x \in q^{\Lambda_X}(\mathbb{F})$ and $y \in q^{\Lambda_X}(\mathbb{G})$ and $\mathbb{F}_A,\mathbb{G}_A$ exist. Then $x \in q^{\Lambda_X}([\mathbb{F}_A])$ and $y \in q^{\Lambda_X}([\mathbb{G}_A])$ and hence $(x,y) \in q^{\Lambda_X}\times q^{\Lambda_X}([\mathbb{F}_A\times\mathbb{G}_A])$. If $u \in \mathbb{U}^{(x,y)}$, then $u\wedge\top_{A\times A} \in [\mathbb{F}_A\times\mathbb{G}_A]$ and hence $\bigvee_{(z_1,z_2)\in A\times A} u(z_1,z_2) = \bigvee_{(z_1,z_2)\in X\times X}(u\wedge\top_{A\times A})(z_1,z_2) = \top$. Therefore the trace $\mathbb{U}_{A\times A}^{(x,y)}$ exists.

Let now $d \in \mathbb{U}(\Phi)$. Then $\widehat{\sigma_{\mathbb{U}}}(d) \in \Phi$. Then $\top = \bigvee_{(x,y)\in X\times X} \bigvee_{e\in\mathbb{U}^{(x,y)}} [e,d] \le \bigvee_{(x,y)\in X\times X} \bigvee_{e\in[\mathbb{U}^{(x,y)}_{A\times A}]} [e,d]$. We have $e \in [\mathbb{U}^{(x,y)}_{A\times A}]$ if and only if $(i_A \times i_A)^{\leftarrow}(e) \in \mathbb{U}^{(x,y)}_{A\times A}$ and as this is a ⊤-filter, we get $\top = \bigvee_{(a_1,a_2)\in A\times A} e(a_1,a_2)$. We conclude from this

$$\begin{aligned}
\bigvee_{e\in[\mathbb{U}^{(x,y)}_{A\times A}]} [e,d] &\le \bigvee_{e\in[\mathbb{U}^{(x,y)}_{A\times A}]} [e \wedge \top_{A\times A}, d] \\
&\le \bigvee_{e\in[\mathbb{U}^{(x,y)}_{A\times A}]} \bigwedge_{(a_1,a_2)\in A\times A} (e(a_1,a_2) \to d(a_1,a_2)) \\
&\le \bigvee_{e\in[\mathbb{U}^{(x,y)}_{A\times A}]} \left(\left(\bigvee_{(a_1,a_2)\in A\times A} e(a_1,a_2) \right) \to \left(\bigvee_{(a_1,a_2)\in A\times A} d(a_1,a_2) \right) \right) \\
&= \bigvee_{e\in[\mathbb{U}^{(x,y)}_{A\times A}]} \left(\top \to \left(\bigvee_{(a_1,a_2)\in A\times A} d(a_1,a_2) \right) \right) = \bigvee_{(a_1,a_2)\in A\times A} d(a_1,a_2).
\end{aligned}$$

Hence $\mathbb{U}(\Phi)_{A\times A} \in \mathsf{F}^{\top}_L(A\times A)$ and we have $[\mathbb{U}(\Phi)_{A\times A}] \ge \mathbb{U}(\Phi) \in \Lambda_X$, which implies $\mathbb{U}(\Phi)_{A\times A} \in \Lambda_X|_A$. The uniform continuity of φ then yields $(\varphi\times\varphi)(\mathbb{U}(\Phi)_{A\times A}) \in \Lambda_Y$ and uniform ⊤-regularity implies $\overline{(\varphi\times\varphi)(\mathbb{U}(\Phi)_{A\times A})} \in \Lambda_Y$. We therefore show in a second step that $\overline{(\varphi\times\varphi)(\mathbb{U}(\Phi)_{A\times A})} \le (\psi\times\psi)(\Phi)$.

To this end, we introduce the following notation. For $e \in L^{A\times A}$ we define $e_*, e^* \in L^{X\times X}$ by $e_*(x_1,x_2) = e(x_1,x_2)$ if $(x_1,x_2) \in A\times A$ and $e_*(x_1,x_2) = \bot$ otherwise. Similarly, $e^*(x_1,x_2) = e(x_1,x_2)$ if $(x_1,x_2)\in A\times A$ and $e^*(x_1,x_2) = \top$ otherwise.

Before we proceed, we need three technical lemmas.

