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Original paper

Numerical solution for interval initial value problems based on interactive arithmetic

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Abstract

This work studies Interval Initial Value Problems (IIVPs), where the derivative is given by the generalized Hukuhara derivative (gH-derivative) and the initial condition is given by an interval. The focus of the paper is to provide the numerical approximations for the solutions associated with the gH-derivative of IIVPs. This article considers the Euler numerical method, where the classical arithmetic operation is adapted for intervals. The arithmetic considered here is obtained using sup-J extension principle, where J is a particular family of joint possibility distributions. This family gives raise to different types of interactivity and this work shows what kind of interactivity is necessary in the numerical method, in order to approximate the solution via gH-derivative. To illustrate the results, the paper focuses in the decay Malthusian model.

Keywords: Interval initial value problem, generalized hukuhara derivative, euler method, malthusian model, interactivity.

1 Introduction

Ordinary and Partial Differential Equations (ODEs and PDEs) are fields of Mathematics which are widely studied mainly by mathematicians, physicists and engineers. This area can be used to describe several phenomenon of nature, such as physical, chemical and biological processes [11].

In the differential equation models the behaviour of the processes with some conditions, such as the initial quantity of individuals in a population study, are given by initial or boundary conditions. These problems are called Initial Value Problems (IVPs) or Boundary Value Problems (BVPs).

Although nowadays there are several mathematical methods for solving differential equations, the task of solving ODEs and PDEs are not so easy. Hence, numerical methods may be useful tools to study such problems. In order these numerical methods be consistent, the outputs obtained from them must approximate the analytical solution to the problem, that is, the analytical and numerical solutions must be coherent.

A simple and well-known numerical method is Euler's method, which approximates the analytical solution to the IVP. This method is used not only to study classical differential equations, but also considered to study Fuzzy Differential Equations (FDEs). A FDE can be classified as a differential equation that presents some parameter or state with fuzzy values. For example, the derivatives in the differential equation can be given by a fuzzy derivative (such as the Hukuhara derivative and its generalizations [1, 7, 17] or interactive derivatives [5, 6, 20]) or the parameters and additional conditions can be given by fuzzy numbers [12, 18].

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In the context of fuzzy set theory, numerical methods, such as Euler and Runge-Kutta methods, are being considered to provide numerical approximations of analytical solutions to IVPs based on some metric (usually Hausdorff metric [10, 17]). In general this study is given by applying the classical numerical method to the endpoints of fuzzy numbers [2, 3]. Other approach consists to extend the classical arithmetic operations in the method to fuzzy numbers by means of the Zadeh or sup-J extension principles [4, 21].

A particular case of FDE is an Interval Initial Value Problem (IIVP), where the variables present in the problem are given by intervals instead of fuzzy numbers. The same tools of fuzzy set theory can be employed for interval theory, since intervals are particular cases of fuzzy numbers.

This paper focuses on the study of IIVPs, where the derivative is given by the generalized Hukuhara derivative (gH-derivative) and the initial condition is given by an interval. Since there are two types of gH-differentiability, we may obtain two analytical solutions without switching points. This study compares these analytical solutions to the numerical solutions obtained using Euler's method with interactive arithmetic based on a parametrized familiy (J_γ) of Joint Possibility Distributions (JPDs) [13].

In this work we show which types of interactivity are necessary to obtain numerical solutions that are consistent with the analytical solutions under each type of gH-differentiability. In order to illustrate the results, the decay Malthusian model is considered.

The article is divided as follows. Section II provides the background for this study, bringing basics concepts of the classical and fuzzy/interval set theories. Section III presents the numerical approximations for IIVP, which is the main result of the paper. Section IV presents an example for a decay Malthusian model. Finally, Section V provides the final remarks of the article.

2 Mathematical background

This section brings the mathematical concepts required for this paper.

2.1 Classical set theory

We start this section introducing the concepts of IVP which consists of an ordinary differential equation with initial condition given by

$$\begin{cases} \frac{dv}{dt} = g(t, v), \\ v(t_0) = v_0 \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $v : [t_0, T] \to \mathbb{R}^n$ and $g : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $T > t_0$. In this paper, we focus on the unidimensional case, i.e., $n = 1$.

It is possible to obtain a solution for IVPs through the theory of differential equations, which can be a complicated task depending on the equation that describes the field. Numerical methods can be an alternative approach to study an IVP. Euler's method consists of producing a numerical solution for (1) by means of the following recursive formula:

$$v^{(k+1)} = v^{(k)} + hg(t_k, v^{(k)}). \tag{2}$$

The numerical solution $v^{(k)}$ approximates the solution $v(t_k)$, $t_k = t_0 + kh$, with local error $\mathcal{O}(h^2)$.

2.2 Fuzzy set theory

This subsection presents the fuzzy set theory and a mathematical background necessary for this paper.

A fuzzy subset A of a universe X is characterized by its membership function φ_A that maps elements of X in [0,1]. For $0 < \alpha \leq 1$, the α-level of A is defined by the classical set $[A]_\alpha = \{u \in X : \varphi_A(u) \geq \alpha\}$. For the case where X is also a topological space, the 0-level of A is defined by $[A]_0 = cl\{u \in X : \varphi_A(u) > 0\}$, where clY denotes the closure of the subset $Y \subseteq X$.

The definition of a fuzzy number is given as follows [9, 13]. A fuzzy subset A of $\mathbb{R}$ is a fuzzy number if it satisfies the following properties:

- every α-level of A is a nonempty and close interval of $\mathbb{R}$;
- $SuppA = \{u \in \mathbb{R} \mid \varphi_A(u) > 0\}$ is bounded.

The set of fuzzy numbers is denoted by $\mathbb{R}_\mathcal{F}$.

