Iranian Journal of Fuzzy Systems

Volume 19, Number 6, (2022), pp. 75-92

Original paper

University of Sistan and Baluchestan

Fuzzy product rule with applications

M. Zeinali[1] and F. Maheri[2]

[1,2]*Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran*

zeinali@tabrizu.ac.ir, maherival@gmail.com

Abstract

In this paper, we establish the GH-derivative of multiplication of fuzzy functions, the so-called product rule. For the first time, a product rule is constructed while both of the multiplied functions are assumed to be fuzzy without any restriction on the signs of multiplied functions. The rule is extracted based on the MCE-product and its property of distributivity. Then, we propose two important applications of the fuzzy product rule: An integration by parts formula for fuzzy functions and solving a nonlinear fuzzy differential equation. Some illustrative examples are given to verify the theoretical results.

Keywords: Fuzzy product, integration by parts, MCE-representation, product rule, strongly generalized Hukuhara derivative.

1 Introduction

After introducing fuzzy sets by Zadeh [25] and fuzzy numbers by Chang and Zadeh [13], binary arithmetic operations on fuzzy numbers have been one of the most interesting and challenging areas in the field of fuzzy mathematics [3,5,9,10,13-16]. In this area, the Zadeh extension principle has the longest history as the first basic approach. The Zadeh extension principle is rather difficult to use and often computationally expensive. A theorem was proved by Nguyen [22] which facilitates the application of the Zadeh extension principle for continuous functions through the so-called r-cuts concept. The addition operation based on the Zadeh extension principle is well behaved and useful. However, the left three operations including difference, multiplication, and division encounter various difficulties making them less useful. Considering the multiplication operation, Zadeh extension approach, despite its comprehensiveness, has some disadvantages. For instance, it is not distributive and doesn't preserve the shape of multiplied fuzzy numbers, it is computationally expensive and practically difficult to use. This caused the researchers to find alternatives for the multiplication operation. The following are some of the common methods proposed for the multiplication of fuzzy numbers.

Ma et al. [20] introduced a new multiplication operation through a new representation of a typical fuzzy number. They represented a fuzzy number by $u = (m_u, \theta_u^-, \theta_u^+)$ in which the first component is the middle of the core called location index number and the second and third components are the left and right spreads called the left and right fuzziness index functions, respectively. Recently, this kind of representation of a fuzzy number has been called middle-core-ecart representation (MCE-representation, for short) [9], and we use this term in this paper. Using MCE-representation, Ma et al. [20] introduced a new fuzzy arithmetic. The arithmetic operations operate on the middle of the cores, while for the left and right spreads the maximum operation, $\vee$, is used, i.e., for $u = (m_u, \theta_u^-, \theta_u^+)$ and $v = (m_v, \theta_v^-, \theta_v^+)$

$$u * v = (m_u * m_v, \theta_u^- \vee \theta_v^-, \theta_u^+ \vee \theta_v^+), \quad * \in \{+, -, ., \div\}.$$

Although this product is distributive, it has a serious drawback. That is the information of the operand with smaller support is lost and not reflected at all on the resulted product. This drawback recently has been tackled through a modification made by Bica et al. [9]. Their main contribution is the replacement of the maximum operation

Corresponding Author: M. Zeinali
Received: February 2021; Revised: August 2022; Accepted: August 2022.
https://doi.org/10.22111/IJFS.2022.7211

with multiplication. This replacement made an important change. In this way, the information of both operands simultaneously is reflected in the resulted product. Since this product operation is defined by using MCE-representation of fuzzy numbers, it is called MCE-product. The MCE-product is fully distributive with respect to the addition and also, easy to use. But it does not preserve the shape of operands. In the present work, we remove this shortcoming by a slight modification. This is our first contribution to this paper.

Another product operation for fuzzy numbers called cross-product is also introduced in [3, 5]. The cross-product preserves the shape of fuzzy numbers, but it is not distributive. Furthermore, the operands cores are not allowed to include zero. This is a serious disadvantage and limits its application in most situations.

There are also other approaches that modify the Zadeh extension-based arithmetic. For instance, t-norm based and interactive approaches. In the t-norm based arithmetic, the standard minimum t-norm (used in the Zadeh extension principle) is replaced by a general t-norm [10, 15, 16, 18, 19, 24].

Besides wide studies on the fuzzy number calculus [12, 14, 21, 23], substantial efforts have been devoted to calculate the derivative of arithmetic rules including sum, difference, product of fuzzy functions [2, 7, 30]. Bede et al. in [6], showed that the multiplication of a fuzzy number and a crisp function generally is not Hukuhara differentiable. Therefore, they were motivated to introduce the so-called GH-derivative. They proved that if $g : (a, b) \to \mathbb{R}$ is differentiable and g' has at most a finite number of roots in (a, b) and $c \in \mathbb{R}_{\mathcal{F}}$, then $f(x) = cg(x)$ is GH-differentiable. Later, the gH-derivative [8] was presented as the generalization of GH-derivative, and it was proved that $f(x) = cg(x)$ is gH-differentiable without having any limitation on roots of g'. Formulas for derivative of H-difference of two fuzzy functions and derivative of the product of a fuzzy function with a crisp function were extracted in [7]. Also, a formula for derivative of the cross-product of a triangular fuzzy number and a triangular fuzzy number-valued function has been derived in [1]. The derivative of gH-difference of fuzzy functions along with some basic rules such as fuzzy Leibniz's rule, chain rule, and integration by parts are introduced in [2]. These efforts caused significant progress in the field of fuzzy mathematics, particularly in fuzzy differential equations which is a very important issue in modeling and solving non-deterministic engineering and scientific problems [2, 7, 26, 27, 28, 29].

In the associated literature, the formula $(f.g)' = f'.g + f.g'$ is called product rule. In the present work, by using MMCE-product, we extract a formula for the derivative of the multiplication of two fuzzy functions, where either of the operands can be (i)- or (ii)-GH-differentiable. We call this the *fuzzy product rule*. The presented fuzzy product rule is very general and it doesn't depend on the sign of multiplied fuzzy functions. This kind of derivative has the potential of various applications, especially it would open the way to examine nonlinear fuzzy differential equations. This is our main objective in the current work.

An important application of the product rule is its use in deriving the formula of integration by parts. In [2], a formula for integration by parts has been obtained where the only one of the involved functions is fuzzy. In the current work, we remove this serious limitation and obtain the integration by parts formula in the most general form for fuzzy calculus. Another application of the fuzzy product rule and MMCE-product is solving the nonlinear equation $u.u' = g$.

This paper is organized as follows: First, some basic concepts about fuzzy numbers and MCE-representation are presented in Section 2. In Section 3, the modified MCE-product (MMCE-product, for short) is introduced and some results about the GH-derivative and integral of fuzzy number-valued functions using MCE- and MMCE-representation are presented. In Section 4, the fuzzy product rule is proved. In the last section, a formula is extracted for the integration by parts as an important application of the product rule and a nonlinear fuzzy differential equation has been solved as another application of the fuzzy product rule and MMCE-product.

2 Preliminaries

Throughout this paper, the space of fuzzy numbers and the space of triangular fuzzy numbers are denoted by $\mathbb{R}_{\mathcal{F}}$ and $\mathbb{R}_{\tau}$, respectively. The notation $u_r = [u_r^-, u_r^+]$ stands for the r-cut of the fuzzy number u and u_0 is called its support.

Definition 2.1. (MCE-representation [9, 20]) *For $u \in \mathbb{R}_{\mathcal{F}}$, consider the functions $\theta_u^-, \theta_u^+ : [0, 1] \to \mathbb{R}_+$ defined by*

$$\theta_u^-(r) = m_u - u_r^-, \quad \theta_u^+(r) = u_r^+ - m_u,$$

where $m_u = \frac{u_1^- + u_1^+}{2}$. Then, $u = (m_u; \theta_u^-, \theta_u^+)$ is MCE-representation of u. Note that the semicolon symbol makes this different from the well-known notation of a typical triangular fuzzy number denoted by (a, b, c). Hereafter, fuzzy numbers are assumed to be in the form of MCE-representation.

From Theorems 4.8 and 4.9 of [4], clearly, $(m_u; \theta_u^-, \theta_u^+)$ represents a fuzzy number if and only if θ_u^-, θ_u^+ are bounded, positive, non-increasing, left-continuous on $(0, 1]$ and right-continuous at 0.

