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Pseudo L-algebras

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Abstract

We introduce generalized structures of L-algebras, called pseudo L-algebras, which are the multiplication reduct of pseudo hoops and are structures combining two L-algebras with one compatible order. We prove that every pseudo hoop gives rise to a pseudo L-algebra and every pseudo effect algebra gives rise to a pseudo L-algebra. The self-similarity is the most important property of an L-algebra L, which guarantees to induce a multiplication on L. We introduce a notion of self-similar pseudo L-algebras and prove that a self-similar pseudo L-algebra becomes an L-algebra if and only if the multiplication $\odot$ is commutative. We get some interesting results for self-similar pseudo L-algebras: (1) The negative cone G^- of an ℓ-group G can be seen as a self-similar pseudo L-algebra. (2) Every self-similar pseudo L-algebra is a pseudo hoop. Next, we introduce the notion of self-similar closures of pseudo L-algebras and obtain a self-similar closure by a recursive method. Given a pseudo L-algebra $(L, \to, \rightsquigarrow, 1)$, we can generate a free semigroup $(A, *)$ by the set $L \setminus \{1\}$. Furthermore, we let $S(L) = A \cup \{1\}$ and define a binary operation $\odot$ on $S(L)$. Then we extend the operations $\to$ and $\rightsquigarrow$ from L to $S(L)$, and prove that $(S(L), \to, 1)$ and $(S(L), \rightsquigarrow, 1)$ are two cycloids, respectively. Furthermore, under some conditions, $(S(L), \to, \rightsquigarrow, 1)$ becomes a self-similar pseudo L-algebra. Finally, we introduce the notion of the structure group of pseudo L-algebras, and give an interesting example to show how to extend a pseudo L-algebra L into the pseudo self-similar closure $S(L)$, and furthermore, derive it's structure group $G(L)$.

Keywords: Pseudo L-algebra, self-similar closure of pseudo L-algebra, structure group of pseudo L-algebra, pseudo hoop, ℓ-group.

1 Introduction

The quantum Yang-Baxter equation is one of the basic equations in mathematical physics. This equation involves a linear operator $R : V \otimes V \to V \otimes V$, where V is a vector space, and has the form

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12} \text{ in } End(V \otimes V \otimes V),$$

where R^{ij} means R acting on the i-th and j-th components of $V \otimes V \otimes V$ [11].

The equation (L) $(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$ first appeared in algebraic logic [3, 19, 34]. Independently, it was found in connection with Garside groups [7, 8, 28]. Rump proved that every set X with a binary operation $\cdot$ satisfying the equation (L) corresponds to a solution of the quantum Yang-Baxter equation if the left multiplication is bijective [25, 26]. Based on the equation (L), the concept of L-algebras was introduced in [27]. Further, it was proved that for each L-algebra X there is a self-similar closure $S(X)$. At the same time, $S(X)$ admits a left hoop [27]. Recently, it was also proved that each L-algebra with negation admits an MV-algebra [35]. Since L-algebras have been combined with quantum set [32], group theory [28, 29, 30], lattice theory [31] and other field. In this year, Ciungu discussed the relation between L-algebras and BCK-algebras, MV-algebras, bounded Rl-monoids and BE-algebras, respectively [5]. The study of L-algebras has attracted more attention of many scholars.

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The non-commutative generalizations of MV-algebras called pseudo-MV algebras were introduced by G. Georgescu and A. Iorgulescu in [14] and [16] and they can be regarded as algebraic semantics for a non-commutative generalization of a multiple-valued reasoning [24]. A. Dvurečenskij proved in [9] that any pseudo-MV algebra is isomorphic with some interval in an ℓ-group with strong unit, that is, the category of pseudo-MV algebras is equivalent to the category of unital ℓ-groups. G. Georgescu and A. Iorgulescu introduced in [15] the pseudo-BL algebras as a natural generalization of BL-algebras in the non-commutative case. Non-commutative residuated lattices, sometimes called pseudo-residuated lattices or generalized residuated lattices, are the algebraic counterparts of substructural logics, i.e. logics which lack at least one of the three structural rules, namely contraction, weakening and exchange. Another generalization of pseudo-BL algebras, pseudo-hoops were defined and studied in [18]. Pseudo-hoops were originally introduced by Bosbach in [1] and [2] under the name of complementary semigroups. It was proved that a pseudo-hoop has the pseudo-divisibility condition and it is a meet-semilattice, so a bounded Rℓ-monoid can be viewed as a bounded pseudo-hoop together with the join-semilattice property. In other words, a bounded pseudo-hoop is a meet semilattice ordered residuated, integral and divisible monoid. Pseudo-BCK algebras were introduced in 2001 by G. Georgescu and A. Iorgulescu [17] as non-commutative generalizations of BCK-algebras. Properties of pseudo-BCK algebras and their connection with other fuzzy structures were established by A. Iorgulescu in [20, 21, 22, 23]. Ciungu [4] introduced commutative pseudo BE-algebras which are the generalization of commutative BE-algebras in 2016. The author proved that the class of commutative pseudo BE-algebras is equivalent to the class of commutative pseudo BCK-algebras.

Combining above observations, we can see that there are the corresponding pseudo structures for residuated lattices, BL-algebras, MTL-algebras and equality algebras, which have closed relations with L-algebras, but there is no pseudo structure for L-algebras. In order to penetrate the links among several pseudo structures, we have to consider to set up the pseudo L-algebras. This is one of our motivation for this paper. Secondly, it is an interesting thing how to induce the semigroup multiplications in some implication pseudo structures $(X, \rightarrow, \rightsquigarrow)$, such as pseudo BCK-algebras, pseudo equality algebras and pseudo BE-algebras, etc. Through self-similarity of pseudo L-algebras, we can induce the semigroup operations on self-similar pseudo L-algebras. This can provide a general method to induce semigroup operations on some implication pseudo structures, which is another motivation for this paper. Thirdly, it is well-known that the multiplication reducts of EQ-algebras are equality algebras. Motivated by this fact, we set up pseudo L-algebras from the multiplication reduct of pseudo hoops.

2 Pseudo L-algebras

In this section, we intend to give a multiplication reduct of pseudo-hoop, which is called pseudo L-algebras. Moreover, some examples are listed to show that pseudo L-algebras widely exist.

Definition 2.1. [18] *A pseudo-hoop is an algebra $(L, \odot, \rightarrow, \rightsquigarrow, 1)$ of the type (2,2,2,0) such that, for all $x, y, z \in L$:*
$(PSH1)$ $x \odot 1 = 1 \odot x = x$;
$(PSH2)$ $x \rightarrow x = x \rightsquigarrow x = 1$;
$(PSH3)$ $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$;
$(PSH4)$ $(x \odot y) \rightsquigarrow z = y \rightsquigarrow (x \rightsquigarrow z)$;
$(PSH5)$ $(x \rightarrow y) \odot x = (y \rightarrow x) \odot y = x \odot (x \rightsquigarrow y) = y \odot (y \rightsquigarrow x)$.

Definition 2.2. [27] *An L-algebra is an algebra $(L, \rightarrow, 1)$ of type $(2, 0)$ satisfying*
$(L1)$ $x \rightarrow x = x \rightarrow 1 = 1$, $1 \rightarrow x = x$;
$(L2)$ $(x \rightarrow y) \rightarrow (x \rightarrow z) = (y \rightarrow x) \rightarrow (y \rightarrow z)$;
$(L3)$ $x \rightarrow y = y \rightarrow x = 1$ *implies* $x = y$,
for all $x, y, z \in L$.

Supposing that $(L, \odot, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-hoop, it is obvious that $(L, \odot, \rightarrow, 1)$ is a left hoop and $(L, \odot, \rightsquigarrow, 1)$ is right hoop in [33]. A left hoop $(L, \odot, \rightarrow, 1)$ gives rise to an L-algebra $(L, \rightarrow, 1)$ [27]; similarly a right hoop $(L, \odot, \rightsquigarrow, 1)$ also induces an L-algebra $(L, \rightsquigarrow, 1)$. In this way, a pseudo-hoop $(L, \odot, \rightarrow, \rightsquigarrow, 1)$ generates two L-algbras $(L, \rightarrow, 1)$ and $(L, \rightsquigarrow, 1)$. Moreover $a \rightarrow b = 1$ iff $a \rightsquigarrow b = 1$ [18]. Based on these facts, we give the concept of pseudo L-algebra as follows.

