Iranian Journal of Fuzzy Systems

Volume 19, Number 6, (2022), pp. 29-49

Original paper

A new method to solve linear programming problems in the environment of picture fuzzy sets

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Abstract

Picture fuzzy set is characterized by neutral membership function along with the membership and non-membership functions, and is, therefore, more general than the intuitionistic fuzzy set which is only characterized by membership and non-membership functions. In this paper, first, we are going to point out a drawback and try to fix it by the existing trapezoidal picture fuzzy number. Furthermore, we define an LR flat picture fuzzy number, which is a generalization of trapezoidal picture fuzzy numbers. We also discuss a linear programming model with LR flat picture fuzzy numbers as parameters and variables and present a method to solve these type of problems using a generalized ranking function.

Keywords: Picture fuzzy linear programming problem, LR flat picture fuzzy numbers, picture fuzzy numbers, ranking function.

1 Introduction

Fuzzy set theory introduced by Zaheh [20] is a generalization of the crisp set theory to deal with the uncertain and vague information. A fuzzy set is characterized by a membership function which assigns membership grade to each element of the universal set. Fuzzy set is widely used in decision making and can be used to handle the information where a crisp set is not applicable. Bellman and Zadeh [8] introduced the concept of decision making in a fuzzy environment. Tanaka et al. [19] initiated the novel concept of linear programming in a fuzzy environment. Zimmerman [22] presented fuzzy linear programming with several objective functions. Lotfi et al. [14] considered a method for solving a full fuzzy linear programming using lexicography method and fuzzy approximate solution. Kaur and Kumar [11] presented Mehar's method for solving fully fuzzy linear programming problems with LR fuzzy parameters. Pérez-Cañedo et al. [16] proposed a revised version of a lexicographical-based method for solving fully fuzzy linear programming problems with inequality constraints. To extend or explore the fuzzy set, Atanassov [7] introduced the concept of intuitionistic fuzzy set (IFS), which is characterized by non-membership function along with membership function. IFS is more general than a fuzzy set which can deal with uncertain information in a better way than a fuzzy set does. Angelov [6] introduced an optimization technique in an intuitionistic fuzzy environment. Singh and Yadav [18] introduced unrestricted LR-type intuitionistic fuzzy mathematical programming problems. Pérez-Cañedo and Concepción-Morales [15] studied an LR-type fully intuitionistic fuzzy linear programming with inequality constraints and gave solutions with unique optimal values. Akram et al. [2] introduced Pythagorean fuzzy linear programming problems with equality constraints. For further terminologies and notions, the reader may concern [3, 4, 5, 12].

In many real life problems, one may wish to suggest neutral grades along with membership and non-membership grades. This kind of ambiguous information cannot be handled through fuzzy set and IFS. To overcome this difficulty, the novel concept of picture fuzzy set (PFS) was first initiated by Cuong [9]. In PFS, there is a neutral grade along with membership and non-membership grades for each element of the universal set. This novel concept of PFS can

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Received: June 2021; Revised: August 2022; Accepted: August 2022.

https://doi.org/10.22111/IJFS.2022.7208

handle the vague information in a best way when someone suggests the grades as: yes to some extent, abstain to some extent and no to some extent. Picture fuzzy set got the attention of many researchers globally. Here is a glimpse of the recent work in the picture fuzzy set. Qiyas et al. [17] studied triangular picture fuzzy linguistic induced ordered weighted aggregation operators and their application on decision making problems. Recently, Akram et al. [1] presented an optimization study based on the Dijkstra algorithm for a network with trapezoidal picture fuzzy numbers. In the literature, there are many techniques which deal with linear programming problems using uncertain fuzzy set or intuitionistic fuzzy set. In all the existing methods, there are limitations in the sense that either they have variables or parameters as fuzzy numbers or intuitionistic fuzzy numbers. Even if they have all the parameters and variables as fuzzy numbers or intuitionistic fuzzy numbers, yet there is a limitation of not having a neutral membership function along with positive membership function and negative membership function. This limitation urged us to develop a new method to deal with such uncertain information. In this paper, we point out a drawback in the definition of existing trapezoidal picture fuzzy numbers (TrPFNs) and triangular picture fuzzy numbers (TPFNs) and fix it. We define a more general type of picture fuzzy numbers (PFNs) called LR flat PFNs. We also propose a solution technique for the linear programming problems using LR flat PFNs as variables and parameters and call these problems as fully picture fuzzy linear programming problems (FPFLPPs). This model deals better than fuzzy set and IFS in uncertain situations.
The article is arranged as follows: Section 2 explains some basic concepts regarding TrPFNs and LR flat PFNs, in Section 3, the linear programming problems with LR flat PFNs is discussed, in section 4, we conclude the manuscript.

2 Basic definitions and concepts

In this section, we discuss some basic concepts regarding TrPFNs and LR flat PFNs.

2.1 Basic definitions associated with TrPFNs

Definition 2.1. [9] *A picture fuzzy set (PFS) X on a universe of discourse Z is an object of the form*

$$X = \{(z, \mu_X(z), \eta_X(z), \nu_X(z)) | z \in Z\},$$

where $\mu_X(z), \eta_X(z), \nu_X(z)$ are positive, neutral and negative membership functions, respectively, of element $z \in Z$ such that $\mu_X(z), \eta_X(z), \nu_X(z) \in [0,1]$ and $0 \leq \mu_X(z) + \eta_X(z) + \nu_X(z) \leq 1$, for every $z \in Z$. Further, $\Pi_X(z) = 1 - \mu_X(z) - \eta_X(z) - \nu_X(z)$ is called refusal degree of z to the set X.

Definition 2.2. *Let X be a PFS on Z, then its $\lambda-cut$, $\delta-cut$ and $\theta-cut$ are defined as $X_\lambda = \{z \in Z : \mu_X(z) \geq \lambda\}$, $X^\delta = \{z \in Z : \eta_X(z) \geq \delta\}$ and ${}^\theta X = \{z \in Z : \nu_X(z) \leq \theta\}$, $\forall \lambda, \delta, \theta \in [0,1]$ and $0 \leq \lambda + \delta + \theta \leq 1$.*

At certain situations, this is defined as:

Definition 2.3. *Let X be a PFS on Z, then its $\lambda-cut$,$\delta-cut$ and $\theta-cut$ are defined as $X_\lambda = \{z \in Z : \mu_X(z) \geq \lambda\}$, $X^\delta = \{z \in Z : \eta_X(z) \leq \delta\}$ and ${}^\theta X = \{z \in Z : \nu_X(z) \leq \theta\}$, $\forall \lambda, \delta, \theta \in [0,1]$ and $0 \leq \lambda + \delta + \theta \leq 1$.*

Now we review some concepts about PFNs. The definitions of PFNs, TrPFNs and TPFNs may be recalled from [1, 17].

Definition 2.4. [17] *A PFN $A = (([p_1, q, r, s_1], \alpha), ([p_2, q, r, s_2], \beta), ([p_3, q, r, s_3], \gamma))$ is of trapezoidal type if*

$$\mu_A(x) = \begin{cases} \dfrac{\alpha(x-p_1)}{q-p_1}, & p_1 \leq x < q, \\ \alpha, & q \leq x < r, \\ \dfrac{\alpha(s_1 - x)}{s_1 - r}, & r \leq x \leq s_1, \\ 0, & x < p_1 \text{ or } x > s_1, \end{cases} \qquad \eta_A(x) = \begin{cases} \dfrac{q - x + \beta(x-p_2)}{q-p_2}, & p_2 \leq x < q, \\ \beta, & q \leq x < r, \\ \dfrac{x - r + \beta(s_2 - x)}{s_2 - r}, & r \leq x \leq s_2, \\ 0, & x < p_2 \text{ or } x > s_2, \end{cases}$$

$$\nu_A(x) = \begin{cases} \dfrac{q - x + \gamma(x-p_3)}{q-p_3}, & p_3 \leq x < q, \\ \gamma, & q \leq x < r, \\ \dfrac{x - r + \beta(s_3 - x)}{s_3 - r}, & r \leq x \leq s_3, \\ 0, & x < p_3 \text{ or } x > s_3. \end{cases}$$

are its positive, neutral and negative membership functions, respectively, such that $\alpha, \beta, \gamma \in [0,1]$, and $0 \leq \alpha + \beta + \gamma \leq 1$.

But there is a major drawback in Definition 2.4 for certain points of this type of numbers, the sum of its positive, neutral and negative memberships exceeds 1.
For example, if we consider

$$A = (([-1, 2, 4, 6], 0.5), ([-1, 2, 4, 6], 0.1), ([-1, 2, 4, 6], 0.2)),$$

and calculate the membership, neutral and non-membership degrees at $x = 0.5$, it comes out to be $0.25, 0.55$ and 0.6, respectively. Sum of these three degrees is 1.4 which is greater than 1. Similarly, there are infinitely many points at which this definition fails. This number is shown graphically in Figure 1.
We now modify Definition 2.4 in two possible ways which are given in Definition 2.5 and in Definition 2.6.

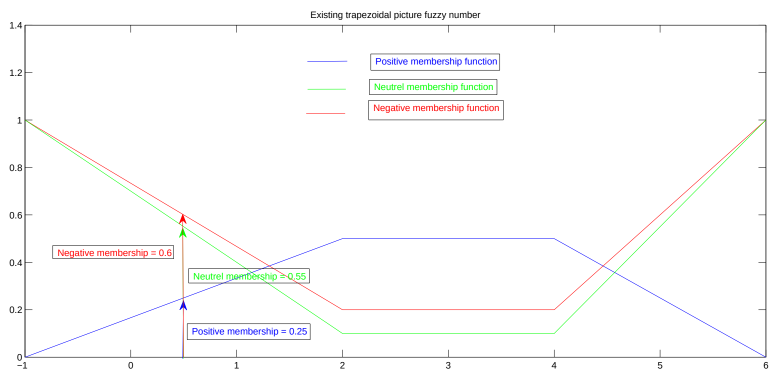

Figure 1: Graphical representation of the existing trapezoidal picture fuzzy number

Definition 2.5. *A simple TrPFN* $A = [(p_3, p_2, p_1, m, n, q_1, q_2, q_3); \alpha, \beta, \gamma]$ *is a PFS defined on* $\mathbb{R}$*, whose positive membership* (μ_A)*, neutral membership* (η_A) *and negative membership*(ν_A) *functions are, respectively, defined as*

$$\mu_A(x) = \begin{cases} \dfrac{(x-p_1)\alpha}{m-p_1}, & p_1 \le x \le m, \\ \alpha, & m \le x \le n, \\ \dfrac{(q_1-x)\alpha}{q_1-n}, & n \le x \le q_1, \\ 0, & otherwise, \end{cases} \qquad \eta_A(x) = \begin{cases} \dfrac{(x-p_2)\beta}{m-p_2}, & p_2 \le x \le m, \\ \beta, & m \le x \le n, \\ \dfrac{(q_2-x)\beta}{q_2-n}, & n \le x \le q_2, \\ 0, & otherwise, \end{cases}$$

$$\nu_A(x) = \begin{cases} \dfrac{m-x+\gamma(x-p_3)}{m-p_3}, & p_3 \le x \le m, \\ \gamma, & m \le x \le n, \\ \dfrac{x-n+\gamma(q_3-x)}{q_3-n}, & n \le x \le q_3, \\ 1, & otherwise, \end{cases}$$

where $p_3 \le p_2 \le p_1 \le m \le n \le q_1 \le q_2 \le q_3$. *The values* α, β and γ *describe the maximum degree of positive membership* (μ_A)*, maximum degree of neutral membership* (η_A) *and minimum degree of negative membership* (ν_A)*, respectively, such that* $\alpha, \beta, \gamma \in [0, 1]$*, and* $0 \le \alpha + \beta + \gamma \le 1$.

Definition 2.6. *A general TrPFN* $A = [(p_3, p_2, p_1, m, n, q_1, q_2, q_3); \alpha, \beta, \gamma]$ *is a PFS defined on* $\mathbb{R}$*, whose positive membership* (μ_A)*,neutral membership* (η_A) *and negative membership*(ν_A) *functions are, respectively, defined as*

$$\mu_A(x)=\begin{cases}\dfrac{(x-p_1)\alpha}{m-p_1}, & p_1\le x\le m,\\ \alpha, & m\le x\le n,\\ \dfrac{(q_1-x)\alpha}{q_1-n}, & n\le x\le q_1,\\ 0, & otherwise,\end{cases}\qquad \eta_A(x)=\begin{cases}\dfrac{(m-x)(\omega-\beta)+\beta(m-p_2)}{m-p_2}, & p_2\le x\le m,\\ \beta, & m\le x\le n,\\ \dfrac{(x-n)(\omega'-\beta)+\beta(q_2-n)}{q_2-n}, & n\le x\le q_2,\\ 0, & otherwise,\end{cases}$$

$$\nu_A(x)=\begin{cases}\dfrac{m-x+\gamma(x-p_3)}{m-p_3}, & p_3\le x\le m,\\ \gamma, & m\le x\le n,\\ \dfrac{x-n+\gamma(q_3-x)}{q_3-n}, & n\le x\le q_3,\\ 1, & otherwise,\end{cases}$$

where $\omega=\min\{1-(\alpha+\gamma),1-\nu_A(p_2)\}$, $\omega'=\min\{1-(\alpha+\gamma),1-\nu_A(q_2)\}$ *and* $p_3\le p_2\le p_1\le m\le n\le q_1\le q_2\le q_3$. *The values* α, β and γ *describe the maximum degree of positive membership* (μ_A), *maximum/minimum degree of neutral membership* (η_A) *and minimum degree of negative membership* (ν_A), *respectively, such that* $\alpha,\beta,\gamma\in[0,1]$, *and* $0\le\alpha+\beta+\gamma\le 1$.

Example 2.7. *We consider a simple TrPFN* $A=[(2,4,6,10,12,16,18,20);0.5,0.2,0.3]$ *as in Definition 2.5. The positive, neutral and negative membership functions of simple TrPFN are shown graphically in Figure 2. It can be seen from the figure that this modified Definition 2.5 remedies in the best way to the mentioned drawback of Definition 2.4.*

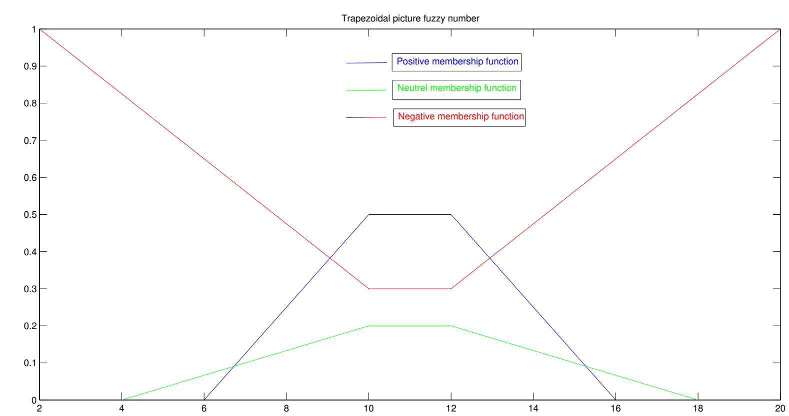

Figure 2: Graphical representation of simple trapezoidal picture fuzzy number as in Definition 2.5

Example 2.8. *We consider a general TrPFN* $A=[(2,4,6,10,12,16,18,20);0.3,0.04,0.3]$ *as in Definition 2.6. The positive, neutral and negative membership functions of general TrPFN are shown graphically in Figure 3. It is easy to see that this modified Definition 2.6 can also work as remedy to the mentioned drawback of Definition 2.4.*

It totally depends upon the decision maker which of the both, the simple TrPFN (Definition 2.5) or general TrPFN (Definition 2.6) is used in a problem. From now onwards, whenever we shall refer to TrPFN, we mean simple TrPFN (Definition 2.5), unless otherwise stated. Similar operations hold for general TrPFN (Definition 2.6) and we shall not discuss them separately.