Definition 2.2. *The function* $d : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}$ *defined by*

$$d(u, v) = |m_u - m_v| + \sup_{r\in[0,1]} \max\{|\theta_u^-(r) - \theta_v^-(r)|, |\theta_u^+(r) - \theta_v^+(r)|\},$$

is a metric on $\mathbb{R}_{\mathcal{F}}$*. Since this metric is equivalent to the metric* D_∞ ([4]), *it follows that d is complete* (see [9, 20]). *All limits in this paper are in the metric d.*

In [9], a new product of fuzzy numbers has been introduced which is fully distributive with respect to the addition and easy to use in comparison with the previous products. It uses MCE-representation of fuzzy numbers and, is called MCE-product. The details of MCE-product are as follows:

Definition 2.3. *Let* $u = (m_u; \theta_u^-, \theta_u^+)$, $v = (m_v; \theta_v^-, \theta_v^+)$*. The MCE-product of u and v is defined as*

$$u \odot v = (m_u m_v; \theta_u^- \theta_v^-, \theta_u^+ \theta_v^+).$$

The r-cut of $u \odot v$ is $(u \odot v)_r = [m_u m_v - \theta_u^- \theta_v^-, m_u m_v + \theta_u^+ \theta_v^+]$ and its support is

$$supp\ u \odot v = [m_u m_v - \theta_u^-(0)\theta_v^-(0), m_u m_v + \theta_u^+(0)\theta_v^+(0)]. \tag{1}$$

For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\alpha \in \mathbb{R}$, the sum and scalar multiplication are defined as

$$u + v = (m_u + m_v; \theta_u^- + \theta_v^-, \theta_u^+ + \theta_v^+),$$

$$\alpha u = \begin{cases} (\alpha m_u; \alpha\theta_u^-, \alpha\theta_u^+), & \alpha \geq 0, \\ (\alpha m_u; -\alpha\theta_u^+, -\alpha\theta_u^-), & \alpha < 0. \end{cases}$$

Theorem 2.4. *Let* $u = (m_u; \theta_u^-, \theta_u^+)$, $v = (m_v; \theta_v^-, \theta_v^+)$ *and* $w = (m_w; \theta_w^-, \theta_w^+)$ *are three fuzzy numbers,* $\overline{0} = (0; 0, 0)$ *and* $\overline{1} = (1; 1, 1)$*. Then we have the following properties:*

(*i*) *Commutativity:* $u \odot v = v \odot u$,

(*ii*) *Associativity:* $(u \odot v) \odot w = u \odot (v \odot w)$,

(*iii*) *Distributivity:* $(u + v) \odot w = (u \odot w) + (v \odot w)$,

(*iv*) *Neutral member:* $u \odot \overline{1} = u$,

(*v*) $u \odot u = \overline{0}$ *iff* $u = \overline{0}$.

For more details about MCE-product, see [9].

3 Modified MCE-product

The MCE-product is easy to use and satisfies the interesting properties presented in Theorem 2.4. However, it doesn't preserve the shapes of triangular and trapezoidal fuzzy numbers. For instance, let $u = (0, 2, 5)$ and $v = (-2, 3, 4)$, then $u = \big(2; 2(1-r), 3(1-r)\big)$, $v = \big(3; 5(1-r), 1-r\big)$ and $u \odot v = \big(6; 10(1-r)^2, 3(1-r)^2\big)$, which shows that $u \odot v$ is not a triangular fuzzy number. In what follows, we remove this drawback by a slight modification. First, we note that for a triangular fuzzy number $u = (a, b, c)$, MCE-representation is in the form

$$u = \Big(b; \big(b - a\big)(1 - r), \big(c - b\big)(1 - r)\Big).$$

This means that if $u \in \mathbb{R}_\tau$, then u can be presented by $\big(m_u; k_u^-(1-r), k_u^+(1-r)\big)$, where $k_u^-, k_u^+ \in \mathbb{R}^+$. Now, the modification of MCE-product can be done as follows:

Definition 3.1. *Let* $u = (m_u; k_u^-(1-r), k_u^+(1-r))$ *and* $v = (m_v; k_v^-(1-r), k_v^+(1-r))$ *be two triangular fuzzy numbers. The modified MCE-product (MMCE-product, for short) is defined as follows*

$$u \circledast v = \big(m_u m_v; k_u^- k_v^-(1-r), k_u^+ k_v^+(1-r)\big).$$

Note that to distinguish between MCE- and MMCE-products, we use the notation "$\circledast$" for the second one. In Fig. 1, the MCE- and MMCE-products are shown for the above mentioned example.

The r-cut of $u \circledast v$ is $(u \circledast v)_r = [m_u m_v - k_u^- k_v^-(1-r), m_u m_v + k_u^+ k_v^+(1-r)]$ and its support is

$$supp\ u \circledast v = [m_u m_v - k_u^- k_v^-, m_u m_v + k_u^+ k_v^+]. \tag{2}$$

The triple representation for $u \circledast v$ as a triangular fuzzy number is

$$u \circledast v = (m_u m_v - k_u^- k_v^-, m_u m_v, m_u m_v + k_u^+ k_v^+).$$

The MMCE-product is invertible and the inverse of $u = \big(m_u; k_u^-(1-r), k_u^+(1-r)\big)$ is $u^{-1} = \big(\frac{1}{m_u}; \frac{1}{k_u^-}(1-r), \frac{1}{k_u^+}(1-r)\big)$.

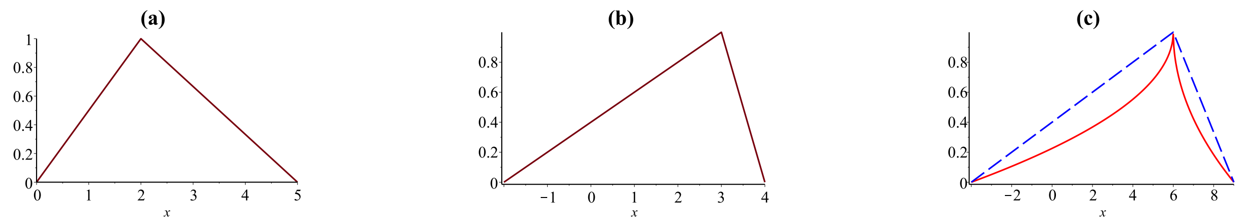

Figure 1: **(a)** and **(b)** denote u and v, respectively and **(c)** denote $u \odot v$ (solid line) and $u \circledast v$ (dash line) of u and v.

Remark 3.2. *It can be easily shown that the properties stated in Theorem 2.4 hold true for MMCE-product, as well, provided that the neutral member is changed to* $\bar{1} = (1; 1-r, 1-r)$.

In the following, the MCE-representation of (i)- and (ii)-GH-derivative of a fuzzy function are presented. Let us denote the MCE-representation of an arbitrary fuzzy function $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ by $f(x) = (m_f(x); \theta_f^-(x,r), \theta_f^+(x,r))$. Using the definitions of MCE-representation and H-difference, the H-difference of $u = (m_u; \theta_u^-, \theta_u^+)$ and $v = (m_v; \theta_v^-, \theta_v^+)$ is

$$u \ominus_H v = (m_u - m_v; \theta_u^- - \theta_v^-, \theta_u^+ - \theta_v^+).$$

Theorem 3.3. [4] *Let* $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$.

(*i*) *If f is (i)-GH-differentiable, then* $f'(x) = \big(m'_f(x); (\theta_f^-)'(x,r), (\theta_f^+)'(x,r)\big)$.

(*ii*) *If f is (ii)-GH-differentiable, then* $f'(x) = \big(m'_f(x); -(\theta_f^+)'(x,r), -(\theta_f^-)'(x,r)\big)$.

Here, the symbol "$\,'\,$" denotes derivation with respect to x.

The following results can be deduced from Theorem 3.3, immediately.

Corollary 3.4. *If f is (i)- or (ii)-GH-differentiable at x, then we have*

(a) $f'(x) = \big(m'_f(x); \max\{-\theta_f^{+\prime}(x,r), \theta_f^{-\prime}(x,r)\}, \max\{\theta_f^{+\prime}(x,r), -\theta_f^{-\prime}(x,r)\}\big)$,

(b) $\theta_f^{+\prime}(x,r)\theta_f^{-\prime}(x,r) \geq 0$.

Theorem 3.5. *Let* $f : (a,b) \to \mathbb{R}_{\mathcal{F}}$, $f(x) = \big(m_f(x); k_f^-(x)(1-r), k_f^+(x)(1-r)\big)$ *and* $m_f, k_f^-, k_f^+ \in C^1(a,b)$.

- *If* $\big(k_f^-(x)\big)' > 0$ *and* $\big(k_f^+(x)\big)' > 0$, *then f is (i)-GH-differentiable and*

$$f'(x) = \big((m_f(x))'; (k_f^-(x))'(1-r), (k_f^+)'(x)(1-r)\big).$$

- *If* $\big(k_f^-(x)\big)' < 0$ *and* $\big(k_f^+(x)\big)' < 0$, *then f is (ii)-GH-differentiable and*

$$f'(x) = \big((m_f(x))'; -(k_f^+(x))'(1-r), -(k_f^-)'(x)(1-r)\big).$$

Proof. The proof is immediate consequence of Theorems 3 and 4 of [11] and of the LU-representation theorem of Goetschel-Voxman [17]. □

In the following, integral of a fuzzy function and its connection with the GH-differentiablity is given by using the MCE-representation. Throughout this paper, we will use the fuzzy Riemann integral for the concept of integral [4]. Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ be fuzzy Riemann integrable. Then,

$$\int_a^b f(x)dx = \Big(\int_a^b m_f(x)dx; \int_a^b \theta_f^-(x,r)dx, \int_a^b \theta_f^+(x,r)dx\Big), \tag{3}$$

stands for the MCE-representation of fuzzy Riemann integral of f.

