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Original paper

On some categories of triangular norms on the real unit interval

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Abstract

In this work, we introduce some categories of triangular norms in which truth values belong to the real unit interval, where arrows are a generalization of automorphisms. We investigate the existence of products, coproducts, equalizers and coequalizers in these categories. Moreover, we show that Theorems 2.29, 2.30 in [22] are false by providing counterexamples.

Keywords: Categories of t-norms, T-norm, fuzzy implication, R-implications.

1 Introduction

Fuzzy logic connectives play an important role in the theory of fuzzy sets and fuzzy logic. In fuzzy logic, many operators are defined. In particular, triangular norms (in short, t-norms) and fuzzy implications have an essential role both in theory and applications. They generalize the classical logical connectives, which take values in $\{0, 1\}$, to fuzzy logic, where the truth values belong to the real unit interval $[0, 1]$. In particular, t-norms are a generalization of the classical binary conjunction and fuzzy implications are a generalization of the classical implication. These operators were widely examined in [3, 9, 15]. In this article we concentrate mainly on triangular norms. Triangular norms were introduced by Karl Menger in [18] with the goal of constructing metric spaces using probabilistic distributions, i.e., values in the real unit interval, instead of the real number set, to describe the distance between two elements. The t-norm axioms as used today are given in [20]. In this work, we introduce some categories of triangular norms in which truth values belong to the real unit interval $[0, 1]$, where arrows are a generalization of automorphisms.

A categorical setting in fuzzy logic has been used in many articles. Mainly for fuzzy sets and algebras (see [4, 7, 10, 11, 19]). In article [5], the category of bounded lattice t-norms has been considered. Bounded lattice t-norms are a generalization of the t-norms. In this category, bounded lattice t-norms are the objects and generalizations of automorphisms are the arrows of the category.

The paper is organised as follows. Section 2 is divided into two parts. The first part concerns the basic concepts of the category theory used in the article. The second part involves some crucial facts and definitions of fuzzy logic used in the sequel. Section 3 is devoted to the study of categories of triangular norms. We define some categories of t-norms on $[0, 1]$. Next, we investigate the existence of products, coproducts, equalizers and coequalizers in these categories. In Section 4 we introduce some concepts of the category of fuzzy implications on a bounded lattice and we present counter-examples for Theorems 2.29, 2.30 from article [22]. Finally, in the last section, we summarize the whole work.

2 Preliminaries

2.1 The basics of category theory

We shall need some basic concepts from category theory. For more notions concerning category theory we refer the reader to [1],[2].

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A **category** $\mathfrak{C}$ is identified with its class of **arrows** (**morphisms**) and with its class of **objects**. Obj($\mathfrak{C}$) will denote its class of objects. Given A, $B \in$ Obj($\mathfrak{C}$) the collection of $\mathfrak{C}$-arrows from A to B will be denoted by $\mathrm{Hom}_{\mathfrak{C}}(A, B)$. If f is an arrow in $\mathfrak{C}$ then we write $f\colon A \to B$ instead of $f \in \mathrm{Hom}_{\mathfrak{C}}(A, B)$, that is, A is the domain of f and B is the codomain of f. Morphisms are equipped with a partial binary operation $\circ$ called composition. The composition of $f\colon A \to B$ and $g\colon B \to C$ is written as $g \circ f$, governed by two axioms:

- Associativity: If $f\colon A \to B$, $g\colon B \to C$ and $h\colon C \to D$ then $h \circ (g \circ f) = (h \circ g) \circ f$.
- Identity: For every object X, there exists an arrow $1_X\colon X \to X$ called the identity arrow for X, such that for every arrow $f\colon A \to B$, we have $1_B \circ f = f = f \circ 1_A$.

In any category $\mathfrak{C}$, an object 0 is **initial** if for any object C there is a unique morphism $0 \longrightarrow C$, analogously 1 is **terminal** if for any object C there is a unique morphism $C \longrightarrow 1$.

Definition 2.1. [1, 2] *In any category $\mathfrak{C}$, an arrow $f\colon A \to B$ is called :*

- *a **monomorphism** (or **monic**) if given any $g, h\colon C \to A$, $f \circ g = f \circ h$ implies $g = h$,*
- *an **epimorphism** (or **epic**) if given any $i, j\colon B \to D$, $i \circ f = j \circ f$ implies $i = j$.*
- *an **isomorphism** if there is an arrow $g\colon B \to A$ in $\mathfrak{C}$ such that*

$$g \circ f = 1_A \qquad \text{and} \qquad f \circ g = 1_B.$$

  We say that A is isomorphic to B and write $A \cong B$, if there exists an isomorphism between them.

Example 2.2. *Examples of categories:*

1. *The category **Set**, where objects are sets and arrows are functions;*
2. *The category **Top**, where objects are topological spaces and arrows are continuous functions;*
3. *The category **Grp**, where objects are groups and arrows are homomorphisms of groups;*
4. *The category **Rng**, where objects are rings and arrows are homomorphisms of rings;*
5. *The category **Poset**, where objects are partial order sets and arrows are homomorphisms of partial order sets;*
6. *Let $\mathfrak{C}$ be a category, the dual (or opposite) category of $\mathfrak{C}$ is the category $\mathfrak{C}^{op}$ where Obj($\mathfrak{C}$) = Obj($\mathfrak{C}^{op}$) and $f\colon A \to B$ is an arrow in Obj($\mathfrak{C}^{op}$) iff $f\colon B \to A$ is an arrow in $\mathfrak{C}$.*

Definition 2.3. [1, Definition 4.1] *A category $\mathfrak{A}$ is said to be a **subcategory** of a category $\mathfrak{B}$ provided that the following conditions are satisfied:*

(i) Obj($\mathfrak{A}$) $\subseteq$ Obj($\mathfrak{B}$).

(ii) *for each $A, B \in$ Obj($\mathfrak{A}$), $Hom_{\mathfrak{A}}(A, B) \subseteq Hom_{\mathfrak{B}}(A, B)$.*

(iii) *for each $f \in Hom_{\mathfrak{A}}(A, B)$ and $g \in Hom_{\mathfrak{A}}(B, C)$, the composition $g \circ f \in Hom_{\mathfrak{A}}(A, C)$.*

(iv) *every identity arrow in $\mathfrak{A}$ is identity arrow in $\mathfrak{B}$.*

*Moreover, $\mathfrak{A}$ is called a **full subcategory** of $\mathfrak{B}$, if for each $A, B \in$ Obj($\mathfrak{A}$) we have*

(v) $Hom_{\mathfrak{A}}(A, B) = Hom_{\mathfrak{B}}(A, B)$.

Definition 2.4. [2, Definition 1.2, Definition 7.1] *If $\mathfrak{C}$ and $\mathfrak{D}$ are categories, then a **functor** F from $\mathfrak{C}$ to $\mathfrak{D}$ ($F\colon \mathfrak{C} \to \mathfrak{D}$) is a map that for each $A \in$ Obj($\mathfrak{C}$), $F(A) \in$ Obj($\mathfrak{C}$) and for each arrow $f \in Hom_{\mathfrak{C}}(A, B)$, $F(f) \in Hom_{\mathfrak{D}}(F(A), F(B))$ in such a way that*

1. *F preserves composition; i.e., $F(f \circ g) = F(f) \circ F(g)$ whenever $f \circ g$ is defined.*
2. *F preserves identity morphisms; i.e., $F(1_A) = 1_{F(A)}$ for every object $A \in$ Obj($\mathfrak{C}$).*

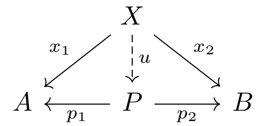
Definition 2.5. [2, Definition 2.15] *In any category $\mathfrak{C}$, a **product** for the objects A and B is an object P equipped with a pair of arrows*

$$A \xleftarrow{p_1} P \xrightarrow{p_2} B$$

satisfying the following UMP (Universal morphism property): Given any diagram of the form:

$$A \xleftarrow{x_1} X \xrightarrow{x_2} B$$

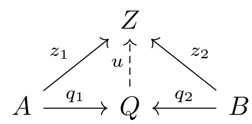
there exists a unique mapping $u\colon X \to P$*, making the diagram:*



commutes, i.e. such that $x_1 = p_1 \circ u$ *and* $x_2 = p_2 \circ u$.

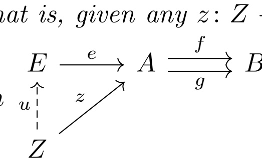
Definition 2.6. [2, Definition 3.3] *In any category* $\mathfrak{C}$*, a* ***coproduct*** *for the objects* A *and* B *is an object* Q *equipped with a pair of arrows* $A \xrightarrow{q_1} Q \xleftarrow{q_2} B$ *satisfying the following UMP: Given any diagram of the form:*

$A \xrightarrow{z_1} Z \xleftarrow{z_2} B$ *there exists a unique mapping* $u\colon X \to P$*, making the diagram:*



commutes, i.e. such that $z_1 = u \circ q_1$ *and* $z_2 = u \circ q_2$.

Definition 2.7. [2, Definition 3.13] *In any category* $\mathfrak{C}$*, given parallel arrows* $A \overset{f}{\underset{g}{\rightrightarrows}} B$ *an* ***equalizer*** *of* f *and* g *consists of an object* E *and an arrow* $e\colon E \to A$*, universal such that* $f \circ e = g \circ e$*. That is, given any* $z\colon Z \to A$ *with* $f \circ z = g \circ z$ *there is a unique mapping* $u\colon Z \to E$ *with* $e \circ u = z$*, all as in the diagram*



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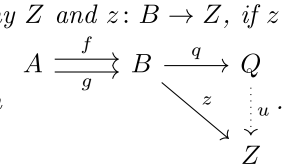
Definition 2.8. [2, Definition 3.18] *In any category* $\mathfrak{C}$*, given parallel arrows* $A \overset{f}{\underset{g}{\rightrightarrows}} B$ *, a* ***coequalizer*** *consists of* Q *and* $q\colon B \to Q$*, universal with the property* $q \circ f = q \circ g$*. That is, given any* Z *and* $z\colon B \to Z$*, if* $z \circ f = z \circ g$*, then there exists a unique mapping* $u\colon Q \to Z$ *with* $u \circ q = z$*, all as in the diagram*



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2.2 Fuzzy connectives

To make this work more self-contained, we place some basic definitions concerning fuzzy connectives here. First of all, we introduce the notions of automorphism and conjugate (see [17]).

Definition 2.9. *A function* $\varphi\colon [0,1] \to [0,1]$ *is an* ***automorphism*** *if it is continuous and strictly increasing and satisfies the boundary conditions* $\varphi(0) = 0$ *and* $\varphi(1) = 1$*, i.e., if it is an increasing bijection in* $[0,1]$*. The family of all increasing bijections* $\varphi\colon [0,1] \to [0,1]$ *will be denoted by the symbol* Φ.