Since the α-levels of a fuzzy number A are intervals, we represent them by $[A]_\alpha = [\underline{a}_\alpha, \overline{a}_\alpha]$. The diameter of $[A]_\alpha$ is defined by $diam([A]_\alpha) = \overline{a}_\alpha - \underline{a}_\alpha$. A translation of a fuzzy number A by the midpoint of $[A]_1$, i.e., $a = 0.5(\underline{a}_1 + \overline{a}_1)$, is the fuzzy number A^t with α-level given by $[A^t]_\alpha = [\underline{a}^t_\alpha, \overline{a}^t_\alpha]$. This translation leads to two interesting properties: $\underline{a}^t_\alpha \leq 0 \leq \overline{a}^t_\alpha$, for all $\alpha \in [0,1]$ and $[A]_\alpha = [A^t]_\alpha + a$. The following example illustrates this concept for intervals.

Example 2.1. *Let $A = [\underline{a}, \overline{a}] = [2,4]$ be an interval. The translated version of A by its midpoint $a = 3$ is the interval $A^t = [\underline{a}^t, \overline{a}^t] = [-1,1]$. Note that $A = A^t + a$ and $\underline{a}^t \leq 0 \leq \overline{a}^t$.*

The notion of interactivity arises whenever a joint distribution of possibility (JPD) is given [14, 22]. A JPD for fuzzy numbers $A_1, \ldots, A_n$ is noting more than a fuzzy set J of $\mathbb{R}^n$ such that $\varphi_{A_i}(w) = \bigvee_{v \in \mathbb{R}^n, v_i = w} \varphi_J(v)$ for all $w \in \mathbb{R}$ and $i = 1, \ldots, n$. The sup-J extension of a function $m : \mathbb{R}^n \to \mathbb{R}$ at $A_1, \ldots, A_n$ is a fuzzy set $m_J(A_1, \ldots, A_n)$ of $\mathbb{R}$ with membership function

$$\varphi_{m_J(A_1,\ldots,A_n)}(u) = \bigvee_{(x_1,\ldots,x_n) \in m^{-1}(u)} \varphi_J(x_1, \ldots, x_n), \tag{3}$$

where $m^{-1}(u) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : m(x_1, \ldots, x_n) = u\}$ [14]. It is wort observing that the sup-J extension principle boils down to the Zadeh's extension principle whenever $J = J_\wedge$, where $\varphi_{J_\wedge}(x_1, \ldots, x_n) = \varphi_{A_1}(x_1) \wedge \ldots \wedge \varphi_{A_n}(x_n)$ for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Here, the symbols $\vee$ and $\wedge$ denote the supremum and infimum operators, respectively.

The notion of interactivity is defined in similar way that dependence of random variables as follows. Let J be a given JPD for fuzzy numbers $A_1, \ldots, A_n$. If $J = J_\wedge$, we say that $A_1, \ldots, A_n$ are noninteractive, ohterwise, they are said to be interactive.

Given $A, B \in \mathbb{R}_\mathcal{F}$. If there exist a fuzzy number H such that

$$\begin{cases} (I)\ A = B + H \text{ or} \\ (II)\ B = A - H \end{cases}, \tag{4}$$

then H is said to be the generalized Hukuhara difference (or, for short, the gH-difference) of A and B, which is denoted by $A -_{gH} B$. Here, the symbols "+" and "-" correspond to the standard addition and subtraction on $\mathbb{R}_\mathcal{F}$ [7]. If A and B are intervals, say $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$, then $A -_{gH} B$ always exists and is given by

$$A -_{gH} B = \left[(\underline{a} - \underline{b}) \wedge (\overline{a} - \overline{b}), (\underline{a} - \underline{b}) \vee (\overline{a} - \overline{b}) \right].$$

Definition 2.2. [7] *A function $v : (a,b) \to \mathbb{R}_\mathcal{F}$ is said to be generalized Hukuhara differentiable (or, for short, gH-differentiable) at $t_0 \in (a,b)$ if there exists a fuzzy number $v'_{gH}(t_0) \in \mathbb{R}_\mathcal{F}$ such that*

$$v'_{gH}(t_0) = \lim_{h \to 0} \frac{v(t_0 + h) -_{gH} v(t_0)}{h},$$

where the limit is taken using the (Hausdorff) metric d_∞ [9].

Moreover, if the gH-difference $v(t_0 + h) -_{gH} v(t_0)$ satisfies the case (I) for all sufficient small $|h|$, then the function v is said to be gH-differentiable of type (I) or, simply, (I)-gH-differentiable at t_0. Similarly, if the gH-difference $v(t_0 + h) -_{gH} v(t_0)$ satisfies the case (II) for all sufficient small $|h|$, then the function v is said to be gH-differentiable of type (II) or, simply, (II)-gH-differentiable at t_0.

This paper considers a family of JPDs given by (J_γ), $\gamma \in [0,1])$, which was employed by Esmi et al. [13]. The construction of these distributions will be omitted here, but, in the interval case, every J_γ corresponds to a crisp subset of $\mathbb{R}^2$. In Figures 1, 2 and 3, one can see graphical representations of the joint possibility distributions J_γ for intervals $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$ for $\gamma = 1, 0.5, 0$, respectively. The gray regions represent J_γ, $\gamma = 1, 0.5, 0$, i.e., the set of all pairs (x_1, x_2), with $x_1 \in A$ and $x_2 \in B$, such that $(x_1, x_2) \in J_\gamma$. Note that the gray region decreases with respect to γ which means intuitively that the smaller γ the higher the interactivity between A and B.