Remark 2.9. *If we set* $m=n$ *in Definition 2.5, then it becomes TPFN.*

Definition 2.10. *A TrPFN* $A=[(p_3,p_2,p_1,m,n,q_1,q_2,q_3);\alpha,\beta,\gamma]$ *is non-negative(respectively non-positive), denoted as* $A\ge 0$*(respectively* $A\le 0$*), if* $p_3\ge 0$*(respectively* $q_3\le 0$*) and A is unrestricted if* p_3 *is a real number.*

Definition 2.11. *Let* $A_1=[(p_3,p_2,p_1,m,n,q_1,q_2,q_3);\alpha,\beta,\gamma]$ *and* $A_2=[(p'_3,p'_2,p'_1,m',n',q'_1,q'_2,q'_3);\alpha',\beta',\gamma']$ *be two TrPFNs and* λ *be a real number, then:*

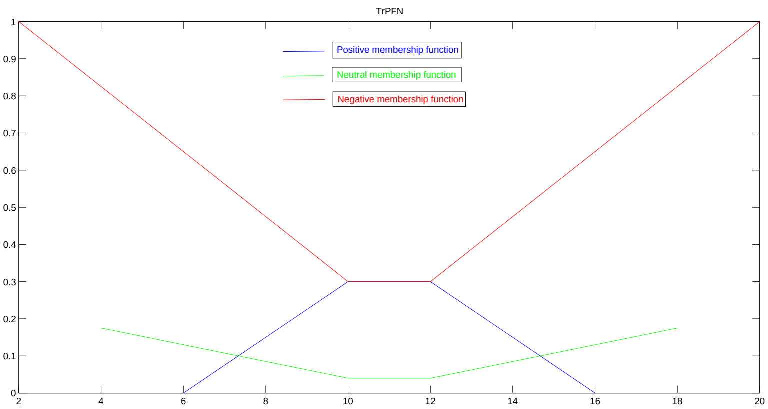

Figure 3: Graphical representation of general trapezoidal picture fuzzy number as in Definition 2.6

1. $A_1 \oplus A_2 = [(p_3+p_3', p_2+p_2', p_1+p_1', m+m', n+n', q_1+q_1', q_2+q_2', q_3+q_3'); \alpha+\alpha'-\alpha\alpha', \beta\beta', \gamma\gamma']$,
2. $-A_1 = [(-q_3, -q_2, -q_1, -n, -m, -p_1, -p_2, -p_3); \alpha, \beta, \gamma]$,
3. $A_1 \ominus A_2 = A_1 \oplus (-A_2) = [(p_3-q_3', p_2-q_2', p_1-q_1', m-n', n-m', q_1-p_1', q_2-p_2', q_3-p_3'); \alpha+\alpha'-\alpha\alpha', \beta\beta', \gamma\gamma']$,
4. $\lambda A_1 = \begin{cases} [(\lambda p_3, \lambda p_2, \lambda p_1, \lambda m, \lambda n, \lambda q_1, \lambda q_2, \lambda q_3); \alpha^\lambda, \beta^\lambda, 1-(1-\gamma)^\lambda], & \lambda \geq 0, \\ [(\lambda q_3, \lambda q_2, \lambda q_1, \lambda n, \lambda m, \lambda p_1, \lambda p_2, \lambda p_3); \alpha^{-\lambda}, \beta^{-\lambda}, 1-(1-\gamma)^{-\lambda}], & \lambda < 0. \end{cases}$
5. $A_1 \otimes A_2 = [(P_3, P_2, P_1, M, N, Q_1, Q_2, Q_3); \alpha\alpha', \beta\beta', \gamma+\gamma'-\gamma\gamma']$, *where*

$$P_1 = \begin{cases} \min\{p_1p_1', q_1p_1'\}, & p_1 \geq 0, \\ \min\{p_1q_1', q_1p_1'\}, & p_1 < 0, q_1 \geq 0, \\ \min\{p_1q_1', q_1q_1'\}, & q_1 < 0, \end{cases} \qquad P_3 = \begin{cases} (\min\{p_3p_3', q_3p_3'\}, & p_3 \geq 0, \\ (\min\{p_3q_3', q_3p_3'\}, & p_3 < 0, q_3 \geq 0, \\ (\min\{p_3q_3', q_3q_3'\}, & q_3 < 0, \end{cases}$$

$$P_2 = \begin{cases} (\min\{p_2p_2', q_2p_2'\}, & p_2 \geq 0, \\ (\min\{p_2q_2', q_2p_2'\}, & p_2 < 0, q_2 \geq 0, \\ (\min\{p_2q_2', q_2q_2'\}, & q_2 < 0, \end{cases} \qquad M = \begin{cases} \min\{mm', nm'\}, & m \geq 0, \\ \min\{mn', nm'\}, & m < 0, n \geq 0, \\ \min\{mn', nn'\}, & n < 0, \end{cases}$$

$$N = \begin{cases} \max\{mn', nn'\}, & m \geq 0, \\ \max\{mm', nn'\}, & m < 0, n_1 \geq 0, \\ \max\{mm', nm'\}, & n_1 < 0, \end{cases} \qquad Q_2 = \begin{cases} \max\{p_2q_2', q_2q_2'\}), & p_2 \geq 0, \\ \max\{p_2p_2', q_2q_2'\}), & p_2 < 0, q_2 \geq 0, \\ \max\{p_2p_2', q_2p_2'\}), & q_2 < 0, \end{cases}$$

$$Q_1 = \begin{cases} \max\{p_1q_1', q_1q_1'\}), & p_1 \geq 0, \\ \max\{p_1p_1', q_1q_1'\}), & p_1 < 0, q_1 \geq 0, \\ \max\{p_1p_1', q_1p_1'\}), & q_1 < 0, \end{cases} \qquad Q_3 = \begin{cases} \max\{p_3q_3', q_3q_3'\}), & p_3 \geq 0, \\ \max\{p_3p_3', q_3q_3'\}), & p_3 < 0, q_3 \geq 0, \\ \max\{p_3p_3', q_3p_3'\}), & q_3 < 0. \end{cases}$$

Definition 2.12. *Two TrPFNs* $A_1 = [(p_3, p_2, p_1, m, n, q_1, q_2, q_3); \alpha, \beta, \gamma]$ *and* $A_2 = [(p_3', p_2', p_1', m', n', q_1', q_2', q_3')$ $; \alpha', \beta', \gamma']$ *are said to be equal if* $p_3 = p_3'$, $p_2 = p_2'$, $p_1 = p_1'$, $m = m'$, $n = n'$, $q_1 = q_1'$, $q_2 = q_2'$, $q_3 = q_3'$, $\alpha = \alpha'$, $\beta = \beta'$ *and* $\gamma = \gamma'$.

Definition 2.13. *A TrPFN* $A_1 = [(p_3, p_2, p_1, m, n, q_1, q_2, q_3); \alpha, \beta, \gamma]$ *is said to be zero if* $p_3 = 0$, $p_2 = 0$, $p_1 = 0$, $m = 0$, $n = 0$, $q_1 = 0$, $q_2 = 0$, $q_3 = 0$, $\alpha = 0$, $\beta = 0$ *and* $\gamma = 0$.

We now present a useful theorem.

Theorem 2.14. *Let* $A_1 = [(p_3, p_2, p_1, m, n, q_1, q_2, q_3); \alpha, \beta, \gamma]$ *and* $A_2 = [(p_3', p_2', p_1', m', n', q_1', q_2', q_3'); \alpha', \beta', \gamma']$ *be two TrPFNs then:*

1. $A_1 \oplus A_2 = A_2 \oplus A_1$,
2. $A_1 \otimes A_2 = A_2 \otimes A_1$.

Proof. Let $A_1 = [(p_3, p_2, p_1, m, n, q_1, q_2, q_3); \alpha, \beta, \gamma]$ and $A_2 = [(p'_3, p'_2, p'_1, m', n', q'_1, q'_2, q'_3); \alpha', \beta', \gamma']$ be two TrPFNs, then

$$\begin{aligned} 1.\ A_1 \oplus A_2 &= [(p_3, p_2, p_1, m, n, q_1, q_2, q_3); \alpha, \beta, \gamma] \oplus [(p'_3, p'_2, p'_1, m', n', q'_1, q'_2, q'_3); \alpha', \beta', \gamma'] \\ &= [(p_3 + p'_3, p_2 + p'_2, p_1 + p'_1, m + m', n + n', q_1 + q'_1, q_2 + q'_2, q_3 + q'_3); \alpha + \alpha' - \alpha\alpha', \beta\beta', \gamma\gamma'] \\ &= [(p'_3 + p_3, p'_2 + p_2, p'_1 + p_1, m' + m, n' + n, q'_1 + q_1, q'_2 + q_2, q'_3 + q_3); \alpha' + \alpha - \alpha'\alpha, \beta'\beta, \gamma'\gamma] \\ &= [(p'_3, p'_2, p'_1, m', n', q'_1, q'_2, q'_3); \alpha', \beta', \gamma'] \oplus [(p_3, p_2, p_1, m, n, q_1, q_2, q_3); \alpha, \beta, \gamma] \\ &= A_2 \oplus A_1. \end{aligned}$$

2. If $p_3, p'_3 \geq 0$ then

$$\begin{aligned} A_1 \otimes A_2 &= [(p_3, p_2, p_1, m, n, q_1, q_2, q_3); \alpha, \beta, \gamma] \otimes [(p'_3, p'_2, p'_1, m', n', q'_1, q'_2, q'_3); \alpha', \beta', \gamma'] \\ &= [(p_3p'_3, p_2p'_2, p_1p'_1, mm', nn', q_1q'_1, q_2q'_2, q_3q'_3); \alpha\alpha', \beta\beta', \gamma + \gamma' - \gamma\gamma'] \\ &= [(p'_3p_3, p'_2p_2, p'_1p_1, m'm, n'n, q'_1q_1, q'_2q_2, q'_3q_3); \alpha'\alpha, \beta'\beta, \gamma' + \gamma - \gamma'\gamma] \\ &= [(p'_3, p'_2, p'_1, m', n', q'_1, q'_2, q'_3); \alpha', \beta', \gamma'] \otimes [(p_3, p_2, p_1, m, n, q_1, q_2, q_3); \alpha, \beta, \gamma] \\ &= A_2 \otimes A_1. \end{aligned}$$

Similarly, for all the possible cases, the result can be proved. □

Theorem 2.15. *Let $A = [(p_3, p_2, p_1, m, n, q_1, q_2, q_3); \alpha, \beta, \gamma]$ be a simple TrPFN, then its $\lambda - cut$, $\delta - cut$ and $\theta - cut$ are $A_\lambda = [\frac{(m-p_1)(\lambda)}{\alpha} + p_1, q_1 - \frac{(q_1-n)(\lambda)}{\alpha}]$, $A^\delta = [\frac{(m-p_2)(\delta)}{\beta} + p_2, q_2 - \frac{(q_2-n)(\delta)}{\beta}]$ and ${}^\theta A = [\frac{m-\gamma p_3-\theta(m-p_3)}{1-\gamma}, \frac{\theta(q_3-n)+n-\gamma q_3}{1-\gamma}]$, $\forall\lambda \in [0, \alpha]$, $\forall\delta \in [0, \beta]$, $\forall\theta \in [\gamma, 1]$.*

Proof. This is the simple consequence of the Definition 2.2. □

Theorem 2.16. *Let $A = [(p_3, p_2, p_1, m, n, q_1, q_2, q_3); \alpha, \beta, \gamma]$ be a general TrPFN, then its $\lambda-cut$, $\delta-cut$ and $\theta-cut$ are $A_\lambda = [\frac{(m-p_1)(\lambda)}{\alpha} + p_1, q_1 - \frac{(q_1-n)(\lambda)}{\alpha}]$, $A^\delta = [m - \frac{(m-p_2)(\delta-\beta)}{\omega-\beta}, \frac{(q_2-n)(\delta-\beta)}{\omega'-\beta} + n]$ and ${}^\theta A = [\frac{m-\gamma p_3-\theta(m-p_3)}{1-\gamma}, \frac{\theta(q_3-n)+n-\gamma q_3}{1-\gamma}]$, $\forall\lambda \in [0, \alpha]$, $\forall\delta \in [0, \beta]$, $\forall\theta \in [\gamma, 1]$, where $\omega = \min\{1 - (\alpha + \gamma), 1 - \nu_A(p_2)\}$, $\omega' = \min\{1 - (\alpha + \gamma), 1 - \nu_A(q_2)\}$.*

Proof. This is the consequence of the Definition 2.2 if its neutral membership is first increasing and lastly decreasing. If its neutral membership is first decreasing and lastly increasing, then it is the consequence of Definition 2.3. □

2.2 Basic definitions associated with LR flat PFNs

The possible definitions for the LR flat PFNs are Definition 2.17 and Definition 2.18.

Definition 2.17. *A PFN $A = [(m, n; l_1, r_1; l_2, r_2; l_3, r_3); \alpha, \beta, \gamma]_{LR}$ is defined as a simple LR flat PFN, if its membership (μ_A),neutral (η_A) and non-membership (ν_A) functions are given by:*

$$\mu_A(x) = \begin{cases} L_1((\frac{m-x}{l_1})\alpha), & x \leq m, l_1 > 0, \\ \alpha, & m \leq x \leq n, \\ R_1((\frac{x-n}{r_1})\alpha), & x \geq n, r_1 > 0, \end{cases} \qquad \eta_A(x) = \begin{cases} L_2((\frac{m-x}{l_2})\beta), & x \leq m, l_2 > 0, \\ \beta, & m \leq x \leq n, \\ R_2((\frac{x-n}{r_2})\beta), & x \geq n, r_2 > 0, \end{cases}$$

and

$$\nu_A(x) = \begin{cases} L_3((\frac{m-x}{l_3})\gamma), & x \leq m, l_3 > 0, \\ \gamma, & m \leq x \leq n, \\ R_3((\frac{x-n}{r_3})\gamma), & x \geq n, r_3 > 0, \end{cases}$$

where $l_1 \leq l_2 \leq l_3$, $r_1 \leq r_2 \leq r_3$, and $0 \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1$. L_1, R_1 are monotone, continuous, decreasing functions from $[0, \infty)$ to $[0, \alpha]$, L_2, R_2 are monotone, continuous, decreasing functions from $[0, \infty)$ to $[0, \beta]$ and L_3, R_3 are monotone, continuous, increasing functions from $[0, \infty)$ to $[\gamma, 1]$ such that

1. $L_1(0) = R_1(0) = \alpha$,
2. $lim_{y\to\infty} R_1(y) = lim_{y\to\infty} L_1(y) = 0$,
3. $L_2(0) = R_2(0) = \beta$,
4. $lim_{y\to\infty} R_2(y) = lim_{y\to\infty} L_2(y) = 0$,
5. $L_3(0) = R_3(0) = \gamma$,
6. $lim_{y\to\infty} R_3(y) = lim_{y\to\infty} L_3(y) = 1$.