Theorem 3.6. [4] *If f is (i)- or (ii)-GH-differentiable over the entire interval $[a,b]$ then we have*

$$\int_a^b f'(x)dx = f(b) \ominus_{gH} f(a).$$

Lemma 3.7. [4] *Let $x_0 \in \mathbb{R}$, $f : \mathbb{R}\times\mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ be continuous and u be (i)-GH-differentiable. Then the fuzzy differential equation $u'(x) = f(x,u), u(x_0) = u_0 \in \mathbb{R}_{\mathcal{F}}$ is equivalent to the integral equation*

$$u(t) = u_0 + \int_{x_0}^{x} f(s,u(s))ds,$$

on some interval $[x_0, x_1] \subset \mathbb{R}$.

4 Fuzzy product rule

The aim of this section is to obtain a rule for the derivative of the product of two fuzzy functions.

Lemma 4.1. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}}$. Then*

$$\begin{aligned} -f(x) + 2m_f(x) &= \big(m_f(x); \theta_f^+(x,r), \theta_f^-(x,r)\big), \\ f(x) - 2m_f(x) &= \big(-m_f(x); \theta_f^-(x,r), \theta_f^+(x,r)\big). \end{aligned}$$

Proof. Since m_f is a crisp function, $m_f(x) = \big(m_f(x); 0, 0\big)$. Thus,

$$\begin{aligned} -f(x) + 2m_f(x) &= \big(-m_f(x); \theta_f^+(x,r), \theta_f^-(x,r)\big) + \big(2m_f(x); 0, 0\big) \\ &= \big(m_f(x); \theta_f^+(x,r), \theta_f^-(x,r)\big), \end{aligned}$$

and similarly the second equality can be shown. □

Let we denote a triangular fuzzy number-valued function, f, by $f(x) = \big(m_f(x); k_f^-(x)(1-r), k_f^+(x)(1-r)\big)$, where k_f^- and k_f^+ are functions of only x.

Lemma 4.2. *Let $f : [a,b] \to \mathbb{R}_{\mathcal{T}}$ be a triangular fuzzy number-valued function, where $f(x) = \big(m_f(x); k_f^-(x)(1-r), k_f^+(x)(1-r)\big)$. For $x_0 \in [a,b]$, if $\lim_{x\to x_0} m_f(x)$, $\lim_{x\to x_0} k_f^-(x)$ and $\lim_{x\to x_0} k_f^+(x)$ exist, then*

$$\lim_{x\to x_0} f(x) = \big(\lim_{x\to x_0} m_f(x), \lim_{x\to x_0} k_f^-(x)(1-r), \lim_{x\to x_0} k_f^+(x)(1-r)\big).$$

The limit is taken with respect to the metric d defined by Definition 2.2.

Proof. Suppose that $\lim_{x\to x_0} m_f(x) = m_u$, $\lim_{x\to x_0} k_f^-(x) = k_u^-$ and $\lim_{x\to x_0} k_f^+(x) = k_u^+$. Let us define $u := \big(m_u; k_u^-(1-r), k_u^+(1-r)\big)$. Since for every x in the domain of f, $k_f^-(x), k_f^+(x) \geq 0$, thus $\lim_{x\to x_0} k_f^-(x) = k_u^- \geq 0$ and $\lim_{x\to x_0} k_f^+(x) = k_u^+ \geq 0$. This means that $u \in \mathbb{R}_{\mathcal{F}}$. Furthermore,

$$\begin{aligned} \lim_{x\to x_0} d\big(f(x), u\big) &= \lim_{x\to x_0} d\Big(\big(m_f(x); k_f^-(x)(1-r), k_f^+(x)(1-r)\big), \big(m_u; k_u^-(1-r), k_u^+(1-r)\big)\big(m_u; k_u^-(1-r), k_u^+(1-r)\big)\Big) \\ &= \lim_{x\to x_0} \Big(|m_f(x) - m_u| + \sup_{r\in[0,1]} \max\{|(k_f^-(x) - k_u^-)(1-r)|, |(k_f^+(x) - k_u^+)(1-r)|\} \\ &= \lim_{x\to x_0} |m_f(x) - m_u| + \lim_{x\to x_0} \sup_{r\in[0,1]} (1-r)\max\{|k_f^-(x) - k_u^-|, |k_f^+(x) - k_u^+|\}. \end{aligned} \tag{4}$$

Since for every x sufficiently near to x_0, $\max\{|k_f^-(x) - k_u^-|, |k_f^+(x) - k_u^+|\}$ is bounded and positive, we have

$$\sup_{r\in[0,1]} (1-r)\max\{|k_f^-(x) - k_u^-|, |k_f^+(x) - k_u^+|\} = \max\{|k_f^-(x) - k_u^-|, |k_f^+(x) - k_u^+|\}. \tag{5}$$

From (4) and (5) and considering that $m_f(x)$, $k_f^-(x)$ and $k_f^+(x)$ tend to m_u, k_u^- and k_u^+, respectively, it can be simply concluded $\lim_{x\to x_0} d\big(f(x), u\big) = 0$. □

Having the Lemma 4.2 in the mind, all limits in the following theorem are calculated in the metric d.

Theorem 4.3. (Fuzzy product rule) *Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}_\tau$ defined as*

$$f(x) = \big(m_f(x); k_f^-(x)(1-r), k_f^+(x)(1-r)\big),$$
$$g(x) = \big(m_g(x); k_g^-(x)(1-r), k_g^+(x)(1-r)\big).$$

Then

(i) If f and g are (i)-GH-differentiable at x, then $f \circledast g$ is (i)-GH-differentiable at x and,

$$\big(f \circledast g\big)'(x) = \big(f' \circledast g\big)(x) + \big(f \circledast g'\big)(x). \tag{6}$$

(ii) If f and g are (ii)-GH-differentiable at x, then $f \circledast g$ is (ii)-GH-differentiable at x and,

$$\big(f \circledast g\big)'(x) = \big(f' \circledast (-g + 2m_g)\big)(x) + \big((-f + 2m_f) \circledast g'\big)(x). \tag{7}$$

(iii) If f is (ii)-GH-differentiable and g is (i)-GH-differentiable at x, then we have different cases as below:

(a) if $\Big(k_f^-(x)k_g^-(x)\Big)' > 0$ and $\Big(k_f^+(x)k_g^+(x)\Big)' > 0$, then $f \circledast g$ is (i)-GH-differentiable and,

$$(f \circledast g)'(x) = \big(g' \circledast f\big)(x) \ominus_H (-1)\big(f' \circledast (-g + 2m_g)\big)(x), \tag{8}$$

(b) if $\Big(k_f^-(x)k_g^-(x)\Big)' < 0$ and $\Big(k_f^+(x)k_g^+(x)\Big)' < 0$, then $f \circledast g$ is (ii)-GH-differentiable and,

$$(f \circledast g)'(x) = \big(f' \circledast (-g + 2m_g)\big)(x) \ominus_H (-1)(g' \circledast f)(x). \tag{9}$$

(c) if $\Big(k_f^-(x)k_g^-(x)\Big)' = 0$ and $\Big(k_f^+(x)k_g^+(x)\Big)' = 0$, then $(f \circledast g)'(x) \in \mathbb{R}$.

(d) if $\Big(k_f^-(x)k_g^-(x)\Big)'\Big(k_f^+(x)k_g^+(x)\Big)' < 0$, then $f \circledast g$ is not (i)- or (ii)-GH-differentiable.