Definition 2.3. *A pseudo L-algebras is an algebra $(L, \rightarrow, \rightsquigarrow, 1)$ with two binary operations $\rightarrow, \rightsquigarrow$ and one constant 1 such that: for all $a, b, c \in L$,*
$(PL1)$ $1 \rightarrow a = a = 1 \rightsquigarrow a$, $a \rightarrow 1 = 1$;
$(PL2)$ $a \rightarrow a = 1$;
$(PL3)$ $(a \rightarrow b) \rightarrow (a \rightarrow c) = (b \rightarrow a) \rightarrow (b \rightarrow c)$;

$(PL4)$ $(a \rightsquigarrow b) \rightsquigarrow (a \rightsquigarrow c) = (b \rightsquigarrow a) \rightsquigarrow (b \rightsquigarrow c)$;
$(PL5)$ $a \to b = b \to a = 1 \Rightarrow a = b$;
$(PL6)$ $a \to b = 1$ *iff* $a \rightsquigarrow b = 1$.

It is well-known that L-algbras are the reduct of left hoops [27]. Similarly, pseudo L-algbras are the reduct of pseudo-hoops. Although the operations $\to$ and $\rightsquigarrow$ are not determined by each other, they share the same partial order, which is called a compatible order. In the following, from the logical viewpoint, implications $\to$ and $\rightsquigarrow$ are soundness under the compatible order.

Remark 2.4. *Let* $(L, \to, \rightsquigarrow, 1)$ *be a pseudo L-algebra. Then we can see that the reducts* $(L, \to, 1)$ *and* $(L, \rightsquigarrow, 1)$ *of* $(L, \to, \rightsquigarrow, 1)$ *are both L-algebras.*

Proposition 2.5. *Let L be a pseudo L-algebra. Define a binary relation "$\leq$" as follows:*

$$x \leq y \Leftrightarrow x \to y = 1 \Leftrightarrow x \rightsquigarrow y = 1.$$

Then "$\leq$" is a partial order on L.

Proof. It is similar to that of pseudo-hoops [18]. □

Now, we give some examples of pseudo L-algebras to show that pseudo L-algebras widely exist.

Example 2.6. *Every pseudo hoop* $(L, \odot, \to, \rightsquigarrow, 1)$ *gives rise to a pseudo L-algebra* $(L, \to, \rightsquigarrow, 1)$.

Example 2.7. *Let* $(L, \leq)$ *be a partially ordered set with the greatest element* 1 *and* $L \setminus \{1\}$ *be discrete. Define the binary operations* $\to$ *and* $\rightsquigarrow$ *as following:*

$$x \to y = \begin{cases} y, & x, y \in L \setminus \{1\} \\ y, & x=1 \\ 1, & y=1, \end{cases} \quad \text{and} \quad x \rightsquigarrow y = \begin{cases} x, & x, y \in L \setminus \{1\} \\ y, & x=1 \\ 1, & y=1. \end{cases}$$

Then we can easily check $(L, \to, \rightsquigarrow, 1)$ *is a pseudo L-algebra.*

Example 2.8. *Let* $L = \{0, a, b, c, 1\}$ *be a lattice such that* $0 < a < b, c < 1$, *b and c are incomparable. Define the operations* $\to$ *and* $\rightsquigarrow$ *by the following two tables.*

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	a	a	1	1
1	0	a	b	c	1

Then we can check that $(L, \to, \rightsquigarrow, 1)$ *is a pseudo L-algebra.*

In order to give rise to a pseudo L-algebra by a pseudo effect algebra, we recall some definitions and results about pseudo effect algebras.

Definition 2.9. [10] *A structure* $(E, +, 0, 1)$, *where* $+$ *is a partial binary operation, is called a pseudo effect algebra if, for all* $a, b, c \in E$, *the following hold.*
$(E1)$ $a + b$ *and* $(a + b) + c$ *exist if and only if* $b + c$ *and* $a + (b + c)$ *exist, and in this case,* $(a + b) + c = a + (b + c)$.
$(E2)$ *There is exactly one* $d \in E$ *and exactly one* $e \in E$ *such that* $a + d = e + a = 1$.
$(E3)$ *If* $a + b$ *exists, then there are elements* $d, e \in E$ *such that* $a + b = d + a = b + e$.
$(E4)$ *If* $1 + a$ *or* $a + 1$ *exists, then* $a = 0$.

In a pseudo effect algebra E, we may define a partial order in the following way:
$a \leq b$ iff $c + a = b$ for some $c \in E$.
Equivalently,
$a \leq b$ iff $a + d = b$ for some $d \in E$.

Proposition 2.10. [10] *Let $(E,+,0,1)$ be a pseudo effect algebra. For all $a,b,c \in E$, we have the following:*
(1) $a+0=0+a=a$*(i.e., 0 is a neutral element).*
(2) $a+b=0$ *implies* $a=b=0$ *(positivity).*
(3) *Let* $c+a, c+b$, *and* $(c+a)\wedge(c+b)$ *exist. Then* $a\wedge b$ *and* $c+(a\wedge b)$ *exist, and we have* $c+(a\wedge b)=(c+a)\wedge(c+b)$.
(4) *Let* $a+c, b+c$, *and* $(a+c)\wedge(b+c)$ *exist. Then* $a\wedge b$ *and* $(a\wedge b)+c$ *exist, and we have* $(a\wedge b)+c=(a+c)\wedge(b+c)$.

If a pseudo effect algebra is a lattice under the partial order $\leq$, we call it a lattice pseudo effect algebra. A pseudo effect algebra E admits two partial subtractions ("right" and "left") as follows: $b\backslash a$ is defined and equals x iff $b=x+a$, and a/b is defined and equals y iff $b=a+y$. Moreover, for the elements d and e in axiom (E2), we write $1\backslash a =: a^-$ (the "left" complement) and $a/1 := a^\sim$ (the "right" complement). Clearly, $0^\sim=1=0^-$ and $1^\sim=0=1^-$ (see [12]).

Proposition 2.11. [12] *Let E be a pseudo effect algebra. Then we have the following:*
(1) $b\backslash a$ *and* a/b *are defined iff* $a\leq b$.
(2) $(b\backslash a)+a=b=a+(a/b)$.
(3) $a+b$ *exists iff* $a\leq b^-$ *iff* $b\leq a^\sim$.
(4) $a^{-\sim}=a^{\sim-}=a$.
If $a\leq b$, *then*
(5) $a\leq b\Rightarrow b^-\leq a^-, b^\sim\leq a^\sim$,
(6) $a/b=a^\sim\backslash b^\sim$,
(7) $b\backslash a=b^-/a^-$.
If $a\leq b^-$, *then*
(8) $(a+b)^-=b^-\backslash a$,
(9) $(a+b)^\sim=b/a^\sim$.