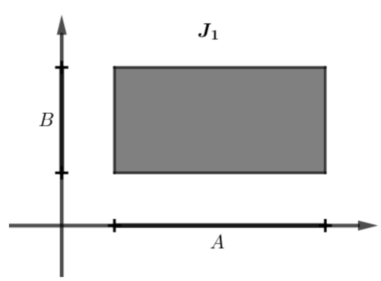

Figure 1: Graphical representation of the joint possibility distributions J_1 of the intervals A and B.

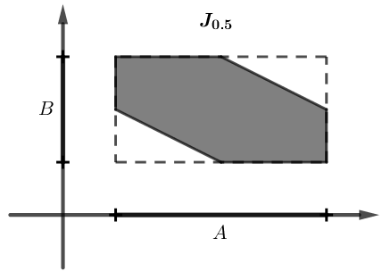

Figure 2: Graphical representation of the joint possibility distributions $J_{0.5}$ of the intervals A and B.

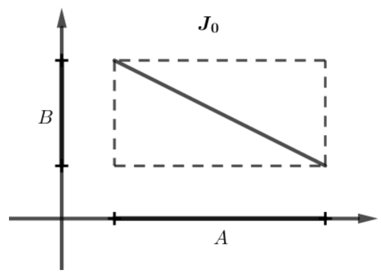

Figure 3: Graphical representation of the joint possibility distributions J_0 of the intervals A and B.

As we can observe in Figure 1, for $\gamma = 1$, the distribution J_1 is equivalent to the Cartesian product $A \times B$. In this case, from the perspective of fuzzy set theory, the intervals are called non-interactive. For $\gamma = 0$ we obtain the smallest region. In this case the intervals A and B have the highest "level" of interactivity, which is similar to the case of completely correlation [8] (see Figure 3).

In [19], authors provided a general formula for the α-levels to the interactive sum of two fuzzy numbers based on JPD J_γ, for any value of $\gamma \in [0, 1]$. The interval case of this formula is described in the following proposition.

Proposition 2.3. [19] *Let $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$ be two intervals and $\gamma \in [0, 1]$. Consider $A^t = [\underline{a}^t, \overline{a}^t]$ and $B^t = [\underline{b}^t, \overline{b}^t]$ the translated versions of A and B by their midpoints $a = 0.5(\underline{a} + \overline{a})$ and $b = 0.5(\underline{b} + \overline{b})$, respectively. The interactive sum $C = A +_\gamma B$ for any $\gamma \in [0, 1]$ is given by $C = [\underline{c}, \overline{c}]$, where*

$$\underline{c} = \left(\underline{a}^t + \overline{b}^t - \gamma(\overline{b}^t - \underline{b}^t)\right) \wedge \left(\overline{a}^t + \underline{b}^t - \gamma(\overline{a}^t - \underline{a}^t)\right) \wedge \left(\gamma(\underline{a}^t + \underline{b}^t)\right) + \{a + b\},$$

and

$$\overline{c} = \left(\underline{a}^t + \overline{b}^t + \gamma(\overline{a}^t - \underline{a}^t)\right) \vee \left(\overline{a}^t + \underline{b}^t + \gamma(\overline{b}^t - \underline{b}^t)\right) \vee \left(\gamma(\overline{a}^t + \overline{b}^t)\right) + \{a + b\}.$$

However, as we shall see in the following theorem, the formula presented in Proposition 2.3 can be described in a reduce form.

Theorem 2.4. *Let $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$ be two intervals and $\gamma \in [0, 1]$. The interactive sum $C = A +_\gamma B$ for any $\gamma \in [0, 1]$ is given by $C = [\underline{c}, \overline{c}]$, where*

$$\underline{c} = \left(\underline{a} + \overline{b} - \gamma(\overline{b} - \underline{b})\right) \wedge \left(\overline{a} + \underline{b} - \gamma(\overline{a} - \underline{a})\right),$$

and

$$\overline{c} = \left(\underline{a} + \overline{b} + \gamma(\overline{a} - \underline{a})\right) \vee \left(\overline{a} + \underline{b} + \gamma(\overline{b} - \underline{b})\right).$$

Moreover, one of the following cases occurs:

$$\underline{c} = \underline{a} + \overline{b} - \gamma(\overline{b} - \underline{b}) \text{ and } \overline{c} = \overline{a} + \underline{b} + \gamma(\overline{b} - \underline{b}),$$

or

$$\underline{c} = \overline{a} + \underline{b} - \gamma(\overline{a} - \underline{a}) \text{ and } \overline{c} = \underline{a} + \overline{b} + \gamma(\overline{a} - \underline{a}).$$

Proof. If $A^t = [\underline{a}^t, \overline{a}^t]$ and $B^t = [\underline{b}^t, \overline{b}^t]$ be the translated versions of A and B by their midpoints $a = 0.5(\underline{a} + \overline{a})$ and $b = 0.5(\underline{b} + \overline{b})$, then

$$\underline{a}^t = \underline{a} - a = -(\overline{a} - a) = -\overline{a}^t,$$

and

$$\underline{b}^t = \underline{b} - b = -(\overline{b} - b) = -\overline{b}^t.$$

Consider the formulas of $\underline{c}$ and $\overline{c}$ given in Proposition 2.3. Suppose that the minimum in the right side occurs at

$$\underline{a}^t + \overline{b}^t - \gamma(\overline{b}^t - \underline{b}^t).$$

Thus, we have

$$\begin{aligned} \underline{a}^t + \overline{b}^t - \gamma(\overline{b}^t - \underline{b}^t) &\leq \overline{a}^t + \underline{b}^t - \gamma(\overline{a}^t - \underline{a}^t) \\ -\underline{a}^t - \overline{b}^t + \gamma(\overline{b}^t - \underline{b}^t) &\geq -\overline{a}^t - \underline{b}^t + \gamma(\overline{a}^t - \underline{a}^t) \\ \Rightarrow \overline{a}^t + \underline{b}^t + \gamma(\overline{b}^t - \underline{b}^t) &\geq \underline{a}^t + \overline{b}^t + \gamma(\overline{a}^t - \underline{a}^t). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \underline{a}^t + \overline{b}^t - \gamma(\overline{b}^t - \underline{b}^t) &\leq \gamma(\underline{a}^t + \underline{b}^t) \\ \Rightarrow -\underline{a}^t - \overline{b}^t + \gamma(\overline{b}^t - \underline{b}^t) &\geq \gamma(-\underline{a}^t - \underline{b}^t) \\ \Rightarrow \overline{a}^t + \underline{b}^t + \gamma(\overline{b}^t - \underline{b}^t) &\geq \gamma(\overline{a}^t - \overline{b}^t). \end{aligned}$$