Proof. To avoid repeating the details of the proof for each case **(i)-(iii)**, first we briefly explain the idea of the proof. Let x be given. To verify that $f \circledast g$ is (i)-GH-differentiable at x, we have to prove the existence of the following H-differences

$$(f \circledast g)(x + h) \ominus_H (f \circledast g)(x), \tag{10}$$
$$(f \circledast g)(x) \ominus_H (f \circledast g)(x - h). \tag{11}$$

Also, the existence of the H-differences $(f \circledast g)(x) \ominus_H (f \circledast g)(x + h)$ and $(f \circledast g)(x - h) \ominus_H (f \circledast g)(x)$ are needed for (ii)-GH-differentiability of $f \circledast g$ at x. In this proof, in each case, we show the existence of the only one of the two required H-differences and the other one can be shown similarly. Suppose that we want to prove (10) exists. Since

$$(f \circledast g)(x + h) = \Big(m_f(x + h)m_g(x + h); k_f^-(x + h)k_g^-(x + h)(1 - r), k_f^+(x + h)k_g^+(x + h)(1 - r)\Big),$$
$$(f \circledast g)(x) = \Big(m_f(x)m_g(x); k_f^-(x)k_g^-(x)(1 - r), k_f^+(x)k_g^+(x)(1 - r)\Big),$$

we have

$$(f \circledast g)(x + h) \ominus_H (f \circledast g)(x) = \Big(m_f(x + h)m_g(x + h) - m_f(x)m_g(x);$$
$$\big(k_f^-(x + h)k_g^-(x + h) - k_f^-(x)k_g^-(x)\big)(1 - r), \big(k_f^+(x + h)k_g^+(x + h) - k_f^+(x)k_g^+(x)\big)(1 - r)\Big).$$

For having $(f \circledast g)(x + h) \ominus_H (f \circledast g)(x) \in \mathbb{R}_\mathcal{F}$, it is necessary to show

$$\big(k_f^-(x + h)k_g^-(x + h) - k_f^-(x)k_g^-(x)\big)(1 - r) \qquad \& \qquad \big(k_f^+(x + h)k_g^+(x + h) - k_f^+(x)k_g^+(x)\big)(1 - r),$$

are positive, bounded, non-increasing with respect to r, left-continuous with respect to r on $(0, 1]$ and right-continuous at $r = 0$. They inherit boundedness and continuity conditions from boundedness and continuity conditions of fuzzy numbers $f(x)$, $g(x)$, $f(x+h)$ and $g(x+h)$. Thus, it is enough to show positivity and non-increasing properties. Since $(1-r)$ is positive and non-increasing with respect to r, it is enough to show

$$k_f^-(x+h)k_g^-(x+h) - k_f^-(x)k_g^-(x) \geq 0, \tag{12}$$

$$k_f^+(x+h)k_g^+(x+h) - k_f^+(x)k_g^+(x) \geq 0. \tag{13}$$

In a similar way, for verifying the existence of $(f \circledast g)(x) \ominus_H (f \circledast g)(x+h)$, it is enough to show

$$k_f^-(x)k_g^-(x) - k_f^-(x+h)k_g^-(x+h) \geq 0, \tag{14}$$

$$k_f^+(x)k_g^+(x) - k_f^+(x+h)k_g^+(x+h) \geq 0. \tag{15}$$

Now, having this in mind, we investigate (i)-(iii).

(i) Let f and g be (i)-GH-differentiable at x. Following the above reasoning, to verify (i)-GH-differentiability of $f \circledast g$ at x, we have to show (12) and (13). By adding and subtracting the term $k_f^-(x)k_g^-(x+h)$ to the left hand side of the inequality (12), we have

$$\begin{aligned}(k_f^- k_g^-)(x+h) - (k_f^- k_g^-)(x) =& k_f^-(x+h)k_g^-(x+h) - k_f^-(x)k_g^-(x+h) + k_f^-(x)k_g^-(x+h) - k_f^-(x)k_g^-(x)\\ =& \big(k_f^-(x+h) - k_f^-(x)\big)k_g^-(x+h) + \big(k_g^-(x+h) - k_g^-(x)\big)k_f^-(x). \end{aligned} \tag{16}$$

Since $f(x+h) \ominus_H f(x)$, $g(x+h) \ominus_H g(x)$, $g(x+h)$ and $f(x)$ are fuzzy numbers, the terms $k_f^-(x+h) - k_f^-(x)$, $k_g^-(x+h)$, $k_g^-(x+h) - k_g^-(x)$ and $k_f^-(x)$ are positive. Thus,

$$\big(k_f^-(x+h) - k_f^-(x)\big)k_g^-(x+h) + \big(k_g^-(x+h) - k_g^-(x)\big)k_f^-(x) \geq 0. \tag{17}$$

From (16) and (17), we conclude (12) (and the inequality (13) can be proved in a similar way). Therefore, the H-differences presented in (10) and (11) are exist.

Now, we have

$$\begin{aligned}(f \circledast g)'(x) =& \lim_{h \searrow 0} \frac{f(x+h) \circledast g(x+h) \ominus_H f(x) \circledast g(x)}{h}\\
=& \lim_{h \searrow 0} \Big(\frac{m_f(x+h)m_g(x+h) - m_f(x)m_g(x)}{h}; \frac{k_f^-(x+h)k_g^-(x+h) - k_f^-(x)k_g^-(x)}{h}(1-r),\\
& \frac{k_f^+(x+h)k_g^+(x+h) - k_f^+(x)k_g^+(x)}{h}(1-r)\Big)\\
=& \Big(\lim_{h \searrow 0} \frac{m_f(x+h)m_g(x+h) - m_f(x)m_g(x)}{h}; \lim_{h \searrow 0} \frac{k_f^-(x+h)k_g^-(x+h) - k_f^-(x)k_g^-(x)}{h}(1-r),\\
& \lim_{h \searrow 0} \frac{k_f^+(x+h)k_g^+(x+h) - k_f^+(x)k_g^+(x)}{h}(1-r)\Big)\\
=& \Big(m'_f(x)m_g(x) + m'_g(x)m_f(x); \big[\big(k_f^-(x)\big)'k_g^-(x) + \big(k_g^-(x)\big)'k_f^-(x)\big](1-r),\\
& \big[\big(k_f^+(x)\big)'k_g^+(x) + \big(k_g^+(x)\big)'k_f^+(x)\big](1-r)\Big)\\
=& \Big(m'_f(x); \big(k_f^-(x)\big)'(1-r), \big(k_f^+(x)\big)'(1-r)\Big) \circledast \Big(m_g(x); k_g^-(x)(1-r), k_g^+(x)(1-r)\Big)\\
+& \Big(m'_g(x); \big(k_g^-(x)\big)'(1-r), \big(k_g^+(x)\big)'(1-r)\Big) \circledast \Big(m_f(x); k_f^-(x)(1-r), k_f^+(x)(1-r)\Big)\\
=& f'(x) \circledast g(x) + g'(x) \circledast f(x).\end{aligned}$$

This limit has been taken with respect to the metric d defined in Definition 2.2.

(ii) Let f and g be (ii)-GH-differentiable. Based on the reasoning stated at the beginning of the proof, to show the existence of required H-differences for (ii)-GH-differentiability of $f \circledast g$ at x, it is enough to show (14) and (15). We show the inequality (14) and the inequality (15) can be shown similarly. By adding and subtracting the term $k_f^-(x+h)k_g^-(x)$ to the left hand side of (14), we have

$$\begin{aligned}(k_f^- k_g^-)(x) - (k_f^- k_g^-)(x+h) =& k_f^-(x)k_g^-(x) - k_f^-(x+h)k_g^-(x) + k_f^-(x+h)k_g^-(x) - k_f^-(x+h)k_g^-(x+h)\\ =& \big(k_f^-(x) - k_f^-(x+h)\big)k_g^-(x) + \big(k_g^-(x) - k_g^-(x+h)\big)k_f^-(x+h).\end{aligned} \tag{18}$$

Since $f(x)\ominus_H f(x+h), g(x)\ominus_H g(x+h), g(x+h), f(x) \in \mathbb{R}_\tau$, the terms $k_f^-(x)-k_f^-(x+h)$, $k_g^-(x)$, $k_g^-(x)-k_g^-(x+h)$ and $k_f^-(x+h)$ are positive. Thus, (18) is positive and consequently (14) holds.

Proceeding the proof, similar to the corresponding term in the case (i), we have

$$\begin{aligned}(f\circledast g)'(x) =& \lim_{h\searrow 0}\frac{f(x)\circledast g(x)\ominus_H f(x+h)\circledast g(x+h)}{-h}\\ =&\Big(m_f'(x)m_g(x)+m_g'(x)m_f(x);\big[-(k_f^+(x))'k_g^+(x)-(k_g^+(x))'k_f^+(x)\big](1-r),\\ &\big[-(k_f^-(x))'k_g^-(x)-(k_g^-(x))'k_f^-(x)\big]\Big)\\ =&\big(m_f'(x);-(k_f^+(x))'(1-r),-(k_f^-(x))'(1-r)\big)\circledast\big(m_g(x);k_g^+(x)(1-r),k_g^-(x)(1-r)\big)+\\ &\big(m_g'(x);-(k_g^+(x))'(1-r),-(k_g^-(x))'(1-r)\big)\circledast\big(m_f(x);k_f^+(x)(1-r),k_f^-(x)(1-r)\big)\\ =&f'(x)\circledast\big(-g(x)+2m_g(x)\big)+g'(x)\circledast\big(-f(x)+2m_f(x)\big).\end{aligned}$$

The last equality follows by Theorem 3.3 and Lemma 4.1.

(*iii*) Let f be (ii)-GH-differentiable and g be (i)-GH-differentiable. Based on the reasoning explained at the beginning of the proof, in order to prove that $f\circledast g$ is (i)-GH-differentiable, we have to show (12) and (13) are true and also, for having (ii)-GH-differentiability of $f\circledast g$, we have to show (14) and (15) are true.