Definition 2.10. *Let* $\varphi\colon [0,1] \to [0,1]$ *be an automorphism. We say that functions* $f, g\colon [0,1]^n \to [0,1]$ *are* Φ***-conjugate*** *if* $g = f_\varphi$*, where*

$$f_\varphi(x_1, \ldots, x_n) := \varphi^{-1}(f(\varphi(x_1), \ldots, (\varphi(x_n))),$$

for every $x_1, \ldots, x_n \in [0,1]$.

Definition 2.11. [9, 15] *Let* L *be a bounded lattice. A function* $T\colon L^2 \to L$ *is called a* ***triangular norm (t-norm*** *in short), if it satisfies, for all* $x, y, z \in L$*, the following conditions:*

(T1) $T(x,y) = T(y,x)$,

(T2) $T(x, T(y,z)) = T(T(x,y), z)$,

(T3) $T(x,y) \le T(x,z)$ *for* $y \le z$*, i.e.,* $T(x,\cdot)$ *is non-decreasing,*

(T4) $T(x, 1_L) = x$.

We will mainly consider the case of $L = [0,1]$ (real unit interval). Then, from (T1), (T3) and (T4) every t-norm T satisfies the following

$$T(x,y) \leq \min(x,y), \qquad \text{for all } x,y \in [0,1]. \tag{T5}$$

A function $T\colon [0,1]^2 \to [0,1]$ which satisfies (T1), (T2), (T3) and (T5) is called a **t-subnorm** (see [13]).

Example 2.12. *We present some basic examples of t-subnorms in Table 1. Note that $T_{\mathbf{P}}, T_{\mathbf{M}}, T_{\mathbf{LK}}$ are continuous t-norms, $T_{\mathbf{nM}}$ is left-continuous t-norm, $T_{\mathbf{D}}$ is non-continuous t-norm and T_0 is not a t-norm.*

Table 1: Examples of t-subnorms

function	formula
the product t-norm	$T_{\mathbf{P}}(x,y) = xy$
the Łukasiewicz t-norm	$T_{\mathbf{LK}}(x,y) = \max(x+y-1,0)$
the minimum t-norm	$T_{\mathbf{M}}(x,y) = \min(x,y)$
the drastic t-norm	$T_{\mathbf{D}}(x,y) = \begin{cases} 0, & x,y \in [0,1) \\ \min(x,y), & \text{otherwise} \end{cases}$
the nilpotent minimum t-norm	$T_{\mathbf{nM}}(x,y) = \begin{cases} 0, & x+y \leq 1 \\ \min(x,y), & \text{otherwise} \end{cases}$
the t-subnorm T_0	$T_0 \equiv 0$

Note, if T is a t-norm, then T_φ is also a t-norm, for every $\varphi \in \Phi$ (see [15]) .

Proposition 2.13. [15, Proposition 2.31] *Let T be a t-norm. The following are equivalent:*

(i) For every $\varphi \in \Phi$, $T = T_\varphi$.

(ii) $T = T_{\mathbf{M}}$ or $T = T_{\mathbf{D}}$.

In the next definition, we need the concept of the power notation. The power notation $x_T^{[n]}$ (where T is a t-norm and $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of non-negative integers), is defined by

$$x_T^{[n]} = \begin{cases} 1, & n = 0 \\ x, & n = 1 \\ T(x, x_T^{[n-1]}), & n > 1 \end{cases}.$$

Definition 2.14. [15, Definition 2.13] *Let T be a t-norm.*

1. *T is called* ***Archimedean*** *if, for all $x,y \in (0,1)$, there is an $n \in \mathbb{N}$ such that $x_T^{[n]} \leq y$.*
2. *T is called* ***strict****, if it is continuous and strictly monotonic.*
3. *T is called* ***nilpotent*** *if, it is continuous and any $x \in (0,1)$ is its nilpotent element i.e., $x_T^{[n]} = 0$ for some $n \in \mathbb{N}$.*
4. *T has* ***zero divisors****, if there exist $x,y \in (0,1)$ such that $T(x,y) = 0$.*

Let us recall that a continuous t-norm T is **Archimedean** if and only if $T(x,x) < x$, for every $x \in (0,1)$ (see [15, Proposition 5.1.2]).

We need the concept of an ordinal sum to characterize t-norms. Ordinal sums of abstract semigroups were introduced by A. H. Clifford in [6]. Let us recall that a set X with a map $*\colon X \times X \to X$ (for $X \neq \emptyset$) is a **semigroup** if it is associative. We recall this fundamental result.

Theorem 2.15. [6] *Let $(A, \preceq)$ be a linearly ordered set with $A \neq \emptyset$ and $((X_\alpha, *_\alpha))_{\alpha \in A}$ a family of semigroups such that $X_\alpha \cap X_\beta = \emptyset$ whenever $\alpha \neq \beta$. Put $X = \bigcup_{\alpha \in A} X_\alpha$ and define the operation $*\colon X^2 \to X$ by*

$$x * y = \begin{cases} x *_\alpha y, & (x,y) \in X_\alpha^2, \\ x, & (x,y) \in X_\alpha \times X_\beta \text{ and } \alpha \prec \beta, \\ y, & (x,y) \in X_\alpha \times X_\beta \text{ and } \beta \prec \alpha. \end{cases}$$

Then $(X, *)$ is a semigroup, and it will be called the ***ordinal sum of the semigroups*** *$((X_\alpha, *_\alpha))_{\alpha \in A}$.*

Definition 2.16. *Let I be a non-empty subinterval of $[0,1]$ (real unit interval). A totally ordered abelian semigroup $(I,*)$ where the semigroup operation $*$ is bounded from above by the minimum, i.e., which satisfies $x*y \le \min(x,y)$, for all $x,y \in I$, will be called a* ***tosab****.*

The following theorem is the modification of Theorem 2.15, where the resulting t-norm T will be referred to as an ordinal sum of t-subnorms.

Theorem 2.17. [16, Theorem 2.4],[14, Corrolary 2] *Let $(A,\preceq)$ be a linearly ordered set, $(V_\alpha)_{\alpha\in A}$ be a family of t-subnorms and $((a_\alpha,b_\alpha))_{\alpha\in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. Further, if $b_{\alpha_0}=1$ for some $\alpha_0 \in A$, then assume that V_{α_0} is a t-norm, and if $b_{\alpha_0}=a_{\beta_0}$ for some $\alpha_0,\beta_0\in A$ then assume that V_{α_0} is a t-norm or that V_{β_0} has no zero divisors. Then a function $T\colon [0,1]^2 \to [0,1]$ defined by*

$$T(x,y)=\begin{cases} a_\alpha+(b_\alpha-a_\alpha)V_\alpha\left(\frac{x-a_\alpha}{b_\alpha-a_\alpha},\frac{y-a_\alpha}{b_\alpha-a_\alpha}\right), & \text{if } x,y\in(a_\alpha,b_\alpha],\\ \min(x,y), & \text{otherwise}\end{cases}$$

is a t-norm. T is called the ***ordinal sum*** *of the* ***summands*** *$((a_\alpha,b_\alpha),V_\alpha)$, $\alpha\in A$ and we shall write*

$$T=\{((a_\alpha,b_\alpha),V_\alpha)\mid \alpha\in A\}.$$

In the case of $A=\emptyset$, we obtain $T_{\mathbf{M}}$. From [16, Theorem 3.1] we know if T is a t-norm, then T is an ordinal sum of t-subnorms in the sense of Theorem 2.17 iff $([0,1],T)$ is an ordinal sum of semigroups (in the sense of Theorem 2.15). At the moment, when we will use the notion of ordinal sum for a t-norm we will understand that this t-norm is an ordinal sum in the sense of Theorem 2.17. Moreover, we assume that $(A,\preceq)$ is compatible with the usual order $\le$ on $[0,1]$, i.e., for $\alpha,\beta\in A$ we have $\alpha\prec\beta$ if and only if $x<y$ for all $x\in(a_\alpha,b_\alpha)$ and $y\in(a_\beta,b_\beta)$. The concept of an ordinal sum is useful to characterize all continuous t-norms.

Theorem 2.18. [15, Theorem 5.11] *For a function $T\colon[0,1]^2\to[0,1]$ the following statements are equivalent:*

1. *T is a continuous t-norm.*
2. *T is uniquely representable as an ordinal sum of continuous Archimedean t-norms, i.e., there exists a uniquely determined (finite or countably infinite) index set A, a family of uniquely determined pairwise disjoint open subintervals $((a_\alpha,b_\alpha))_{\alpha\in A}$ of $[0,1]$ and a family of uniquely determined continuous Archimedean t-norms $(T_\alpha)_{\alpha\in A}$ such that $T=\{((a_\alpha,b_\alpha),T_\alpha)\mid\alpha\in A\}$.*

Definition 2.19. [16, Definition 4.1] *A tosab is called* ***ordinally irreducible*** *if it cannot be expressed as an ordinal sum of two or more non-singleton tosabs.*

In this spirit, a t-norm T is considered to be ordinally irreducible if the tosab $([0,1],T)$ is ordinally irreducible. We have the following characterization for ordinally irreducible t-norms.

Proposition 2.20. [16, Proposition 4.2] *Let T be a t-norm. Then the following are equivalent:*

(i) *T is ordinally irreducible.*

(ii) *For each $x\in(0,1)$ there exist $y,z\in(0,1)$ with $y<x<z$ and $T(y,z)<y$.*

We will need the following lemma.

Lemma 2.21. *Let T_1,T_2 be t-norms such that $T_2=(T_1)_\varphi$, for some $\varphi\in\Phi$. Then T_1 is ordinally irreducible if and only if T_2 is ordinally irreducible.*

Proof. When $T_2=(T_1)_\varphi$, then $T_1=(T_2)_{\varphi^{-1}}$, so it is enough to prove the part $\Rightarrow$ of the lemma. Assume that T_1 is ordinally irreducible. Let $x\in(0,1)$ and let $x_0=\varphi(x)$. Of course, $x_0\in(0,1)$. Let $y_0,z_0\in(0,1)$ be such that $y_0<x_0<z_0$ and $T_1(y_0,z_0)<y_0$. Thus, there exist $y,z\in(0,1)$ such that $\varphi(y)=y_0$ and $\varphi(z)=z_0$. Hence, $y<x<z$, and

$$T_2(y,z)<y \Leftrightarrow \varphi^{-1}(T_1(\varphi(y),\varphi(z)))<y \Leftrightarrow T_1(\varphi(y),\varphi(z))<\varphi(y),$$

this proves this lemma. □

As a consequence (see [16]), for each t-norm T there is a family of pairwise disjoint open subintervals $((a_\alpha, b_\alpha))_{\alpha\in A}$ of $[0,1]$ such that $\bigcup_{\alpha\in A}(a_\alpha, b_\alpha)$ is a dense subset of $[0,1]$ (of course in inherited topology from the topology of a Euclidean space real line) and such that T can be expressed as the ordinal sum of t-subnorms ($T = \{((a_\alpha, b_\alpha), V_\alpha) \mid \alpha \in A\}$) which are ordinally irreducible or equal to minimum t-norm, whenever $b_\alpha = a_\beta$ for some $\alpha, \beta \in A$, then $V_\alpha \neq \min$ or $V_\beta \neq \min$. For t-norms of this form we obtain the following remark.