Theorem 2.12. *Every lattice-ordered pseudo effect algebra $(E,+,0,1)$ gives rise to a pseudo L-algebra $(E,\to,\rightsquigarrow,1)$, where $x\to y:=(x\wedge y)+x^\sim$ and $x\rightsquigarrow y:=x^-+(x\wedge y)$ for $x,y\in E$.*

Proof. First we have $1\to x=(1\wedge x)+1^\sim=x+0=x, x\to 1=(x\wedge 1)+x^\sim=x+x^\sim=1, x\to x=(x\wedge x)+x^\sim=x+x^\sim=1$. Similarly we can check that $1\rightsquigarrow x=x, x\rightsquigarrow 1=1, x\rightsquigarrow x=1$. Hence 1 is a logical unit. Thus we have verified (PL1) and (PL2) in the definition of a pseudo L-algebra. Now we have

$$\begin{aligned}
&(x\to y)\to(x\to z)\\
&=((x\to y)\wedge(x\to z))+(x\to y)^\sim && \text{by Definition of }\to\\
&=(((x\wedge y)+x^\sim)\wedge((x\wedge z)+x^\sim))+((x\wedge y)+x^\sim)^\sim && \text{by Definition of }\to\\
&=(((x\wedge y)\wedge(x\wedge z))+x^\sim)+((x\wedge y)+x^\sim)^\sim && \text{by Proposition 2.10(4)}\\
&=(x\wedge y\wedge z)+x^\sim+((x\wedge y)+x^\sim)^\sim\\
&=(x\wedge y\wedge z)+x^\sim+(x^\sim/(x\wedge y)^\sim) && \text{by Proposition 2.11(9)}\\
&=(x\wedge y\wedge z)+(x^\sim+(x^\sim/(x\wedge y)^\sim))\\
&=(x\wedge y\wedge z)+(x\wedge y)^\sim. && \text{by Proposition 2.11(2)}
\end{aligned}$$

Interchanging the roles of x and y, we get

$$(y\to x)\to(y\to z)=(y\wedge x\wedge z)+(y\wedge x)^\sim.$$

Hence

$$(x\to y)\to(x\to z)=(x\wedge y\wedge z)+(x\wedge y)^\sim=(y\to x)\to(y\to z).$$

Similarly, we can prove that:

$$(x\rightsquigarrow y)\rightsquigarrow(x\rightsquigarrow z)=(y\rightsquigarrow x)\rightsquigarrow(y\rightsquigarrow z).$$

This verifies (PL3) and (PL4) in the definition of a pseudo L-algebra. □

3 Self-similar pseudo L-algebras

In the above section, we saw that pseudo L-algebras are reduct of pseudo hoops. Conversely given a special pseudo L-algebra, how does the pseudo L-algebra become a pseudo hoop? In order to discuss the issue, we introduce the notion of self-similarity into a pseudo L-algebra. Self-similarity can permit us to induce a multiplication operation on L. In this way, a self-similar pseudo L-algebra can be a pseudo hoop.

Definition 3.1. *We call a pseudo L-algebra $(L, \to, \rightsquigarrow, 1)$ self-similar if it satisfies the following:*
$(SPL1)$ for each $x \in L$, the maps $\downarrow x \to L$, defined by $y \mapsto (x \to y)$ and $y \mapsto (x \rightsquigarrow y)$, induce two bijections;
$(SPL2)$ for all $x, y \in L$, there exists some $c \in L$ such that $x \to c = y$ iff $y \rightsquigarrow c = x$.

Now we give an example of self-similar pseudo L-algebra. Let $S = Z \times Z \times Z$, where Z is the set of all integrals. Define a partial order in S as follows:

$(x, y, z) \leq (a, b, c)$ if $x < a$, or $x = a$, $y \leq b$ and $z \leq c$.

Define a binary operation $+$ by

$$(x, y, z) + (a, b, c) = \begin{cases} (x + a, y + b, z + c), & \text{if a is even} \\ (x + a, z + b, y + c), & \text{if a is odd.} \end{cases}$$

In the structure $(S, +)$, there is a unit $0 = (0, 0, 0)$. For any $(x, y, z) \in S$, there exists an inverse, denoted by $-(x, y, z)$, and is defined as

$$-(x, y, z) = \begin{cases} (-x, -y, -z), & \text{if } x \text{ is even} \\ (-x, -z, -y), & \text{if } x \text{ is odd.} \end{cases}$$

Then $(S, \leq, +)$ is an l-group, which is called Scrimger 2-group([6]). Denote the negative cone of S by S^-, where $S^- = \{x \in S | x \leq 0\}$.
Define operations $\to$ and $\rightsquigarrow$ on S^- as follows: for any $a, b \in S^-$

$$a \to b = (b - a) \wedge 0, \qquad a \rightsquigarrow b = (-a + b) \wedge 0.$$

So we have the following result.

Example 3.2. $(S^-, \to, \rightsquigarrow, 0)$ *is a self-similar pseudo L-algebra.*

Let L be a self-similar pseudo L-algebra, and let $a, b \in L$. Then the restrictions of $c \mapsto (b \to c)$ and $c \mapsto (a \rightsquigarrow c)$ are two bijections $\downarrow b \to L$ and $\downarrow a \to L$, respectivily. We define $a \odot b \in (\downarrow a \cap \downarrow b)$ as the following:

$a \odot b \leq b, \qquad b \to (a \odot b) = a,$ (M1)

$a \odot b \leq a, \qquad a \rightsquigarrow (a \odot b) = b.$ (M2)

In the following, we always assume that for a self-similar pseudo L-algebra $(L, \to, \rightsquigarrow, \odot, 1)$, the operation $\odot$ is defined by (M1) and (M2).

Now we give a self-similar pseudo L-algebras by an l-group.

Proposition 3.3. *Let $\mathbf{G} = (G, \vee, \wedge, +, -, 0)$ be an arbitrary l-group and G^- be the negative cone of G, that is, $G^- = \{x \in G | x \leq 0\}$. On G^- we define the following operations:*

$$x \to y := (y - x) \wedge 0, \qquad x \rightsquigarrow y := (-x + y) \wedge 0.$$

Then $\mathbf{G}^- = (G^-, \to, \rightsquigarrow, 0)$ is a self-similar pseudo L-algebra. Moreover it satisfies (M1) and (M2).

Proof. We can easily prove that $(G^-, \to, \rightsquigarrow, 0)$ is a pseudo L-algebra, and so we omit it.

Now we prove that (SPL1) holds, that is, the map $y \mapsto x \to y$ is bijection from $\downarrow x$ to G^-. Let $y_1, y_2 \in \downarrow x$ and $x \to y_1 = x \to y_2$. Since $y_1 \leq x$, then $y_1 - x \leq x - x = 0$, and hence $x \to y_1 = (y_1 - x) \wedge 0 = y_1 - x$. Similarly we can get $x \to y_2 = y_2 - x$. Hence $y_1 - x = y_2 - x$, and so $y_1 = y_2$. This shows that the map $y \mapsto x \to y$ is injective. Next we prove that $y \mapsto x \to y$ is surjective. For any $z \in G^-$, we have $z + x \leq 0 + x = x$, and thus $z + x \in \downarrow x$. Note that $x \to (z + x) = (z + x) - x = z$, which means that $z + x$ is the inverse image of z on the map $y \mapsto x \to y$. Combining the above arguments, we get that the map $y \mapsto x \to y$ is bijection from $\downarrow x$ to G^-.

By similar method, we can prove that the map $y \mapsto x \rightsquigarrow y$ is bijection from $\downarrow x$ to G^-.

Next, we prove that (SPL2) holds. Let $x, y \in G^-$. Since $y \mapsto x \to y$ is bijection, there is an unique $a \in \downarrow x$ such that $x \to a = y$. Similarly there is an unique $b \in \downarrow y$ such that $y \rightsquigarrow b = x$. Since $x \to a = y$ and $x \to a = a - x$, we get $y = a - x$, and thus $y + x = a$. Similarly by $y \rightsquigarrow b = x$ and $y \rightsquigarrow b = -y + b$, we get $-y + b = x$, and hence $b = y + x$. It follows that $a = b$. Taking $c := a = b$, we get that $x \to c = y$ iff $y \rightsquigarrow c = x$.

Lastly we prove that G^- satisfies (M1) and (M2). Let $a, b \in G^-$. Then we have $a + b \leq 0 + b = b$, that is $a + b \leq b$. Similarly we can get $a + b \leq a$. Therefore $b \to (a + b) = (a + b - b) \wedge 0 = a$, that is $b \to (a + b) = a$. Moreover $a \rightsquigarrow (a + b) = (-a + a + b) \wedge 0 = b$, that is $a \rightsquigarrow (a + b) = b$. So (M1) and (M2) hold. □

Remark 3.4. (1) *Through Proposition 3.3, we can see that there exist self-similar pseudo L-algebras.*
(2) *In the definition of self-similar pseudo L-algebras, we add the condition (SPL2): $x \to c = y \Leftrightarrow y \rightsquigarrow c = x$, where $c \in \downarrow x \cap \downarrow y$. Therefore we need $\downarrow x \cap \downarrow y \neq \emptyset$, which requires that L is a down-directed poset.*

We list some properties of self-similar pseudo L-algebras for the following propositions.