(*a*) First, we investigate (i)-GH-differentiability. By adding and subtracting the term $k_f^-(x)k_g^-(x+h)$ to the left hand side of the inequality (12), we have

$$\begin{aligned}(k_f^-k_g^-)(x+h)-(k_f^-k_g^-)(x) =& k_f^-(x+h)k_g^-(x+h)-k_f^-(x)k_g^-(x+h)+k_f^-(x)k_g^-(x+h)\\ &-k_f^-(x)k_g^-(x)=k_g^-(x+h)\big(k_f^-(x+h)-k_f^-(x)\big)+\big(k_g^-(x+h)-k_g^-(x)\big)k_f^-(x). \qquad (19)\end{aligned}$$

Since $(k_f^-(x))'k_g^-(x)+k_f^-(x)(k_g^-(x))'>0$, for sufficiently small $h>0$, we have

$$\frac{k_f^-(x+h)-k_f^-(x)}{h}k_g^-(x)+k_f^-(x)\frac{k_g^-(x+h)-k_g^-(x)}{h}\geq 0,$$

and consequently

$$\big(k_f^-(x+h)-k_f^-(x)\big)k_g^-(x)+k_f^-(x)\big(k_g^-(x+h)-k_g^-(x)\big)\geq 0. \qquad (20)$$

Eqs. (19) and (20) lead to (12). The inequality (13) can be shown similarly. Thus

$$(f\circledast g)(x+h)\ominus_H(f\circledast g)(x),$$

exists. Now, again similar to the corresponding term in case (i), we have

$$\begin{aligned}(f\circledast g)'(x) =& \lim_{h\searrow 0}\frac{f(x+h)\circledast g(x+h)\ominus_H f(x)\circledast g(x)}{h}\\ =&\Big(m_f'(x)m_g(x)+m_g'(x)m_f(x);\big[(k_f^-(x))'k_g^-(x)+(k_g^-(x))'k_f^-(x)\big](1-r),\\ &\big[(k_f^+(x))'k_g^+(x)+(k_g^+(x))'k_f^+(x)\big](1-r)\Big)\\ =&\Big(m_g'(x)m_f(x);(k_g^-(x))'k_f^-(x)(1-r),(k_g^+(x))'k_f^+(x)(1-r)\Big)\\ &\ominus_H(-1)\Big(m_f'(x)m_g(x);-(k_f^+(x))'k_g^+(x)(1-r),-(k_f^-(x))'k_g^-(x)(1-r)\Big)\\ =&\big(m_g'(x);(k_g^-(x))'(1-r),(k_g^+(x))'(1-r)\big)\circledast\big(m_f(x);k_f^-(x)(1-r),k_f^+(x)(1-r)\big)\\ &\ominus_H(-1)\big(m_f'(x);-(k_f^+(x))'(1-r),-(k_f^-(x))'(1-r)\big)\circledast\big(m_g(x);k_g^+(x)(1-r),k_g^-(x)(1-r)\big)\\ =&g'(x)\circledast f(x)\ominus_H(-1)\big(f'(x)\circledast\big(-g(x)+2m_g(x)\big)\big).\end{aligned}$$

In the last equality, we used Theorem 3.3 and Lemma 4.1.

(*b*) Now, we investigate (ii)-GH-differentiability. We have to show the inequalities (14) and (15). The proofs of (14) and (15) are similar and so we only give the proof of (14). By adding and subtracting the term $k_f^-(x)k_g^-(x+h)$ to the left hand side of inequality (14), we have

$$(k_f^- k_g^-)(x) - (k_f^- k_g^-)(x+h) = k_f^-(x)k_g^-(x) - k_f^-(x)k_g^-(x+h) + k_f^-(x)k_g^-(x+h) - k_f^-(x+h)k_g^-(x+h)$$
$$= -\Big(k_f^-(x)\big(k_g^-(x+h) - k_g^-(x)\big) + k_g^-(x+h)\big(k_f^-(x+h) - k_f^-(x)\big)\Big). \tag{21}$$

Since $(k_f^-(x))'k_g^-(x) + k_f^-(x)(k_g^-(x))' < 0$, for sufficiently small $h > 0$, we have

$$\frac{k_f^-(x+h) - k_f^-(x)}{h}k_g^-(x) + k_f^-(x)\frac{k_g^-(x+h) - k_g^-(x)}{h} \leq 0,$$

and therefore

$$\big(k_f^-(x+h) - k_f^-(x)\big)k_g^-(x) + k_f^-(x)\big(k_g^-(x+h) - k_g^-(x)\big) \leq 0. \tag{22}$$

From (21) and (22), we conclude (14).

Eventually, we have

$$\begin{aligned}
(f \circledast g)'(x) = &\lim_{h\searrow 0} \frac{f(x) \circledast g(x) \ominus_H f(x+h) \circledast g(x+h)}{-h}\\
=&\Big(m_f'(x)m_g(x) + m_g'(x)m_f(x); \big[-(k_f^+(x))'k_g^+(x) - (k_g^+(x))'k_f^+(x)\big](1-r),\\
&\big[-(k_f^-(x))'k_g^-(x) - (k_g^-(x))'k_f^-(x)\big](1-r)\Big)\\
=&\Big(m_f'(x)m_g(x); -(k_f^+(x))'k_g^+(x)(1-r), -(k_f^-(x))'k_g^-(x)(1-r)\Big)\\
&\ominus_H (-1)\Big(m_g'(x)m_f(x); (k_g^-(x))'k_f^-(x)(1-r), (k_g^+(x))'k_f^+(x)(1-r)\Big)\\
=&\big(m_f'(x); -(k_f^+(x))'(1-r), -(k_f^-(x))'(1-r)\big) \circledast \big(m_g(x); k_g^+(x)(1-r), k_g^-(x)(1-r)\big)\\
&\ominus_H (-1)\big(m_g'(x); (k_g^-(x))'(1-r), (k_g^+(x))'(1-r)\big) \circledast \big(m_f(x); k_f^-(x)(1-r), k_f^+(x)(1-r)\big)\\
=&f'(x) \circledast \big(-g(x) + 2m_g(x)\big) \ominus_H (-1)\big(g'(x) \circledast f(x)\big).
\end{aligned}$$

The last equality comes from Theorem 3.3 and Lemma 4.1.

(*c*) In this case, the proof depends on the signs of $\left(k_f^- k_g^-\right)'$ and $\left(k_f^+ k_g^+\right)'$ before and after x. The details of the proof is similar to the previous cases.

(*d*) If $\left(k_f^-(x)k_g^-(x)\right)'\left(k_f^+(x)k_g^+(x)\right)' < 0$, then from Corollary 3.4 part **(b)**, $f \circledast g$ is not (i)- or (ii)-GH-differentiable.

□

Corollary 4.4. *Let f be a fuzzy function and $n \in \mathbb{N}$. Then*

(*i*) *if f is (i)-GH-differentiable, then $f^{\circledast n}$ is (i)-GH-differentiable and*

$$(f^{\circledast n})' = nf' \circledast f^{\circledast(n-1)}, \tag{23}$$

(*ii*) *if f is (ii)-GH-differentiable, then $f^{\circledast n}$ is (ii)-GH-differentiable and*

$$(f^{\circledast n})' = f' \circledast \left[(-f + 2m_f)^{\circledast(n-1)} + \sum_{i=1}^{n-2}(-f^{\circledast(i)} + 2m_{f^{\circledast(i)}}) \circledast (-f + 2m_f)^{\circledast(n-1-i)} + (-f^{\circledast(n-1)} + 2m_{f^{\circledast(n-1)}})\right]. \tag{24}$$

Note that the nth power of f is denoted by $f^{\circledast n}$ to emphasize that the multiplication operation is of MMCE-product type.