Remark 2.22. *Each t-norm can be uniquely represented as ordinal sum of t-subnorms, up to isomorphism of some linearly ordered set.*

Proof. Let T be a t-norm of the following form $\{(P_\alpha, V_\alpha) \mid \alpha \in A\}$, where $P_\alpha = (a_\alpha, b_\alpha) \subseteq [0,1]$, for all $\alpha \in A$. If $|A| = 1$, then $T = T_{\mathbf{M}}$ or T is ordinally irreducible t-norm. Of course, this representation is unique up to isomorphism. Assume that T is not ordinally irreducible and let $\{(P_\alpha, V_\alpha) \mid \alpha \in A\} = T = \{(Q_\beta, W_\beta) \mid \beta \in B\}$. Thus,

$$\forall_{\alpha\in A} \quad \exists_{\beta\in B} \quad P_\alpha = Q_\beta. \tag{1}$$

Indeed, let $\mathcal{P} = \{P_\alpha \cap Q_\beta \mid \alpha \in A, \beta \in B\} \setminus \{\emptyset\}$. If the family $\mathcal{P}$ is not equal to family $\{P_\alpha \mid \alpha \in A\}$, then V_α is neither ordinally irreducible nor equal min, for some $\alpha \in A$. Indeed, assume that for some $\alpha \in A$ we have $|\{P_\alpha \cap Q_\beta \neq \emptyset \mid \beta \in B\}| = 2$, when cardinality is higher than 2 the proof is analogues. By assumption about summands of T, we obtain that $V_\alpha \neq \min$. Thus, there is $c_\alpha \in (a_\alpha, b_\alpha) = P_\alpha$ such that $T|_{(\alpha, c_\alpha]\times(c_\alpha, b_\alpha]} = \min$ and $T|_{(c_\alpha, b_\alpha]\times(\alpha, c_\alpha]} = \min$. Hence, there are pairwise disjoint intervals $X, Y \subseteq [0,1]$ such that $X \cup Y = [0,1]$, $V_\alpha|_{X\times Y} = \min$ and $V_\alpha|_{Y\times X} = \min$. This proves that V_α is not ordinally irreducible. We obtain a contradiction, what proves (1). From (1) we obtain that $A \cong B$ in **Poset**. □

Taking into account the above remark, we can all t-norms unique write as ordinal sum of t-subnorms, up to isomorphism of some linearly ordered set. We will use two methods:

First T can be expressed as the ordinal sum of t-subnorms ($T = \{((a_\alpha, b_\beta), V_\alpha) \mid \alpha \in A\}$, where $A \neq \emptyset$) which are ordinally irreducible or equal to minimum t-norm, whenever $b_\alpha = a_\beta$ for some $\alpha, \beta \in A$, then $V_\alpha \neq \min$ or $V_\beta \neq \min$.

Second We allow the assumption that a set A may be empty ($T_{\mathbf{M}}$ is represented as $\emptyset$). Moreover, when in some summand $((a_\alpha, b_\alpha), V_\alpha)$ t-norm $V_\alpha = \min$, then we omit this summand in representation of t-norm T. In particular, the sum $\bigcup_{\alpha\in A}(a_\alpha, b_\alpha)$ does not have to be dense in $[0,1]$. Of course, $T(x,y) = \min(x,y)$, when $x, y \notin \bigcup_{\alpha\in A}(a_\alpha, b_\alpha)$.

Definition 2.23. [3, 9] *Let L be a bounded lattice. A function $I\colon L^2 \to L$ is called a* ***fuzzy implication****, if it satisfies the following conditions.*

(I1) *I is non-increasing with respect to the first variable,*

(I2) *I is non-decreasing with respect to the second variable,*

(I3) *$I(0_L, 0_L) = I(1_L, 1_L) = 1_L$ and $I(1_L, 0_L) = 0_L$.*

Definition 2.24. [23] *Let L be a complete lattice. A function $I\colon L^2 \to L$ is called a residual implication (R-implication for short) if there exists a t-norm T such that*

$$I(x,y) = \sup\{t \in L \mid T(x,t) \leq y\}, \quad \textit{for all } x, y \in L.$$

If I is generated from a t-norm T, then it will be denoted by I_T.

We will mainly consider the case of $L = [0,1]$ (real unit interval). Then, every R-implication is fuzzy implication (see [3, Theorem 2.5.4]).

Example 2.25. *We present some basic examples of fuzzy implications from $[0,1]^2$ to $[0,1]$ in Table 2.*

Only $I_{\mathbf{RC}}, I_{\mathbf{KD}}$ are not R-implications from Example 2.25.

Example 2.26. *We present some basic examples of R-implications in Table 3.*

The many principles of fuzzy implications are analysed (see [3]). One of them is the ordering property.

Table 2: Examples of fuzzy implications

fuzzy implication	formula
the Gödel implication	$I_{\mathbf{GD}}(x,y)=\begin{cases}1, & x\le y,\\ y, & x>y.\end{cases}$
the Goguen implication	$I_{\mathbf{GG}}(x,y)=\begin{cases}1, & x\le y,\\ \frac{y}{x}, & x>y.\end{cases}$
the Łukasiewicz implication	$I_{\mathbf{LK}}(x,y)=\min\{1,1-x+y\}$
the Reichenbach implication	$I_{\mathbf{RC}}(x,y)=1-x+xy$
the Fodor implication	$I_{\mathbf{FD}}(x,y)=\begin{cases}1, & x\le y,\\ \max\{1-x,y\}, & x>y.\end{cases}$
the Weber implication	$I_{\mathbf{WB}}(x,y)=\begin{cases}1, & x<1,\\ y, & x=1.\end{cases}$
the Kleene-Dienes implication	$I_{\mathbf{KD}}(x,y)=\max\{1-x,y\}$

Table 3: Examples of R-implications

t-norm T	R-implication I_T
$T_{\mathbf{M}}$	$I_{\mathbf{GD}}$
$T_{\mathbf{P}}$	$I_{\mathbf{GG}}$
$T_{\mathbf{LK}}$	$I_{\mathbf{LK}}$
$T_{\mathbf{nM}}$	$I_{\mathbf{FD}}$
$T_{\mathbf{D}}$	$I_{\mathbf{WB}}$

Definition 2.27. [3] *We say that a fuzzy implication* $I\colon [0,1]^2\to[0,1]$ *satisfies the ordering property, if*

$$I(x,y)=1\Leftrightarrow x\le y,\qquad \text{for all } x,y\in[0,1]. \tag{OP}$$

Note that if an R-implication I is generated from a left-continuous t-norm, then I satisfies (OP) (see [3, Proposition 2.5.9]).

Proposition 2.28. [12, Proposition 4.4] *Let* $(T_k)_{k\in\mathcal{K}}$ *be a family of t-norms and let* $T=\{((a_k,b_k),T_k)\mid k\in\mathcal{K}\}$ *be an ordinal sum of t-norms. Then*

$$I_T(x,y)=\begin{cases}\alpha_k+(\beta_k-\alpha_k)I_{T_k}\left(\frac{x-\alpha_k}{\beta_k-\alpha_k},\frac{y-\alpha_k}{\beta_k-\alpha_k}\right), & \alpha_k\le y<x\le\beta_k,\\ I_{\mathbf{GD}}(x,y), & \text{otherwise},\end{cases}$$

is the R-implication generated from the t-norm T.

3 Categories of triangular norms

3.1 The subcategory of t-norms on a bounded lattice

First we start with definition of lattice homomorphism (see [8]).

Definition 3.1. [8] *Let* L *and* K *be bounded lattices. A map* $\phi\colon L\to K$ *is said to be a lattice homomorphism if*

1. $\phi(0_L)=0_K$,
2. $\phi(1_L)=1_K$,
3. *for each* $x,y\in L$

$$\phi(x\vee y)=\phi(x)\vee\phi(y)\quad\text{and}\quad \phi(x\wedge y)=\phi(x)\wedge\phi(y).$$

Note that in 3.1, if $L=K=[0,1]$ the condition (3) means that ϕ is non-decreasing function.

Definition 3.2. [5] *Let L and M be bounded lattices.* ***T-NORM*** *is the category, whose objects are all t-norms on a bounded lattice and a lattice homomorphism $\rho\colon L \to M$ is a* ***t-norm morphism*** *from T_L into T_M if for each $x, y \in L$*

$$\rho(T_L(x,y)) = T_M(\rho(x), \rho(y)), \tag{2}$$

where T_L, T_M are t-norms such that T_L acts on L and T_M on M. Clearly the composition of two t-norm morphisms is also a t-norm morphism. Moreover, for any bounded lattice L, the identity id_L is an identity of every t-norm on L.

Consider the subcategory of the category **T-NORM**, where objects are t-norms defined on the real unit interval $[0,1]$ and arrows are lattice homomorphisms from $[0,1]$ to $[0,1]$ satisfying equation (2) above. We will denote this category by $\mathbf{T}_{[0,1]}$. For every t-norms $T_1, T_2\colon [0,1] \to [0,1]$ if $\rho \in \mathrm{Hom}_{\mathbf{T\text{-}NORM}}(T_1, T_2)$, then from Definition 2.3 $\rho \in \mathrm{Hom}_{\mathbf{T}_{[0.1]}}(T_1, T_2)$. Thus, $\mathbf{T}_{[0,1]}$ is full subcategory of **T-NORM**. From [5] we know that a morphism φ in $\mathbf{T}_{[0,1]}$ is an isomorphism iff φ is a bijection i.e., $\varphi \in \Phi$. Thus, if $T_2 = (T_1)_\varphi$ for some $\varphi \in \Phi$, then φ is an isomorphism from T_2 to T_1. Let T_1, T_2 be any t-norms and let $\alpha \in [0,1)$. Then the function

$$\varphi_\alpha(x) = \begin{cases} 0, & x \leq \alpha \\ 1, & \text{otherwise} \end{cases}$$

is always an arrow from T_1 to T_2. Thus, the category $\mathbf{T}_{[0,1]}$ has neither terminal objects nor initial objects.