These inequalities above ensure that

$$\underline{c} = \left(\underline{a}^t + \overline{b}^t - \gamma(\overline{b}^t - \underline{b}^t)\right) + \{a + b\},$$

and

$$\overline{c} = \left(\overline{a}^t + \underline{b}^t + \gamma(\overline{b}^t - \underline{b}^t)\right) + \{a + b\}.$$

Since $\overline{a}^t - \underline{a}^t = \overline{a} - \underline{a}$, $\overline{b}^t - \underline{b}^t = \overline{b} - \underline{b}$, $\underline{a}^t + a = \underline{a}$, $\overline{a}^t + a = \overline{a}$, $\underline{b}^t + b = \underline{b}$, and $\overline{b}^t + b = \overline{b}$, we conclude

$$\underline{c} = \underline{a} + \overline{b} - \gamma(\overline{b} - \underline{b}) \text{ and } \overline{c} = \overline{a} + \underline{b} + \gamma(\overline{b} - \underline{b}).$$

Analogously, if the minimum in the right side occurs at

$$\overline{a}^t + \underline{b}^t - \gamma(\overline{a}^t - \underline{a}^t),$$

then we have

$$\underline{c} = \overline{a} + \underline{b} - \gamma(\overline{a} - \underline{a}) \text{ and } \overline{c} = \underline{a} + \overline{b} + \gamma(\overline{a} - \underline{a}).$$

Since A^t and B^t are centered at the origin, we have $\underline{a}^t \leq 0 \leq \overline{a}^t$ and $\underline{b}^t \leq 0 \leq \overline{b}^t$. If $\underline{a}^t \leq \underline{b}^t = -\overline{b}^t$, then the following inequalities hold true:

$$\begin{aligned} 0 &\geq (1-\gamma)(\underline{a}^t + \overline{b}^t) \\ \Leftrightarrow 0 &\geq -\gamma(\underline{a}^t + \underline{b}^t) + \underline{a}^t + \overline{b}^t - \gamma(\overline{b}^t - \underline{b}^t) \\ \Leftrightarrow \gamma(\underline{a}^t + \underline{b}^t) &\geq \underline{a}^t + \overline{b}^t - \gamma(\overline{b}^t - \underline{b}^t). \end{aligned}$$

Now, if $\underline{a}^t > \underline{b}^t = -\overline{b}^t$, then the following inequalities are satisfied:

$$\begin{aligned} 0 &> (1-\gamma)(\underline{b}^t + \overline{a}^t) \\ \Leftrightarrow 0 &> -\gamma(\underline{a}^t + \underline{b}^t) + \overline{a}^t + \underline{b}^t - \gamma(\overline{a}^t - \underline{a}^t) \\ \Leftrightarrow \gamma(\underline{a}^t + \underline{b}^t) &> \overline{a}^t + \underline{b}^t - \gamma(\overline{a}^t - \underline{a}^t). \end{aligned}$$

Hence,

$$\gamma(\underline{a}^t + \underline{b}^t) \not< \left(\underline{a}^t + \overline{b}^t - \gamma(\overline{b}^t - \underline{b}^t)\right) \wedge \left(\overline{a}^t + \underline{b}^t - \gamma(\overline{a}^t - \underline{a}^t)\right).$$

Similarly, we can show that

$$\left(\underline{a}^t + \overline{b}^t + \gamma(\overline{a}^t - \underline{a}^t)\right) \vee \left(\overline{a}^t + \underline{b}^t + \gamma(\overline{b}^t - \underline{b}^t)\right) \not< \gamma(\overline{a}^t + \overline{b}^t).$$

□

From Theorem 2.4, the diameter of the sum of the intervals A and B with $\gamma = 0$ is less or equal the diameters of A and B: $diam(A +_0 B) \leq \max\{diam(A), diam(B)\}$. This property contrasts with the standard addition since $diam(A + B) \geq \max\{diam(A), diam(B)\}$. In addition, for $\gamma = 1$, we have $A +_1 B = A + B$. This observation and the definition of gH-difference reveal that $(A +_1 B) -_{gH} A = B$. The next corollary establishes this connection with the gH-difference for $\gamma = 0$.

Corollary 2.5. *Let $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$ be two intervals. One of the following cases occurs*

$$\begin{cases} (A +_0 B) -_{gH} A = B & , \text{ if } diam(A) \geq dim(B) \\ (A +_0 B) -_{gH} B = A & , \text{ if } diam(A) \leq dim(B) \end{cases} \tag{5}$$

Proof. Let $C = A +_0 B$. From Theorem 2.4, we have:

$$\underline{c} = \underline{a} + \overline{b} \text{ and } \overline{c} = \overline{a} + \underline{b} \quad \text{ or } \quad \underline{c} = \overline{a} + \underline{b} \text{ and } \overline{c} = \underline{a} + \overline{b}.$$

If the first case occurs, then

$$\underline{a} + \overline{b} \leq \overline{a} + \underline{b} \Rightarrow diam(B) \leq diam(A).$$

In this case, we have $C -_{gH} A = B$ since $A = C + (-B)$. Similarly, if the second case above is satisfied, then we have $C -_{gH} B = A$ since $B = C + (-A)$. □

The interactive difference between intervals is defined by $A -_\gamma B = A +_\gamma (-B) = [\underline{a}, \overline{a}] +_\gamma [-\overline{b}, -\underline{b}]$. The following corollary is an immediate consequence of Theorem 2.4.