Definition 2.18. *A PFN $A = [(m, n; l_1, r_1; l_2, r_2; l_3, r_3); \alpha, \beta, \gamma]_{LR}$ is defined as a general LR flat PFN, if its membership (μ_A),neutral (η_A) and non-membership (ν_A) functions are given by:*

$$\mu_A(x) = \begin{cases} L_1((\frac{m-x}{l_1})\alpha), & x \le m, l_1 > 0, \\ \alpha, & m \le x \le n, \\ R_1((\frac{x-n}{r_1})\alpha), & x \ge n, r_1 > 0, \end{cases} \qquad \eta_A(x) = \begin{cases} L_2((\frac{m-x}{l_2})\beta), & x \le m, l_2 > 0, \\ \beta, & m \le x \le n, \\ R_1((\frac{x-n}{r_2})\beta), & x \ge n, r_2 > 0, \end{cases}$$

and

$$\nu_A(x) = \begin{cases} L_3((\frac{m-x}{l_3})\gamma), & x \le m, l_3 > 0, \\ \gamma, & m \le x \le n, \\ R_3((\frac{x-n}{r_3})\gamma), & x \ge n, r_3 > 0, \end{cases}$$

where $l_1 \le l_2 \le l_3$, $r_1 \le r_2 \le r_3$, and $0 \le \mu_A(x) + \eta_A(x) + \nu_A(x) \le 1$. L_1, R_1 are monotone, continuous, decreasing functions from $[0, \infty)$ to $[0, \alpha]$, L_2 is monotone, continuous, increasing/decreasing function from $[0, \infty)$ to $[\beta, \min\{1-(\alpha+\gamma), 1-\nu_A(m-l_2)\}]/[\min\{1-(\alpha+\gamma), 1-\nu_A(m-l_2)\}, \beta]$, R_2 is monotone, continuous, increasing/decreasing function from $[0, \infty)$ to $[\beta, \min\{1-(\alpha+\gamma), 1-\nu_A(n+r_2)\}]/[\min\{1-(\alpha+\gamma), 1-\nu_A(n+r_2)\}, \beta]$ and L_3, R_3 are monotone, continuous, increasing functions from $[0, \infty)$ to $[\gamma, 1]$ such that

1. $L_1(0) = R_1(0) = \alpha$,
2. $lim_{y\to\infty} L_1(y) = lim_{y\to\infty} R_1(y) = 0$,
3. $L_2(0) = R_2(0) = \beta$,
4. $lim_{y\to\infty} L_2(y) = lim_{y\to\infty} R_2(y) = 0$,
5. $L_3(0) = R_3(0) = \gamma$,
6. $lim_{y\to\infty} L_3(y) = lim_{y\to\infty} R_3(y) = 1$.

Example 2.19. *Let us consider a simple LR flat PFN as $A = [(10, 12; 4, 4; 6, 6; 8, 8); 0.5, 0.2, 0.3]_{LR}$ as in Definition 2.17 with $L_1(y) = R_1(y) = \max\{0, (\alpha - y)^2\}$, $L_2(y) = R_2(y) = \max\{0, \beta - y\}$, $L_3(y) = R_3(y) = \min\{1, \dfrac{y - \gamma y + \gamma^2}{\gamma}\}$. The membership, neutral and non-membership functions of this sample LR flat PFN are shown graphically in Figure 4.*

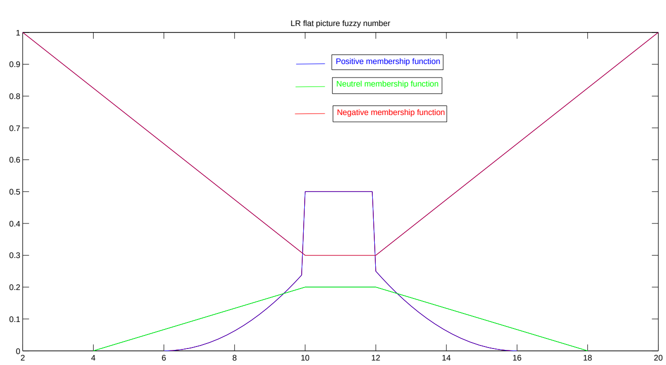

Figure 4: Graphical representation of the simple LR flat picture fuzzy number as in Definition 2.17

Example 2.20. *Let us consider a general LR flat PFN* $A = [(10, 12; 4, 4; 6, 6; 8, 8); 0.3, 0.05, 0.4]_{LR}$ *as in Definition 2.18 with* $L_1(y) = R_1(y) = \max\{0, (\alpha - y)^2\}$, $L_2(y) = \max\{\min\{1 - (\alpha + \gamma), 1 - \nu_A(m - l_2)\}, \beta - y\}$, $R_2(y) = \max\{\min\{1 - (\alpha + \gamma), 1 - \nu_A(n + r_2)\}, \beta - y\}$, $L_3(y) = R_3(y) = \min\{1, \dfrac{y - \gamma y + \gamma^2}{\gamma}\}$. *The membership, neutral and non-membership functions of this general LR flat PFN are shown graphically in Figure 5.*

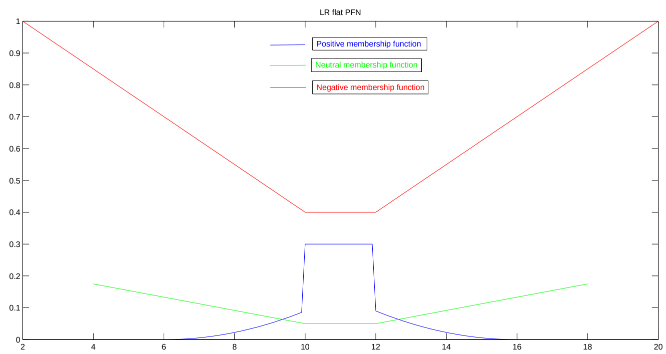

Figure 5: Graphical representation of the general LR flat picture fuzzy number as in Definition 2.18

From now onwards, when we say LR flat PFN, we mean simple LR flat PFN (Definition 2.17), unless otherwise stated. Similar results hold for general LR flat PFN (Definition 2.18).

Remark 2.21. *If we take* $L_1(y) = R_1(y) = \begin{cases} \alpha - y, & 0 \leq y \leq \alpha, \\ 0, & otherwise, \end{cases}$ $L_2(y) = R_2(y) = \begin{cases} \beta - y, & 0 \leq y \leq \beta, \\ 0, & otherwise, \end{cases}$ *and*

$$L_3(y) = R_3(y) = \begin{cases} \frac{y - \gamma y + \gamma^2}{\gamma}, & 0 \leq y \leq \gamma, \\ 1, & otherwise, \end{cases}$$

in Definition 2.17, then A becomes TrPFN.

Definition 2.22. *An LR flat PFN* $A = [(m, n; l_1, r_1; l_2, r_2; l_3, r_3); \alpha, \beta, \gamma]_{LR}$ *is non-negative(respectively non-positive), denoted as* $A \geq 0$*(respectively* $A \leq 0$*), if* $m - l_3 \geq 0$*(respectively* $n + r_3 \leq 0$*) and A is unrestricted if m or n belong to real numbers.*

Definition 2.23. *An LR flat PFN* $A = [(m, n; l_1, r_1; l_2, r_2; l_3, r_3); \alpha, \beta, \gamma]_{LR}$ *is positive if* $m - l_3 > 0$ *and negative if* $n + r_3 < 0$.

Definition 2.24. *An LR flat PFN* $A = [(m, n; l_1, r_1; l_2, r_2; l_3, r_3); \alpha, \beta, \gamma]_{LR}$ *is zero if and only if* $m = 0$, $n = 0$, $l_1 = 0$, $r_1 = 0$, $l_2 = 0$, $r_2 = 0$, $l_3 = 0$, *and* $r_3 = 0$.

Definition 2.25. *Two LR flat PFNs* $A_1 = [(m, n; l_1, r_1; l_2, r_2; l_3, r_3); \alpha, \beta, \gamma]_{LR}$ *and* $A_2 = [(m', n'; l'_1, r'_1; l'_2, r'_2; l'_3, r'_3); \alpha', \beta', \gamma']_{LR}$ *are equal if* $m = m'$, $n = n'$ $l_1 = l'_1$, $r_1 = r'_1$, $l_2 = l'_2$, $r_2 = r'_2$ $l_3 = l'_3$,$r_3 = r'_3$.

Theorem 2.26. *Let* $A = [(m, n; l_1, r_1; l_2, r_2; l_3, r_3); \alpha, \beta, \gamma]_{LR}$ *be an LR flat PFN, then its* $\lambda - cut$, $\delta - cut$ *and* $\theta - cut$ *are* $A_\lambda = [m - \frac{l_1}{\alpha} L_1^{-1}(\lambda), n + \frac{r_1}{\alpha} R_1^{-1}(\lambda)]$, $A^\delta = [m - \frac{l_2}{\beta} L_2^{-1}(\delta), n + \frac{r_2}{\beta} R_2^{-1}(\delta)]$ *and* ${}^\theta A = [m - \frac{l_3}{\gamma} L_3^{-1}(\theta), n + \frac{r_3}{\gamma} R_3^{-1}(\theta)]$, $\forall \lambda \in [0, \alpha]$, $\forall \delta \in [0, \beta]$, $\forall \theta \in [\gamma, 1]$.

Proof. The result is straightforward as a consequence of Definition 2.2. □

Theorem 2.27. *Let* $A_1 = [(m, n; l_1, r_1; l_2, r_2; l_3, r_3); \alpha, \beta, \gamma]_{LR}$ *and* $A_2 = [(m', n'; l'_1, r'_1; l'_2, r'_2; l'_3, r'_3); \alpha', \beta', \gamma']_{LR}$ *be two LR flat PFNs, then*
$A_1 \oplus A_2 = [(m + m', n + n'; l_1 + l'_1, r_1 + r'_1; l_2 + l'_2, r_2 + r'_2; l_3 + l'_3, r_3 + r'_3); \alpha + \alpha' - \alpha\alpha', \beta\beta', \gamma\gamma']_{LR}$.

Proof. Let $A_1 = [(m, n; l_1, r_1; l_2, r_2; l_3, r_3); \alpha, \beta, \gamma]_{LR}$ and $A_2 = [(m', n'; l'_1, r'_1; l'_2, r'_2; l'_3, r'_3); \alpha', \beta', \gamma']_{LR}$ be two LR flat PFNs, then their $\lambda - cut$, $\delta - cut$, $\theta - cut$, $\lambda' - cut$, $\delta' - cut$ and $\theta' - cut$ are given as:
$A_{1_\lambda} = [m - \frac{l_1}{\alpha} L_1^{-1}(\lambda), n + \frac{r_1}{\alpha} R_1^{-1}(\lambda)]$, $A_1^\delta = [m - \frac{l_2}{\beta} L_2^{-1}(\delta), n + \frac{r_2}{\beta} R_2^{-1}(\delta)]$,
${}^\theta A_1 = [m - \frac{l_3}{\gamma} L_3^{-1}(\theta), n + \frac{r_3}{\gamma} R_3^{-1}(\theta)]$, $A_{2_{\lambda'}} = [m' - \frac{l'_1}{\alpha'} L_1^{-1}(\lambda'), n' + \frac{r'_1}{\alpha'} R_1^{-1}(\lambda')]$,
$A_2^{\delta'} = [m' - \frac{l'_2}{\beta'} L_2^{-1}(\delta'), n' + \frac{r'_2}{\beta'} R_2^{-1}(\delta')]$, ${}^{\theta'} A_2 = [m' - \frac{l'_3}{\gamma'} L_3^{-1}(\theta'), n' + \frac{r'_3}{\gamma'} R_3^{-1}(\theta')]$.
Thus,

$$A_{1_\lambda} + A_{2_{\lambda'}} = [m - \frac{l_1}{\alpha} L_1^{-1}(\lambda) + m' - \frac{l'_1}{\alpha'} L_1^{-1}(\lambda'), n + \frac{r_1}{\alpha} R_1^{-1}(\lambda) + n' + \frac{r'_1}{\alpha'} R_1^{-1}(\lambda')]. \quad (1)$$

By taking $\lambda = \alpha$ and $\lambda' = \alpha'$ in Equation (1), we have

$$A_{1_{\lambda=\alpha}} + A_{2_{\lambda'=\alpha'}} = [m + m', n + n']. \quad (2)$$

By taking $\lambda = 0$ and $\lambda' = 0$ in Equation (1), we have

$$A_{1_{\lambda=0}} + A_{2_{\lambda'=0}} = [m + m' - l_1 - l'_1, n + n' + r_1 + r'_1]. \quad (3)$$

Now

$$A_1^\delta + A_2^{\delta'} = [m - \frac{l_2}{\beta} L_2^{-1}(\delta) + m' - \frac{l'_2}{\beta'} L_2^{-1}(\delta'), n + \frac{r_2}{\beta} R_2^{-1}(\delta) + n' + \frac{r'_2}{\beta'} R_2^{-1}(\delta')]. \quad (4)$$

By taking $\delta = \beta$ and $\delta' = \beta'$ in Equation (4), we have

$$A_1^{\delta=\beta} + A_2^{\delta'=\beta'} = [m + m', n + n']. \quad (5)$$

By taking $\delta = 0$ and $\delta' = 0$ in Equation (4), we have

$$A_1^{\delta=0} + A_2^{\delta'=0} = [m + m' - l_2 - l'_2, n + n' + r_2 + r'_2]. \quad (6)$$

Now

$${}^\theta A_1 +^{\theta'} A_2 = [m - \frac{l_3}{\gamma} L_3^{-1}(\theta) + m' - \frac{l'_3}{\gamma'} L_3^{-1}(\theta'), n + \frac{r_3}{\gamma} R_3^{-1}(\theta) + n' + \frac{r'_3}{\gamma'} R_3^{-1}(\theta')]. \quad (7)$$

By taking $\theta = \gamma$ and $\theta' = \gamma'$ in Equation (7), we have

$${}^{\theta=\gamma} A_1 +^{\theta'=\gamma'} A_2 = [m + m', n + n']. \quad (8)$$

By taking $\theta = 1$ and $\theta' = 1$ in Equation (7), we have

$${}^{\theta=1} A_1 +^{\theta'=1} A_2 = [m + m' - l_3 - l'_3, n + n' + r_3 + r'_3]. \quad (9)$$

By combining Equations (2,3,5,6,8,9), the result follows. □

Theorem 2.28. *Let* $A_1 = [(m, n; l_1, r_1; l_2, r_2; l_3, r_3); \alpha, \beta, \gamma]_{LR}$ *and* $A_2 = [(m', n'; l'_1, r'_1; l'_2, r'_2; l'_3, r'_3); \alpha', \beta', \gamma']_{LR}$ *be two LR flat PFNs, then*
$A_1 \ominus A_2 = [(m - n', n - m'; l_1 + r'_1, r_1 + l'_1; l_2 + r'_2, r_2 + l'_2; l_3 + r'_3, r_3 + l'_3); \alpha + \alpha' - \alpha\alpha', \beta\beta', \gamma\gamma']_{LR}$.

Proof. By using similar arguments as used in Theorem 2.27, the result can be proved easily. □

Theorem 2.29. *Let* $A_1 = [(m, n; l_1, r_1; l_2, r_2; l_3, r_3); \alpha, \beta, \gamma]_{LR}$ *be an LR flat PFN in which* $m - l_3 < 0$, $m - l_2 \geq 0$, *and* $A_2 = [(m', n'; l'_1, r'_1; l'_2, r'_2; l'_3, r'_3); \alpha', \beta', \gamma']_{LR}$ *be an unrestricted LR flat PFN, then*

$$A_1 \otimes A_2 = [(M, N; L_1, R_1; L_2, R_2; L_3, R_3); \alpha\alpha', \beta\beta', \gamma + \gamma' - \gamma\gamma']_{LR},$$

where
$M = \min\{mm', nm'\}$, $N = \max\{mn', nn'\}$,
$L_1 = \min\{mm', nm'\} - \min\{mm' - ml'_1 - m'l_1 + l_1 l'_1, nm' - nl'_1 + m'r_1 - r_1 l'_1\}$,
$R_1 = \max\{mn' + mr'_1 - n'l_1 - l_1 r'_1, nn' + nr'_1 + n'r_1 + r_1 r'_1\} - \max\{mn', nn'\}$,
$L_2 = \min\{mm', nm'\} - \min\{mm' - ml'_2 - m'l_2 + l_2 l'_2, nm' - nl'_2 + m'r_2 - r_2 l'_2\}$,
$R_2 = \max\{mn' + mr'_2 - n'l_2 - l_2 r'_2, nn' + nr'_2 + n'r_2 + r_1 r'_1\} - \max\{mn', nn'\}$,
$L_3 = \min\{mm', nm'\} - \min\{mn' + mr'_3 - n'l_3 - l_3 r'_3, nm' - nl'_3 + m'r_3 - l'_3 r_3\}$,
$R_3 = \max\{mm' - ml'_3 - m'l_3 + l_3 l'_3, nn' + nr'_3 + n'r_3 + r_3 r'_3\} - \max\{mn', nn'\}$.