The concept of t-norm homomorphism is helpful to characterize continuous t-norms. For instance (see [15, Proposition 5.9]), T is a strict t-norm iff T is isomorphic to $T_{\mathbf{P}}$ in $\mathbf{T}_{[0,1]}$, i.e., there exists a bijection $\varphi \in \Phi$, such that

$$T(x,y) = (T_{\mathbf{P}})_\varphi(x,y) = \varphi^{-1}(\varphi(x) \cdot \varphi(y)), \quad x, y \in [0,1].$$

Moreover (see [15, Proposition 5.10]), T is a nilpotent t-norm iff T is isomorphic to $T_{\mathbf{LK}}$ in $\mathbf{T}_{[0,1]}$, i.e., there exists a bijection $\varphi \in \Phi$, such that for every $x, y \in [0,1]$

$$T(x,y) = (T_{\mathbf{LK}})_\varphi(x,y) = \varphi^{-1}(\max\{\varphi(x) + \varphi(y) - 1, 0\}).$$

In the next example we present non-trivial example of an arrow in $\mathbf{T}_{[0,1]}$.

Example 3.3. *Let $T = \{((a_\alpha, b_\alpha), T_\alpha) \mid \alpha \in A\}$, where every T_α is an ordinally irreducible t-norm and*

$$\varphi_\alpha(x) = \begin{cases} 0, & x < a_\alpha \\ \frac{x - a_\alpha}{b_\alpha - a_\alpha}, & x \in [a_\alpha, b_\alpha] \\ 1, & x > b_\alpha \end{cases},$$

then $\varphi_\alpha\colon T \to T_\alpha$ is a morphism. Indeed, when $x < a_\alpha$ or $y < a_\alpha$, then $T(x,y) \leq \min(x,y) < a_\alpha$, so

$$\varphi_\alpha(T(x,y)) = 0 = T_\alpha(\varphi_\alpha(x), \varphi_\alpha(y)).$$

When $x > b_\alpha$, then from the form of T we have $T(x,y) = y$ for $y < b_\alpha$ and $T(x,y) \geq b_\alpha$ for $y \geq b_\alpha$, so

$$\varphi_\alpha(T(x,y)) = \varphi_\alpha(y) = T_\alpha(1, \varphi_\alpha(y)) = T_\alpha(\varphi_\alpha(x), \varphi_\alpha(y))$$

and when $y > b_\alpha$, then $T(x,y) = x$ for $x < b_\alpha$ and $T(x,y) \geq b_\alpha$ for $x \geq b_\alpha$, so

$$\varphi_\alpha(T(x,y)) = \varphi_\alpha(x) = T_\alpha(\varphi_\alpha(x), 1) = T_\alpha(\varphi_\alpha(x), \varphi_\alpha(y)).$$

For $x, y \in [a_\alpha, b_\alpha]$ again from the form of T we have

$$\varphi_\alpha(T(x,y)) = \frac{T - a_\alpha}{b_\alpha - a_\alpha} = T_\alpha\left(\frac{x - a_\alpha}{b_\alpha - a_\alpha}, \frac{y - a_\alpha}{b_\alpha - a_\alpha}\right) = T_\alpha(\varphi_\alpha(x), \varphi_\alpha(y)).$$

As shown in [5, Theorem 1] the category **T-NORM** has products for every pair of morphisms. In particular, the product of t-norms $T_1\colon L \times L \to L$, $T_2\colon M \times M \to M$ is the t-norm $T_1 \times T_2\colon (L \times M)^2 \to L \times M$ of the following form

$$T_1 \times T_2((x_1, x_2), (y_1, y_2)) := (T_1(x_1, y_1), T_2(x_2, y_2)), \qquad \text{for all} \quad x_1, x_2 \in L, y_1, y_2 \in M.$$

One can verify that if $L = M = [0,1]$, then $T_1 \times T_2$ is not a product in $\mathbf{T}_{[0,1]}$. In fact the category $\mathbf{T}_{[0,1]}$ does not have products as shown below.

Proposition 3.4. *In fact the category* $\boldsymbol{T}_{[0,1]}$ *has neither products nor coproducts for every pair of t-norms.*

Proof. Let $\varphi\colon [0,1] \to [0,1]$ be of the form $\varphi(x) = x^2$, for all $x \in [0,1]$. Let T be a t-norm such that $T = T_\varphi$ (for instance $T_{\mathbf{M}}$, $T_{\mathbf{D}}$, $T_{\mathbf{P}}$). Suppose P is a t-norm such that the following diagram $T \xleftarrow{p_1} P \xrightarrow{p_2} T$ satisfies UMP. Consider the following diagrams $T \xleftarrow{\mathrm{id}_{[0,1]}} T \xrightarrow{\mathrm{id}_{[0,1]}} T$ and $T \xleftarrow{\mathrm{id}_{[0,1]}} T \xrightarrow{\varphi} T$ Thus, there are u, v such that $p_1 \circ u = p_2 \circ u = \mathrm{id}_{[0,1]}$, $p_1 \circ v = \mathrm{id}_{[0,1]}$ and $p_2 \circ v = \varphi$. Hence, functions u, v are injective. Suppose now that $u(x) \neq v(x)$, for all $x \in (0,1)$. Hence, the set

$$J := \{I_x \mid x \in (0,1)\}, \qquad \text{where} \qquad I_x = (\min(u(x), v(x)), \max(u(x), v(x))),$$

is uncountable family of pairwise disjoint open subsets of the interval $[0,1]$. Indeed, it is enough to note that $p_1|_{I_x} \equiv x$, for all $x \in (0,1)$. Hence, if $x, y \in (0,1)$ are different, then $p_1|_{I_x} \equiv x \neq y \equiv p_1|_{I_y}$, so $I_x \cap I_y = \emptyset$. This is a contradiction, because every family of pairwise disjoint open subsets of [0,1] is countable. Thus, there exists $x_0 \in (0,1)$ such that $u(x_0) = v(x_0)$. Thus, $x_0 = p_1(v(x_0)) = p_1(u(x_0)) = p_2(u(x_0)) = p_2(v(x_0)) = x_0^2$. This is a contradiction, so the pair (T, T) has no product in $\mathbf{T}_{[0,1]}$.

Suppose now that Q is a t-norm such that the following diagram $T \xrightarrow{q_1} Q \xleftarrow{q_2} T$ satisfies UMP. Consider the following diagrams $T \xrightarrow{\mathrm{id}_{[0,1]}} T \xleftarrow{\mathrm{id}_{[0,1]}} T$ and $T \xrightarrow{\varphi} T \xleftarrow{\mathrm{id}_{[0,1]}} T$ Thus, there are u, v such that $u \circ q_1 = u \circ q_2 = \mathrm{id}_{[0,1]}$, $v \circ q_1 = \mathrm{id}_{[0,1]}$ and $v \circ q_2 = \varphi$. Analogues to the first part of the proof, we obtain that $q_1(x_0) = q_2(x_0)$, for some $x_0 \in (0,1)$. This leads to a contradiction, so the pair (T, T) also has no coproduct in $\mathbf{T}_{[0,1]}$. □

Proposition 3.5. *The category* $\boldsymbol{T}_{[0,1]}$ *has neither equalizers nor coequalizers for every pair of t-norms.*

Proof. Let $\varphi_1 = \mathrm{id}_{[0,1]}$ and let $\varphi_2\colon [0,1] \to [0,1]$ be of the form $\varphi_2(x) = x^2$, for all $x \in [0,1]$. Let T be a t-norm such that $T = T_{\varphi_2}$ (for instance $T_{\mathbf{M}}$, $T_{\mathbf{D}}$, $T_{\mathbf{P}}$). Suppose that $e\colon E \to T$ is an equalizer for parallel arrows $\varphi_1, \varphi_2\colon T \to T$, so $e(x) = e(x)^2$, for all $x \in [0,1]$. Hence, $e(x) = 0$ or $e(x) = 1$, for all $x \in [0,1]$. Since, $e \circ e = e \circ \mathrm{id}_{[0,1]}$ and every equalizer must be monic, from Definition 2.1(ii) we obtain $e = \mathrm{id}_{[0,1]}$. This is contradiction with fact that $e(x) = 0$ or $e(x) = 1$, for all $x \in [0,1]$.

Assume now that $\varphi_{up}, \varphi_{down}\colon T_1 \to T_2$ are t-norm homomorphisms of the following forms

$$\varphi_{down}(x) = \begin{cases} 0, & x < 1 \\ 1, & x = 1 \end{cases} \qquad \text{and} \qquad \varphi_{up}(x) = \begin{cases} 0, & x = 0 \\ 1, & x > 0 \end{cases},$$

where T_1, T_2 are any t-norms. Suppose now that $q\colon T_2 \to Q$ is a coequalizer. Hence, $q \circ \varphi_{up} = q \circ \varphi_{down}$, so $1 = q(1) = q \circ \varphi_{up}(x) = q \circ \varphi_{down}(x) = q(0) = 0$, for all $x \in (0,1)$. This is a contradiction. □

3.2 Categories of t-norms on the real unit interval

In subsection 2.2 we presented two uniquely representation methods of a t-norm as an ordinal sum of t-subnorms. In the first method each linearly ordered set is not empty and in the second method linearly ordered sets may be empty. Based on these two methods, we will introduce two categories of t-norms on $[0,1]$ in the following definition.

Definition 3.6. *By the symbol of* $\mathfrak{T}_1$ *(*$\mathfrak{T}_0$*) we will denote the category of t-norms, whose objects are t-norms on* $[0,1]$*, which are represented by the first method from subsection 2.2 (represented by the second method from subsection 2.2). We define morphisms of categories* $\mathfrak{T}_1$*,* $\mathfrak{T}_0$ *and composition of morphisms in the following way.*

- *Let* T_1, T_2 *be t-norms and have the following forms* $T_1 = \{(A_\alpha, V_\alpha) \mid \alpha \in A\}$*,* $T_2 = \{(B_\beta, W_\beta) \mid \beta \in B\}$*. We write that* $f = (f^*, \pi_f)$*, where* $\pi_f\colon B \to A$ *and* $f^*\colon B \to \Phi$ *belongs to* $Hom^*_{\mathfrak{C}}(T_1, T_2)$ *(*$\mathfrak{C} = \mathfrak{T}_1$ *or* $\mathfrak{C} = \mathfrak{T}_0$*) if and only if*

$$\forall_{\beta \in B} \quad \forall_{x,y \in [0,1]} \qquad W_\beta(f^*(\beta)(x), f^*(\beta)(y)) = f^*(\beta)(V_{\pi_f(\beta)}(x, y)), \ \textit{i.e.,} \tag{3}$$

Hom$^*_{\mathfrak{C}}(T_1, T_2) = \{(f^*, \pi_f) \mid$ *where* $f^*\colon B \to \Phi, \pi_f\colon B \to A$ *satisfy* (3)$\}$*, for* $\mathfrak{C} = \mathfrak{T}_1$ *or* $\mathfrak{C} = \mathfrak{T}_0$*. Assume now that* $A \cong A_1$ *and* $B \cong B_1$ *in* ***Poset*** *and let* $\psi_A\colon A \to A_1$*,* $\psi_B\colon B \to B_1$ *be isomorphisms in* ***Poset****. Let* $g = (g^*, \pi_g)$*, where* $\pi_g\colon B_1 \to A_1$ *and* $g^*\colon B_1 \to \Phi$ *satisfies*

$$g^* = f^* \circ \psi_B^{-1} \qquad \textit{and} \qquad \pi_g = \psi_A \circ \pi_f \circ \psi_B^{-1}.$$