Corollary 2.6. *Let $C = [\underline{c}, \overline{c}]$ and $B = [\underline{b}, \overline{b}]$ be two intervals and $\gamma \in [0, 1]$. The interactive diference $C -_\gamma B$ for any $\gamma \in [0, 1]$ is given by an interval $A = [\underline{a}, \overline{a}]$, where*

$$\underline{a} \quad = \quad \left(\underline{c} - \underline{b} - \gamma(\overline{b} - \underline{b})\right) \wedge \left(\overline{c} - \overline{b} - \gamma(\overline{c} - \underline{c})\right),$$

and

$$\overline{a} \quad = \quad \left(\underline{c} - \underline{b} + \gamma(\overline{c} - \underline{c})\right) \vee \left(\overline{c} - \overline{b} + \gamma(\overline{b} - \underline{b})\right).$$

Moreover, one of the following cases occurs:

$$\underline{a} = \underline{c} - \underline{b} - \gamma(\overline{b} - \underline{b}) \text{ and } \overline{a} = \overline{c} - \overline{b} + \gamma(\overline{b} - \underline{b}),$$

or

$$\underline{a} = \overline{c} - \overline{b} - \gamma(\overline{c} - \underline{c}) \text{ and } \overline{a} = \underline{c} - \underline{b} + \gamma(\overline{c} - \underline{c}).$$

From Corollary 2.6, we can observe that the interactive difference $A = C -_\gamma B$ coincides to the gH-differece of the intervals C and B whenever $\gamma = 0$.

The following example shows an interactive addition and subtraction of two intervals using the formula provided by Theorem 2.4 and Corollary 2.6.

Example 2.7. *Let $A = [1, 2.5]$ and $B = [2, 4]$ be intervals. For $\gamma \in [0, 1]$, the intervals $C = A +_\gamma B$ and $D = A -_\gamma B$ are given by*

$$\begin{aligned} C &= [(1 + 4 - \gamma(4 - 2)) \wedge (2.5 + 2 - \gamma(2.5 - 1)), (1 + 4 + \gamma(2.5 - 1)) \vee (2.5 + 2 + \gamma(4 - 2)] \\ &= [4.5 - 1.5\gamma, 5 + 1.5\gamma]. \end{aligned}$$

and

$$\begin{aligned} D &= [(1 - 2 - \gamma(4 - 2)) \wedge (2.5 - 4 - \gamma(2.5 - 1)), (1 - 2 + \gamma(2.5 - 1)) \vee (2.5 - 4 + \gamma(4 - 2)] \\ &= [-1.5 - 1.5\gamma, -1 + 1.5\gamma]. \end{aligned}$$

Hence,

$$\begin{array}{ll} A +_0 B = [4.5, 5], & A -_0 B = [-1.5, -1], \\ A +_{0.5} B = [3.75, 5.75], & A -_{0.5} B = [-2.25, -0.25], \\ A +_1 B = [3, 6.5], & A -_1 B = [-3, -0.5]. \end{array}$$

Note that for $\gamma = 1$, we obtain the usual addition and subtraction on intervals.

For more properties and details on the arithmetic based on J_γ, the reader can refer to [13].

The definition of interactive subtraction gives raise to the interactive derivative for interval-valued functions via J_γ [5].

Definition 2.8. [5] *Let v be an interval-valued function defined on (a, b). The function v is J_γ-differentiable at $t_0 \in (a, b)$ if there exists an interval $v'_\gamma(t_0)$ satisfying the limit*

$$v'_\gamma(t_0) = \lim_{h \to 0} \frac{v(t_0 + h) -_{\gamma(h)} v(t_0)}{h}.$$

The interval $v'_\gamma(t_0)$ is called J_γ-derivative of v at t_0.

It is worth observing that $\gamma(h)$ depends on h, since for each h we may have a different interactivity between $v(t_0)$ and $v(t_0 + h)$. Consequently, $\gamma(h)$ can be view as function in the form of $\gamma : (-\delta, \delta) \to [0, 1]$, $\delta > 0$. The following theorem provides a formula to compute the J_γ-derivative of interval-valued function if the function $\gamma : (-\delta, \delta) \to [0, 1]$ satisfies some properties.

Theorem 2.9. *Let $v(t) = [\underline{v}(t), \overline{v}(t)]$ be a continuous interval-valued function defined on (a, b). Suppose that v is J_γ-differentiable at some $t \in (a, b)$ and that the right and left derivatives of $\underline{v}$ and $\overline{v}$ at t exist, which are denoted by $\underline{v}'_+(t)$, $\underline{v}'_-(t)$, $\overline{v}'_+(t)$, and $\overline{v}'_+(t)$, respectively. Let $\delta > 0$ and $\gamma : (-\delta, \delta) \to [0, 1]$ be such that*

$$v'_\gamma(t) = \lim_{h \to 0} \frac{v(t + h) -_{\gamma(h)} v(t)}{h}.$$

If $\gamma(0) = 0$ and $\gamma'(0)$ exists, then the functions $\underline{v}$ and $\overline{v}$ are differentiable at t and

$$v'_\gamma(t) = [\underline{v}'_\pm(t) \wedge \overline{v}'_\pm(t) - \gamma'(0)(\overline{v}(t) - \underline{v}(t)), \underline{v}'_\pm(t) \vee \overline{v}'_\pm(t) + \gamma'(0)(\overline{v}(t) - \underline{v}(t))].$$