Proof. Let $A_1 = [(m, n; l_1, r_1; l_2, r_2; l_3, r_3); \alpha, \beta, \gamma]_{LR}$ and $A_2 = [(m', n'; l'_1, r'_1; l'_2, r'_2; l'_3, r'_3); \alpha', \beta', \gamma']_{LR}$ be two LR flat PFNs, then their $\lambda - cut$, $\delta - cut$, $\theta - cut$, $\lambda' - cut$, $\delta' - cut$ and $\theta' - cut$ are given as:
$A_{1_\lambda} = [m - \frac{l_1}{\alpha} L_1^{-1}(\lambda), n + \frac{r_1}{\alpha} R_1^{-1}(\lambda)]$, $A_1^\delta = [m - \frac{l_2}{\beta} L_2^{-1}(\delta), n + \frac{r_2}{\beta} R_2^{-1}(\delta)]$,
${}^\theta A_1 = [m - \frac{l_3}{\gamma} L_3^{-1}(\theta), n + \frac{r_3}{\gamma} R_3^{-1}(\theta)]$, $A_{2_{\lambda'}} = [m' - \frac{l'_1}{\alpha'} L_1^{-1}(\lambda'), n' + \frac{r'_1}{\alpha'} R_1^{-1}(\lambda')]$,
$A_2^{\delta'} = [m' - \frac{l'_2}{\beta'} L_2^{-1}(\delta'), n' + \frac{r'_2}{\beta'} R_2^{-1}(\delta')]$, ${}^{\theta'} A_2 = [m' - \frac{l'_3}{\gamma'} L_3^{-1}(\theta'), n' + \frac{r'_3}{\gamma'} R_3^{-1}(\theta')]$.
Thus,

$$\begin{aligned} A_{1_\lambda} A_{2_{\lambda'}} \quad &= \quad [\min\{(m - \frac{l_1}{\alpha} L_1^{-1}(\lambda))(m' - \frac{l'_1}{\alpha'} L_1^{-1}(\lambda')), (n + \frac{r_1}{\alpha} R_1^{-1}(\lambda))(m' - \frac{l'_1}{\alpha'} L_1^{-1}(\lambda'))\}, \\ & \max\{(m - \frac{l_1}{\alpha} L_1^{-1}(\lambda))(n' + \frac{r'_1}{\alpha'} R_1^{-1}(\lambda')), (n + \frac{r_1}{\alpha} R_1^{-1}(\lambda))(n' + \frac{r'_1}{\alpha'} R_1^{-1}(\lambda'))\}]. \end{aligned} \tag{10}$$

By taking $\lambda = \alpha$ and $\lambda' = \alpha'$ in Equation (10), we have

$$A_{1_{\lambda=\alpha}} A_{2_{\lambda'=\alpha'}} = [\min\{mm', nm'\}, \max\{mn', nn'\}]. \tag{11}$$

By taking $\lambda = 0$ and $\lambda' = 0$ in Equation (10), we have

$$\begin{aligned} A_{1_{\lambda=0}} A_{2_{\lambda'=0}} \quad &= \quad [\min\{mm' - ml'_1 - m'l_1 + l_1 l'_1, nm' - nl'_1 + m'r_1 - r_1 l'_1\}, \\ & \max\{mn' + mr'_1 - n'l_1 - l_1 r'_1, nn' + nr'_1 + n'r_1 + r_1 r'_1\}]. \end{aligned} \tag{12}$$

Now

$$\begin{aligned} A_1^\delta A_2^{\delta'} \quad &= \quad [\min\{(m - \frac{l_2}{\beta} L_2^{-1}(\delta))(m' - \frac{l'_2}{\beta'} L_2^{-1}(\delta')), (n + \frac{r_2}{\beta} R_2^{-1}(\delta))(m' - \frac{l'_2}{\beta'} L_2^{-1}(\delta'))\}, \\ & \max\{(m - \frac{l_2}{\beta} L_2^{-1}(\delta))(n' + \frac{r'_2}{\beta'} R_2^{-1}(\delta')), (n + \frac{r_2}{\beta} R_2^{-1}(\delta))(n' + \frac{r'_2}{\beta'} R_2^{-1}(\delta'))\}]. \end{aligned} \tag{13}$$

By taking $\delta = \beta$ and $\delta' = \beta'$ in Equation (13), we have

$$A_1^{\delta=\beta} A_2^{\delta'=\beta'} = [\min\{mm', nm'\}, \max\{mn', nn'\}]. \tag{14}$$

By taking $\delta = 0$ and $\delta' = 0$ in Equation (13), we have

$$\begin{aligned} A_1^{\delta=0} A_2^{\delta'=0} \quad &= \quad [\min\{mm' - ml'_2 - m'l_2 + l_2 l'_2, nm' - nl'_2 + m'r_2 - r_2 l'_2\}, \\ & \max\{mn' + mr'_2 - n'l_2 - l_2 r'_2, nn' + nr'_2 + n'r_2 + r_1 r'_1\}]. \end{aligned} \tag{15}$$

Now the following two cases arise:
$m - \frac{l_3}{\gamma} L_3^{-1}(\theta) < 0$ for $\theta > L_3(\frac{mr_1}{l_3})$ and $m - \frac{l_3}{\gamma} L_3^{-1}(\theta) \geq 0$ for $\theta \leq L_3(\frac{mr_1}{l_3})$.
For the case $(m - \frac{l_3}{\gamma} L_3^{-1}(\theta) < 0$ for $\theta > L_3(\frac{mr_1}{l_3}))$:

$$\begin{aligned} {}^\theta A_1 . {}^{\theta'} A_2 \quad &= \quad [\min\{(m - \frac{l_3}{\gamma} L_3^{-1}(\theta))(m' - \frac{l'_3}{\gamma'} L_3^{-1}(\theta')), (n + \frac{r_3}{\gamma} R_3^{-1}(\theta))(m' - \frac{l'_3}{\gamma'} L_3^{-1}(\theta'))\}, \\ & \max\{(m - \frac{l_3}{\gamma} L_3^{-1}(\theta))(n' + \frac{r'_3}{\gamma'} R_3^{-1}(\theta')), (n + \frac{r_3}{\gamma} R_3^{-1}(\theta))(n' + \frac{r'_3}{\gamma'} R_3^{-1}(\theta'))\}]. \end{aligned} \tag{16}$$

By taking $\theta = \gamma$ and $\theta' = \gamma'$ in Equation (16), we have

$${}^{\theta=\gamma} A_1 . {}^{\theta'=\gamma'} A_2 = [\min\{mm', nm'\}, \max\{mn', nn'\}]. \tag{17}$$

For the case ($m - \frac{l_3}{\gamma}L_3^{-1}(\theta) < 0$ for $\theta > L_3(\frac{mr_1}{l_3})$):

$$^{\theta}A_1.^{\theta'}A_2 = [\min\{(m - \frac{l_3}{\gamma}L_3^{-1}(\theta))(n' + \frac{r_3'}{\gamma'}R_3^{-1}(\theta')), (n + \frac{r_3}{\gamma}R_3^{-1}(\theta))(m' - \frac{l_3'}{\gamma'}L_3^{-1}(\theta'))\},$$

$$\max\{(m - \frac{l_3}{\gamma}L_3^{-1}(\theta))(m' - \frac{l_3'}{\gamma'}L_3^{-1}(\theta')), (n + \frac{r_3}{\gamma}R_3^{-1}(\theta))(n' + \frac{r_3'}{\gamma'}R_3^{-1}(\theta'))\}]. \tag{18}$$

By taking $\theta = 1$ and $\theta' = 1$ in Equation (18), we have

$$^{\theta=1}A_1.^{\theta'=1}A_2 = [\min\{mn' + mr_3' - n'l_3 - l_3r_3', nm' - nl_3' + m'r_3 - l_3'r_3\},$$

$$\max\{mm' - ml_3' - m'l_3 + l_3l_3', nn' + nr_3' + n'r_3 + r_3r_3'\}]. \tag{19}$$

By combining Equations (11,12,14,15,17,19), the result follows. □

By using the similar arguments as used in Theorem 2.29, the following results can be easily proved.

Theorem 2.30. *Let $A_1 = [(m, n; l_1, r_1; l_2, r_2; l_3, r_3); \alpha, \beta, \gamma]_{LR}$ be an LR flat PFN in which $m - l_2 < 0$, $m - l_1 \geq 0$, and $A_2 = [(m', n'; l_1', r_1'; l_2', r_2'; l_3', r_3'); \alpha', \beta', \gamma']_{LR}$ be an unrestricted LR flat PFN, then*

$$A_1 \otimes A_2 = [(M, N; L_1, R_1; L_2, R_2; L_3, R_3); \alpha\alpha', \beta\beta', \gamma + \gamma' - \gamma\gamma']_{LR},$$

where
$M = \min\{mm', nm'\}$, $N = \max\{mn', nn'\}$,
$L_1 = \min\{mm', nm'\} - \min\{mm' - ml_1' - m'l_1 + l_1l_1', nm' - nl_1' + m'r_1 - r_1l_1'\}$,
$R_1 = \max\{mn' + mr_1' - n'l_1 - l_1r_1', nn' + nr_1' + n'r_1 + r_1r_1'\} - \max\{mn', nn'\}$,
$L_2 = \min\{mm', nm'\} - \min\{mn' + mr_2' - l_2n' - l_2r_2', nm' - nl_2' + r_2m' - r_2l_2'\}$,
$R_2 = \max\{mm' - ml_2' - l_2m' + l_2l_2', nn' + nr_2' + r_2n' + r_2r_2'\} - \max\{mn', nn'\}$,
$L_3 = \min\{mm', nm'\} - \min\{mn' + mr_3' - n'l_3 - l_3r_3', nm' - nl_3' + m'r_3 - l_3'r_3\}$,
$R_3 = \max\{mm' - ml_3' - m'l_3 + l_3l_3', nn' + nr_3' + n'r_3 + r_3r_3'\} - \max\{mn', nn'\}$.

Theorem 2.31. *Let $A_1 = [(m, n; l_1, r_1; l_2, r_2; l_3, r_3); \alpha, \beta, \gamma]_{LR}$ be an LR flat PFN in which $m - l_1 < 0$, $m \geq 0$, and $A_2 = [(m', n'; l_1', r_1'; l_2', r_2'; l_3', r_3'); \alpha', \beta', \gamma']_{LR}$ be an unrestricted LR flat PFN, then*

$$A_1 \otimes A_2 = [(M, N; L_1, R_1; L_2, R_2; L_3, R_3); \alpha\alpha', \beta\beta', \gamma + \gamma' - \gamma\gamma']_{LR},$$

where
$M = \min\{mm', nm'\}$, $N = \max\{mn', nn'\}$,
$L_1 = \min\{mm', nm'\} - \min\{mn' + mr_1' - l_1n' - l_1r_1', nm' - nl_1' + r_1m' - r_1l_1'\}$,
$R_1 = \max\{mm' - ml_1' - l_1m' + l_1l_1', nn' + nr_1' + r_1n' + r_1r_1'\} - \max\{mn', nn'\}$,
$L_2 = \min\{mm', nm'\} - \min\{mn' + mr_2' - l_2n' - l_2r_2', nm' - nl_2' + r_2m' - r_2l_2'\}$,
$R_2 = \max\{mm' - ml_2' - l_2m' + l_2l_2', nn' + nr_2' + r_2n' + r_2r_2'\} - \max\{mn', nn'\}$,
$L_3 = \min\{mm', nm'\} - \min\{mn' + mr_3' - n'l_3 - l_3r_3', nm' - nl_3' + m'r_3 - l_3'r_3\}$,
$R_3 = \max\{mm' - ml_3' - m'l_3 + l_3l_3', nn' + nr_3' + n'r_3 + r_3r_3'\} - \max\{mn', nn'\}$.

Theorem 2.32. *Let $A_1 = [(m, n; l_1, r_1; l_2, r_2; l_3, r_3); \alpha, \beta, \gamma]_{LR}$ be an LR flat PFN in which $m < 0$, $n \geq 0$, and $A_2 = [(m', n'; l_1', r_1'; l_2', r_2'; l_3', r_3'); \alpha', \beta', \gamma']_{LR}$ be an unrestricted LR flat PFN, then*

$$A_1 \otimes A_2 = [(M, N; L_1, R_1; L_2, R_2; L_3, R_3); \alpha\alpha', \beta\beta', \gamma + \gamma' - \gamma\gamma']_{LR},$$

where
$M = \min\{mn', nm'\}$, $N = \max\{mm', nn'\}$,
$L_1 = \min\{mn', nm'\} - \min\{mn' + mr_1' - l_1n' - l_1r_1', nm' - nl_1' + r_1m' - r_1l_1'\}$,
$R_1 = \max\{mm' - ml_1' - l_1m' + l_1l_1', nn' + nr_1' + r_1n' + r_1r_1'\} - \max\{mm', nn'\}$,
$L_2 = \min\{mn', nm'\} - \min\{mn' + mr_2' - l_2n' - l_2r_2', nm' - nl_2' + r_2m' - r_2l_2'\}$,
$R_2 = \max\{mm' - ml_2' - l_2m' + l_2l_2', nn' + nr_2' + r_2n' + r_2r_2'\} - \max\{mm', nn'\}$,
$L_3 = \min\{mn', nm'\} - \min\{mn' + mr_3' - n'l_3 - l_3r_3', nm' - nl_3' + m'r_3 - l_3'r_3\}$,
$R_3 = \max\{mm' - ml_3' - m'l_3 + l_3l_3', nn' + nr_3' + n'r_3 + r_3r_3'\} - \max\{mm', nn'\}$.

Theorem 2.33. *Let* $A_1 = [(m, n; l_1, r_1; l_2, r_2; l_3, r_3); \alpha, \beta, \gamma]_{LR}$ *be an LR flat PFN in which* $n < 0$, $n + r_1 \geq 0$, *and* $A_2 = [(m', n'; l'_1, r'_1; l'_2, r'_2; l'_3, r'_3); \alpha', \beta', \gamma']_{LR}$ *be an unrestricted LR flat PFN, then*

$$A_1 \otimes A_2 = [(M, N; L_1, R_1; L_2, R_2; L_3, R_3); \alpha\alpha', \beta\beta', \gamma + \gamma' - \gamma\gamma']_{LR},$$

where
$M = \min\{mn', nn'\}$, $N = \max\{mm', nm'\}$,
$L_1 = \min\{mn', nn'\} - \min\{mn' + mr'_1 - l_1n' - l_1r'_1, nm' - nl'_1 + r_1m' - r_1l'_1\}$,
$R_1 = \max\{mm' - ml'_1 - l_1m' + l_1l'_1, nn' + nr'_1 + r_1n' + r_1r'_1\} - \max\{mm', nm'\}$,
$L_2 = \min\{mn', nn'\} - \min\{mn' + mr'_2 - l_2n' - l_2r'_2, nm' - nl'_2 + r_2m' - r_2l'_2\}$,
$R_2 = \max\{mm' - ml'_2 - l_2m' + l_2l'_2, nn' + nr'_2 + r_2n' + r_2r'_2\} - \max\{mm', nm'\}$,
$L_3 = \min\{mn', nn'\} - \min\{mn' + mr'_3 - n'l_3 - l_3r'_3, nm' - nl'_3 + m'r_3 - l'_3r_3\}$,
$R_3 = \max\{mm' - ml'_3 - m'l_3 + l_3l'_3, nn' + nr'_3 + n'r_3 + r_3r'_3\} - \max\{mm', nm'\}$.