Taking into account Remark 2.22 g belongs to $Hom^*_{\mathfrak{C}}(T_1, T_2)$*. In this case we will write* $f \approx g$*. It is easy to show that the relation* $\approx$ *is an equivalence relation in* $Hom^*_{\mathfrak{C}}(T_1, T_2)$*. We define* $Hom_{\mathfrak{C}}(T_1, T_2) = Hom^*_{\mathfrak{C}}(T_1, T_2) / \approx$ *(the quotient set), where* $\mathfrak{C} = \mathfrak{T}_1$ *or* $\mathfrak{C} = \mathfrak{T}_0$*.*

- *Let T_1, T_2, T_3 be t-norms and have the following forms $T_1 = \{(A_\alpha, V_\alpha) \mid \alpha \in A\}$, $T_2 = \{(B_\beta, W_\beta) \mid \beta \in B\}$, $T_3 = \{(C_\gamma, X_\gamma) \mid \gamma \in C\}$. Moreover, let $f = (f^*, \pi_f)$ and $g = (g^*, \pi_g)$ be morphisms such that $f\colon T_1 \to T_2$ and $g\colon T_2 \to T_3$. Then the composition $h = g \circ f\colon T_1 \to T_3$ is defined such as the pair (h^*, π_h), where $\pi_h = \pi_f \circ \pi_g$ and $h^*(\gamma) = g^*(\gamma) \circ f^*(\pi_g(\gamma))$, for all $\gamma \in C$. Again it is easy to show that $\approx$ is a congruence relation on the composition i.e., if $f' \approx f$ and $g' \approx g$, then $g \circ f \approx g' \circ f'$. Therefore, we will write $f = g$ instead of $f \approx g$ without loss of generality. A morphism $1_{T_1}\colon T_1 \to T_1$ is of the form $1_{T_1} = (1^*_{T_1}, \pi_{1_{T_1}})$, where $1^*_{T_1} \equiv id_{[0,1]}$ and $\pi = id_A$.*

Remark 3.7. *The composition in categories $\mathfrak{T}_1$, $\mathfrak{T}_0$ is well-defined.*

Proof. Let T_1, T_2, T_3, T_4 be t-norms and have the following forms $T_1 = \{(A_\alpha, V_\alpha) \mid \alpha \in A\}$, $T_2 = \{(B_\beta, W_\beta) \mid \beta \in B\}$, $T_3 = \{(C_\gamma, X_\gamma) \mid \gamma \in C\}$, $T_4 = \{(D_\delta, Y_\delta) \mid \delta \in D\}$. Moreover, let $f = (f^*, \pi_f)$, $g = (g^*, \pi_g)$ and $h = (h^*, \pi_h)$ be morphisms such that $f\colon T_1 \to T_2$, $g\colon T_2 \to T_3$ and $h\colon T_3 \to T_4$. First we show that the composition $g \circ f$ belongs to $\mathrm{Hom}_{\mathfrak{C}}(T_1, T_3)$ ($\mathfrak{C} = \mathfrak{T}_1$ or $\mathfrak{C} = \mathfrak{T}_0$). Indeed, for all $x, y, \mu, \nu \in [0,1]$ and for all $\gamma \in C$, $\beta \in B$ we have

$$W_\beta(f^*(\beta)(x), f^*(\beta)(y)) = f^*(\beta)(V_{\pi_f(\beta)}(x,y)),$$
$$X_\gamma(g^*(\gamma)(\mu), g^*(\gamma)(\nu)) = g^*(\gamma)(W_{\pi_g(\gamma)}(\mu,\nu)),$$

from bijectivity of every function from Φ ($\mu = f^*(\pi_g(\gamma))(x)$, $\nu = f^*(\pi_g(\gamma))(y)$) we obtain

$$\begin{aligned} X_\gamma(g^*(\gamma) \circ f^*(\pi_g(\gamma))(x), g^*(\gamma) \circ f^*(\pi_g(\gamma))(y)) &= g^*(\gamma)(W_{\pi_g(\gamma)}(f^*(\pi_g(\gamma))(x), f^*(\pi_g(\gamma))(y))) \\ &= g^*(\gamma) \circ f^*(\pi_g(\gamma))(V_{\pi_f(\pi_g(\gamma))}(x,y)). \end{aligned}$$

Next we show the condition of associativity. From Definition 3.6 we obtain for every $\delta \in D$

$$h \circ (g \circ f) = (h^*(\delta) \circ (g^*(\pi_h(\delta)) \circ f^*(\pi_g(\pi_h(\delta)))), (\pi_f \circ \pi_g) \circ \pi_h),$$
$$(h \circ g) \circ f = ((h^*(\delta) \circ g^*(\pi_h(\delta))) \circ f^*(\pi_g(\pi_h(\delta))), \pi_f \circ (\pi_g \circ \pi_h)),$$

from associativity of functions we obtain the condition of associativity. Again from Definition 3.6 for all $\beta \in B$ we obtain

$$1_{T_2} \circ f = (\mathrm{id}_{[0,1]} \circ f^*(\mathrm{id}_B(\beta)), \pi_f \circ \mathrm{id}_B) = (f^*(\beta), \pi_f) = f = (f^*(\beta) \circ \mathrm{id}_{[0,1]}, \mathrm{id}_A \circ \pi_f) = f \circ 1_{T_1},$$

this proves the condition of identity. □

Proposition 3.8. *The category $\mathfrak{T}_0$ has only one terminal object (minimum t-norm). The category $\mathfrak{T}_0$ does not have initial objects. However, the category $\mathfrak{T}_1$ has neither terminal objects nor initial objects.*

Proof. $T_{\mathbf{M}}$ is the terminal object in $\mathfrak{T}_0$, because for every t-norm T every morphism from T to $T_{\mathbf{M}}$ is the pair of empty sets. Suppose now that T is terminal object in $\mathfrak{T}_1$. Thus, there exist morphisms $f_1\colon T_{\mathbf{P}} \to T$ and $f_1\colon T_{\mathbf{LK}} \to T$. This is a contradiction, because there does not exist such $\varphi \in \Phi$ that $T_{\mathbf{P}} = (T_{\mathbf{LK}})_\varphi$. Suppose now that T is initial object in $\mathfrak{T}_1$ (in $\mathfrak{T}_0$ the proof is analogous). Thus, there exists one morphism $f\colon T \to T_{\mathbf{P}}$. Hence, in summands of T there exists the t-norm $(T_{\mathbf{P}})_\varphi$, so $f^* \equiv \varphi^{-1}$. Let $g\colon T \to T_{\mathbf{P}}$ be such that $\pi_g = \pi_f$ and $g^* \equiv \psi$, where $\psi = (\varphi^{-1})^2$. Since,

$$\psi^{-1}(\psi(x) \cdot \psi(y)) = \varphi(\sqrt{(\varphi^{-1}(x))^2 \cdot (\varphi^{-1}(x))^2}) = \varphi(\varphi^{-1}(x) \cdot \varphi^{-1}(y)),$$

so g is the morphism from T to $T_{\mathbf{P}}$ different than f. □

Example 3.9. *Some morphisms in categories $\mathfrak{T}_0$, $\mathfrak{T}_1$:*

1. *Let T be an ordinally irreducible t-norm and let $\varphi \in \Phi$. Then $f\colon T \to T_\varphi$ is a morphism, where $f^*(1) = \varphi^{-1}$ and $\pi_f = id_{\{1\}}$.*

2. *Let $T = \{(A_\alpha, T_\alpha) \mid \alpha \in A\}$ be ordinal sum of ordinally irreducible t-norms. Then, $pr_{\alpha_0}\colon T \to T_{\alpha_0}$ is a morphism, where $\pi_{pr_{\alpha_0}} \equiv \alpha_0$ and $pr^*_{\alpha_0} \equiv id_{[0,1]}$, for some $\alpha_0 \in A$.*

3. *Let T be an ordinally irreducible t-norm and let $T' = \{(A_\alpha, T_{\varphi(\alpha)}) \mid \alpha \in A\}$, where $\varphi(\alpha) \in \Phi$, for all $\alpha \in A$. Then, $\iota\colon T \to T'$ is a morphism, where $\pi_\iota(\alpha) = 1$ and $\iota^*(\alpha) = (\varphi(\alpha))^{-1}$, for all $\alpha \in A$.*

4. *Let $\pi\colon A \to A$ be a bijection and let $T = \{(A_\alpha, T_\alpha) \mid \alpha \in A\}$, $T' = \{(B_\alpha, T_{\pi(\alpha)}) \mid \alpha \in A\}$ be t-norms. Then, $Perm_\pi\colon T \to T'$ is a morphism, where $\pi_{Perm_\pi} = \pi$ and $Perm^*_\pi \equiv id_{[0,1]}$.*

Lemma 3.10. *Let $f\colon T_1 \to T_2$ be a morphism in the category $\mathfrak{T}_0$ and in the category $\mathfrak{T}_1$, respectively. Then the following sentences are true in the category $\mathfrak{T}_1$:*

(i) f is epic iff π_f is injective.

(ii) f is monic iff π_f is surjective.

(iii) f is isomorphism iff π_f is bijective.

Moreover, (i) and (iii) are true in $\mathfrak{T}_0$, but (ii) is true except the case when, we have a morphism $f\colon T \to T_{\mathbf{M}}$, where $T \neq T_{\mathbf{M}}$.

Proof. (i) Let g, h be morphisms from a t-norm T_2 to a t-norm $T_3 = \{(C_\gamma, R_\gamma) \mid \gamma \in C\}$. Assume that f is epic. Thus, if $g \circ f = h \circ f$, then $g = h$. This means that if $\pi_f \circ \pi_g = \pi_f \circ \pi_h$, then $\pi_g = \pi_h$. Hence, π_f is injective, it can prove similarly as in the category **Set**.

Assume now that π_f is injective and let $g \circ f = h \circ f$. Thus, $\pi_f \circ \pi_g = \pi_f \circ \pi_h$ and $g^*(\gamma) \circ f^*(\pi_g(\gamma)) = h^*(\gamma) \circ f^*(\pi_h(\gamma))$, for all $\gamma \in C$. From injectivty of π_f, we obtain that $\pi_g = \pi_h$ and $g^* = h^*$, so $g = h$.

(ii) The proof is analogues to the proof of (i).