Proposition 3.5. *Let $(L, \rightarrow, \rightsquigarrow, \odot, 1)$ be a self-similar pseudo L-algebra. Then we have the following: for all $a, b, c \in L$;*
(1) $(L, \odot, 1)$ *is a monoid;*
(2) $(a \odot b) \rightarrow c = a \rightarrow (b \rightarrow c)$;
(3) $(a \odot b) \rightsquigarrow c = b \rightsquigarrow (a \rightsquigarrow c)$;
(4) $a \rightsquigarrow b = (c \odot a) \rightsquigarrow (c \odot b)$;
(5) $(a \rightarrow b) \odot a = (b \rightarrow a) \odot b$,
(6) $a \odot (a \rightsquigarrow b) = b \odot (b \rightsquigarrow a)$;
(7) $(a \rightarrow b) \odot a = a \wedge b = a \odot (a \rightsquigarrow b)$;
(8) $a \rightarrow b = a \rightarrow (a \wedge b)$; $a \rightsquigarrow b = a \rightsquigarrow (a \wedge b)$.

Proof. (1) We show first that this product is associative. For $a, b, c \in L$, we have $a \odot (b \odot c) \leq (b \odot c) \leq c$ and $(a \odot b) \odot c \leq c$. Therefore, $a \odot (b \odot c) = (a \odot b) \odot c$ is equivalent to $c \rightarrow (a \odot (b \odot c)) = c \rightarrow ((a \odot b) \odot c)$, i.e. $c \rightarrow (a \odot (b \odot c)) = a \odot b$. By Remark 2.4, we can use the properties of L-algebras into pseudo L-algebras, and thus $c \rightarrow (a \odot (b \odot c)) \leq c \rightarrow (b \odot c) = b$. Hence $c \rightarrow (a \odot (b \odot c)) = a \odot b$ is equivalent to $b \rightarrow (c \rightarrow (a \odot (b \odot c))) = a$. Now $b \rightarrow (c \rightarrow (a \odot (b \odot c))) = (c \rightarrow (b \odot c)) \rightarrow (c \rightarrow (a \odot (b \odot c))) = ((b \odot c) \rightarrow c) \rightarrow ((b \odot c) \rightarrow (a \odot (b \odot c))) = 1 \rightarrow a = a$, which yields $a \odot (b \odot c) = (a \odot b) \odot c$. This proves that L is a semigroup. Furthermore, $a \rightarrow (1 \odot a) = 1$ implies that $a \leq (1 \odot a) \leq a$, whence $1 \odot a = a$. Since $a \odot 1 = 1 \rightarrow (a \odot 1) = a$, we infer that L is a monoid.
(3) By the condition (M2) and (PL4), we have
$(a \odot b) \rightsquigarrow c = 1 \rightsquigarrow ((a \odot b) \rightsquigarrow c) = ((a \odot b) \rightsquigarrow a) \rightsquigarrow ((a \odot b) \rightsquigarrow c) = (a \rightsquigarrow (a \odot b)) \rightsquigarrow (a \rightsquigarrow c) = b \rightsquigarrow (a \rightsquigarrow c)$.
(4) From Proposition 3.5(3) and the condition (M2), we have $(c \odot a) \rightsquigarrow (c \odot b) = a \rightsquigarrow (c \rightsquigarrow (c \odot b)) = a \rightsquigarrow b$.
(6) By Proposition 3.5(3) and (PL4), we get $(a \odot (a \rightsquigarrow b)) \rightsquigarrow c = (a \rightsquigarrow b) \rightsquigarrow (a \rightsquigarrow c) = (b \rightsquigarrow a) \rightsquigarrow (b \rightsquigarrow c) = (b \odot (b \rightsquigarrow a)) \rightsquigarrow c$. By Corollary 2 in [27], we have $a \odot (a \rightsquigarrow b) = b \odot (b \rightsquigarrow a)$.
The proofs of other items can be seen in [27].

□

Proposition 3.6. *Let L be a self-similar pseudo L-algebra. Then $\rightarrow = \rightsquigarrow$ if and only if the multiplication $\odot$ is commutative.*

Proof. ($\Leftarrow$) Let $\odot$ be commutative and $x, a \in L$. We will prove $x \rightarrow a = x \rightsquigarrow a$. Assume that $x \rightarrow a = b$ and $x \rightsquigarrow a = c$. By Proposition 3.5(8), we have $x \rightarrow (x \wedge a) = b$ and $x \rightsquigarrow (x \wedge a) = c$. Then $b \odot x = x \wedge a$ and $x \odot c = x \wedge a$ by (M1) and (M2). It follows that $b \odot x = x \odot c$. Since $\odot$ is commutative, we get $x \odot b = x \odot c$. By Proposition 3.5(4), $b \rightsquigarrow c = (x \odot b) \rightsquigarrow (x \odot c) = 1$, and hence $b \leq c$. On the other hand, $c \rightsquigarrow b = (x \odot c) \rightsquigarrow (x \odot b) = 1$, which implies $c \leq b$. Hence $b = c$, which means $x \rightarrow a = x \rightsquigarrow a$.
($\Rightarrow$) Let $\rightarrow = \rightsquigarrow$. By (M1) and (M2), we have $b \rightarrow (a \odot b) = a$ and $a \rightsquigarrow (a \odot b) = b$. Therefore $b \rightarrow (b \odot a) = b \rightsquigarrow (b \odot a) = a = b \rightarrow (a \odot b)$. Since $c \mapsto (b \rightarrow c)$ is injective, then $a \odot b = b \odot a$. □

Theorem 3.7. *Every self-similar pseudo L-algebra is a pseudo hoop.*

Proof. It follows from Proposition 3.5(1) that (PSH1) holds.
(PSH2) follows from (PL2) and (PL6).
(PSH3) follows from Proposition 3.5(2).
(PSH4) follows from Proposition 3.5(3).
(PSH5) follows from (5), (6) and (7) of Proposition 3.5. □

Proposition 3.8. *Let L be a pseudo L-algebra and let $y, z \leq x$. Then the following are equivalent:*
(1) $y \leq z$;
(2) $x \rightarrow y \leq x \rightarrow z$;
(3) $x \rightsquigarrow y \leq x \rightsquigarrow z$.

Proof. We only prove (1) $\Leftrightarrow$ (2). By Remark 2.4, we can use the properties of L-algebras into pseudo L-algebras, and thus (1) $\Rightarrow$ (2) is true.
(2) $\Rightarrow$ (1), take $x = 1$ in (2), we get $y \leq z$. □

From Proposition 3.8, the mappings $y \mapsto (x \rightarrow y)$ and $y \mapsto (x \rightsquigarrow y)$: $\downarrow x \rightarrow L$ are always injective for any pseudo L-algebra. Thus, characterizations of self-similarity in pseudo L-algebra can be obtained by surjection of the above two mappings.

Proposition 3.9. *Let L be a pseudo hoop, and $a \in L$. The following are equivalent:*
(1) *The maps $b \mapsto (b \odot a)$ and $b \mapsto (a \odot b)$ are injective;*
(2) *The maps $b \mapsto (a \to b)$ and $b \mapsto (a \rightsquigarrow b)$ are surjective;*
(3) *$a \to (b \odot a) = b$ and $a \rightsquigarrow (a \odot b) = b$ for all $b \in L$;*
(4) *$(b \odot a) \to (c \odot a) = b \to c$ and $(a \odot b) \rightsquigarrow (a \odot c) = b \rightsquigarrow c$ for all $b, c \in L$;*
(5) *$a \to (b \odot c) = ((c \to a) \to b) \odot (a \to c)$ and $a \rightsquigarrow (c \odot b) = (a \rightsquigarrow c) \odot ((c \rightsquigarrow a) \rightsquigarrow b)$,*
hold for all $a, b, c \in L$.