Proof. The proof is by induction on n.
(i) Let f be (i)-GH-differentiable. The equation (23) obviously is satisfied for $n = 2$. Now assume $(f^{\circledast n})' = nf' \circledast f^{\circledast(n-1)}$, for some n. To compute $(f^{\circledast(n+1)})'$, we use Theorem 4.3 along with the commutativity and distributivity of the MMCE-product. Thus,

$$\begin{aligned}(f^{\circledast(n+1)})' &= (f \circledast f^{\circledast(n)})' \\ &= f' \circledast f^{\circledast(n)} + f \circledast (f^{\circledast(n)})' \\ &= f' \circledast f^{\circledast(n)} + f \circledast \left(nf' \circledast f^{\circledast(n-1)}\right) \\ &= f' \circledast f^{\circledast(n)} + nf' \circledast f^{\circledast(n)} \\ &= (n+1)f' \circledast f^{\circledast(n)}.\end{aligned}$$

(ii) Similar to the case of (i), obviously, (24) holds for $n = 2$. We assume then that (24) holds for some n, and prove that it is true for $n + 1$.

$$\begin{aligned}(f^{\circledast(n+1)})' =&(f \circledast f^{\circledast(n)})' \\ =&f' \circledast (-f^{\circledast(n)} + 2m_{f^{\circledast(n)}}) + (-f + 2m_f) \circledast (f^{\circledast(n)})' \\ =&f' \circledast (-f^{\circledast(n)} + 2m_{f^{\circledast(n)}}) + (-f + 2m_f) \circledast f' \circledast \Big[(-f + 2m_f)^{\circledast(n-1)}+ \\ &\sum_{i=1}^{n-2}(-f^{\circledast(i)} + 2m_{f^{\circledast(i)}}) \circledast (-f + 2m_f)^{\circledast(n-1-i)} + (-f^{\circledast(n-1)} + 2m_{f^{\circledast(n-1)}})\Big] \\ =&f' \circledast \Big[(-f + 2m_f)^{\circledast(n)} + \sum_{i=1}^{n-1}(-f^{\circledast(i)} + 2m_{f^{\circledast(i)}}) \circledast (-f + 2m_f)^{\circledast(n-i)} + (-f^{\circledast(n)} + 2m_{f^{\circledast(n)}})\Big].\end{aligned}$$

□

Remark 4.5. *Note that the fuzzy product rule (Theorem 4.3) and its result (Corollary 4.4) were proved for triangular fuzzy number-valued functions. However, this assumption was used just for saving the shape-preserving property. It should be emphasized they are, also, hold for non-triangular fuzzy number-valued functions provided that the MMCE-product is replaced with the MCE-product (of course, the shape-preserving property will not be satisfied anymore). The basic idea and details of the proof are the same as that of Theorem 4.2. As an instance, for the case (i), the only difference is that, now, we have to prove*

$$\theta_f^-(x+h,r)\theta_g^-(x+h,r) - \theta_f^-(x,r)\theta_g^-(x,r), \tag{25}$$
$$\theta_f^+(x+h,r)\theta_g^+(x+h,r) - \theta_f^+(x,r)\theta_g^+(x,r), \tag{26}$$

are non-increasing with respect to r. By adding and subtracting the term $\theta_f^-(x+h,r)\theta_g^-(x,r)$ to (25), we have

$$\begin{aligned}\theta_f^-(x+h,r)\theta_g^-(x+h,r) - \theta_f^-(x,r)\theta_g^-(x,r) =&\theta_f^-(x+h,r)\theta_g^-(x+h,r) - \theta_f^-(x+h,r)\theta_g^-(x,r) \\ &+ \theta_f^-(x+h,r)\theta_g^-(x,r) - \theta_f^-(x,r)\theta_g^-(x,r) \\ =&\theta_f^-(x+h,r)\Big(\theta_g^-(x+h,r) - \theta_g^-(x,r)\Big) \\ &+ \Big(\theta_f^-(x+h,r) - \theta_f^-(x,r)\Big)\theta_g^-(x,r).\end{aligned} \tag{27}$$

Since $f(x+h), g(x), f(x+h) \ominus_H f(x), g(x+h) \ominus_H g(x) \in \mathbb{R}_{\mathcal{F}}$, it follows that $\theta_f^-(x+h,r)$, $\theta_g^-(x,r)$, $\theta_f^-(x+h,r) - \theta_f^-(x,r)$ and $\theta_g^-(x+h,r) - \theta_g^-(x,r)$ are positive and non-increasing with respect to r, and since the product of positive and non-increasing functions is non-increasing, and the sum of non-increasing functions is non-increasing, we can concluded that

$$\theta_f^-(x+h,r)\Big(\theta_g^-(x+h,r) - \theta_g^-(x,r)\Big) + \Big(\theta_f^-(x+h,r) - \theta_f^-(x,r)\Big)\theta_g^-(x,r), \tag{28}$$

is non-increasing with respect to r. This fact along with (27) leads to the non-increase property of (25). In the same way, non-increase property can be proven for (26). Similar reasoning can be used for the other cases of Theorem 4.3.

We provide an example to verify different cases of Theorem 4.3.

Example 4.6. *Consider* $f(x) = \big(1-\cos(\frac{\pi}{6}x-\frac{\pi}{6})\big)(1;4(1-r),1-r)$ *and* $g(x) = (x+1)^2(1;1-r,1-r)$*. First, we calculate* $(f \circledast g)'(x)$*, directly. The MMCE-product of* f *and* g *is*

$$(f \circledast g)(x) = h(x)\Big(1;4(1-r),(1-r)\Big),$$

where, $h(x) = (x+1)^2\big(1-\cos(\frac{\pi}{6}x-\frac{\pi}{6})\big)$*. By noting that*

$$\begin{cases} h(x)h'(x) > 0, & x \in (1,2) \cup (-1, 0.0113),\\ h(x)h'(x) < 0, & x \in (-2,-1) \cup (0.0113, 1),\\ h(x)h'(x) = 0, & x \in \{-1, 0.0113, x = 1\},\end{cases}$$

and using Theorem 2 in [7], $f \circledast g$ *is (i)-GH-differentiable on* $(1,2)\cup(-1,0.0113)$ *and (ii)-GH-differentiable on* $(-2,-1)\cup(0.0113,1)$*. Note that* $x=-1, x=0.0113$ *and* $x=1$ *are the so-called switching points. Furthermore, from Theorem 3.3, we have*

$$(f \circledast g)'(x) = \begin{cases} \Big(h'(x); 4h'(x)(1-r), h'(x)(1-r)\Big), & x \in (1,2)\cup(-1,0.0113),\\ \Big(h'(x); -h'(x)(1-r), -4h'(x)(1-r)\Big), & x \in (-2,-1)\cup(0.0113,1),\\ 0, & x \in \{-1, 0.0113, x=1\}.\end{cases} \tag{29}$$

Now, we calculate $(f \circledast g)'(x)$ *by using the proposed fuzzy product rule, Theorem 4.3 and compare with* (29)*. Note that this example is designed in a way that covers all cases of the theorem. To avoid prolonging the content, let's consider only the case* **(iii)***, which is rather complicated than the other ones. First we note that by a simple calculation, it can be shown that:*

$$-g(x) + 2m_g(x) = \Big((x+1)^2; (x+1)^2(1-r), (x+1)^2(1-r)\Big), \tag{30}$$

$$-f(x) + 2m_f(x) = \Big(\big(1-\cos(\frac{\pi}{6}x-\frac{\pi}{6})\big); \big(1-\cos(\frac{\pi}{6}x-\frac{\pi}{6})\big)(1-r), 4\big(1-\cos(\frac{\pi}{6}x-\frac{\pi}{6})\big)(1-r)\Big). \tag{31}$$

Now, let us to calculate $(f \circledast g)'(x)$ *by using the formula given in Eqs.* (8) *and* (9)*. Obviously,* f *is (ii)-GH-differentiable and* g *is (i)-GH-differentiable on* $[-1,1]$ *and,*

a) Since on $[-1, 0.0113]$*,* $\Big(k_f^-(x)k_g^-(x)\Big)'$ *and* $\Big(k_f^+(x)k_g^+(x)\Big)'$ *are positive,* $(f \circledast g)$ *is (i)-GH-differentiable and* $(f \circledast g)'(x)$ *reads:*

$$\begin{aligned} g'(x) \circledast f(x) \ominus_H (-1)f'(x) \circledast \Big(-g(x) + 2m_g(x)\Big) &= \Big(2(x+1); 2(x+1)(1-r), 2(x+1)(1-r)\Big)\\ &\circledast \Big(1-\cos(\frac{\pi}{6}x-\frac{\pi}{6}); \big[4-4\cos(\frac{\pi}{6}x-\frac{\pi}{6})\big](1-r), \big[1-\cos(\frac{\pi}{6}x-\frac{\pi}{6})\big](1-r)\Big)\\ &\ominus_H (-1)\Big(\frac{\pi}{6}\sin(\frac{\pi}{6}x-\frac{\pi}{6}); -\frac{\pi}{6}\sin(\frac{\pi}{6}x-\frac{\pi}{6})(1-r), -4\frac{\pi}{6}\sin(\frac{\pi}{6}x-\frac{\pi}{6})(1-r)\Big)\\ &\circledast \Big((x+1)^2; (x+1)^2(1-r), (x+1)^2(1-r)\Big)\\ &= \Bigg(2(x+1)\big(1-\cos(\frac{\pi}{6}x-\frac{\pi}{6})\big) + (x+1)^2\frac{\pi}{6}\sin(\frac{\pi}{6}x-\frac{\pi}{6});\\ &\quad 8(x+1)\big(1-\cos(\frac{\pi}{6}x-\frac{\pi}{6})\big)(1-r) + 4(x+1)^2\frac{\pi}{6}\sin(\frac{\pi}{6}x-\frac{\pi}{6})(1-r),\\ &\quad 2(x+1)\big(1-\cos(\frac{\pi}{6}x-\frac{\pi}{6})\big)(1-r) + (x+1)^2\frac{\pi}{6}\sin(\frac{\pi}{6}x-\frac{\pi}{6})(1-r)\Bigg) \qquad (32)\\ &= \Big(h'(x); 4h'(x)(1-r), h'(x)(1-r)\Big), \qquad (33)\end{aligned}$$