(iii) Let $T_1 = \{(A_\alpha, V_\alpha) \mid \alpha \in A\}$ and $T_2 = \{(B_\beta, W_\beta) \mid \beta \in B\}$. If f is an isomorphism, then f is epic and monic (see [2, Proposition 2.6]), so from (i) and (ii) π_f is bijective.

Assume now that π_f is bijective, then let g be an inverse arrow to f, where $\pi_g = (\pi_f)^{-1}$ and $g^*(\alpha) = (f^*(\pi_g(\alpha)))^{-1}$, for all $\alpha \in A$. Indeed, $g \circ f = (g^*(\alpha) \circ f^*(\pi_g(\alpha)), \pi_f \circ \pi_g) = (\alpha \longmapsto \mathrm{id}_{[0,1]}, \mathrm{id}_A) = 1_{T_1}$, for all $\alpha \in A$ and $f \circ g = (f^*(\beta) \circ g^*(\pi_f(\beta)), \pi_g \circ \pi_f) = (f^*(\beta) \circ (f^*(\beta))^{-1}), \mathrm{id}_B) = (\beta \longmapsto \mathrm{id}_{[0,1]}, \mathrm{id}_B) = 1_{T_2}$, for all $\beta \in B$.

If $T_1 \neq T_{\mathbf{M}} \neq T_2$, the proof is identical, for the category $\mathfrak{T}_0$. Note that if $T_2 = T_{\mathbf{M}}$, then $T_1 = T_{\mathbf{M}}$ in the category $\mathfrak{T}_0$ and the sentences (i), (iii) are also true in $\mathfrak{T}_0$. Consider the t-norm $T = \{((0, \frac{1}{2}), T_{\mathbf{P}}), ((\frac{1}{2}, 1), T_{\mathbf{P}})\}$ and morphisms $g, h\colon T \to T$ such that $g = 1_T$, $h^* \equiv \mathrm{id}_{[0,1]}$, $\pi_h(1) = 2$ and $\pi_h(2) = 1$. Thus for a morphism $f\colon T \to T_{\mathbf{M}}$, $f \circ g = f \circ h$, but $g \neq h$. □

From the above lemma the first and 4th morphism (Example 3.9) are isomorphisms and when the set A contains at least two elements, the second morphism is epic and the third morphism is monic.

Similarly as categories $\mathfrak{T}_0$, $\mathfrak{T}_1$, we can define categories $(\mathfrak{T}_1)^{\mathrm{op}}$, $(\mathfrak{T}_0)^{\mathrm{op}}$. A pair $f = (f^*, \pi_f)$, where $\pi_f\colon A \to B$ and $f^*\colon A \to \Phi$ belongs to $\mathrm{Hom}_{\mathfrak{C}}(T_1, T_2)$ (the quotient set, similarly as in Definition 3.6), where $\mathfrak{C} \in \{(\mathfrak{T}_1)^{\mathrm{op}}, (\mathfrak{T}_0)^{\mathrm{op}}\}$ if and only if

$$\forall_{\alpha \in A} \quad \forall_{x,y \in [0,1]} \qquad W_{\pi_f(\alpha)}(f^*(\alpha)(x), f^*(\alpha)(y)) = f^*(\alpha)(V_\alpha(x, y)).$$

Moreover, the composition $h = g \circ f\colon T_1 \to T_3$ is defined such as the pair (h^*, π_h), where $\pi_h = \pi_g \circ \pi_f$ and $h^*(\gamma) = f^*(\alpha) \circ g^*(\pi_f(\alpha))$, for all $\alpha \in A$. Thus, $T_{\mathbf{M}}$ is the only initial element of the category $(\mathfrak{T}_0)^{\mathrm{op}}$. The category $(\mathfrak{T}_0)^{\mathrm{op}}$ has no terminal elements and the category $(\mathfrak{T}_1)^{\mathrm{op}}$ has neither initial elements nor terminal elements.

Theorem 3.11. *Let T_1, T_2 be t-norms and $\varphi \in \Phi$. Then, if $T_2 = (T_1)_\varphi$, then there exists an isomorphism $f\colon T_1 \to T_2$ in the category $\mathfrak{T}_1$ ($\mathfrak{T}_0$).*

Proof. We have 3 cases. First when $T_1 = T_{\mathbf{M}}$, then from Proposition 2.13 $T_2 = T_{\mathbf{M}}$. Thus, in the category $\mathfrak{T}_1$ it is enough take $f = (f^*, \pi_f)$, where $f^* \equiv \mathrm{id}_{[0,1]}$ and $\pi_f = id_{\{1\}}$. In the category $\mathfrak{T}_0$, we have $f^* = \emptyset$ and $\pi_f = \emptyset$. Second when T_1 is ordinally irreducible, then from Lemma 2.21 $(T_1)_\varphi$ (for some $\varphi \in \Phi$) is also ordinally irreducible. Thus, in categories $\mathfrak{T}_1$, $\mathfrak{T}_0$ it is enough to take $f = (f^*, \pi_f)$, where $f^* \equiv \varphi^{-1}$ and $\pi_f = id_{\{1\}}$. Indeed, we have for all $x, y \in [0, 1]$

$$T_2(\varphi^{-1}(x), \varphi^{-1}(y)) = \varphi^{-1}(T_1(\varphi(\varphi^{-1}(x)), \varphi(\varphi^{-1}(y)))),$$
$$T_2(\varphi^{-1}(x), \varphi^{-1}(y)) = \varphi^{-1}(T_1(x, y)),$$

this is compatible with the equation (3). Third, let $T_1 = \{(A_\alpha, V_\alpha) \mid \alpha \in A\}$ and let $(T_1)_\varphi = T_2 = \{(B_\alpha, W_\alpha) \mid \alpha \in A'\}$. Since $\varphi^{-1} \in \Phi$, we obtain that $A \cong A'$ in **Poset**, so without loss of generality we assume that $A = A'$. Furthermore, from monotonicity of φ we have $\varphi[B_\alpha] = A_\alpha$, for all $\alpha \in A$ ($\varphi[B_\alpha]$ is image of the set B_α). In categories $\mathfrak{T}_1$, $\mathfrak{T}_0$, it is enough to take $f = (f^*, \pi_f)$, where $\pi_f = \mathrm{id}_A$ and $f^*(\alpha) = \varphi_{B_\alpha} \circ \varphi^{-1} \circ \varphi_{A_\alpha}^{-1}$. The functions $\varphi_{A_\alpha}\colon \mathrm{cl}A_\alpha \to [0, 1]$, $\varphi_{B_\alpha}\colon \mathrm{cl}B_\alpha \to [0, 1]$, where $\mathrm{cl}A_\alpha = [a_\alpha, b_\alpha]$ and $\mathrm{cl}B_\alpha = [c_\alpha, d_\alpha]$ are defined as follow

$$\varphi_{A_\alpha}(x) = \frac{x - a_\alpha}{b_\alpha - a_\alpha}, \qquad \varphi_{B_\alpha}(x) = \frac{x - c_\alpha}{d_\alpha - c_\alpha}.$$

Indeed, for all $x, y \in (a_\alpha, b_\alpha]$ and for all $z, w \in (c_\alpha, d_\alpha]$ we obtain

$$T_1(x,y) = \varphi_{A_\alpha}^{-1} V_\alpha(\varphi_{A_\alpha}(x), \varphi_{A_\alpha}(y)),$$
$$T_2(z,w) = \varphi_{B_\alpha}^{-1} W_\alpha(\varphi_{B_\alpha}(z), \varphi_{B_\alpha}(w)).$$

Hence, for all $\mu, \nu \in (0,1]$

$$\varphi_{A_\alpha}(T_1(\varphi_{A_\alpha}^{-1}(\mu), \varphi_{A_\alpha}^{-1}(\nu)) = V_\alpha(\mu,\nu),$$
$$\varphi_{B_\alpha}(T_2(\varphi_{B_\alpha}^{-1}(\mu), \varphi_{B_\alpha}^{-1}(\nu)) = W_\alpha(\mu,\nu).$$

$$\begin{aligned} W_\alpha(f^*(\alpha)(\mu), f^*(\alpha)(\nu)) &= \varphi_{B_\alpha}(T_2(\varphi^{-1} \circ \varphi_{A_\alpha}^{-1}(\mu), \varphi^{-1} \circ \varphi_{A_\alpha}^{-1}(\nu)) \\ &= \varphi_{B_\alpha} \circ \varphi^{-1}(T_1(\varphi \circ \varphi^{-1} \circ \varphi_{A_\alpha}^{-1}(\mu), \varphi \circ \varphi^{-1} \circ \varphi_{A_\alpha}^{-1}(\nu)) \\ &= \varphi_{B_\alpha} \circ \varphi^{-1}(T_1(\varphi_{A_\alpha}^{-1}(\mu), \varphi_{A_\alpha}^{-1}(\nu)) \\ &= f^*(\alpha)(V_\alpha(\mu,\nu)). \end{aligned}$$

The above equation is also true when $\mu = 0$ or $\nu = 0$, what proves the theorem. □

Example 3.12. *Let $T = \{((0, \frac{1}{2}), T_{\mathbf{nM}}), ((\frac{1}{2}, 1), T_{\mathbf{LK}})\}$ and let $\varphi(x) = \sqrt{x}$, for all $x \in [0,1]$. Then, $T_\varphi = \{((0, \frac{1}{4}), (T_{\mathbf{nM}})_\varphi), ((\frac{1}{4}, 1), (T_{\mathbf{LK}})_\varphi)\}$, where*

$$(T_{\mathbf{nM}})_\varphi(x,y) = \begin{cases} 0, & \sqrt{x} + \sqrt{y} \le 1 \\ \min(x,y), & \textit{otherwise} \end{cases}, \qquad (T_{\mathbf{LK}})_\varphi(x,y) = (\max(\sqrt{x} + \sqrt{y} - 1, 0))^2.$$

From Theorem 3.11 a morphism $f \colon T \to T_\varphi$ of the from

$$f^*(\alpha)(x) = \begin{cases} x^2, & \alpha = 1 \\ \frac{4(\frac{x+1}{2})^2 - 1}{3}, & \alpha = 2 \end{cases}, \qquad \pi_f = id_{\{1,2\}}$$

is an isomorphism.