Theorem 2.34. *Let* $A_1 = [(m, n; l_1, r_1; l_2, r_2; l_3, r_3); \alpha, \beta, \gamma]_{LR}$ *be an LR flat PFN in which* $n + r_1 < 0$, $n + r_2 \geq 0$, *and* $A_2 = [(m', n'; l'_1, r'_1; l'_2, r'_2; l'_3, r'_3); \alpha', \beta', \gamma']_{LR}$ *be an unrestricted LR flat PFN, then*

$$A_1 \otimes A_2 = [(M, N; L_1, R_1; L_2, R_2; L_3, R_3); \alpha\alpha', \beta\beta', \gamma + \gamma' - \gamma\gamma']_{LR},$$

where
$M = \min\{mn', nn'\}$, $N = \max\{mm', nm'\}$,
$L_1 = \min\{mn', nn'\} - \min\{mn' + mr'_1 - l_1n' - l_1r'_1, nn' + nr'_1 + r_1n' + r_1r'_1\}$,
$R_1 = \max\{mm' - ml'_1 - l_1m' + l_1l'_1, nm' - nl'_1 + r_1m' - r_1l'_1\} - \max\{mm', nm'\}$,
$L_2 = \min\{mn', nn'\} - \min\{mn' + mr'_2 - l_2n' - l_2r'_2, nm' - nl'_2 + r_2m' - r_2l'_2\}$,
$R_2 = \max\{mm' - ml'_2 - l_2m' + l_2l'_2, nn' + nr'_2 + r_2n' + r_2r'_2\} - \max\{mm', nm'\}$,
$L_3 = \min\{mn', nn'\} - \min\{mn' + mr'_3 - n'l_3 - l_3r'_3, nm' - nl'_3 + m'r_3 - l'_3r_3\}$,
$R_3 = \max\{mm' - ml'_3 - m'l_3 + l_3l'_3, nn' + nr'_3 + n'r_3 + r_3r'_3\} - \max\{mm', nm'\}$.

Theorem 2.35. *Let* $A_1 = [(m, n; l_1, r_1; l_2, r_2; l_3, r_3); \alpha, \beta, \gamma]_{LR}$ *be an LR flat PFN in which* $n + r_2 < 0$, $n + r_3 \geq 0$, *and* $A_2 = [(m', n'; l'_1, r'_1; l'_2, r'_2; l'_3, r'_3); \alpha', \beta', \gamma']_{LR}$ *be an unrestricted LR flat PFN, then*

$$A_1 \otimes A_2 = [(M, N; L_1, R_1; L_2, R_2; L_3, R_3); \alpha\alpha', \beta\beta', \gamma + \gamma' - \gamma\gamma']_{LR},$$

where
$M = \min\{mn', nn'\}$, $N = \max\{mm', nm'\}$,
$L_1 = \min\{mn', nn'\} - \min\{mn' + mr'_1 - l_1n' - l_1r'_1, nn' + nr'_1 + r_1n' + r_1r'_1\}$,
$R_1 = \max\{mm' - ml'_1 - l_1m' + l_1l'_1, nm' - nl'_1 + r_1m' - r_1l'_1\} - \max\{mm', nm'\}$,
$L_2 = \min\{mn', nn'\} - \min\{mn' + mr'_2 - l_2n' - l_2r'_2, nn' + nr'_2 + r_2n' + r_2r'_2\}$,
$R_2 = \max\{mm' - ml'_2 - l_2m' + l_2l'_2, nm' - nl'_2 + r_2m' - r_2l'_2\} - \max\{mm', nm'\}$,
$L_3 = \min\{mn', nn'\} - \min\{mn' + mr'_3 - n'l_3 - l_3r'_3, nm' - nl'_3 + m'r_3 - l'_3r_3\}$,
$R_3 = \max\{mm' - ml'_3 - m'l_3 + l_3l'_3, nn' + nr'_3 + n'r_3 + r_3r'_3\} - \max\{mm', nm'\}$.

Theorem 2.36. *Let* $A_1 = [(m, n; l_1, r_1; l_2, r_2; l_3, r_3); \alpha, \beta, \gamma]_{LR}$ *be an LR flat PFN in which* $n + r_3 < 0$ *and* $A_2 = [(m', n'; l'_1, r'_1; l'_2, r'_2; l'_3, r'_3); \alpha', \beta', \gamma']_{LR}$ *be an unrestricted LR flat PFN, then*

$$A_1 \otimes A_2 = [(M, N; L_1, R_1; L_2, R_2; L_3, R_3); \alpha\alpha', \beta\beta', \gamma + \gamma' - \gamma\gamma']_{LR},$$

where
$M = \min\{mn', nn'\}$, $N = \max\{mm', nm'\}$,
$L_1 = \min\{mn', nn'\} - \min\{mn' + mr'_1 - l_1n' - l_1r'_1, nn' + nr'_1 + r_1n' + r_1r'_1\}$,
$R_1 = \max\{mm' - ml'_1 - l_1m' + l_1l'_1, nm' - nl'_1 + r_1m' - r_1l'_1\} - \max\{mm', nm'\}$,
$L_2 = \min\{mn', nn'\} - \min\{mn' + mr'_2 - l_2n' - l_2r'_2, nn' + nr'_2 + r_2n' + r_2r'_2\}$,
$R_2 = \max\{mm' - ml'_2 - l_2m' + l_2l'_2, nm' - nl'_2 + r_2m' - r_2l'_2\} - \max\{mm', nm'\}$,
$L_3 = \min\{mn', nn'\} - \min\{mn' + mr'_3 - l_3n' - l_3r'_3, nn' + nr'_3 + r_3n' + r_3r'_3\}$,
$R_3 = \max\{mm' - ml'_3 - l_3m' + l_3l'_3, nm' - nl'_3 + r_3m' - r_3l'_3\} - \max\{mm', nm'\}$.

Theorem 2.37. *Let* $A_1 = [(m, n; l_1, r_1; l_2, r_2; l_3, r_3); \alpha, \beta, \gamma]_{LR}$ *be an LR flat PFN in which* $m - l_3 \geq 0$ *and* $A_2 = [(m', n'; l'_1, r'_1; l'_2, r'_2; l'_3, r'_3); \alpha', \beta', \gamma']_{LR}$ *be an unrestricted LR flat PFN, then*

$$A_1 \otimes A_2 = [(M, N; L_1, R_1; L_2, R_2; L_3, R_3); \alpha\alpha', \beta\beta', \gamma + \gamma' - \gamma\gamma']_{LR},$$

where
$M=\min\{mm',nm'\}$, $N=\max\{mn',nn'\}$,
$L_1=\min\{mm',nm'\}-\min\{mm'-ml'_1-m'l_1+l_1l'_1, nm'-nl'_1+m'r_1-r_1l'_1\}$,
$R_1=\max\{mn'+mr'_1-n'l_1-l_1r'_1, nn'+nr'_1+n'r_1+r_1r'_1\}-\max\{mn',nn'\}$,
$L_2=\min\{mm',nm'\}-\min\{mm'-ml'_2-m'l_2+l_2l'_2, nm'-nl'_2+m'r_2-r_2l'_2\}$,
$R_2=\max\{mn'+mr'_2-n'l_2-l_2r'_2, nn'+nr'_2+n'r_2+r_1r'_1\}-\max\{mn',nn'\}$,
$L_3=\min\{mm',nm'\}-\min\{mm'-ml'_3-l_3m'+l_3l'_3, nm'-nl'_3+r_3m'-r_3l'_3\}$,
$R_3=\max\{mn'+mr'_3-l_3n'-l_3r'_3, nn'+nr'_3+r_3n'+r_3r'_3\}-\max\{mn',nn'\}$.

Theorem 2.38. *Let $A_1=[(m,n;l_1,r_1;l_2,r_2;l_3,r_3);\alpha,\beta,\gamma]_{LR}$ and $A_2=[(m',n';l'_1,r'_1;l'_2,r'_2;l'_3,r'_3);\alpha',\beta',\gamma']_{LR}$ be two non-negative LR flat PFNs, then*

$$A_1\otimes A_2=[(M,N;L_1,R_1;L_2,R_2;L_3,R_3);\alpha\alpha',\beta\beta',\gamma+\gamma'-\gamma\gamma']_{LR},$$

where
$M=mm'$, $N=nn'$,
$L_1=ml'_1+l_1m'-l_1l'_1$, $R_1=nr'_1+r_1n'+r_1r'_1$,
$L_2=ml'_2+l_2m'-l_2l'_2$, $R_2=nr'_2+r_2n'+r_2r'_2$,
$L_3=ml'_3+l_3m'-l_3l'_3$, $R_3=nr'_3+r_3n'+r_3r'_3$.

Theorem 2.39. *Let $A_1=[(m,n;l_1,r_1;l_2,r_2;l_3,r_3);\alpha,\beta,\gamma]_{LR}$ be a non-negative LR flat PFN and $A_2=[(m',n';l'_1,r'_1;l'_2,r'_2;l'_3,r'_3);\alpha',\beta',\gamma']_{LR}$ be a non-positive LR flat PFN, then*

$$A_1\otimes A_2=[(M,N;L_1,R_1;L_2,R_2;L_3,R_3);\alpha\alpha',\beta\beta',\gamma+\gamma'-\gamma\gamma']_{LR},$$

where
$M=nm'$, $N=mn'$,
$L_1=nl'_1-r_1m'+r_1l'_1$, $R_1=mr'_1-l_1n'-l_1r'_1$,
$L_2=nl'_2-r_2m'+r_2l'_2$, $R_2=mr'_2-l_2n'-l_2r'_2$,
$L_3=nl'_3-r_3m'+r_3l'_3$, $R_3=mr'_3-l_3n'-l_3r'_3$.

Theorem 2.40. *Let $A=[(m,n;l_1,r_1;l_2,r_2;l_3,r_3);\alpha,\beta,\gamma]_{LR}$ be an LR flat PFN and c be any real number, then*

$$cA=\begin{cases}[(cm,cn;cl_1,cr_1;cl_2,cr_2;cl_3,cr_3);\alpha^c,\beta^c,1-(1-\gamma)^c]_{LR}, & c\geq 0,\\ [(cn,cm;-cr_1,-cl_1;-cr_2,-cl_2;-cr_3,-cl_3);\alpha^{-c},\beta^{-c},1-(1-\gamma)^{-c}]_{LR}, & c<0.\end{cases}$$

Definition 2.41. *Let $A=[(m,n;l_1,r_1;l_2,r_2;l_3,r_3);\alpha,\beta,\gamma]_{LR}$ be an LR flat PFN, then ranking of A, denoted $\Re(A)$, can be defined as*

$$\begin{aligned}\Re(A) &= \frac{1}{2}[\int_0^\alpha(m-\frac{l_1}{\alpha}L_1^{-1}(x))dx+\int_0^\alpha(n+\frac{r_1}{\alpha}R_1^{-1}(x))dx+\int_0^\beta(m-\frac{l_2}{\beta}L_2^{-1}(x))dx\\ &+\int_0^\beta(n+\frac{r_2}{\beta}R_2^{-1}(x))dx+\int_\gamma^1(m-\frac{l_3}{\gamma}L_3^{-1}(x))dx+\int_\gamma^1(n+\frac{r_3}{\gamma}R_3^{-1}(x))dx].\end{aligned}$$

Definition 2.42. *Let A_1 and A_2 be two LR flat PFNs, then*
(1) $A_1\prec A_2$ *if* $\Re(A_1)<\Re(A_2)$, (2) $A_1\succ A_2$ *if* $\Re(A_1)>\Re(A_2)$, (3) $A_1\approx A_2$ *if* $\Re(A_1)=\Re(A_2)$.

Remark 2.43. *For a TrPFN ($L_1(x)=R_1(x)=\max\{0,\alpha-x\}$, $L_2(x)=R_2(x)=\max\{0,\beta-x\}$, $L_3(x)=R_3(x)=\min\{1,\frac{x-\gamma x+\gamma^2}{\gamma}\}$), the ranking function as in Definition 2.41, reduces to* $\frac{\alpha(p_1+m+n+q_1)+\beta(p_2+m+n+q_2)+(1-\gamma)(p_3+m+n+q_3)}{4}$.

Proof. For any LR flat PFN $A=[(m,n;l_1,r_1;l_2,r_2;l_3,r_3);\alpha,\beta,\gamma]_{LR}$, the ranking is given as:

$$\begin{aligned}\Re(A) &= \frac{1}{2}[\int_0^\alpha(m-\frac{l_1}{\alpha}L_1^{-1}(x))dx+\int_0^\alpha(n+\frac{r_1}{\alpha}R_1^{-1}(x))dx+\int_0^\beta(m-\frac{l_2}{\beta}L_2^{-1}(x))dx\\ &+\int_0^\beta(n+\frac{r_2}{\beta}R_2^{-1}(x))dx+\int_\gamma^1(m-\frac{l_3}{\gamma}L_3^{-1}(x))dx+\int_\gamma^1(n+\frac{r_3}{\gamma}R_3^{-1}(x))dx].\end{aligned}$$

By taking $L_1(x)=R_1(x)=\max\{0,\alpha-x\},\ L_2(x)=R_2(x)=\max\{0,\beta-x\},\ L_3(x)=R_3(x)=\min\{1,\dfrac{x-\gamma x+\gamma^2}{\gamma}\}$ in the above equation, we have

$$\begin{aligned}\Re(A) &= \frac{1}{2}[\int_0^\alpha (m-\frac{l_1}{\alpha}(\alpha-x))dx+\int_0^\alpha (n+\frac{r_1}{\alpha}(\alpha-x))dx+\int_0^\beta (m-\frac{l_2}{\beta}(\beta-x))dx \\ &\quad +\int_0^\beta (n+\frac{r_2}{\beta}(\beta-x))dx+\int_\gamma^1 (m-\frac{l_3}{1-\gamma}(x-\gamma))dx+\int_\gamma^1 (n+\frac{r_3}{1-\gamma}(x-\gamma))dx] \\ &= \frac{1}{2}[\alpha(m-\frac{l_1}{2}+n+\frac{r_1}{2})+\beta(m-\frac{l_2}{2}+n+\frac{r_2}{2})+(1-\gamma)(m-\frac{l_3}{2}+n+\frac{r_3}{2})].\end{aligned}$$

For a TrPFN, $l_1=m-p_1$, $r_1=q_1-n$, $l_2=m-p_2$, $r_2=q_2-n$, $l_3=m-p_3$, $r_3=q_3-n$. Thus we have

$$\Re(A) = \frac{\alpha(p_1+m+n+q_1)+\beta(p_2+m+n+q_2)+(1-\gamma)(p_3+m+n+q_3)}{4}.$$

□

3 FPFLPP with LR flat PFNs

In this section, we discuss the linear programming problem with LR flat PFNs.

3.1 Methodology of FPFLPP with LR flat PFNs

Let us consider FPFLPP with LR flat PFNs.

$$\text{Max/Min}\sum_{j=1}^{n} C_j\otimes X_j, \tag{20}$$

$$\text{subject to}\quad \sum_{j=1}^{n} A_{ij}\otimes X_j \quad \preceq,=,\succeq \quad B_i \quad \forall i=1,2,\cdots,m,$$

where A_{ij},X_j,B_i, and C_j are all LR flat PFNs.
Now we present a method to solve FPFLPP (20). Firstly we present a criterion for the optimal solution.

Definition 3.1. *An LR flat picture fuzzy optimal solution of the FPFLPP (20) with LR flat PFNs will be LR flat PFNs X_j if:*

- *X_j are LR flat PFNs.*
- *$\Re(\sum_{j=1}^{n} A_{ij}\otimes X_j)\le,=,\ge \Re(B_i)$, for all $i=1,2,\ldots,m$.*
- *If there exist any LR flat PFNs X'_j satisfying step 2, then*
  $\Re(\sum_{j=1}^{n} C_j\otimes X_j)\ge \Re(\sum_{j=1}^{n} C_j\otimes X'_j)$ in maximization problem and
  $\Re(\sum_{j=1}^{n} C_j\otimes X_j)\le \Re(\sum_{j=1}^{n} C_j\otimes X'_j)$ in minimization problem.