which is exactly the same as Eq. (29) *(the first line).*

b) Since $\Big(k_f^-(x)k_g^-(x)\Big)'$ *and* $\Big(k_f^+(x)k_g^+(x)\Big)'$ *are negative on* $[0.0113, 1]$, $(f \circledast g)$ *is (ii)-GH-differentiable and by Eq.* (9), $(f \circledast g)'(x)$ *equals to:*

$$f'(x) \circledast \big(-g(x) + 2m_g(x)\big) \ominus_H (-1)g'(x) \circledast f(x) =$$
$$\Big(\frac{\pi}{6}\sin(\frac{\pi}{6}x - \frac{\pi}{6}); -\frac{\pi}{6}\sin(\frac{\pi}{6}x - \frac{\pi}{6})(1-r), -4\frac{\pi}{6}\sin(\frac{\pi}{6}x - \frac{\pi}{6})(1-r)\Big)$$
$$\circledast \Big((x+1)^2; (x+1)^2(1-r), (x+1)^2(1-r)\Big)$$
$$\ominus_H (-1)\Big(2(x+1); 2(x+1)(1-r), 2(x+1)(1-r)\Big)$$
$$\circledast \Big(1 - \cos(\frac{\pi}{6}x - \frac{\pi}{6}); \big[4 - 4\cos(\frac{\pi}{6}x - \frac{\pi}{6})\big](1-r), \big[1 - \cos(\frac{\pi}{6}x - \frac{\pi}{6})\big](1-r)\Big) \tag{34}$$

$$= \Big(\frac{\pi}{6}\sin(\frac{\pi}{6}x - \frac{\pi}{6})(x+1)^2 + 2(x+1)\big(1 - \cos(\frac{\pi}{6}x - \frac{\pi}{6});$$
$$-\frac{\pi}{6}\sin(\frac{\pi}{6}x - \frac{\pi}{6})(x+1)^2(1-r) - 2(x+1)\big[1 - \cos(\frac{\pi}{6}x - \frac{\pi}{6})\big](1-r),$$
$$-4\frac{\pi}{6}\sin(\frac{\pi}{6}x - \frac{\pi}{6})(x+1)^2(1-r) - 8(x+1)\big[1 - \cos(\frac{\pi}{6}x - \frac{\pi}{6})\big](1-r)\Big) \tag{35}$$
$$= \Big(h'(x); -h'(x)(1-r), -4h'(x)(1-r)\Big), \tag{36}$$

which is exactly the same as Eq. (29) *(the second line).*

In Figure 2, the functions f, g *and* $f \circledast g$ *are plotted. The behavior of the uncertainty conforms the kind of differentiability mentioned above. As expected, in the switching points, the kind of differentiability is changed.*

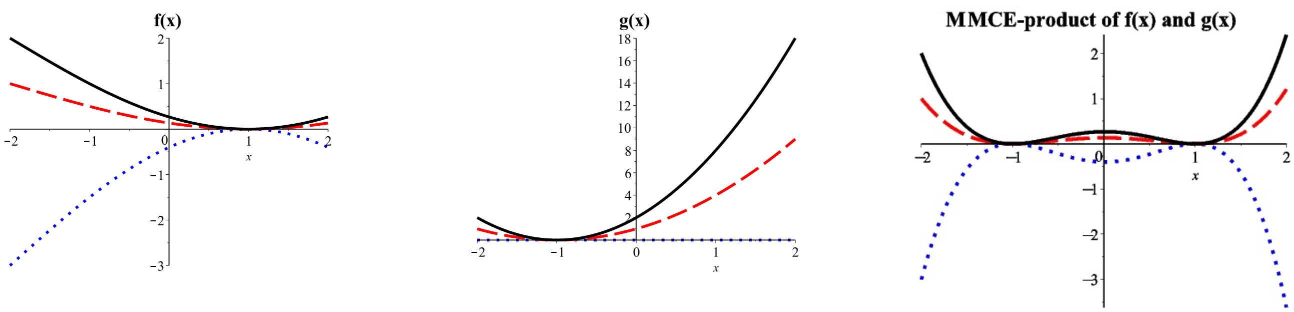

Figure 2: Solid, dash and dot lines denote upper endpoint of 0-cuts, 1-cuts and lower endpoint of 0-cut, respectively.

5 Applications

It is well known that the product rule is a basic rule in the calculus and has many applications. Here, we propose two cases. First, we obtain an integration by parts formula for fuzzy functions. Then, we solve a nonlinear fuzzy differential equation.

5.1 Integration by parts

As an important application of fuzzy product rule (Theorem 4.3), we obtain an integration by parts formula for fuzzy functions.

Theorem 5.1. *Let* $f, g : \mathbb{R} \longrightarrow \mathbb{R}_\tau$. *Then*

(*i*) *If* f *and* g *are (i)-GH-differentiable, then*

$$\int_a^b (f' \circledast g)(x)dx = \Big((f \circledast g)(b) \ominus_{gH} (f \circledast g)(a)\Big) \ominus_H \int_a^b (f \circledast g')(x)dx. \tag{37}$$

(ii) *If f and g are (ii)-GH-differentiable, then*

$$\int_a^b \big(f' \circledast (-g)\big)(x)dx = \Big((f \circledast g)(b) \ominus_{gH} (f \circledast g)(a)\Big) \ominus_H \int_a^b \big((-f + 2m_f) \circledast g'\big)(x)d \ominus_H \int_a^b \big(f' \circledast (2m_g)\big)(x)dx. \tag{38}$$

(iii) *If f is (ii)-GH-differentiable and g is (i)-GH-differentiable, then we have two different cases as bellows:*

a) *if $f \circledast g$ is (i)-GH-differentiable, then*

$$\int_a^b (g' \circledast f)(x)dx = \Big((f \circledast g)(b) \ominus_{gH} (f \circledast g)(a)\Big) + (-1) \int_a^b \big(f' \circledast (-g + 2m_g)\big)(x)dx, \tag{39}$$

b) *if $f \circledast g$ is (ii)-GH-differentiable, then*

$$\int_a^b \big(g' \circledast f\big)(x)dx = (-1) \int_a^b \big(f' \circledast (-g + 2m_g)\big)(x)dx \ominus_H (-1)\Big((f \circledast g)(b) \ominus_{gH} (f \circledast g)(a)\Big). \tag{40}$$

Proof. (i) According to Theorem 4.3, we have

$$(f \circledast g)'(x) = (f' \circledast g)(x) + (f \circledast g')(x). \tag{41}$$

Integrating both sides of (41) with respect to x and using Theorem 3.6, we find that

$$(f \circledast g)(b) \ominus_{gH} (f \circledast g)(a) = \int_a^b (f' \circledast g)(x)dx + \int_a^b (f \circledast g')(x)dx,$$

and so

$$\int_a^b (f' \circledast g)(x)dx = \Big((f \circledast g)(b) \ominus_{gH} (f \circledast g)(a)\Big) \ominus_H \int_a^b (f \circledast g')(x)dx,$$

which is the required result. Similar to (i) by using Theorem 4.3, the proofs of (ii) and (iii) are straightforward. □

Example 5.2. *In this example, we illustrate the effectiveness of the proposed integration by parts formula.*

(i) *Let $f(x) = e^x\big(1; (1-r), (1-r)\big)$ and $g(x) = x^2\big(1; 2(1-r), (1-r)\big)$. Clearly f and g are (i)-GH-differentiable for $x > 0$ and*

$$f'(x) = e^x\big(1; (1-r), (1-r)\big),\ \ g'(x) = 2x\big(1; 2(1-r), (1-r)\big).$$

From (3), we have

$$\begin{aligned}\int_0^1 (f' \circledast g)(x)dx &= \int_0^1 e^x\big(1; (1-r), (1-r)\big) \circledast x^2\big(1; 2(1-r), (1-r)\big)dx \\ &= (e-2)\big(1; 2(1-r), (1-r)\big).\end{aligned} \tag{42}$$

Also,

$$\begin{aligned}\int_0^1 (f \circledast g')(x)dx &= \int_0^1 e^x\big(1; (1-r), (1-r)\big) \circledast 2x\big(1; 2(1-r), (1-r)\big)(x)dx \\ &= 2\big(1; 2(1-r), (1-r)\big).\end{aligned} \tag{43}$$

On the other hand,

$$(f \circledast g)(1) \ominus_H (f \circledast g)(0) = e\big(1; 2(1-r), (1-r)\big). \tag{44}$$

Finally, (42)-(44) confirm (37).