Theorem 3.13. *The category $\mathfrak{T}_0$ has products for every pair of t-norms. In particular, the product for t-norms $T_1 = \{(A_\alpha, V_\alpha) \mid \alpha \in A\}$, $T_2 = \{(B_\beta, W_\beta) \mid \beta \in B\}$ is of the form*

$$T_1 \times T_2 := \{(C_{(\gamma,\delta)}, Z_{(\gamma,\delta)}) \mid (\gamma,\delta) \in \{0\} \times A \cup \{1\} \times B\}, \tag{4}$$

where $\{0\} \times A \cup \{1\} \times B$ is orderly with lexicographical order, $C_{(\gamma,\delta)} = \begin{cases} \frac{1}{2} A_\delta, & \gamma = 0, \delta \in A \\ \frac{1}{2} + \frac{1}{2} B_\delta, & \gamma = 1, \delta \in B \end{cases}$ *and* $Z_{(\gamma,\delta)} = \begin{cases} V_\delta, & \gamma = 0, \delta \in A \\ W_\delta, & \gamma = 1, \delta \in B \end{cases}$, *where for every $a, b \in \mathbb{R}$ and $X \subseteq \mathbb{R}$, the nation $a + bX$ means $a + bX := \{a + bx \mid x \in X\}$. Moreover, $\mathfrak{T}_1$ has no products for every pair of t-norms, but has products for every pair of t-norms, for which it does not occur $T_{\mathbf{M}}$ in summands.*

Proof. Assume that T_1 and T_2 are t-norms such as in the thesis of the theorem. We will show that $T_1 \times T_2$ is the product for T_1 and T_2. From Theorem 2.17 $T_1 \times T_2$ is well-defined. Let $pr_1 \colon T_1 \times T_2 \to T_1$ be of the form $pr_1 = (pr_1^*, \pi_{pr_1})$, where $pr_1^* \equiv \mathrm{id}_{[0,1]}$ and $\pi_{pr_1}((0,\alpha)) = \alpha$, for all $\alpha \in A$. Similarly, let $pr_2 \colon T_1 \times T_2 \to T_2$ be of the form $pr_2 = (pr_2^*, \pi_{pr_2})$, where $pr_2^* \equiv \mathrm{id}_{[0,1]}$ and $\pi_{pr_2}((1,\beta)) = \beta$, for all $\beta \in B$. Finally, we prove that $T_1 \times T_2$ satisfies the universal property. Let X be a t-norm and let x_1 and x_2 be morphisms from X to T_1 and T_2, respectively. Let $\rho \colon X \to T_1 \times T_2$ be a morphism of the form $\rho = (\rho*, \pi_\rho)$, where

$$\pi_\rho((\gamma,\delta)) = \begin{cases} \pi_{x_1}((0,\delta)), & \gamma = 0, \delta \in A \\ \pi_{x_2}((1,\delta)), & \gamma = 0, \delta \in B \end{cases} \quad \text{and} \quad \rho^*((\gamma,\delta)) = \begin{cases} x_1^*((0,\delta)), & \gamma = 0, \delta \in A \\ x_2^*((1,\delta)), & \gamma = 0, \delta \in B \end{cases}.$$

The following compositions fulfil $x_1 = pr_1 \circ \rho$ and $pr_2 \circ \rho = x_2$. Indeed, $pr_1 \circ \rho = (x_1^*, \pi_{x_1}) = x_1$, similarly for x_2. Now suppose that ρ' is another morphism from X to $T_1 \times T_2$ such that $x_1 = pr_1 \circ \rho'$ and $pr_2 \circ \rho' = x_2$. From the forms pr_1, pr_2 we have $\rho = \rho'$, so ρ is unique.

In the case of the category $\mathfrak{T}_1$, when $T_{\mathbf{M}}$ does not occur in summands of T_1 and T_2 the proof is the same. However, the category $\mathfrak{T}_1$ has no product for $T_1 = T_2 = T_{\mathbf{M}}$. Suppose that P is a t-norm such that the following diagram $T_{\mathbf{M}} \xleftarrow{p_1} P \xrightarrow{p_2} T_{\mathbf{M}}$ satisfies UMP. Consider the following diagrams $T_{\mathbf{M}} \xleftarrow{\varphi} T_{\mathbf{M}} \xrightarrow{\mathrm{id}_{[0,1]}} T_{\mathbf{M}}$ and $T_{\mathbf{M}} \xleftarrow{\psi} T_{\mathbf{M}} \xrightarrow{\mathrm{id}_{[0,1]}} T_{\mathbf{M}}$ where $\varphi, \psi \in \Phi \setminus \{\mathrm{id}_{[0,1]}\}$, $\varphi \neq \psi$. Thus, $P = T_{\mathbf{M}}$ and there are $u, v\colon T_{\mathbf{M}} \to P$ such that $pr_1^*(1) \circ u^*(1) = \varphi$, $pr_2^*(1) \circ u^*(1) = \mathrm{id}_{[0,1]}$, $pr_1^*(1) \circ v^*(1) = \psi$ and $pr_2^*(1) \circ v^*(1) = \mathrm{id}_{[0,1]}$. Thus, $\varphi = pr_1^*(1) \circ (pr_2^*(1))^{-1} = \psi$, this is a contradiction. □

Note that from above theorem the product for $T_{\mathbf{P}}, T_{\mathbf{LK}}$ is of the form

$$T_{\mathbf{P}} \times T_{\mathbf{LK}} = \{((0, \frac{1}{2}), T_{\mathbf{P}}), ((\frac{1}{2}, 1), T_{\mathbf{LK}})\},$$

in categories $\mathfrak{T}_1$, $\mathfrak{T}_0$. Furthermore, products are unique up to isomorphism, so for example

$$\{((0, \frac{1}{3}), T_{\mathbf{LK}}), ((\frac{1}{3}, 1), T_{\mathbf{P}})\},$$

is also a product in categories $\mathfrak{T}_1$, $\mathfrak{T}_0$. However, $\{((0, \frac{1}{3}), T_{\mathbf{LK}}), ((\frac{1}{3}, \frac{2}{3}), T_{\mathbf{P}})\}$ is a product for $T_{\mathbf{P}}, T_{\mathbf{LK}}$ only in the category $\mathfrak{T}_0$.

Corollary 3.14. *For every continuous t-norm T there exists epimorphism to at least one of 3 t-norms: $T_{\mathbf{M}}$, $T_{\mathbf{P}}$, $T_{\mathbf{LK}}$ in categories $\mathfrak{T}_1$, $\mathfrak{T}_0$.*

Proposition 3.15. *Categories $\mathfrak{T}_0$, $\mathfrak{T}_1$ has no coproducts for every pair of t-norms.*

Proof. Suppose that there exists a coproduct T for $T_{\mathbf{P}}$ and $T_{\mathbf{LK}}$. Thus, there exist morphisms $\iota_1\colon T_{\mathbf{P}} \to T$ and $\iota_2\colon T_{\mathbf{LK}} \to T$. This is a contradiction, because there does not exist such $\varphi \in \Phi$ that $T_{\mathbf{P}} = (T_{\mathbf{LK}})_\varphi$. □

Proposition 3.16. *Categories $\mathfrak{T}_0$, $\mathfrak{T}_1$ has no equalizers for every t-norms.*

Proof. Let $T = T_{\mathbf{D}}$ (for $T_{\mathbf{M}}$ or strict t-norm the proof is analogues) and let $f, g\colon T \to T$ be such that $f^* \equiv \varphi_1 \neq \varphi_2 \equiv g^*$, for some $\varphi_1, \varphi_2 \in \Phi$ such that $\varphi_1 \neq \varphi_2$. Suppose that $e\colon E \to T$ is an equalizer. This is a contradiction, because $f \circ e \neq g \circ e$. □

Theorem 3.17. *Let t-norms T_1, T_2 be of the following forms $T_1 = \{(A_\alpha, V_\alpha) \mid \alpha \in A\}$ and $T_2 = \{(B_\beta, W_\beta) \mid \beta \in B\}$. Let f, g be morphisms from t-norm T_1 to t-norm T_2. Then $q\colon T_2 \to Q$ is a coequalizer in the category $\mathfrak{T}_0$, where*

$$Q := \{(B_\beta, W_\beta) \mid \beta \in B_1\} \qquad \text{and} \qquad B_1 = \{\beta \in B \mid \pi_f(\beta) = \pi_g(\beta) \wedge f^*(\beta) = g^*(\beta)\}.$$

Moreover, the category $\mathfrak{T}_1$ has no coequalizer for every parallel arrows.

Proof. Assume that t-norms T_1, T_2, Q are of the same forms as in the thesis of the theorem. Then the morphism $q\colon T_2 \to Q$, where $q^* \equiv \mathrm{id}_{[0,1]}$ and $\pi_q(\beta) = \beta$, for all $\beta \in B_1$ satisfies the equation $q \circ f = q \circ g$, from the assumption. Assume now that $z\colon T_2 \to Z$ is the morphism such that $z \circ f = z \circ g$, where $Z = \{(C_\gamma, X_\gamma) \mid \gamma \in C\}$. Thus, $\pi_z[C] \subseteq B_1$. Indeed, $\pi_f \circ \pi_z = \pi_g \circ \pi_z$ and $z^*(\gamma) \circ f^*(\pi_z(\gamma)) = z^*(\gamma) \circ g^*(\pi_z(\gamma))$, for all $\gamma \in C$. Next let $\rho\colon Q \to Z$ be the morphism such that $\rho^* = z^*$ and $\pi_\rho(\gamma) = \pi_z(\gamma)$, for all $\gamma \in C$. Of course, $\rho \circ q = z$ and assume that a morphism $\sigma\colon Q \to Z$ satisfies the equation $\sigma \circ q = z$. Firstly, we obtain that $\sigma^*(\gamma) \circ q^*(\pi_\sigma(\gamma)) = z^*(\gamma)$, for all $\gamma \in C$, so $\sigma^* = z^*$. Secondly, we have $\pi_q \circ \pi_\sigma = \pi_z$, so $\pi_\sigma(\gamma) = \pi_z(\gamma)$, for all $\gamma \in C$, because $\pi_q|_{B_1} = \mathrm{id}_{B_1}$. Thus, $\rho = \sigma$, so ρ is unique.

Suppose now that the category $\mathfrak{T}_1$ has coequalizers for every parallel arrows. Then, there exists a coequalizer for $f, g\colon T \to T$, where $T = T_0 \times T_{\mathbf{P}}$ and

$$f = 1_T, \qquad \pi_g = \mathrm{id}_{\{1,2\}}, \quad g^*(\alpha) = \begin{cases} \mathrm{id}_{[0,1]}, & \alpha = 1 \\ x \longmapsto x^2, & \alpha = 2 \end{cases}.$$

Let T_2 be this coequalizer with a morphism $q\colon T \to T_2$. Thus, $q \circ f = q \circ g$, so $q^*(\gamma) \circ f^*(\pi_q(\gamma)) = q^*(\gamma) \circ g^*(\pi_q(\gamma))$. Hence, $\pi_q \equiv 1$, so T_2 is an ordinal sum, where only T_0 occurs in summands. This is a contradiction, because T_2 is a t-norm, but an ordinal sum, where only T_0 occurs in summands is not a t-norm (does not satisfy Theorem 2.17, every t-norm must satisfies the thesis Theorem 2.17, see [16]). □

Corollary 3.18. *The category $(\mathfrak{T}_0)^{op}$ has coproducts for every pair of t-norms and equalizers for every parallel arrows. The category $(\mathfrak{T}_1)^{op}$ has no product, coproducts for every pair of t-norms and has no equalizers, coequalizers for every parallel arrows. Furthermore, $(\mathfrak{T}_0)^{op}$ has no products for every pair of t-norms and has no coequalizers for every parallel arrows.*

4 The category of R-implications defined on bounded lattices

In this section, we show that Theorems 2.29, 2.30 (given by authors without proofs) from article [22]:

A. Youse, M. Mashinchi, and R. Mesiar. *Some notes on the category of fuzzy implications on bounded lattices.* Kybernetika, 57:332–351, 2021.

are **not correct**.