3.2 Steps involved in finding the solution

Here are the steps to find the optimal solution of FPFLPP (20) with LR flat PFNs.
Step 1: Separating all the constraints into three categories, $\sum_{j=1}^{k} A_{uj}\otimes X_j\preceq B_u,\quad \forall u\in M_1$, $\sum_{j=1}^{k} A_{vj}\otimes X_j = B_v,\quad \forall v\in M_2$ and $\sum_{j=1}^{k} A_{wj}\otimes X_j\succeq B_w,\quad \forall w\in M_3$, the FPFLPP (20) can be rewritten as:

$$\text{Max/Min}\sum_{j=1}^{k} C_j\otimes X_j, \tag{21}$$

subject to
$\sum_{j=1}^{k} A_{uj} \otimes X_j \preceq B_u, \quad \forall u \in M_1,$
$\sum_{j=1}^{k} A_{vj} \otimes X_j = B_v, \quad \forall v \in M_2,$
$\sum_{j=1}^{k} A_{wj} \otimes X_j \succeq B_w, \quad \forall w \in M_3,$

where X_j are LR PFNs and $M_1 = \{i : 1 \leq i \leq m, \sum_{j=1}^{k} A_{ij} \otimes X_j \preceq B_i\}$, $M_2 = \{i : 1 \leq i \leq m, \sum_{j=1}^{k} A_{ij} \otimes X_j = B_i\}$, $M_3 = \{i : 1 \leq i \leq m, \sum_{j=1}^{k} A_{ij} \otimes X_j \succeq B_i\}$.
Step 2: By assuming $A_{ij} = [(a_{ij}^m, a_{ij}^n; p_{ij}^1, q_{ij}^1; p_{ij}^2, q_{ij}^2; p_{ij}^3, q_{ij}^3); \alpha_{ij}, \beta_{ij}, \gamma_{ij}]_{LR}$, $X_j = [(x_j^m, x_j^n; y_j^1, z_j^1; y_j^2, z_j^2; y_j^3, z_j^3); \theta_j, \phi_j, \zeta_j]_{LR}$, $B_i = [(b_i^m, b_i^n; f_i^1, g_i^1; f_i^2, g_i^2; f_i^3, g_i^3); \delta_i, \lambda_i, \sigma_i]_{LR}$, and $C_j = [(c_j^m, c_j^n; d_j^1, e_j^1; d_j^2, e_j^2; d_j^3, d_j^3); \kappa_j, \omega_j, \epsilon_j]_{LR}$, the FPFLPP (21) can be rewritten as:

$$\text{Max/Min} \sum_{j=1}^{k} [(c_j^m, c_j^n; d_j^1, e_j^1; d_j^2, e_j^2; d_j^3, d_j^3); \kappa_j, \omega_j, \epsilon_j]_{LR} \otimes [(x_j^m, x_j^n; y_j^1, z_j^1; y_j^2, z_j^2; y_j^3, z_j^3); \theta_j, \phi_j, \zeta_j]_{LR}, \tag{22}$$

subject to
$\sum_{j=1}^{k} [(a_{lj}^m, a_{lj}^n; p_{lj}^1, q_{lj}^1; p_{lj}^2, q_{lj}^2; p_{lj}^3, q_{lj}^3); \alpha_{lj}, \beta_{lj}, \gamma_{lj}]_{LR} \otimes [(x_j^m, x_j^n; y_j^1, z_j^1; y_j^2, z_j^2; y_j^3, z_j^3); \theta_j, \phi_j, \zeta_j]_{LR}$
$\preceq [(b_l^m, b_l^n; f_l^1, g_l^1; f_l^2, g_l^2; f_l^3, g_l^3); \delta_l, \lambda_l, \sigma_l]_{LR}, \quad \forall l \in M_1,$

$\sum_{j=1}^{k} [(a_{sj}^m, a_{sj}^n; p_{sj}^1, q_{sj}^1; p_{sj}^2, q_{sj}^2; p_{sj}^3, q_{sj}^3); \alpha_{sj}, \beta_{sj}, \gamma_{sj}]_{LR} \otimes [(x_j^m, x_j^n; y_j^1, z_j^1; y_j^2, z_j^2; y_j^3, z_j^3); \theta_j, \phi_j, \zeta_j]_{LR}$
$\preceq [(b_s^m, b_s^n; f_s^1, g_s^1; f_s^2, g_s^2; f_s^3, g_s^3); \delta_s, \lambda_s, \sigma_s]_{LR}, \quad \forall s \in M_3,$

$\sum_{j=1}^{k} [(a_{rj}^m, a_{rj}^n; p_{rj}^1, q_{rj}^1; p_{rj}^2, q_{rj}^2; p_{rj}^3, q_{rj}^3); \alpha_{rj}, \beta_{rj}, \gamma_{rj}]_{LR} \otimes [(x_j^m, x_j^n; y_j^1, z_j^1; y_j^2, z_j^2; y_j^3, z_j^3); \theta_j, \phi_j, \zeta_j]_{LR}$
$= [(b_r^m, b_r^n; f_r^1, g_r^1; f_r^2, g_r^2; f_r^3, g_r^3); \delta_r, \lambda_r, \sigma_r]_{LR}, \quad \forall r \in M_2,$

where $[(x_j^m, x_j^n; y_j^1, z_j^1; y_j^2, z_j^2; y_j^3, z_j^3); \theta_j, \phi_j, \zeta_j]_{LR}$ is an LR flat PFN.
Step 3: By using the product as discussed in Section 2.2 and taking $[(a_{ij}^m, a_{ij}^n; p_{ij}^1, q_{ij}^1; p_{ij}^2, q_{ij}^2; p_{ij}^3, q_{ij}^3); \alpha_{ij}, \beta_{ij}, \gamma_{ij}]_{LR} \otimes [(x_j^m, x_j^n; y_j^1, z_j^1; y_j^2, z_j^2; y_j^3, z_j^3); \theta_j, \phi_j, \zeta_j]_{LR} = [(a_{ij}^{m'}, a_{ij}^{n'}; p_{ij}^{1'}, q_{ij}^{1'}; p_{ij}^{2'}, q_{ij}^{2'}; p_{ij}^{3'}, q_{ij}^{3'}); \alpha'_{ij}, \beta'_{ij}, \gamma'_{ij}]_{LR}$, and $[(c_j^m, c_j^n; d_j^1, e_j^1; d_j^2, e_j^2; d_j^3, d_j^3); \kappa_j, \omega_j, \epsilon_j]_{LR} \otimes [(x_j^m, x_j^n; y_j^1, z_j^1; y_j^2, z_j^2; y_j^3, z_j^3); \theta_j, \phi_j, \zeta_j]_{LR} = [(x_j^{m'}, x_j^{n'}; y_j^{1'}, z_j^{1'}; y_j^{2'}, z_j^{2'}; y_j^{3'}, z_j^{3'}); \theta'_j, \phi'_j, \zeta'_j]_{LR}$ the FPFLPP (22) can be written as:

$$\text{Max/Min} \sum_{j=1}^{k} [(x_j^{m'}, x_j^{n'}; y_j^{1'}, z_j^{1'}; y_j^{2'}, z_j^{2'}; y_j^{3'}, z_j^{3'}); \theta'_j, \phi'_j, \zeta'_j]_{LR}, \tag{23}$$

subject to
$\sum_{j=1}^{k} [(a_{lj}^{m'}, a_{lj}^{n'}; p_{lj}^{1'}, q_{lj}^{1'}; p_{lj}^{2'}, q_{lj}^{2'}; p_{lj}^{3'}, q_{lj}^{3'}); \alpha'_{lj}, \beta'_{lj}, \gamma'_{lj}]_{LR} \preceq [(b_l^m, b_l^n; f_l^1, g_l^1; f_l^2, g_l^2; f_l^3, g_l^3); \delta_l, \lambda_l, \sigma_l]_{LR},$
$\forall l \in M_1,$

$\sum_{j=1}^{k} [(a_{sj}^{m'}, a_{sj}^{n'}; p_{sj}^{1'}, q_{sj}^{1'}; p_{sj}^{2'}, q_{sj}^{2'}; p_{sj}^{3'}, q_{sj}^{3'}); \alpha'_{sj}, \beta'_{sj}, \gamma'_{sj}]_{LR} \succeq [(b_s^m, b_s^n; f_s^1, g_s^1; f_s^2, g_s^2; f_s^3, g_s^3); \delta_s, \lambda_s, \sigma_s]_{LR},$
$\forall s \in M_3,$

$\sum_{j=1}^{k} [(a_{rj}^{m'}, a_{rj}^{n'}; p_{rj}^{1'}, q_{rj}^{1'}; p_{rj}^{2'}, q_{rj}^{2'}; p_{rj}^{3'}, q_{rj}^{3'}); \alpha'_{rj}, \beta'_{rj}, \gamma'_{rj}]_{LR} = [(b_r^m, b_r^n; f_r^1, g_r^1; f_r^2, g_r^2; f_r^3, g_r^3); \delta_r, \lambda_r, \sigma_r]_{LR},$
$\forall r \in M_2,$

where $[(x_j^m, x_j^n; y_j^1, z_j^1; y_j^2, z_j^2; y_j^3, z_j^3); \theta_j, \phi_j, \zeta_j]_{LR}$ is an LR flat PFN.
Step 4: Using arithmetic operations as discussed in Section 2.2, and using Definition 2.25, the FPFLPP (23) can be rewritten as:

$$\text{Max/Min} \sum_{j=1}^{k} [(x_j^{m'}, x_j^{n'}; y_j^{1'}, z_j^{1'}; y_j^{2'}, z_j^{2'}; y_j^{3'}, z_j^{3'}); \theta'_j, \phi'_j, \zeta'_j]_{LR}, \tag{24}$$

subject to
$\sum_{j=1}^{k} [(a_{lj}^{m'}, a_{lj}^{n'}; p_{lj}^{1'}, q_{lj}^{1'}; p_{lj}^{2'}, q_{lj}^{2'}; p_{lj}^{3'}, q_{lj}^{3'}); \alpha'_{lj}, \beta'_{lj}, \gamma'_{lj}]_{LR} \preceq [(b_l^m, b_l^n; f_l^1, g_l^1; f_l^2, g_l^2; f_l^3, g_l^3); \delta_l, \lambda_l, \sigma_l]_{LR},$

$\forall l \in M_1$,

$\sum_{j=1}^{k}[(a_{sj}^{m'}, a_{sj}^{n'}; p_{sj}^{1'}, q_{sj}^{1'}; p_{sj}^{2'}, q_{sj}^{2'}; p_{sj}^{3'}, q_{sj}^{3'}); \alpha'_{sj}, \beta'_{sj}, \gamma'_{sj}]_{LR} \succeq [(b_s^m, b_s^n; f_s^1, g_s^1; f_s^2, g_s^2; f_s^3, g_s^3); \delta_s, \lambda_s, \sigma_s]_{LR}$, $\forall s \in M_3$,

$\sum_{j=1}^{k} a_{rj}^{m'} = b_r^m, \quad \forall r \in M_2,$
$\sum_{j=1}^{k} a_{rj}^{n'} = b_r^n, \quad \forall r \in M_2,$
$\sum_{j=1}^{k} p_{rj}^{1'} = f_r^1, \quad \forall r \in M_2,$
$\sum_{j=1}^{k} q_{rj}^{1'} = g_r^1, \quad \forall r \in M_2,$
$\sum_{j=1}^{k} p_{rj}^{2'} = f_r^2, \quad \forall r \in M_2,$
$\sum_{j=1}^{k} q_{rj}^{2'} = g_r^2, \quad \forall r \in M_2,$
$\sum_{j=1}^{k} p_{rj}^{3'} = f_r^3, \quad \forall r \in M_2,$
$\sum_{j=1}^{k} q_{rj}^{3'} = g_r^3, \quad \forall r \in M_2,$
$\sum_{j=1}^{k} \alpha'_{rj} = \delta_r, \quad \forall r \in M_2,$
$\sum_{j=1}^{k} \beta'_{rj} = \lambda_r, \quad \forall r \in M_2,$
$\sum_{j=1}^{k} \gamma'_{rj} = \sigma_r, \quad \forall r \in M_2,$

where $[(x_j^m, x_j^n; y_j^1, z_j^1; y_j^2, z_j^2; y_j^3, z_j^3); \theta_j, \phi_j, \zeta_j]_{LR}$ is an LR flat PFN.
Step 5: By applying ranking as discussed in Definition 2.41, the FPFLPP (24) takes the form:

$$\text{Max/Min } \Re[\sum_{j=1}^{k}[(x_j^{m'}, x_j^{n'}; y_j^{1'}, z_j^{1'}; y_j^{2'}, z_j^{2'}; y_j^{3'}, z_j^{3'}); \theta'_j, \phi'_j, \zeta'_j]_{LR}], \tag{25}$$

subject to
$\Re[\sum_{j=1}^{k}[(a_{lj}^{m'}, a_{lj}^{n'}; p_{lj}^{1'}, q_{lj}^{1'}; p_{lj}^{2'}, q_{lj}^{2'}; p_{lj}^{3'}, q_{lj}^{3'}); \alpha'_{lj}, \beta'_{lj}, \gamma'_{lj}]_{LR}] \leq \Re[(b_l^m, b_l^n; f_l^1, g_l^1; f_l^2, g_l^2; f_l^3, g_l^3); \delta_l, \lambda_l, \sigma_l]_{LR}, \quad \forall l \in M_1$,

$\Re[\sum_{j=1}^{k}[(a_{sj}^{m'}, a_{sj}^{n'}; p_{sj}^{1'}, q_{sj}^{1'}; p_{sj}^{2'}, q_{sj}^{2'}; p_{sj}^{3'}, q_{sj}^{3'}); \alpha'_{sj}, \beta'_{sj}, \gamma'_{sj}]_{LR}] \geq \Re[(b_s^m, b_s^n; f_s^1, g_s^1; f_s^2, g_s^2; f_s^3, g_s^3); \delta_s, \lambda_s, \sigma_s]_{LR}, \quad \forall s \in M_3$,

$\sum_{j=1}^{k} a_{rj}^{m'} = b_r^m, \quad \forall r \in M_2,$
$\sum_{j=1}^{k} a_{rj}^{n'} = b_r^n, \quad \forall r \in M_2,$
$\sum_{j=1}^{k} p_{rj}^{1'} = f_r^1, \quad \forall r \in M_2,$
$\sum_{j=1}^{k} q_{rj}^{1'} = g_r^1, \quad \forall r \in M_2,$
$\sum_{j=1}^{k} p_{rj}^{2'} = f_r^2, \quad \forall r \in M_2,$
$\sum_{j=1}^{k} q_{rj}^{2'} = g_r^2, \quad \forall r \in M_2,$
$\sum_{j=1}^{k} p_{rj}^{3'} = f_r^3, \quad \forall r \in M_2,$
$\sum_{j=1}^{k} q_{rj}^{3'} = g_r^3, \quad \forall r \in M_2,$
$\sum_{j=1}^{k} \alpha'_{rj} = \delta_r, \quad \forall r \in M_2,$
$\sum_{j=1}^{k} \beta'_{rj} = \lambda_r, \quad \forall r \in M_2,$
$\sum_{j=1}^{k} \gamma'_{rj} = \sigma_r, \quad \forall r \in M_2,$

$x_j^m \leq x_j^n, y_j^1 \geq 0, y_j^1 \leq y_j^2, y_j^2 \leq y_j^3, z_j^1 \geq 0, z_j^1 \leq z_j^2, z_j^2 \leq z_j^3, \theta_j \geq 0, \phi_j \geq 0, \zeta_j \geq 0, 0 \leq \theta_j + \phi_j + \zeta_j \leq 1, \quad \forall j = 1, 2, \cdots, k.$
Step 6: Solve the crisp linear/non-linear programming Problem (25) by any existing method to find the optimal solution $\{x_j^{m*}, x_j^{n*}, y_j^{1*}, y_j^{2*}, y_j^{3*}, z_j^{1*}, z_j^{2*}, z_j^{3*}, \theta_j^*, \phi_j^* \zeta_j^*\}$.
Step 7: Find the LR flat picture fuzzy optimal solution X_j^* of the FPFLPP (20) by substituting the values of $x_j^{m*}, x_j^{n*}, y_j^{1*}, y_j^{2*}, y_j^{3*}, z_j^{1*}, z_j^{2*}, z_j^{3*}, \theta_j^*, \phi_j^*$ and ζ_j^* in $X_j^* = [(x_j^{m*}, x_j^{n*}; y_j^{1*}, z_j^{1*}; y_j^{2*}, z_j^{2*}; y_j^{3*}, z_j^{3*}); \theta_j^*, \phi_j^*, \zeta_j^*]_{LR}$.
Step 8: Find the LR flat picture fuzzy optimal value of the FPFLPP (20) by substituting the values of X_j^*, as calculated in Step (8), in $\sum_{j=1}^{k} C_j \otimes X_j$.