Proof. (1)⇒(3) By (PSH5) and (PSH3) in Definition 2.1, we get $(a \to (b \odot a)) \odot a = ((b \odot a) \to a) \odot b \odot a = (b \to (a \to a)) \odot b \odot a = b \odot a$. By (1), we have $a \to (b \odot a) = b$. Similarly we can prove $a \rightsquigarrow (a \odot b) = b$, which imply that (3) is true.
(3)⇒(4) By (PSH3) in Definition 2.1 and (3), we have $(b \odot a) \to (c \odot a) = b \to (a \to (c \odot a)) = b \to c$. Similarly, we can Prove the second equality in (4).
(4)⇒(2) Taking $b = 1$ in the first equality of (4), we get $(1 \odot a) \to (c \odot a) = 1 \to c$ and thus $a \to (c \odot a) = c$. This means that the maps $b \mapsto (a \to b)$ is surjective. Similarly, we can prove that $b \mapsto (a \rightsquigarrow b)$ is surjective.
(2)⇒(1) Let $b \odot a = c \odot a$. Then $(b \odot a) \to d = (c \odot a) \to d$. By (PSH3) of Definition 2.1, we have $b \to (a \to d) = c \to (a \to d)$ for any $d \in L$. Since the map $d \mapsto (a \to d)$ is surjective, then $b \to e = c \to e$ for all $e \in L$, which implies that $b = c$. Moreover let $a \odot b = a \odot c$. We have $(a \odot b) \rightsquigarrow d = (a \odot c) \rightsquigarrow d$. By (PSH4) of Definition 2.1, we have $b \rightsquigarrow (a \rightsquigarrow d) = c \rightsquigarrow (a \rightsquigarrow d)$ for any $d \in L$. Since the map $d \mapsto (a \rightsquigarrow d)$ is surjective, then $b \rightsquigarrow e = c \rightsquigarrow e$ for all $e \in L$, which implies that $b = c$.
(2)⇒(5) Assume that (2) holds. Then we have the following:

$$\begin{aligned}
&(a \to (b \odot c)) \to (a \to d) \\
&= ((b \odot c) \to a) \odot ((b \odot c) \to d) && \text{by (PL3)} \\
&= (b \to (c \to a)) \to (b \to (c \to d)) && \text{by (PSH3)} \\
&= ((c \to a) \to b) \to ((c \to a) \to (c \to d)) && \text{by (PL3)} \\
&= ((c \to a) \to b) \to ((a \to c) \to (a \to d)) && \text{by (PL3)} \\
&= (((c \to a) \to b) \odot (a \to c)) \to (a \to d) && \text{by (PSH3)}
\end{aligned}$$

By (2), we get $(a \to (b \odot c)) \to e = (((c \to a) \to b) \odot (a \to c)) \to e$ for all $e \in L$. It follows that $a \to (b \odot c) = ((c \to a) \to b) \odot (a \to c)$.
Similarly, the second equation is true and we omit it. □

Now we turn our attention to self-similarity of L-algebra. Rump [27] introduced a self-similarity in L-algebra as follows. An L-algebra $(L, \to, 1)$ is self-similar if $y \mapsto (x \to y)$ is a bijection $\downarrow x \to L$. Then we define the first operation $a \odot_1 b \in \downarrow b$ as the following:

$a \odot_1 b \le b, \qquad b \to (a \odot_1 b) = a.$

In this way, self-similar L-algebra can induce left hoop.

In fact we can define the second operation $a \odot_2 b \in \downarrow a$ as the following:

$a \odot_2 b \le a, \qquad a \to (a \odot_2 b) = b.$

In this way, self-similar L-algebra can induce right hoop. Similar to [27], we could discuss the corresponding self-similar closures and stucture groups. Generally, $\odot_1 \ne \odot_2$ (see Example 3.10). Similar to Proposition 3.6, $\odot_1 = \odot_2$ iff $\odot_1$ or $\odot_2$ is commutative.

Furthermore, we can define the third operation (see (M1) and (M2)) $a \odot_3 b \in (\downarrow a \cap \downarrow b)$ as the following:

$a \odot_3 b \le b, \qquad b \to (a \odot_3 b) = a,$
$a \odot_3 b \le a, \qquad a \to (a \odot_3 b) = b.$

In this way, it follows from Proposition 3.6 that $\odot_3$ is commutative. Thus self-similar L-algebra can induce a hoop.

Example 3.10. *Let $\mathbf{G} = (G, \vee, \wedge, +, -, 0)$ be an arbitrary l-group and G^- be the negative cone of G, that is, $G^- = \{x \in G | x \le 0\}$. On G^- we define the following operation:*

$x \to y := (y - x) \wedge 0.$

Then $\mathbf{G}^- = (G^-, \to, 0)$ is a self-similar L-algebra. Moreover it satisfies $a \odot_1 b = a + b$ and $a \odot_2 b = b + a$, where $\odot_1$ and $\odot_2$ are defined in the above way.

4 The self-similar closure of pseudo L-algebras

In this section, we will introduce the notion of self-similar closures of pseudo L-algebras. Moreover, with the help of the recursive method, we obtain a self-similar closure of a pseudo L-algebra, under some conditions.

Definition 4.1. *We define a morphism $f : X \to Y$ between pseudo L-algebras X and Y to be a map which satisfies $f(1) = 1$ and $f(x \to_1 y) = f(x) \to_2 f(y)$, $f(x \rightsquigarrow_1 y) = f(x) \rightsquigarrow_2 f(y)$ for all $x, y \in X$. If f is an inclusion $X \hookrightarrow Y$, we call X a pseudo L-subalgebra of Y. In case H is a self-similar pseudo L-algebra with an pseudo L-subalgebra X which generates H as a monoid, we call H a pseudo self-similar closure of X.*

Let $(L, \to, \rightsquigarrow, 1)$ be a pseudo L-algebra. From the set $L - \{1\}$, we can generate a free semigroup $(A, *)$. Now we consider $S(L) = A \cup \{1\}$. Define a binary operation $\odot$ on $S(L)$ as follows:

$$a \odot b = \begin{cases} a * b, & \text{if } a, b \in A \\ b, & \text{if a=1} \\ a, & \text{if b=1.} \end{cases}$$

Then we can check that $(S(L), \odot, 1)$ is a monoid with identity 1. Next we extend the operations $\to$ and $\rightsquigarrow$ from L to $S(L)$ as follows:
$(a \odot b) \to c = a \to (b \to c)$, $(a \odot b) \rightsquigarrow c = b \rightsquigarrow (a \rightsquigarrow c)$, (A)
$a \to (b \odot c) = ((c \to a) \to b) \odot (a \to c)$, $a \rightsquigarrow (b \odot c) = (a \rightsquigarrow b) \odot ((b \rightsquigarrow a) \rightsquigarrow c)$. (S)

Theorem 4.2. *Let $(L, \to, \rightsquigarrow, 1)$ be a pseudo L-algebra and $S(L)$ a semigroup given in above statements. Then the operations $\to, \rightsquigarrow$ can be uniquely extended from L to $(S(L), \odot, \to, \rightsquigarrow)$ satisfying (A) and (S).*

Proof. It is similar to the proof of [27, Theorem 2]. □

Corollary 4.3. *Let $(L, \to, \rightsquigarrow, 1)$ be a pseudo L-algebra and $S(L)$ a semigroup given in Theorem 4.2. Then 1 is the logic unit on $S(L)$.*

Proof. We first prove that $a \to 1 = 1$ and $a \rightsquigarrow 1 = 1$ for any $a \in S(L)$. Clearly, there is $a_0 \in S(L)$ such that $a_0 \to 1 = 1$. Now for $x \in L$, by (A) we have:

$$(x \odot a_0) \to 1 = x \to (a_0 \to 1) = x \to 1 = 1,$$
$$(a_0 \odot x) \to 1 = a_0 \to (x \to 1) = a_0 \to 1 = 1.$$

Next we prove $a \rightsquigarrow 1 = 1$ for any $a \in S(L)$. Also there is $a_0 \in S(L)$ such that $a_0 \rightsquigarrow 1 = 1$. For $x \in L$, by (A) again, we have the following:

$$(x \odot a_0) \rightsquigarrow 1 = a_0 \rightsquigarrow (x \rightsquigarrow 1) = a_0 \rightsquigarrow 1 = 1,$$
$$(a_0 \odot x) \rightsquigarrow 1 = x \rightsquigarrow (a_0 \rightsquigarrow 1) = x \rightsquigarrow 1 = 1.$$