Definition 4.1. [8] *Let P and Q be ordered sets. A map $\phi\colon P \to Q$ is said to be order-embedding if*

$$x \leq y \quad in\ P \quad \Leftrightarrow \quad \phi(x) \leq \phi(y) \quad in\ Q.$$

Definition 4.2. [22, Remark 2] *Let L and M be bounded lattices. Let $I_L\colon L \times L \to L$ and $J_M\colon M \times M \to M$ be two fuzzy implications and φ be a lattice homomorphism such that satisfies the property for each $x, y \in L$*

$$\varphi(I_L(x, y)) = J_M(\varphi(x), \varphi(y)). \tag{5}$$

We denote this φ with extra property by $\varphi^*\colon I_L \to J_M$. Consequently, φ^* is just φ with the extra property, (5).

Definition 4.3. [22, Lemma 2.24] *Let $\varphi^*\colon I_L \to J_M$ and $\psi^*\colon J_M \to K_N$ be two morphisms defined in Definition 4.2, where $\psi^* \circ \varphi^* = (\psi \circ \varphi)^*$ and 1^* is identity lattice homomorphism, where I_L, J_M and K_N are fuzzy implications on bounded lattices L, M and N respectively. Then $\mathfrak{FI}$ is the category (see [22, Lemma 2.24]), whose objects are all fuzzy implications on bounded lattices and morphisms satisfy the above principles. We call $\mathfrak{FI}$ the category of fuzzy implications.*

Definition 4.4. [22, Lemma 2.28] *$\mathfrak{FRI}$ is the subcategory of $\mathfrak{FI}$ (see [22, Lemma 2.28]), whose objects are all R-implications on complete lattices and φ^* be a t-norm morphism in the category **T-NORM** and φ be order-embedding on the lattice.*

The above definition is imprecise, when $\varphi^*\colon I_{T_1} \to I_{T_2}$ is a morphism in $\mathfrak{FRI}$, then φ^* must be also a morphism in **T-NORM**, but it does not mention for which t-norms. Therefore, we will use this definition with the additional assumption:

If $\varphi^*\colon I_{T_1} \to I_{T_2}$ is a morphism in $\mathfrak{FRI}$, then φ^* must be a morphism from T_1 to T_2, where T_1 and T_2 are some generators of I_{T_1} and I_{T_2}, respectively.

Theorem 4.5. [21, Theorem 18] *Let F be a map from $\mathfrak{FRI}$ to **T-NORM** such that $F(I_T) = T$ and $F(\varphi) = \varphi$. Then F is a functor.*

It is worth mentioning that the above theorem appeared early in work [21] (as Theorem 18, also without any proof):

A. Youse and M. Mashinchi. *Categories of fuzzy implications and R-implications on bounded lattices.* In 6th Iranian Joint Congress on Fuzzy and Intelligent Systems (CFIS), pages 40–42. IEEE 2018, 2018.

Theorem 4.6. [22, Theorem 2.30] *Let F be a functor as in Theorem 4.5 and G be a map from **T-NORM** to $\mathfrak{FRI}$ such that $G(T) = I_T$ and $G(\varphi) = \varphi$. Then G is a functor and $F \circ G = 1_{\textbf{T-NORM}}$, $G \circ F = 1_{\mathfrak{FRI}}$ and so F is an isomorphism.*

We will show that the above theorems are false. Before that let consider the following t-norm

$$T_{\mathbf{nM}^*}(x, y) = \begin{cases} 0, & x + y < 1, \\ \min(x, y), & \text{otherwise.} \end{cases}$$

$I_{T_{\mathbf{nM}^*}} = I_{T_{\mathbf{nM}}}$ (see [12]), but $T_{\mathbf{nM}} < T_{\mathbf{nM}^*}$. This shows that the operation $T \longmapsto I_T$ from the category **T-NORM** to the category $\mathfrak{FRI}$ is not injective (is injective if we limit to left-continuous t-norms and R-implications generated from these t-norms, see [3]). Thus, we obtain immediately that a map F from Theorem 4.5 is not well defined. Hence, we will understand this theorem with the additional assumption: for every an R-implication I, the value $F(I)$ is equal to some t-norm T such that $I_T = I$.

Remark 4.7. *Firstly we will show that Theorem 4.5 is not correct. Let F be a function from $\mathfrak{FRI}$ to **T-NORM** such that $F(I_T) = T$ and $F(\varphi) = \varphi$. Moreover, let $F(I_{T_{\mathbf{nM}}}) = T_{\mathbf{nM}^*}$ and let*
$F(I_{T_{\mathbf{nM}} \times T_{\mathbf{M}}}) = T_{\mathbf{nM}} \times T_{\mathbf{M}}$, where $T_{\mathbf{nM}} \times T_{\mathbf{M}}$ is defined as in (4). From Proposition 2.28 we know that

$$I_{T_{\mathbf{nM}} \times T_{\mathbf{M}}}(x, y) = \begin{cases} \frac{1}{2} I_{T_{\mathbf{nM}}}(2x, 2y), & 0 \leq x, y \leq \frac{1}{2}, \\ I_{\mathbf{GD}}(x, y), & \textit{otherwise.} \end{cases}$$

Consider a morphism $\varphi^*\colon I_{T_{\mathbf{nM}}} \to I_{T_{\mathbf{nM}} \times T_{\mathbf{M}}}$ *such that* $\varphi\colon [0,1] \to [0,1]$ *is order-embedding of the form*

$$\varphi(x) = \begin{cases} \frac{x}{2}, & x \in [0,1) \\ 1, & x = 1 \end{cases}.$$

φ^* *is well-defined. Indeed,* $I_{T_{\mathbf{nM}}}$ *satisfies* (OP)*, so*

$$\varphi(I_{T_{\mathbf{nM}}}(x,y)) = 1 \Leftrightarrow I_{T_{\mathbf{nM}}}(x,y) = 1 \Leftrightarrow x \le y \Leftrightarrow \varphi(x) \le \varphi(y) \Leftrightarrow I_{T_{\mathbf{nM}}}(\varphi(x), \varphi(y)) = 1,$$

when $1 > x > y$*, then*

$$\varphi(I_{T_{\mathbf{nM}}}(x,y)) = \frac{I_{T_{\mathbf{nM}}}(x,y)}{2} = \frac{1}{2} I_{T_{\mathbf{nM}}}\left(\frac{\varphi(x)}{\frac{1}{2}}, \frac{\varphi(y)}{\frac{1}{2}}\right) = I_{T_{\mathbf{nM}} \times T_{\mathbf{M}}}(\varphi(x), \varphi(y)),$$

and when $1 = x > y$*, then*

$$\varphi(I_{T_{\mathbf{nM}}}(1,y)) = \varphi(y) = I_{\mathbf{GD}}(1, \varphi(y)) = I_{T_{\mathbf{nM}} \times T_{\mathbf{M}}}(\varphi(1), \varphi(y)).$$

Moreover, φ *is also t-norm morphism from* $T_{\mathbf{nM}}$ *to* $T_{\mathbf{nM}} \times T_{\mathbf{M}}$*. Indeed, for every t-norm* T*,* $T(x,y) = 1$ *iff* $x = 1 = y$*. For* $x, y < 1$ *we have*

$$\varphi(T_{\mathbf{nM}}(x,y)) = \frac{T_{\mathbf{nM}}(x,y)}{2} = \frac{1}{2} T_{\mathbf{nM}}\left(\frac{\varphi(x)}{\frac{1}{2}}, \frac{\varphi(y)}{\frac{1}{2}}\right) = T_{\mathbf{nM}} \times T_{\mathbf{M}}(\varphi(x), \varphi(y)).$$

When, either $x = 1$ *or* $y = 1$*, then*

$$\varphi(T_{\mathbf{nM}}(x,y)) = \frac{\min(x,y)}{2} = \min(\varphi(x), \varphi(y)) = T_{\mathbf{nM}} \times T_{\mathbf{M}}(\varphi(x), \varphi(y)).$$

Thus, φ *is correctly defined. However,* $\varphi^*\colon T_{\mathbf{nM}^*} \to T_{\mathbf{nM}} \times T_{\mathbf{M}}$ *is not a morphism. Indeed,*

$$\frac{1}{4} = \varphi\left(\frac{1}{2}\right) = \varphi\left(T_{\mathbf{nM}^*}\left(\frac{1}{2}, \frac{1}{2}\right)\right) = T_{\mathbf{nM}} \times T_{\mathbf{M}}\left(\varphi\left(\frac{1}{2}\right), \varphi\left(\frac{1}{2}\right)\right) = \frac{T_{\mathbf{nM}}(\frac{1}{2}, \frac{1}{2})}{2} = 0,$$

this is a contradiction, so Theorem 4.5 is false.

Secondly, we will show that Theorem 4.6 is not correct. Indeed, in Theorem 4.6 we assume that F *is a functor as in Theorem 4.5. However, such a functor does not exist from the above, so Theorem 4.6 is not correct.*

5 Conclusions

We have introduced some categories of t-norms on the real unit interval $[0,1]$. Furthermore, we have investigated mainly the existence of products, coproducts, equalizers and coequalizers in these categories. In particular, we have shown that if we limit the category **T-NORM** only to t-norms on $[0,1]$ then, the category $\mathbf{T}_{[0,1]}$ has neither product, coproducts for every pair of t-norms nor equalizers, coequalizers for every parallel arrows. Next, we have introduced two categories of t-norms based on two methods expressed of t-norms as the ordinal sum of t-subnorms. Especially, similar to the category $\mathbf{T}_{[0,1]}$, the category $\mathfrak{T}_1$ has neither products, coproduct for every pair of t-norms nor equalizers, coequalizers for every parallel arrows. However, the category $\mathfrak{T}_0$ has products for every pair of t-norms and has coequalizers for every parallel arrows. Furthermore, the category $\mathfrak{T}_0$ has neither coproducts for every pair of t-norms nor equalizers for every parallel arrows. For categories $(\mathfrak{T}_0)^{op}$ and $(\mathfrak{T}_1)^{op}$ we have obtained analogues properties to the category $\mathfrak{T}_0$ and the category $\mathfrak{T}_1$, respectively. Finally, In section 4 we have presented counter-examples for Theorems 2.29, 2.30 in [22], are false by providing counterexamples.

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