Remark 3.2. *The optimal solution of the FPFLPP (20) with LR flat PFNs as variables and parameters, obtained by using the method discussed in Section 3.1 is LR flat PFN.*

Proof. It can be seen from the discussed methodology that when the problem is translated from picture fuzzy problem to crisp problem, additional constraints (i.e., $x_j^m \leq x_j^n, y_j^1 \geq 0, y_j^1 \leq y_j^2, y_j^2 \leq y_j^3, z_j^1 \geq 0, z_j^1 \leq z_j^2, z_j^2 \leq z_j^3, \theta_j \geq 0, \phi_j \geq 0, \zeta_j \geq 0, 0 \leq \theta_j + \phi_j + \zeta_j \leq 1, \quad \forall j = 1, 2, \cdots, k.$) are considered, which in fact guarantee the fact that the obtained optimal solution is an LR flat PFN. □

3.3 Illustrative example

Example 3.3. *An auto workshop needs around* $[(430, 480; 13, 14; 16, 18; 19, 21); 0.79, 0.018, 0.018]_{LR}$ *cans of paints per month, the supply consists of three different colors: black, blue and green. Cost of black can is Rs.* $[(140, 160; 4, 5; 5, 6; 6, 7); 0.4, 0.2, 0.3]_{LR}$ *per can , cost of blue can is Rs.* $[(150, 180; 3, 3; 4, 4; 5, 5); 0.5, 0.3, 0.2]_{LR}$ *per can and cost of green*

can is Rs. $[(160,170;2,3;3,4;4,5);0.4,0.2,0.2]_{LR}$ *per can. The painting work requires at least* $[(60,70;3,3;4,4;5,5);0.4,0.3,0.2]_{LR}$ *cans of black paint, not more than* $[(200,220;2,2;3,3;5,5);0.4,0.3,0.3]_{LR}$ *cans of blue paint and at least* $[(100,120;4,4;5,5;6,6);0.4,0.3,0.2]_{LR}$ *cans of green paint. How many cans of each of these paints should be purchased to minimize the total cost? [taking linear L and R functions, i. e., if* $X=[(m,n;l,r;l,r;l,r);\alpha,\beta,\gamma]_{LR}$*, then* $L_1(y)=R_1(y)=\max\{0,\alpha-y\},\ L_2(y)=R_2(y)=\max\{0,\beta-y\},\ L_3(y)=R_3(y)=\min\{1,\frac{y-\gamma y+\gamma^2}{\gamma}\}$*].*
Here we note that all the numbers are LR flat PFNs and not crisp one. They explain the fuzzy quantities. For example, $[(430,480;13,14;16,18;19,21);0.79,0.018,0.018]_{LR}$ *cans of paints mean the number of cans vary from* 411 *cans to* 501 *cans having certainty, confusion and negativity given by the positive membership, neutral membership and negative membership functions. Similarly all the quantities are fuzzy and their memberships represent how much fuzziness they contain. Let* X_1*,* X_2 *and* X_3 *be the number of black cans, blue cans and green cans, respectively. This problem can be formulated as FPFLPP as:*

$$Max\ [(140,160;4,5;5,6;6,7);0.4,0.2,0.3]_{LR}\otimes X_1\oplus[(150,180;3,3;4,4;5,5);0.5,0.3,0.2]_{LR}\otimes X_2 \oplus[(160,170;2,3;3,4;4,5);0.4,0.2,0.2]_{LR}\otimes X_3$$

subject to
$X_1\succeq[(60,70;3,3;4,4;5,5);0.4,0.3,0.2]_{LR},\ X_2\preceq[(200,220;2,2;3,3;5,5);0.4,0.3,0.3]_{LR},$
$X_3\succeq[(100,120;4,4;5,5;6,6);0.4,0.3,0.2]_{LR},$
$X_1\oplus X_2\oplus X_3=[(430,480;13,14;16,18;19,21);0.79,0.018,0.018]_{LR},$
where X_1*,* X_2 *and* X_3 *are non-negative LR flat PFNs.*

Step 1: *Let* $X_1=[(m^1,n^1;l_1^1,r_1^1;l_2^1,r_2^1;l_3^1,r_3^1);\alpha^1,\beta^1,\gamma^1]_{LR}$*,* $X_2=[(m^2,n^2;l_1^2,r_1^2;l_2^2,r_2^2;l_3^2,r_3^2);\alpha^2,\beta^2,\gamma^2]_{LR}$ *and* $X_3=[(m^3,n^3;l_1^3,r_1^3;l_2^3,r_2^3;l_3^3,r_3^3);\alpha^3,\beta^3,\gamma^3]_{LR}$*, then problem can be written as:*

$$Max\ [(140,160;4,5;5,6;6,7);0.4,0.2,0.3]_{LR}\otimes[(m^1,n^1;l_1^1,r_1^1;l_2^1,r_2^1;l_3^1,r_3^1);\alpha^1,\beta^1,\gamma^1]_{LR}$$
$$\oplus[(150,180;3,3;4,4;5,5);0.5,0.3,0.2]_{LR}\otimes[(m^2,n^2;l_1^2,r_1^2;l_2^2,r_2^2;l_3^2,r_3^2);\alpha^2,\beta^2,\gamma^2]_{LR}$$
$$\oplus[(160,170;2,3;3,4;4,5);0.4,0.2,0.2]_{LR}\otimes[(m^3,n^3;l_1^3,r_1^3;l_2^3,r_2^3;l_3^3,r_3^3);\alpha^3,\beta^3,\gamma^3]_{LR}$$

subject to
$[(m^1,n^1;l_1^1,r_1^1;l_2^1,r_2^1;l_3^1,r_3^1);\alpha^1,\beta^1,\gamma^1]_{LR}\succeq[(60,70;3,3;4,4;5,5);0.4,0.3,0.2]_{LR},$

$[(m^2,n^2;l_1^2,r_1^2;l_2^2,r_2^2;l_3^2,r_3^2);\alpha^2,\beta^2,\gamma^2]_{LR}\preceq[(200,220;2,2;3,3;5,5);0.4,0.3,0.3]_{LR},$

$[(m^3,n^3;l_1^3,r_1^3;l_2^3,r_2^3;l_3^3,r_3^3);\alpha^3,\beta^3,\gamma^3]_{LR}\succeq[(100,120;4,4;5,5;6,6);0.4,0.3,0.2]_{LR},$

$[(m^1,n^1;l_1^1,r_1^1;l_2^1,r_2^1;l_3^1,r_3^1);\alpha^1,\beta^1,\gamma^1]_{LR}\oplus[(m^2,n^2;l_1^2,r_1^2;l_2^2,r_2^2;l_3^2,r_3^2);\alpha^2,\beta^2,\gamma^2]_{LR}$
$\oplus[(m^3,n^3;l_1^3,r_1^3;l_2^3,r_2^3;l_3^3,r_3^3);\alpha^3,\beta^3,\gamma^3]_{LR}=[(430,480;13,14;16,18;19,21);0.79,0.018,0.018]_{LR},$

where $[(m^1,n^1;l_1^1,r_1^1;l_2^1,r_2^1;l_3^1,r_3^1);\alpha^1,\beta^1,\gamma^1]_{LR}$*,* $[(m^2,n^2;l_1^2,r_1^2;l_2^2,r_2^2;l_3^2,r_3^2);\alpha^2,\beta^2,\gamma^2]_{LR}$
and $[(m^3,n^3;l_1^3,r_1^3;l_2^3,r_2^3;l_3^3,r_3^3);\alpha^3,\beta^3,\gamma^3]_{LR}$ *are non-negative LR flat PFNs.*

Step 2: *Using product as discussed in Section 2.2, the FPFLPP, obtained in Step 1, can be written as:*

$Max\ [(140m^1,160n^1;136l_1^1+4m^1,165r_1^1+5n^1;135l_2^1+5m^1,166r_2^1+6n^1;134l_3^1+6m^1,167r_3^1+7n^1);0.4\alpha^1,0.2\beta^1,0.3+\gamma^1-0.3\gamma^1]_{LR}\oplus[(150m^2,180n^2;147l_1^2+3m^2,183r_1^2+3n^2;146l_2^2+4m^2,184r_2^2+4n^2;145l_3^2+5m^2,185r_3^2+5n^2);0.5\alpha^2,0.3\beta^2,0.2+\gamma^2-0.2\gamma^2]_{LR}\oplus[(160m^3,170n^3;158l_1^3+2m^3,173r_1^3+3n^3;157l_2^3+3m^3,174r_2^3+4n^3;156l_3^3+4m^3,175r_3^3+5n^3);0.4\alpha^3,0.2\beta^3,0.2+\gamma^3-0.2\gamma^3]_{LR}$

subject to
$[(m^1,n^1;l_1^1,r_1^1;l_2^1,r_2^1;l_3^1,r_3^1);\alpha^1,\beta^1,\gamma^1]_{LR}\succeq[(60,70;3,3;4,4;5,5);0.4,0.3,0.2]_{LR},$

$[(m^2,n^2;l_1^2,r_1^2;l_2^2,r_2^2;l_3^2,r_3^2);\alpha^2,\beta^2,\gamma^2]_{LR}\preceq[(200,220;2,2;3,3;5,5);0.4,0.3,0.3]_{LR},$

$[(m^3,n^3;l_1^3,r_1^3;l_2^3,r_2^3;l_3^3,r_3^3);\alpha^3,\beta^3,\gamma^3]_{LR}\succeq[(100,120;4,4;5,5;6,6);0.4,0.3,0.2]_{LR},$

$[(m^1,n^1;l_1^1,r_1^1;l_2^1,r_2^1;l_3^1,r_3^1);\alpha^1,\beta^1,\gamma^1]_{LR}\oplus[(m^2,n^2;l_1^2,r_1^2;l_2^2,r_2^2;l_3^2,r_3^2);\alpha^2,\beta^2,\gamma^2]_{LR}$
$\oplus[(m^3,n^3;l_1^3,r_1^3;l_2^3,r_2^3;l_3^3,r_3^3);\alpha^3,\beta^3,\gamma^3]_{LR}=[(430,480;13,14;16,18;19,21);0.79,0.018,0.018]_{LR},$

where $[(m^1,n^1;l_1^1,r_1^1;l_2^1,r_2^1;l_3^1,r_3^1);\alpha^1,\beta^1,\gamma^1]_{LR}$, $[(m^2,n^2;l_1^2,r_1^2;l_2^2,r_2^2;l_3^2,r_3^2);\alpha^2,\beta^2,\gamma^2]_{LR}$ *and* $[(m^3,n^3;l_1^3,r_1^3;l_2^3,r_2^3;l_3^3,r_3^3);\alpha^3,\beta^3,\gamma^3]_{LR}$ *are non-negative LR flat PFNs.*

Step 3: *By using arithmetic operations discussed in Section 2.2 and using Definition 2.25, the FPFLPP, obtained in Step 2, can be rewritten as:*

$$Max\ [(140m^1+150m^2+160m^3,160n^1+180n^2+170n^3;136l_1^1+4m^1+147l_1^2+3m^2+158l_1^3+2m^3,165r_1^1+5n^1+183r_1^2+3n^2+173r_1^3+3n^3;135l_2^1+5m^1+146l_2^2+4m^2+157l_2^3+3m^3,166r_2^1+6n^1+184r_2^2+4n^2+174r_2^3+4n^3;134l_3^1+6m^1+145l_3^2+5m^2+156l_3^3+4m^3,167r_3^1+7n^1+185r_3^2+5n^2+175r_3^3+5n^3);0.4\alpha^1+0.5\alpha^2+0.4\alpha^3-0.2\alpha^1\alpha^2-0.16\alpha^1\alpha^3-0.2\alpha^2\alpha^3+0.08\alpha^1\alpha^2\alpha^3,0.012\beta^1\beta^2\beta^3,0.012+0.028\gamma^1+0.04\gamma^2+0.048\gamma^3+0.112\gamma^1\gamma^2+0.16\gamma^2\gamma^3+0.112\gamma^1\gamma^3+0.448\gamma^1\gamma^2\gamma^3]_{LR}$$

subject to

$[(m^1,n^1;l_1^1,r_1^1;l_2^1,r_2^1;l_3^1,r_3^1);\alpha^1,\beta^1,\gamma^1]_{LR}\succeq[(60,70;3,3;4,4;5,5);0.4,0.3,0.2]_{LR},$

$[(m^2,n^2;l_1^2,r_1^2;l_2^2,r_2^2;l_3^2,r_3^2);\alpha^2,\beta^2,\gamma^2]_{LR}\preceq[(200,220;2,2;3,3;5,5);0.4,0.3,0.3]_{LR},$

$[(m^3,n^3;l_1^3,r_1^3;l_2^3,r_2^3;l_3^3,r_3^3);\alpha^3,\beta^3,\gamma^3]_{LR}\succeq[(100,120;4,4;5,5;6,6);0.4,0.3,0.2]_{LR},$

$m^1+m^2+m^3=430,$
$n^1+n^2+n^3=480,$
$l_1^1+l_1^2+l_1^3=13,$
$r_1^1+r_1^2+r_1^3=14,$
$l_2^1+l_2^2+l_2^3=16,$
$r_2^1+r_2^2+r_2^3=18,$
$l_3^1+l_3^2+l_3^3=19,$
$r_3^1+r_3^2+r_3^3=21,$
$\alpha^1+\alpha^2+\alpha^3-\alpha^1\alpha^2-\alpha^1\alpha^3-\alpha^2\alpha^3+\alpha^1\alpha^2\alpha^3=0.79,$
$\beta^1\beta^2\beta^3=0.018,$
$\gamma^1\gamma^2\gamma^3=0.018,$

where $[(m^1,n^1;l_1^1,r_1^1;l_2^1,r_2^1;l_3^1,r_3^1);\alpha^1,\beta^1,\gamma^1]_{LR}$, $[(m^2,n^2;l_1^2,r_1^2;l_2^2,r_2^2;l_3^2,r_3^2);\alpha^2,\beta^2,\gamma^2]_{LR}$ *and* $[(m^3,n^3;l_1^3,r_1^3;l_2^3,r_2^3;l_3^3,r_3^3);\alpha^3,\beta^3,\gamma^3]_{LR}$ *are non-negative LR flat PFNs.*