Proof. First of all, since $\gamma(0) = 0$ and v is continuous, we have

$$\lim_{h \to 0} \frac{\gamma(h)(\overline{v}(t) - \underline{v}(t))}{h} = \gamma'(0)(\overline{v}(t) - \underline{v}(t)),$$

and

$$\lim_{h \to 0} \frac{\gamma(h)(\overline{v}(t + h) - \underline{v}(t + h))}{h} = \gamma'(0)(\overline{v}(t) - \underline{v}(t)).$$

Let $[\underline{c}_h, \overline{c}_h] = v(t + h) -_{\gamma(h)} v(t)$. Suppose there exists a decreasing sequence (h_n) converging to zero such that

$$\begin{cases} \underline{c}_{h_n} = \underline{v}(t + h_n) - \underline{v}(t) - \gamma(h_n)(\overline{v}(t) - \underline{v}(t)) \\ \overline{c}_{h_n} = \overline{v}(t + h_n) - \overline{v}(t) + \gamma(h_n)(\overline{v}(t) - \underline{v}(t)) \end{cases} \tag{6}$$

This implies that

$$\lim_{n \to \infty} \frac{\underline{c}_{h_n}}{h} = \underline{v}'_+(t) - \gamma'(0)(\overline{v}(t) - \underline{v}(t)),$$

and

$$\overline{c} = \lim_{h \to 0} \frac{\overline{c}_h}{h} = \overline{v}'_+(t) + \gamma'(0)(\overline{v}(t) - \underline{v}(t)).$$

Note that if the sequence (h_n) is increasing then the above limits are given in terms of the left derivative of $\underline{v}$ and $\overline{v}$ at t.

Similarly, if there exists a monotonic sequence (h_n) converging to zero such that

$$\begin{cases} \underline{c}_{h_n} = \overline{v}(t + h_n) - \overline{v}(t) - \gamma(h_n)(\overline{v}(t + h_n) - \underline{v}(t + h_n)) \\ \overline{c}_{h_n} = \underline{v}(t + h_n) - \underline{v}(t) + \gamma(h_n)(\overline{v}(t + h_n) - \underline{v}(t + h_n)) \end{cases} \tag{7}$$

then

$$\lim_{n \to \infty} \frac{\underline{c}_{h_n}}{h_n} = \overline{v}'_\pm(t) - \gamma'(0)(\overline{v}(t) - \underline{v}(t)),$$

and

$$\lim_{n \to \infty} \frac{\overline{c}_{h_n}}{h_n} = \underline{v}'_\pm(t) + \gamma'(0)(\overline{v}(t) - \underline{v}(t)).$$

$\square$

If we additionally assume that $\gamma'(0) = 0$ in Theorem 2.9, then the J_γ-derivative of f at t coincides with gH-derivative of f at t whenever it exists. Note that, from Corollary 2.6, the gH-difference coincides if the interactive derivative whenever $\gamma = 0$. Thus, the notion of J_γ-differentiability for interval-valued functions extends that of gH-differentiability which arises whenever we consider the trivial choice $\gamma(h) = 0$ for all h.

Consider y such that

$$y = \frac{x(t+h) -_{\gamma(h)} x(t)}{h} \Leftrightarrow yh = x(t+h) -_{\gamma(h)} x(t).$$

Using the hypotheses of continuity of x and γ with $\gamma(0) = 0$, one can prove that, for some h sufficiently close to zero, $x(t+h)$ can be approximated an interactive sum of $x(t)$ and yh, i.e.,

$$x(t+h) \approx x(t) +_{\tilde{\gamma}} yh,$$

where $\tilde{\gamma} \in [0,1]$. In this case, y is an approximation for the interactive derivative x'_γ, if it exists. The values that $\tilde{\gamma}$ assume can be adjusted or estimated using a dataset. For example, Sussner *et al.* [19] considered a chemical degradation problem, where the "level" (i.e., the value $\tilde{\gamma} \in [0,1]$) of interactivity has been adjusted by a dataset. In the next section we discuss the consistence between the analytical solution via gH-derivative and numerical solution via J_γ-interactive arithmetic. To this end, we consider interval Euler's method, i.e., the arithmetic operations consist of interactive operations on intervals.

3 Numerical approximation to the IVP with interval initial value condition

Consider the following interval IVP

$$\begin{cases} x'_{gH}(t) = f(t,x) \\ x(t_0) = x_0 = [\underline{x}_0, \overline{x}_0] \end{cases} \tag{8}$$

where $f(t,x)$ is an interval of the form $[\underline{f}(t,\underline{x},\overline{x}), \overline{f}(t,\underline{x},\overline{x})]$. For the sake of simplicity, we simply denote the function f by $[\underline{f}, \overline{f}]$.

The purpose here is to provide a numerical approximation for the analytical solution of (8). In this context, one can defined the Euler-type method as follows

$$x_{k+1} = x_k +_\gamma hf(t, x_k), \tag{9}$$

for $k = 0, 1, \ldots$.

Note that for $\gamma = 1$ the addition $+_1$ corresponds to the usual addition of intervals and the difference $x_{k+1} -_{gH} x_k = hf(t, x_k)$ since the case (I) of (4) occurs. In contrast, for $\gamma = 0$, from Corollary 2.5, we may also have $x_{k+1} -_{gH} x_k = hf(t, x_k)$, where the case (II) of (4) is satisfied. In both cases, $\frac{x_{k+1} -_{gH} x_k}{h} = f(t, x_k)$ approaches $x'_{gH}(t_k)$ whenever x_k approaches $x(t_k)$. This indicates a certain consistence between the solution of (8) with the recursive formula . Below, we provide the proof of Theorem 3.1 that clarifies this connection.