Secondly, we prove that $1 \to a = a$ and $1 \rightsquigarrow a = a$ for any $a \in S(L)$. It is clear that there is $a_0, b_0 \in S(L)$ such that $1 \to a_0 = a_0$ and $1 \rightsquigarrow b_0 = b_0$. Let $x \in L$. By (S), we have the following:
$1 \to (x \odot a_0) = ((a_0 \to 1) \to x) \odot (1 \to a_0) = x \odot a_0$;
$1 \to (a_0 \odot x) = ((x \to 1) \to a_0) \odot (1 \to x) = a_0 \odot x$;
$1 \rightsquigarrow (x \odot b_0) = (1 \rightsquigarrow x) \odot ((x \rightsquigarrow 1) \rightsquigarrow b_0) = x \odot b_0$;
$1 \rightsquigarrow (b_0 \odot x) = (1 \rightsquigarrow b_0) \odot ((b_0 \rightsquigarrow 1) \rightsquigarrow x) = b_0 \odot x$.

Finally, we prove that $a \to a = 1$ and $a \rightsquigarrow a = 1$ for all $a \in S(L)$. We know that there is $a_0, b_0 \in S(L)$ such that $a_0 \to a_0 = 1$ and $b_0 \rightsquigarrow b_0 = 1$. Let $x \in L$, by (A) and (S), we have the following:
$(x \odot a_0) \to (x \odot a_0) = x \to (a_0 \to (x \odot a_0)) = x \to (((a_0 \to a_0) \to x) \odot (a_0 \to a_0)) = x \to (x \odot 1) = x \to x = 1$;
$(a_0 \odot x) \to (a_0 \odot x) = a_0 \to (x \to (a_0 \odot x)) = a_0 \to (((x \to x) \to a_0) \odot (x \to x)) = a_0 \to (a_0 \odot 1) = a_0 \to a_0 = 1$;
$(x \odot b_0) \rightsquigarrow (x \odot b_0) = b_0 \rightsquigarrow (x \rightsquigarrow (x \odot b_0)) = b_0 \rightsquigarrow ((x \rightsquigarrow x) \odot ((x \rightsquigarrow x) \rightsquigarrow b_0)) = b_0 \rightsquigarrow (1 \odot b_0) = 1$;
$(b_0 \odot x) \rightsquigarrow (b_0 \odot x) = x \rightsquigarrow (b_0 \rightsquigarrow (b_0 \odot x)) = x \rightsquigarrow ((b_0 \rightsquigarrow b_0) \odot (b_0 \rightsquigarrow b_0) \rightsquigarrow x)) = x \rightsquigarrow (1 \odot x) = 1$.

□

Proposition 4.4. *Let L and $S(L)$ be given in the above arguments. Then $(S(L), \to, \rightsquigarrow, 1)$ satisfies L-equations: (PL3) and (PL4).*

Proof. It similar to the proof of [27, Proposition 7].

□

By Corollary 4.3 and Proposition 4.4, we can see that for a pseudo L-algebra L, $(S(L), \to, \rightsquigarrow, 1)$ satisfies the conditions (PL1)-(PL4), but except (PL5) and (PL6). In the following we give an example to show that, in general, $S(L)$ may not satisfy the conditions (PL5) and (PL6) for a pseudo L-algebra L.

Example 4.5. *Consider a pseudo L-algebra L given in Example 2.8. In $S(L)$, we have $(c\odot b) \to a = c \to (b \to a) = 1$, but $(c\odot b) \rightsquigarrow a = b \rightsquigarrow (c \rightsquigarrow a) = b \rightsquigarrow a = c \neq 1$. Taking $x = c\odot b$ and $y = a$, we have $x \to y = 1$ but $x \rightsquigarrow y \neq 1$, for some $x, y \in S(L)$. It follows that (PL6) is not true in $S(L)$. Moreover, $a \to (c \odot b) = ((b \to a) \to c) \odot (a \to c) = a \to c = 1$ and $(c \odot b) \to a = c \to (b \to a) = 1$, but $a \neq c \odot b$, which means that $S(L)$ does not satisfy (PL5).*

From the above example, we can see that the compatible order $\leq$ in a pseudo L-algebra L can not naturally extend to $S(L)$. It is a key issue how to extend the compatible order $\leq$ into $S(L)$. Therefore, we discuss that when a pseudo L-algebra satisfies some conditions, $S(L)$ satisfies (PL6), that is, for all $a, b \in S(L)$, we have

$a \to b = 1$ iff $a \rightsquigarrow b = 1$.

As you know, in a pseudo L-algebra L, we have

$\{(a, b) \mid a, b \in L, a \to b = 1\} = \{(a, b) \mid a, b \in L, a \rightsquigarrow b = 1\}$. For $A, B \subseteq S(L)$, we define a binary relation $\leq_{A\times B}$ on $A \times B$ as follows:

$\leq_{A\times B}:= \{(a, b) \mid a \in A, b \in B,\ a \to b = 1\} = \{(a, b) \mid a \in A, b \in B,\ a \rightsquigarrow b = 1\}$.

By the above argument, the relation $\leq_{L\times L}$ is well defined. In the following, we will extend the relation to $S(L)\times S(L)$.

We consider the following conditions:

(CP1) for all $n \in N$ and $x_1, x_2, \cdots, x_n, y \in L$,
$(x_1, x_2 \to (x_3 \to (\cdots (x_n \to y) \cdots))) \in \leq_{L\times L}$ iff $(x_n, x_{n-1} \rightsquigarrow (x_{n-2} \rightsquigarrow (\cdots (x_1 \rightsquigarrow y) \cdots))) \in \leq_{L\times L}$.

(CP2) for all $n \in N$ and $x, y_1, y_2, \cdots, y_n, \in L$,
$(x, y_n), (y_n \to x, y_{n-1}), (y_{n-1} \to (y_n \to x), y_{n-2}), \cdots, (y_2 \to (y_3 \to (\cdots (y_n \to x) \cdots)), y_1) \in \leq_{L\times L}$ iff $(x, y_1), (y_1 \rightsquigarrow x, y_2), (y_2 \rightsquigarrow (y_1 \rightsquigarrow x), y_3), \cdots, (y_{n-1} \rightsquigarrow (y_{n-2} \rightsquigarrow (\cdots (y_1 \rightsquigarrow x) \cdots), y_n) \in \leq_{L\times L}$.

(CP3) for all $n \in N$ and $x_1, x_2, \cdots, x_n \in L$ and $b \in S(L)$,
$(x_1, x_2 \to (x_3 \to (\cdots (x_n \to b) \cdots))) \in \leq_{L\times S(L)}$ iff $(x_n, x_{n-1} \rightsquigarrow (x_{n-2} \rightsquigarrow (\cdots (x_1 \rightsquigarrow b) \cdots))) \in \leq_{L\times S(L)}$.

Proposition 4.6. *Let L be a pseudo L-algebra.*
(1) If L satisfies (CP1), then $\leq_{L\times L}$ can be extended to $\leq_{S(L)\times L}$.
(2) If L satisfies (CP2), then $\leq_{L\times L}$ can be extended to $\leq_{L\times S(L)}$.
(3) If L satisfies (CP3), then $\leq_{L\times S(L)}$ can be extended to $\leq_{S(L)\times S(L)}$.

Proof. (1) We only prove the following case: for $x_1, x_2, y \in L$, $(x_1 \odot x_2) \to y = 1$ iff $(x_1 \odot x_2) \rightsquigarrow y = 1$. Let $(x_1 \odot x_2) \to y = 1$ hold. By the condition (A), we have $x_1 \to (x_2 \to y) = 1$. Since $x_2 \to y \in L$, we have that $x_1 \to (x_2 \to y) = 1$ iff $x_1 \rightsquigarrow (x_2 \to y) = 1$. This shows that $(x_1, x_2 \to y) \in \leq_{L\times L}$. By (CP1), we get $(x_2, x_1 \rightsquigarrow y) \in \leq_{L\times L}$, and hence $x_2 \rightsquigarrow (x_1 \rightsquigarrow y) = 1$. By the condition (A), $(x_1 \odot x_2) \rightsquigarrow y = 1$. Similarly, we can prove the inverse implication.