Step 4: *Using Step 5 of the method proposed in Section 3.2, the FPFLPP, obtained in Step 3, can be rewritten as:*

$$Max\ \Re[(140m^1+150m^2+160m^3,160n^1+180n^2+170n^3;136l_1^1+4m^1+147l_1^2+3m^2+158l_1^3+2m^3,165r_1^1+5n^1+183r_1^2+3n^2+173r_1^3+3n^3;135l_2^1+5m^1+146l_2^2+4m^2+157l_2^3+3m^3,166r_2^1+6n^1+184r_2^2+4n^2+174r_2^3+4n^3;134l_3^1+6m^1+145l_3^2+5m^2+156l_3^3+4m^3,167r_3^1+7n^1+185r_3^2+5n^2+175r_3^3+5n^3);0.4\alpha^1+0.5\alpha^2+0.4\alpha^3-0.2\alpha^1\alpha^2-0.16\alpha^1\alpha^3-0.2\alpha^2\alpha^3+0.08\alpha^1\alpha^2\alpha^3,0.012\beta^1\beta^2\beta^3,0.012+0.028\gamma^1+0.04\gamma^2+0.048\gamma^3+0.112\gamma^1\gamma^2+0.16\gamma^2\gamma^3+0.112\gamma^1\gamma^3+0.448\gamma^1\gamma^2\gamma^3]_{LR}$$

subject to

$\Re[(m^1,n^1;l_1^1,r_1^1;l_2^1,r_2^1;l_3^1,r_3^1);\alpha^1,\beta^1,\gamma^1]_{LR}\geq\Re[(60,70;3,3;4,4;5,5);0.4,0.3,0.2]_{LR},$

$\Re[(m^2,n^2;l_1^2,r_1^2;l_2^2,r_2^2;l_3^2,r_3^2);\alpha^2,\beta^2,\gamma^2]_{LR}\leq\Re[(200,220;2,2;3,3;5,5);0.4,0.3,0.3]_{LR},$

$\Re[(m^3,n^3;l_1^3,r_1^3;l_2^3,r_2^3;l_3^3,r_3^3);\alpha^3,\beta^3,\gamma^3]_{LR}\geq\Re[(100,120;4,4;5,5;6,6);0.4,0.3,0.2]_{LR},$

$m^1+m^2+m^3=430,$
$n^1+n^2+n^3=480,$
$l_1^1+l_1^2+l_1^3=13,$
$r_1^1+r_1^2+r_1^3=14,$
$l_2^1+l_2^2+l_2^3=16,$
$r_2^1+r_2^2+r_2^3=18,$
$l_3^1+l_3^2+l_3^3=19,$
$r_3^1+r_3^2+r_3^3=21,$
$\alpha^1+\alpha^2+\alpha^3-\alpha^1\alpha^2-\alpha^1\alpha^3-\alpha^2\alpha^3+\alpha^1\alpha^2\alpha^3=0.79,$
$\beta^1\beta^2\beta^3=0.018,$
$\gamma^1\gamma^2\gamma^3=0.018,$

$m^1\leq n^1,\ l_1^1\geq 0,\ l_1^1\leq l_2^1,\ l_2^1\leq l_3^1,\ r_1^1\geq 0,\ r_1^1\leq r_2^1,\ r_2^1\leq r_3^1,\ \alpha^1\geq 0,\ \beta^1\geq 0,\ \gamma^1\geq 0,\ \alpha^1+\beta^1+\gamma^1\geq 0,$
$m^2\leq n^2,\ l_1^2\geq 0,\ l_1^2\leq l_2^2,\ l_2^2\leq l_3^2,\ r_1^2\geq 0,\ r_1^2\leq r_2^2,\ r_2^2\leq r_3^2,\ \alpha^2\geq 0,\ \beta^2\geq 0,\ \gamma^2\geq 0,\ \alpha^2+\beta^2+\gamma^2\geq 0,$
$m^3\leq n^3,\ l_1^3\geq 0,\ l_1^3\leq l_2^3,\ l_2^3\leq l_3^3,\ r_1^3\geq 0,\ r_1^3\leq r_2^3,\ r_2^3\leq r_3^3,\ \alpha^3\geq 0,\ \beta^3\geq 0,\ \gamma^3\geq 0,\ \alpha^3+\beta^3+\gamma^3\geq 0.$

Step 5: *Using Definition 2.41, the FPFLPP, obtained in Step 4, can be written as:*

$$Max\ \frac{1}{2}[(0.4\alpha^1+0.5\alpha^2+0.4\alpha^3-0.2\alpha^1\alpha^2-0.16\alpha^1\alpha^3-0.2\alpha^2\alpha^3+0.08\alpha^1\alpha^2\alpha^3)(140m^1+150m^2+160m^3+160n^1+180n^2+$$

$170n^3-\frac{1}{2}(136l_1^1+4m^1+147l_1^2+3m^2+158l_1^3+2m^3)+\frac{1}{2}(165r_1^1+5n^1+183r_1^2+3n^2+173r_1^3+3n^3))+(0.012\beta^1\beta^2\beta^3)(140m^1+150m^2+160m^3+160n^1+180n^2+170n^3-\frac{1}{2}(135l_2^1+5m^1+146l_2^2+4m^2+157l_2^3+3m^3)+\frac{1}{2}(166r_2^1+6n^1+184r_2^2+4n^2+174r_2^3+4n^3))+(1-(0.012+0.028\gamma^1+0.04\gamma^2+0.048\gamma^3+0.112\gamma^1\gamma^2+0.16\gamma^2\gamma^3+0.112\gamma^1\gamma^3+0.448\gamma^1\gamma^2\gamma^3))(140m^1+150m^2+160m^3+160n^1+180n^2+170n^3-\frac{1}{2}(134l_3^1+6m^1+145l_3^2+5m^2+156l_3^3+4m^3)+\frac{1}{2}(167r_3^1+7n^1+185r_3^2+5n^2+175r_3^3+5n^3))]$

subject to

$\frac{1}{2}[\alpha^1(m^1+n^1-\frac{l_1^1}{2}+\frac{r_1^1}{2})+\beta^1(m^1+n^1-\frac{l_2^1}{2}+\frac{r_2^1}{2})+(1-\gamma^1)(m^1+n^1-\frac{l_3^1}{2}+\frac{r_3^1}{2})]\geq 97.5,$

$\frac{1}{2}[\alpha^2(m^2+n^2-\frac{l_1^2}{2}+\frac{r_1^2}{2})+\beta^2(m^2+n^2-\frac{l_2^2}{2}+\frac{r_2^2}{2})+(1-\gamma^2)(m^2+n^2-\frac{l_3^2}{2}+\frac{r_3^2}{2})]\leq 294,$

$\frac{1}{2}[\alpha^3(m^3+n^3-\frac{l_1^3}{2}+\frac{r_1^3}{2})+\beta^3(m^3+n^3-\frac{l_2^3}{2}+\frac{r_2^3}{2})+(1-\gamma^3)(m^3+n^3-\frac{l_3^3}{2}+\frac{r_3^3}{2})]\geq 165.$

$m^1+m^2+m^3=430,$
$n^1+n^2+n^3=480,$
$l_1^1+l_1^2+l_1^3=13,$
$r_1^1+r_1^2+r_1^3=14,$
$l_2^1+l_2^2+l_2^3=16,$
$r_2^1+r_2^2+r_2^3=18,$
$l_3^1+l_3^2+l_3^3=19,$
$r_3^1+r_3^2+r_3^3=21,$
$\alpha^1+\alpha^2+\alpha^3-\alpha^1\alpha^2-\alpha^1\alpha^3-\alpha^2\alpha^3+\alpha^1\alpha^2\alpha^3=0.79,$
$\beta^1\beta^2\beta^3=0.018,$
$\gamma^1\gamma^2\gamma^3=0.018,$

$m^1\leq n^1,\ l_1^1\geq 0,\ l_1^1\leq l_2^1,\ l_2^1\leq l_3^1,\ r_1^1\geq 0,\ r_1^1\leq r_2^1,\ r_2^1\leq r_3^1,\ \alpha^1\geq 0,\ \beta^1\geq 0,\ \gamma^1\geq 0,\ \alpha^1+\beta^1+\gamma^1\geq 0,$
$m^2\leq n^2,\ l_1^2\geq 0,\ l_1^2\leq l_2^2,\ l_2^2\leq l_3^2,\ r_1^2\geq 0,\ r_1^2\leq r_2^2,\ r_2^2\leq r_3^2,\ \alpha^2\geq 0,\ \beta^2\geq 0,\ \gamma^2\geq 0,\ \alpha^2+\beta^2+\gamma^2\geq 0,$
$m^3\leq n^3,\ l_1^3\geq 0,\ l_1^3\leq l_2^3,\ l_2^3\leq l_3^3,\ r_1^3\geq 0,\ r_1^3\leq r_2^3,\ r_2^3\leq r_3^3,\ \alpha^3\geq 0,\ \beta^3\geq 0,\ \gamma^3\geq 0,\ \alpha^3+\beta^3+\gamma^3\geq 0.$

Step 6: *The optimal solution of the crisp non-linear programming problem, obtained in Step 5,(using: MATLAB R2014a, solver "fmincon", algorithm "interior point", TolFun= 1, TolX=eps, TolCon= 1) is*
$m^1=100,\ n^1=120,\ l_1^1=2.9,\ r_1^1=3.9,\ l_2^1=3.9,\ r_2^1=4.9,\ l_3^1=5,\ r_3^1=5.9,\ \alpha^1=0.37,\ \beta^1=0.06,\ \gamma^1=0.3,$
$m^2=179.9,\ n^2=199.9,\ l_1^2=4,\ r_1^2=4,\ l_2^2=5,\ r_2^2=5,\ l_3^2=6,\ r_3^2=6,\ \alpha^2=0.4,\ \beta^2=0.4,\ \gamma^2=0.2,$
$m^3=149.9,\ n^3=160,\ l_1^3=5.9,\ r_1^3=6,\ l_2^3=6.9,\ r_2^3=8,\ l_3^3=7.9,\ r_3^3=9,\ \alpha^3=0.27,\ \beta^3=0.13$ *and* $\gamma^3=0.37.$

Step 7: *Substituting the values of* $m^1,\ n^1,\ l_1^1,\ r_1^1,\ l_2^1,\ r_2^1,\ l_3^1,\ r_3^1,\ \alpha^1,\ \beta^1,\ \gamma^1,\ m^2,\ n^2,\ l_1^2,\ r_1^2,\ l_2^2,\ r_2^2,\ l_3^2,\ r_3^2,\ \alpha^2,\ \beta^2,\ \gamma^2,\ m^3,\ n^3,\ l_1^3,\ r_1^3,\ l_2^3,\ r_2^3,\ l_3^3,\ r_3^3,\ \alpha^3,\ \beta^3,$ *and* γ^3 *in*
$X_1=[(m^1,n^1;l_1^1,r_1^1;l_2^1,r_2^1;l_3^1,r_3^1);\alpha^1,\beta^1,\gamma^1]_{LR},$
$X_2=[(m^2,n^2;l_1^2,r_1^2;l_2^2,r_2^2;l_3^2,r_3^2);\alpha^2,\beta^2,\gamma^2]_{LR}$ *and*
$X_3=[(m^3,n^3;l_1^3,r_1^3;l_2^3,r_2^3;l_3^3,r_3^3);\alpha^3,\beta^3,\gamma^3]_{LR},$
the exact LR flat picture fuzzy optimal solution is:
$X_1=[(100,120;2.9,3.9;3.9,4.9;5,5.9);0.37,0.06,0.3]_{LR},$
$X_2=[(179.9,199.9;4,4;5,5;6,6);0.4,0.4,0.2]_{LR}$ *and*
$X_3=[(149.9,160;5.9,6;6.9,8;7.9,9);0.27,0.13,0.37]_{LR}.$

Step 8: *Substituting the values of* X_1, X_2 *and* X_3, *obtained in Step 7, into the objective function, the picture fuzzy optimal value is* $[(64969,82382;3154.1,4153.2;4009.1,5284.6;4871.5,6309.8);0.3920,3.744\times 10^{-5},0.0911]_{LR}$. *Thus, the owner of the workshop should buy* $[(100,120;2.9,3.9;3.9,4.9;5,5.9);0.37,0.06,0.3]_{LR}$ *black cans,* $[(179.9,199.9;4,4;5,5;6,6);0.4,0.4,0.2]_{LR}$ *blue cans and* $[(149.9,160;5.9,6;6.9,8;7.9,9);0.27,0.13,0.37]_{LR}$ *green cans of paints at a minimum cost of Rs.* $[(64969,82382;3154.1,4153.2;4009.1,5284.6;4871.5,6309.8);0.3920,3.744\times 10^{-5},0.0911]_{LR}$. *Here we note that* $[(100,120;2.9,3.9;3.9,4.9;5,5.9);0.37,0.06,0.3]_{LR}$ *black cans means that the owner should buy the black cans ranging from* 95 *to* 125.9 *with the risk given by the positive membership, neutral membership and negative membership. Similarly all the quantities are fuzzy and show how much the range of the quantity should be in order to minimize the risk. The optimal value is shown graphically in the Figure 6. The shape of the membership function explains how much we are sure about the minimum price. Similarly, the neutral membership and negative membership functions explain how much we are confused and not in favor of this minimum cost, respectively.*

If we consider only the membership function in all variables and parameters of the Example 3.3, we get the following problem:
Max $(140,160;4,5)_{LR}\otimes X_1\oplus(150,180;3,3)_{LR}\otimes X_2\oplus(160,170;2,3)_{LR}\otimes X_3,$
subject to
$X_1\succeq(60,70;3,3)_{LR},\ X_2\preceq(200,220;2,2)_{LR},\ X_3\succeq(100,120;4,4)_{LR},$
$X_1\oplus X_2\oplus X_3=(430,480;13,14)_{LR},$

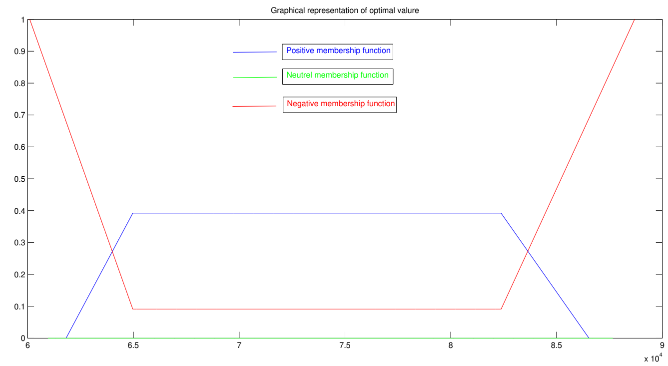

Figure 6: Graphical representation of the optimal value of the Example 3.3 using the method discussed in Section 3.1

where $L(y) = R_(y) = \max\{0, 1 - y\}$ and X_1, X_2 and X_3 are non-negative LR flat fuzzy numbers.
Using the method presented by Kumar and Kaur [13], we get the optimal solution as: $X_1 = (100, 120; 3, 4)_{LR}$, $X_2 = (180, 200; 4, 4)_{LR}$, $X_3 = (150, 160; 6, 6)_{LR}$. The optimal value comes out to be $(65000, 82400; 3184, 4110)_{LR}$. We thus conclude that our method is the true extension of the Kumar and Kaur's method [13] which confirms the justification of the proposed method.

4 Conclusion

In this manuscript, we have studied TrPFNs by pointing out a drawback in the existing definition of TrPFNs and fixed it along with defining their arithmetic operations. We have defined the TrPFNs in two possible ways depending upon the shape of the neutral membership function. Further, we have defined a general type of PFNs, namely, LR flat PFNs and their arithmetic operations. We have defined LR flat PFNs in two possible ways depending upon the shape of the neutral membership function which is the decision maker's ultimate interest. We have defined a generalized ranking function for LR flat PFNs. We have considered FPFLPP with unrestricted LR flat PFNs and used the ranking function to solve such kind of problems. The practical model is solved to support the proposed concept. The proposed method is compared with Kumar and Kaur's method to check the validity of the method. We plan to extend our research work for nonlinear programming problems, fractional programming problems, and transportation problems under a picture fuzzy environment.
Conflict of Interest : The authors declare no conflict of interest.

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