The analytical solution of (8) can be obtained from a system of classical differential equations [2]. If the gH-derivative is the type I, then we obtain

$$\begin{cases} [\underline{x}', \overline{x}'] = [\underline{f}, \overline{f}] \\ x(t_0) = [\underline{x}_0, \overline{x}_0] \end{cases} \Rightarrow \begin{cases} \underline{x}' = \underline{f} \\ \overline{x}' = \overline{f} \\ x(t_0) = [\underline{x}_0, \overline{x}_0] \end{cases}. \tag{10}$$

If the gH-derivative is the type II, then we obtain

$$\begin{cases} [\underline{x}', \overline{x}'] = [\overline{f}, \underline{f}] \\ x(t_0) = [\underline{x}_0, \overline{x}_0] \end{cases} \Rightarrow \begin{cases} \underline{x}' = \overline{f} \\ \overline{x}' = \underline{f} \\ x(t_0) = [\underline{x}_0, \overline{x}_0] \end{cases}. \tag{11}$$

Consider a general value of γ and let us compute the right side of Equation (3). We obtain

$$\begin{aligned} [\underline{x}_{k+1}, \overline{x}_{k+1}] &= [\underline{x}_k, \overline{x}_k] +_\gamma [h\underline{f}_k, h\overline{f}_k] \\ &= [(\underline{x}_k + h\overline{f}_k - \gamma h(\overline{f}_k - \underline{f}_k)) \wedge (\overline{x}_k + h\underline{f}_k - \gamma(\overline{x}_k - \underline{x}_k))\ , \\ &\quad (\underline{x}_k + h\overline{f}_k + \gamma(\overline{x}_k - \underline{x}_k)) \vee (\overline{x}_k + h\underline{f}_k + \gamma h(\overline{f}_k - \underline{f}_k))]. \end{aligned} \tag{12}$$

The task to determinate the left and right sides of the interval (12), depends on the values of $\underline{f}_k$ and $\overline{f}_k$ and $\gamma \in [0,1]$. In particular, for $\gamma = 1$, we obtain:

$$[\underline{x}_{k+1}, \overline{x}_{k+1}] \overset{\text{for } \gamma=1}{=} [\underline{x}_k + h\underline{f}_k, \overline{x}_k + h\overline{f}_k]. \tag{13}$$

We can observe that numerical solution given by $\gamma = 1$ coincides to the numerical solution produced by the classical Euler method for the IVP at right side in Equation (10). Therefore, the intervals x_k recursive formula (3) with $\gamma = 1$ approximates the analytical solution $x(t_k)$ of the interval IVP under gH-derivative of type I.

For $\gamma = 0$, we obtain:

$$[\underline{x}_{k+1}, \overline{x}_{k+1}] \overset{\text{for } \gamma=0}{=} [(\underline{x}_k + h\overline{f}_k) \wedge (\overline{x}_k + h\underline{f}_k), (\underline{x}_k + h\overline{f}_k) \vee (\overline{x}_k + h\underline{f}_k)].$$

Analysing Equation (14), we can observe that only one of the following cases occurs:

(C1) $[\underline{x}_{k+1}, \overline{x}_{k+1}] = [\underline{x}_k + h\overline{f}_k, \overline{x}_k + h\underline{f}_k]$ if $diam(x_k) \geq diam(hf_k)$;

(C2) $[\underline{x}_{k+1}, \overline{x}_{k+1}] = [\overline{x}_k + h\underline{f}_k, \underline{x}_k + h\overline{f}_k]$ if $diam(x_k) \leq diam(hf_k)$.

Note that the formulas for $\underline{x}_{k+1}$ and $\overline{x}_{k+1}$ in (C1) correspond to that obtained using the classical Euler method for the IVP at right side in Equation (11). Moreover, the case (C1) is satisfied if, and only if, $\underline{x}_k + h\overline{f}_k \leq \overline{x}_k + h\underline{f}_k$, which is the condition to guarantee proper intervals in numerical solutions produced by applying the classical Euler method to the IVP at right side in Equation (11).

Theorem 3.1. *Let x be a solution of (8) in the sense of gH-differentiability.*

- *Euler's method, with the J_1-interactive arithmetic, approximates the analytical solution under gH-derivative of type I;*
- *Euler's method, with the J_0-interactive arithmetic, approximates the analytical solution under gH-derivative of type II whenever (C1) is satisfied.*

This means that the numerical solution via interactive arithmetic, with $\gamma = 0$, is compatible with the analytical solution of the second form of gH-derivative, corroborating the result stated in Theorem 3.1.

In Table 1, we present a brief summary of the numerical approximations provided in this article. For the sake of simplicity, in Table 1, we assume that (C1) is satisfied for $\gamma = 0$.

Table 1: Summary of numerical approximations between Euler's method with interactive arithmetic and analytical solutions via gH-derivative

Euler's method	Value of γ	Approximated solution
$x_k +_\gamma hf(t, x_k)$	$\gamma = 1$	(I)-gH-derivative
$x_k +_\gamma hf(t, x_k)$	$\gamma = 0$	(II)-gH-derivative

In the next section we illustrate the result of Theorem 3.1 in the decay Malthusian model under the gH-differentiability.

4 The interval malthusian model

Consider the interval decay Malthusian model, with initial condition given by

$$\begin{cases} x'_{gH}(t) = -\lambda x(t), & \lambda > 0, \\ x(0) = [\underline{x}_0, \overline{x}_0]. \end{cases} \tag{14}$$

For this interval initial value problem we consider solutions that are (I)-gH-differentiable and (II)-gH-differentiable [15].