(2) We only prove the following case: for $x, y_1, y_2 \in L$, $x \to (y_1 \odot y_2) = 1$ iff $x \rightsquigarrow (y_1 \odot y_2) = 1$. Let $x \to (y_1 \odot y_2) = 1$. Then by the condition (S), we have $x \to (y_1 \odot y_2) = ((y_2 \to x) \to y_1) \odot (x \to y_2) = 1$, and thus $(y_2 \to x) \to y_1 = 1$ and $x \to y_2 = 1$. It is $(x, y_2), (y_2 \to x), y_1) \in \leq_{L\times L}$. By (CP2), we have $(x, y_1), (y_1 \rightsquigarrow x, y_2) \in \leq_{L\times L}$. Hence $x \rightsquigarrow y_1 = 1$ and $(y_1 \rightsquigarrow x) \rightsquigarrow y_2 = 1$. It follows that $(x \rightsquigarrow y_1) \odot ((y_1 \rightsquigarrow x) \rightsquigarrow y_2) = 1$. By (S), we have $x \rightsquigarrow (y_1 \odot y_2) = (x \rightsquigarrow y_1) \odot ((y_1 \rightsquigarrow x) \rightsquigarrow y_2) = 1$. Similarly we can get the inverse implication.

(3) We only prove the following case: for $x_1, x_2 \in L$ and $b \in S(L)$, $(x_1 \odot x_2) \to b = 1$ iff $(x_1 \odot x_2) \rightsquigarrow b = 1$. Let $(x_1 \odot x_2) \to b = 1$. Then by (A), we have $x_1 \to (x_2 \to b) = 1$. By Proposition 4.6(2), we get $x_1 \rightsquigarrow (x_2 \to b) = 1$. This means that $(x_1, x_2 \to b) \in \leq_{L\times S(L)}$. By the condition (CP3), $(x_2, x_1 \rightsquigarrow b) \in \leq_{L\times S(L)}$, which implies $x_2 \rightsquigarrow (x_1 \rightsquigarrow b) = 1$. By (A), we get $(x_1 \odot x_2) \rightsquigarrow b = 1$. Similarly we can prove the inverse implication. □

Corollary 4.7. *Let L be a pseudo L-algebra. Then $S(L)$ satisfies (PL6) if and only if L satisfies the conditions (CP1), (CP2) and (CP3).*

Proof. (⇒) Let $S(L)$ satisfy (PL6). Now we check (CP2). We only prove (CP2) for the case $n = 2$, that is, $(x, y_2) \in \leq_{L\times L}$ and $(y_2 \to x, y_1) \in \leq_{L\times L}$ iff $(x, y_1) \in \leq_{L\times L}$ and $(y_1 \rightsquigarrow x, y_2) \in \leq_{L\times L}$. Let $x \to y_2 = 1$, $(y_2 \to x) \to y_1) = 1$. Then $((y_2 \to x) \to y_1) \odot (x \to y_2) = 1$. By (S), $x \to (y_1 \odot y_2) = 1$. Since $S(L)$ satisfies (PL6), we have $x \rightsquigarrow (y_1 \odot y_2) = 1$. By (S) again, we get $(x \rightsquigarrow y_1) \odot ((y_1 \rightsquigarrow x) \rightsquigarrow y_2) = 1$, which implies $x \rightsquigarrow y_1 = 1$ and $(y_1 \rightsquigarrow x) \rightsquigarrow y_2 = 1$. By (PL6), $x \to y_1 = 1$ and $(y_1 \rightsquigarrow x) \to y_2 = 1$, and hence, $(x, y_1) \in \leq_{L\times L}$ and $(y_1 \rightsquigarrow x, y_2) \in \leq_{L\times L}$. The inverse implication is similar. By the similar method, we can check the conditions (CP1) and (CP3) hold.

(⇐) It follows from Proposition 4.6. □

Remark 4.8. *By the above observation, we can see that an order of a pseudo L-algebra L plays an important role in extending L to monoid $S(L)$. So we have an open problem: for an L-algebra L, what are conditions on L such that $S(L)$ admits (PL5)?*

Let $(L, \to, \rightsquigarrow, 1)$ be a pseudo L-algebra and $S(L)$ be given as above. We call L with the condition (AS) if $S(L)$ satisfies (PL5).

Proposition 4.9. *Let $(L, \to, \rightsquigarrow, 1)$ be a pseudo L-algebra satisfying (CP1), (CP2), (CP3) and (AS). Then $S(L)$ forms a self-similar pseudo L-algebra.*

Proof. It follows from Corollary 4.3, Proposition 4.4, Corollary 4.7, and (AS) that $S(L)$ is a pseudo L-algebra. Now we prove that $S(L)$ is self-similar. First we prove that $S(L)$ is a pseudo hoop. For all $a, b, c \in S(L)$, we have $((a \to b) \odot a) \to c = (a \to b) \to (a \to c) = (b \to a) \to (b \to c) = ((b \to a) \odot b) \to c)$. Thus, $(a \to b) \odot a = (b \to a) \odot b$. Similarly, we have $a \odot (a \rightsquigarrow b) = b \odot (b \rightsquigarrow a)$. Define $a \leq b$ iff $a \to b = 1$ for $a, b \in S(L)$. Then we can check $(a \to b) \odot a$ and $a \odot (a \rightsquigarrow b)$ are both infimum of $\{a, b\}$, and thus $(a \to b) \odot a = a \odot (a \rightsquigarrow b)$. This shows that $S(L)$ is a pseudo hoop. By Proposition 3.9 and $S(L)$ with (S), we get the maps $b \mapsto (a \to b)$ and $b \mapsto (a \rightsquigarrow b)$ are both bijections. Therefore $S(L)$ is self-similar. □

Theorem 4.10. *Let L be a pseudo L-algebra with the conditions (CP1), (CP2), (CP3) and (AS). Then the natural morphism $L \to S(L)$ is injective, and up to isomorphism of pseudo hoops, $S(L)$ is the unique self-similar closure of L.*

Proof. It follows from Proposition 4.9 that $(S(L), \odot_1, \to_1, \rightsquigarrow_1)$ is a self-similar pseudo L-algebra and a pseudo hoop. By Definition 4.1, we can see that $S(L)$ is the pseudo self-similar closure of L. Now we prove the uniqueness of pseudo self-similar closures. Suppose that there is another pseudo self-similar closure $(H, \odot_2, \to_2, \rightsquigarrow_2)$ of L. Define $f : S(L) \to H$ by $f(x_1 \odot_1 x_2, \cdots, \odot_1 x_n) = x_1 \odot_2 x_2, \cdots, \odot_2 x_n$ for $x_1, x_2, \cdots, x_n \in L$. Since $S(L)$ is a free monoid, the map f is well-defined. Now we prove that f satisfies $f((x_1 \odot_1 x_2) \to_1 (y_1 \odot_1 y_2)) = f(x_1 \odot_1 x_2) \to_2 f(y_1 \odot_1 y_2)$ and $f((x_1 \odot_1 x_2) \rightsquigarrow_1 (y_1 \odot_1 y_2)) = f(x_1 \odot_1 x_2) \rightsquigarrow_2 f(y_1 \odot_1 y_2)$ for $x_1, x_2, y_1, y_2 \in L$.