Lemma 7.3. *We have $e \in \mathbb{U}(\Phi)_{A\times A}$ if and only if $e^* \in \mathbb{U}(\Phi)$.*

Proof. It is not diffucult to show that for $f \in L^{X\times X}$ and $e \in L^{A\times A}$ we have $[f, e^*]_{X\times X} = [(i_A\times i_A)^{\leftarrow}(f), e]_{A\times A}$. Hence we conclude $e \in \mathbb{U}(\Phi)_{A\times A}$ if and only if $\top = \bigvee_{f\in\mathbb{U}(\Phi)} [(i_A\times i_A)^{\leftarrow}(f), e] = \bigvee_{f\in\mathbb{U}(\Phi)} [f, e^*]$ if and only if $e^* \in \mathbb{U}(\Phi)$. □

Lemma 7.4. *For $e \in L^{A\times A}$ we have $\widehat{\sigma_{\mathbb{U}}}(e^*) \le \overline{e_*}$.*

Proof. We note that if $d \in \mathbb{U}^{(x,y)}$, then $d \in \Psi$ for all Ψ with $(x,y) \in q^{\Lambda_X} \times q^{\Lambda_X}(\Psi)$ and in particular for such Ψ with $\top_{A\times A} \in \Psi$. Note that the denseness of A implies that such Ψ exist. Hence

$$\begin{aligned}
\widehat{\sigma_{\mathbb{U}}}(e^*)(x,y) &\le \bigvee_{(x,y)\in q^{\Lambda_X}\times q^{\Lambda_X}(\Psi), \top_{A\times A}\in\Psi} \bigvee_{d\in\Psi} [d, e^*] \\
&= \bigvee_{(x,y)\in q^{\Lambda_X}\times q^{\Lambda_X}(\Psi), \top_{A\times A}\in\Psi} \bigvee_{d\in\Psi} [d\wedge\top_{A\times A}, e_*] \\
&\le \bigvee_{(x,y)\in q^{\Lambda_X}\times q^{\Lambda_X}(\Psi)} \bigvee_{d\in\Psi} [d, e^*] \\
&= \overline{e_*}(x,y).
\end{aligned}$$

□

Lemma 7.5. *For $e \in L^{A\times A}$ we have $(\varphi\times\varphi)(e) = (\psi\times\psi)(e_*)$.*

Proof. We have by definition of e_*

$$\begin{aligned}
(\psi\times\psi)(e_*)(y_1,y_2) &= \bigvee_{(x_1,x_2)\in A\times A, \varphi(x_1)=y_1, \varphi(x_2)=y_2} e(x_1,x_2) \vee \bigvee_{(x_1,x_2)\notin A\times A, \psi(x_1)=y_1, \psi(x_2)=y_2} \bot \\
&= (\varphi\times\varphi)(e).
\end{aligned}$$

□

We return to the proof of Theorem 7.2. Let $d \in \overline{(\varphi\times\varphi)(\mathbb{U}(\Phi)_{A\times A})}$. Then

$$\top = \bigvee_{e\in\mathbb{U}(\Phi)_{A\times A}} [\overline{(\varphi\times\varphi)(e)}, d] = \bigvee_{e^*\in\mathbb{U}(\Phi)} [\overline{(\psi\times\psi)(e_*)}, d].$$

As the continuity of $\psi : (X, q^{\Lambda_X}) \longrightarrow (Y, q^{\Lambda_Y})$ implies that $\psi\times\psi : (X\times X, q^{\Lambda_X}\times q^{\Lambda_X}) \longrightarrow (Y\times Y, q^{\Lambda_Y}\times q^{\Lambda_Y})$ is continuous, the last expression is

$$\leq \bigvee_{\widehat{\sigma_{\mathbb{U}}}(e^*)\in\Phi} [(\psi\times\psi)(\overline{e_*}), d] \leq \bigvee_{\widehat{\sigma_{\mathbb{U}}}(e^*)\in\Phi} [(\psi\times\psi)(\widehat{\sigma_{\mathbb{U}}}(e^*)), d] \leq \bigvee_{f\in\Phi} [(\psi\times\psi)(f), d],$$

which implies $d \in (\psi\times\psi)(\Phi)$ and the proof is complete. □

8 Conclusions

We introduced uniform diagonal axioms and uniform regularity for ⊤-uniform limit spaces. With the help of these notions, an extension theorem for a uniformly continuous mapping from a dense subspace could be established. This shows that the uniform diagonal conditions are reasonable and useful. We also showed that probabilistic uniform spaces satisfy the uniform ⊤-neighborhood condition and are uniformly ⊤-regular, provided we restrict the lattice background to frames. It would be interesting to know if probabilistic uniform spaces can be characterized using such diagonal conditions. This question seems to be still open also for the classical case, except the negative result that requiring the uniform neighbourhood condition and uniform regularity is not sufficient.

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