From Euler method with interactive arithmetic via J_γ on intervals, which is given by (12), we have

$$\begin{aligned} [\underline{x}_{k+1}, \overline{x}_{k+1}] \quad = \quad & [(\underline{x}_k - h\lambda\underline{x}_k - \gamma h(-\lambda\underline{x}_k + \lambda\overline{x}_k)) \wedge (\overline{x}_k - h\lambda\overline{x}_k - \gamma(-\underline{x}_k + \overline{x}_k)) \ , \\ & (\underline{x}_k - h\lambda\underline{x}_k + \gamma(\overline{x}_k - \underline{x}_k)) \vee (\overline{x}_k - h\lambda\overline{x}_k + \gamma h(-\lambda\underline{x}_k + \lambda\overline{x}_k))]. \end{aligned}$$

Note that $(\underline{x}_k - h\lambda\underline{x}_k - \gamma h(-\lambda\underline{x}_k + \lambda\overline{x}_k)) \leq (\overline{x}_k - h\lambda\overline{x}_k - \gamma(-\underline{x}_k + \overline{x}_k))$ if, and only if $\overline{x}_k((1-\gamma)(1-h\lambda)) \leq \overline{x}_k((1-\gamma)(1-h\lambda))$. For sufficient small $h > 0$, we have $(1-h\lambda) \geq 0$ and, therefore, this inequality holds true. This also guarantees that the case (C1) is satisfied for $\gamma = 0$.

Thus, for sufficient small $h > 0$, it follows that:

$$[\underline{x}_{k+1}, \overline{x}_{k+1}] = [\underline{x}_k - h\lambda\underline{x}_k - \gamma h(-\lambda\underline{x}_k + \lambda\overline{x}_k)\ , \overline{x}_k - h\lambda\overline{x}_k + \gamma h(-\lambda\underline{x}_k + \lambda\overline{x}_k)].$$

It is worth noting that the mapping $\gamma \mapsto \underline{x}_k - h\lambda\underline{x}_k - \gamma h(-\lambda\underline{x}_k + \lambda\overline{x}_k)$ is decreasing whereas the mapping $\gamma \mapsto \overline{x}_k - h\lambda\overline{x}_k + \gamma h(-\lambda\underline{x}_k + \lambda\overline{x}_k)$ is increasing. As an immediate consequence, the length of the numerical solution given by (15) is decreasing (and continuous) with respect to γ. Thus, if any data is available for the corresponding phenomenon, then one can adjust the parameter γ by fitting the numerical solution to data [19].

4.1 (I)-gH-derivative

Considering the (I)-gH-differentiability in the interval Malthusian model, we obtain the following associated system of classical differential equation:

$$\begin{cases} \underline{x}'(t) = -\lambda\overline{x}(t), \quad \lambda > 0 \\ \overline{x}'(t) = -\lambda\underline{x}(t) \\ \underline{x}(0) = \underline{x}_0 \\ \overline{x}(0) = \overline{x}_0. \end{cases} \tag{15}$$

For $\gamma = 1$, the recursive formula (15) becomes

$$[\underline{x}_{k+1}, \overline{x}_{k+1}] \quad = \quad [\underline{x}_k - h\lambda\overline{x}_k\ , \overline{x}_k - h\lambda\underline{x}_k]. \tag{16}$$

One can observe that Equation (16) is equivalent to apply the classical Euler's method to the IVP (15). This means that the numerical solution via interactive arithmetic, for $\gamma = 1$, is compatible with the analytical solution of the first form of gH-derivative, corroborating the result stated in Theorem 3.1.

4.2 (II)-gH-derivative

For gH-derivative of type II, we have the following system

$$\begin{cases} \underline{x}'(t) = -\lambda\underline{x}(t), \quad \lambda > 0 \\ \overline{x}'(t) = -\lambda\overline{x}(t), \\ \underline{x}(0) = \underline{x}_0 \\ \overline{x}(0) = \overline{x}_0 \end{cases} \tag{17}$$

Setting $\gamma = 0$ in Equation (15), we obtain

$$[\underline{x}_{k+1}, \overline{x}_{k+1}] \quad = \quad [\underline{x}_k - h\lambda\underline{x}_k, \overline{x}_k - h\lambda\overline{x}_k]. \tag{18}$$

This coincides with the classical Euler's method for (17).

5 Final remarks

In this article we study numerical methods to approximate analytical solutions for Initial Value Problems, with interval initial conditions. Here we focus on Euler's method, whose classical arithmetic operations are adapted for interval values. To this end, we consider an arithmetic employed by Esmi *et al.* [13], which is obtained from the notion of sup-J extension principle, where J, in this work, refers to a parameterized family of joint possibility distributions J_γ between intervals.

We showed that once considered the interactive arithmetic via J_γ, where $\gamma \in [0,1]$, the obtained numerical solution approximates the analytical solution referring to the approach via gH-derivative. That is, if γ goes to 1, then the numerical solution via Euler's method approximates the analytical solution of gH-derivative of type I (*i.e*, Hukuhara derivative). On the other hand, if γ goes to 0, then the numerical solution approximates the analytical solution of gH-derivative of type II (see TABLE 1). This result is presented in Theorem 3.1.

For a better illustration of the results, we provide more detailed calculations for a decay Malthusian model, which analytical solution is given by an exponential function.

It is important to observe that we prove this connection for an interval initial value problem. As a future work, this result will be extended for a general fuzzy initial value problem. To this end, we believe that formulas (13) and (14) can be applied in each α-level under certain conditions. Also, we intend to study an error analysis for such approximations. In addition, we also intend to apply interactive arithmetic and derivative in fuzzy adaptive systems [16].

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