Note that

$$
\begin{aligned}
& f((x_1 \odot_1 x_2) \to_1 (y_1 \odot_1 y_2)) \\
= {} & f(x_1 \to_1 (x_2 \to_1 (y_1 \odot_1 y_2))) \\
= {} & f(x_1 \to_1 (((y_2 \to_1 x_2) \to_1 y_1) \odot_1 (x_2 \to_1 y_2))) \\
= {} & f((((x_2 \to_1 y_2) \to_1 x_1) \to_1 ((y_2 \to_1 x_2) \to_1 y_1)) \odot_1 (x_1 \to_1 (x_2 \to_1 y_2))) \\
= {} & (((x_2 \to_1 y_2) \to_1 x_1) \to_1 ((y_2 \to_1 x_2) \to_1 y_1)) \odot_2 (x_1 \to_1 (x_2 \to_1 y_2)) \\
= {} & (((x_2 \to_2 y_2) \to_2 x_1) \to_2 ((y_2 \to_2 x_2) \to_2 y_1)) \odot_2 (x_1 \to_2 (x_2 \to_2 y_2)) \quad \text{by } \to_1 = \to = \to_2 \text{ in } L \\
= {} & x_1 \to_2 (((y_2 \to_2 x_2) \to_2 y_1) \odot_2 (x_2 \to_2 y_2)) \\
= {} & x_1 \to_2 (x_2 \to_2 (y_1 \odot_2 y_2)) \\
= {} & (x_1 \odot_2 x_2) \to_2 (y_1 \odot_2 y_2) \\
= {} & f(x_1 \odot_1 x_2) \to_2 f(y_1 \odot_1 y_2).
\end{aligned}
$$

Similarly we can prove that $f((x_1 \odot_1 x_2) \rightsquigarrow_1 (y_1 \odot_1 y_2)) = f(x_1 \odot_1 x_2) \rightsquigarrow_2 f(y_1 \odot_1 y_2)$ for $x_1, x_2, y_1, y_2 \in L$.

Moreover, we prove that f is a bijection. Obviously, f is surjective. Now we prove that f is injective. Define $ker(f) = \{a \in S(L) | f(a) = 1\}$. Let $a \in S(L)$, $a \neq 1$ and $f(a) = 1$. Then $a = x \odot_1 b$ for some $x \in L \setminus \{1\}$ and $b \in S(L)$. By (M1), $f(x) = f(b \to_1 (x \odot_1 b)) = f(b) \to_2 f(x \odot_1 b) = f(b) \to_2 f(a) = f(b) \to_2 1 = 1$, a contradiction.

Combining the above arguments, we get that f is an isomorphism of pseudo hoops. □

5 The structure group of a pseudo L-algebra

Recall that a monoid M is said to fulfil the left Ore condition if for each pair of elements $a, b \in M$, there are $c, d \in M$ with $c \odot a = d \odot b$. If, in addition, $a \odot c = b \odot c$ implies that there is some $d \in M$ with $d \odot a = d \odot b$, we can form a quotient group $Q(M)$ consisting of formal fractions $a^{-1} \odot b$ with $a, b \in M$ (see [13]). For $a, b, c \in M$, the equation $a^{-1} \odot b = (c \odot a)^{-1} \odot (c \odot b)$ holds in $Q(M)$, and the left Ore condition guarantees that in this way, two arbitrary fractions can be transformed into those with a common denominator. They are defined to be equal in $Q(M)$ if the common denominator can be chosen such that the numerators become equal in M.

Proposition 5.1. *Every self-similar pseudo L-algebra L admits a quotient group $Q(L)$.*

Proof. First we prove that L satisfies left Ore condition. Let $a, b \in L$. Then by Proposition 3.5(10), $(a \to b) \odot a = (b \to a) \odot b$. Taking $c = a \to b$ and $d = b \to a$, we get $c \odot a = d \odot b$. Moreover let $a \odot c = b \odot c$. By Proposition 3.5(8), we have $a = b$, and hence, there is some $d \in L$ with $d \odot a = d \odot b$. So that we can form a quotient group $Q(L)$. □

Definition 5.2. *Let L be a pseudo L-algebra. We define the structure group of L to be the quotient group $G(L) := Q(S(L))$ of the self-similar closure $S(L)$.*

Now we will give a special L-algebra to explain how to get the self-similar closure and the structure group of it. Considering Example 3.2, denote $L = [(-1,0,0),(0,0,0)] = \{x \in S \mid (-1,0,0) \leq x \leq (0,0,0)\}$. Then we can see that $L = \{(0,b,c) \mid b \leq 0, c \leq 0\} \cup \{(-1,b,c) \mid b \geq 0, c \geq 0\}$.
So we can get the following results.

Proposition 5.3. *Let L be given as in the above statements.*
(1) *Then $(L, \to, \rightsquigarrow, 0)$ is a pseudo L-algebra and $\to \neq \rightsquigarrow$. Moreover L is not self-similar.*
(2) *The self-similar closure $S(L)$ of L is S^-.*
(3) *The structure group $G(L)$ of L is S.*

Proof. (1) It follows from Example 3.10 in [33] that $(L, \to)$ and $(L, \rightsquigarrow)$ are L-algebras, respectively. Now we prove that L satisfies (PL6). Let $a \to b = 0$, then $(b - a) \wedge 0 = 0$. Hence $b - a \geq 0$, or $b \geq a$. It follows that $-a + b \geq 0$, and hence $a \rightsquigarrow b = (-a + b) \wedge 0 = 0$. The converse implication also holds. This shows that (PL6) is true. Therefore L forms a pseudo L-algebra. Since $(0,-2,-3) \to (-1,0,0) = ((-1,0,0) - (0,-2,-3)) \wedge 0 = ((-1,0,0) + (0,2,3)) \wedge 0 = (-1,2,3)$, but $(0,-2,-3) \rightsquigarrow (-1,0,0) = (-(0,-2,-3) + (-1,0,0)) \wedge 0 = ((0,2,3) + (-1,0,0)) \wedge 0 = (-1,3,2)$. This means that $\to \neq \rightsquigarrow$. Moreover since $(-1,0,0)$ is the least element of L, so $a \mapsto ((-1,0,0) \to a)$ is not bijection from $\downarrow (-1,0,0)$ to L, which shows that L is not self-similar.
(2) We now consider the extension from L to $S(L)$ by use of the operation "+". Since $(0,-1,0) \in L$, by the definition of the operation "+", we have $(0,-1,0) + (0,-1,0) = (0,-2,0)$, and thus $(0,-2,0) \in S(L)$. Recursively, we get $(0,-n,0) \in S(L)$, for all $n \in N$. Similarly we have $(0,0,-m) \in S(L)$ for all $m \in N$. Therefore we get $(0,-n,-m) \in S(L)$ for all $n, m \in N$. Moreover, since $(-1,0,0) \in L$, we know $(-q,0,0) \in S(L)$ for all $q \in N$. Hence for all $m,n,q \in N$, we have $(-q,0,0)+(0,-n,-m) = (-q,-n,-m) \in S(L)$. For any $k-1, m, n \in Z^+$, we have $(-k+1,0,0)+(-1,m,n) = (-k,m,n)$. It follows that $S(L) = S^-$, where the operations $\to$ and $\rightsquigarrow$ are extended from L to $S(L)$ by the (A) and (S) in Theorem 4.2.
(3) In $S(L)$, it is clearly that $(0,0,0)$ is an identity. For any $(x,y,z) \in S(L)$, there is the element $-(x,y,z) \in S$ such that

$$-(x,y,z) = \begin{cases} (-x,-y,-z), & \text{if } x \text{ is even} \\ (-x,-z,-y), & \text{if } x \text{ is odd.} \end{cases}$$

and $(x,y,z) + (-(x,y,z)) = (0,0,0)$. Therefore the structure group of L is $(S, \leq, +)$ and it is an l-group. □

6 Conclusions

In this paper, we introduce the notion of pseudo L-algebras and study their extensions, which are their pseudo self-similar closures and structure groups. From a pseudo L-algebra L, we use the method different to the Rump's one to derive a self-similar pseudo L-algebra $S(L)$, furthermore get a group structure $G(L)$. In this way, we provide a general method for inducing semigroup structures and group structures on other pseudo implication structures, such as, pseudo BCK-algebras, pseudo equality algebras and pseudo BE-algebras, etc.

In the next work, we will set up non-commutative L-logic, which corresponds to the pseudo L-algebras and by use of pseudo L-algebras, we will study diversification reasons in L-logic systems. Moreover we will study the special properties of pseudo L-algebras as the solutions of some special quantum equations.

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