(*ii*) *Let* $f(x) = e^{-x}\big(1;(1-r),(1-r)\big)$ *and* $g(x) = (2-x)\big(1;2(1-r),(1-r)\big)$. *Clearly* f *and* g *are (ii)-GH-differentiable for* $0 < x < 2$ *and*

$$f'(x) = e^{-x}\big(-1;(1-r),(1-r)\big),\ g'(x) = \big(-1;(1-r),2(1-r)\big),$$

then $f \circledast g$ *is (ii)-GH-differentiable, and*

$$\begin{aligned}\int_0^1 \big(f' \circledast (-g)\big)(x)dx &= \int_0^1 e^{-x}\big(-1;(1-r),(1-r)\big) \circledast (x-2)\big(1;2(1-r),(1-r)\big)dx\\ &= \big(1;(1-r),2(1-r)\big).\end{aligned} \tag{45}$$

On the other hand,

$$\begin{aligned}\big(f \circledast g\big)(1) \ominus_{gH} \big(f \circledast g\big)(0) &= (2-1)e^{-1}\big(1;2(1-r),(1-r)\big) \ominus_{gH} (2-0)e^{-0}\big(1;2(1-r),(1-r)\big)\\ &= \big(e^{-1}-2;(2-e^{-1})(1-r),2(2-e^{-1})(1-r)\big),\end{aligned} \tag{46}$$

and

$$\begin{aligned}&\int_0^1 \big((-f+2m_f) \circledast (g')\big)(x)dx \ominus_H \int_0^1 \big(f' \circledast (2m_g)\big)(x)dx\\ &= \int_0^1 \big(e^{-x};(1-r),(1-r)\big) \circledast (-1)\big(1;2(1-r),(1-r)\big)dx\\ &\qquad \ominus_H \int_0^1 -e^{-x}\big(1;(1-r),(1-r)\big) \circledast (2(2-x);0,0)dx\\ &= \big(e^{-1}-3;(1-e^{-1})(1-r),2(1-e^{-1})(1-r)\big).\end{aligned} \tag{47}$$

The Eqs. (45)-(47) *confirm* (38).

(*iii*) a) *Let* $f(x) = e^{-x}\big(1;(1-r),(1-r)\big)$ *and* $g(x) = x^2\big(1;2(1-r),(1-r)\big)$. *Clearly,* f *is (ii)-GH-differentiable and* g *is (i)-GH-differentiable for* $x > 0$ *and*

$$f'(x) = e^{-x}\big(-1;(1-r),(1-r)\big),\ g'(x) = 2x\big(1;2(1-r),(1-r)\big),$$

then $f \circledast g$ *is (i)-GH-differentiable for* $x > 0$, *and*

$$\begin{aligned}\int_0^1 (g' \circledast f)(x)dx &= \int_0^1 2x\big(1;2(1-r),(1-r)\big) \circledast e^{-x}\big(1;(1-r),(1-r)\big)dx\\ &= (2-4e^{-1})\big(1;2(1-r),(1-r)\big).\end{aligned} \tag{48}$$

On the other hand,

$$\big(f \circledast g\big)(1) \ominus_{gH} \big(f \circledast g\big)(0) = e^{-1}\big(1;2(1-r),(1-r)\big), \tag{49}$$

and

$$\begin{aligned}(-1)\int_0^1 \big(f' \circledast (-g+2m_g)\big)(x)dx &= (-1)\int_0^1 \big(-e^{-x};e^{-x}(1-r),e^{-x}(1-r)\big) \circledast \big(x^2;x^2(1-r),2x^2(1-r)\big)dx\\ &= \big(2-5e^{-1};2(2-5e^{-1})(1-r),(2-5e^{-1})(1-r)\big).\end{aligned} \tag{50}$$

The Eqs. (48)-(50) *confirm* (39).

b) *Let* $f(x) = (1-x)\big(1;(1-r),(1-r)\big)$ *and* $g(x) = e^x\big(1;2(1-r),(1-r)\big)$. *Clearly* f *is (ii)-GH-differentiable and* g *is (i)-GH-differentiable for* $x > 0$ *and*

$$f'(x) = \big(-1;(1-r),(1-r)\big),\ g'(x) = e^x\big(1;2(1-r),(1-r)\big),$$

then $f \circledast g$ *is (ii)-GH-differentiable for* $x > 0$, *and*

$$\begin{aligned}\int_0^1 \big(g' \circledast f\big)(x)dx &= \int_0^1 e^x\big(1;2(1-r),(1-r)\big) \circledast (1-x)\big(1;(1-r),(1-r)\big)dx\\ &= (e-2)\big(1;2(1-r),(1-r)\big).\end{aligned} \tag{51}$$

On the other hand,

$$(-1)\Big[\big(f\circledast g\big)(1)\ominus_{gH}\big(f\circledast g\big)(0)\Big]=\big(1;2(1-r),(1-r)\big), \tag{52}$$

and

$$\begin{aligned}(-1)\int_0^1\big(f'\circledast(-g+2m_g)\big)(x)dx=&(-1)\int_0^1 e^x\big(-1;(1-r),2(1-r)\big)dx\\=&(e-1)\big(1;2(1-r),(1-r)\big).\end{aligned} \tag{53}$$

The Eqs. (51)-(53) *confirm* (40).

5.2 Solution of a nonlinear differential equation

Here, we study the following initial value problem

$$\begin{cases}u\circledast u'=g(x),\\ u(x_0)=u_0,\end{cases} \tag{54}$$

where $g:\mathbb{R}\to\mathbb{R}_{\mathcal{F}}$ is continuous and $u_0\in\mathbb{R}_{\mathcal{F}}$.

Definition 5.3. *We say that $u:[x_0,\infty)\to\mathbb{R}_{\mathcal{F}}$ is a solution for problem* (54) *if it is GH-differentiable and satisfies problem* (54) *for $x\geq x_0$. According to the type of GH-differentiability, two types of the solution can be considered:*

- *A fuzzy solution u is called* **(i)-solution***, if u is (i)-GH-differentiable.*
- *A fuzzy solution u is called* **(ii)-solution***, if u is (ii)-GH-differentiable.*

The (i)-solution is obtained by using Corollary 4.4 and the (ii)-solution is obtained by using MMCE-product, directly. **(i)-solution**: Let u be (i)-GH-differentiable and

$$\begin{aligned}u(x)=&\big(m_u(x);k_u^-(x)(1-r),k_u^+(x)(1-r)\big),\\ g(x)=&\big(m_g(x);k_g^-(x)(1-r),k_g^+(x)(1-r)\big),\\ u_0=&\big(m_0;k_0^-(1-r),k_0^+(1-r)\big).\end{aligned}$$

From (23), we know

$$\big(u^{\circledast 2}\big)'=2u\circledast u'.$$

This equality along with (54) yield

$$\big(u^{\circledast 2}\big)'=2g(x).$$

Since u is (i)-GH-differentiable, from the first part of Corollary 4.4, $u^{\circledast 2}$ is (i)-GH-differentiable. This fact along with the definition of MMCE-product and Theorem 3.3 result

$$\big(u^{\circledast 2}\big)'=\big((m_u)^{2'},(k_u^-)^{2'}(1-r),(k_u^+)^{2'}(1-r)\big)=2\big(m_g;k_g^-(1-r),k_g^+(1-r)\big).$$

From Lemma 3.7, we have

$$u^{\circledast 2}(x)=u^{\circledast 2}(0)+2\int_0^x\big(m_g(t);k_g^-(t)(1-r),k_g^+(t)(1-r)\big)dt,$$

thus,

$$\begin{cases}\big(m_u(x)\big)^2=m_0^2+2\int_0^x m_g(t)dt,\\ \big(k_u^-(x)\big)^2=\big(k_0^-\big)^2+2\int_0^x k_g^-(t)dt,\\ \big(k_u^+(x)\big)^2=\big(k_0^+\big)^2+2\int_0^x k_g^+(t)dt.\end{cases}\quad\Rightarrow\quad\begin{cases}m_u(x)=\pm\sqrt{m_0^2+2\int_0^x m_g(t)dt},\\ k_u^-(x)=\pm\sqrt{\big(k_0^-\big)^2+2\int_0^x k_g^-(t)dt},\\ k_u^+(x)=\pm\sqrt{\big(k_0^+\big)^2+2\int_0^x k_g^+(t)dt}.\end{cases}$$

Since k_u^- and k_u^+ must be positive, the positive signs are acceptable, i.e,

$$\begin{cases} m_u(x) = \pm\sqrt{m_0^2 + 2\int_0^x m_g(t)dt}, \\ k_u^-(x) = \sqrt{\left(k_0^-\right)^2 + 2\int_0^x k_g^-(t)dt}, \\ k_u^+(x) = \sqrt{\left(k_0^+\right)^2 + 2\int_0^x k_g^+(t)dt}. \end{cases}$$

Since $(k_u^-(x))', (k_u^-(x))' > 0$, from Theorem 3.5,

$$u_1(x) = \big(\sqrt{m_0^2 + 2\int_0^x m_g(t)dt}, k_u^-(x), k_u^+(x)\big) \quad \& \quad u_2(x) = \big(-\sqrt{m_0^2 + 2\int_0^x m_g(t)dt}, k_u^-(x), k_u^+(x)\big)$$

are (i)-GH-differentiable and clearly satisfy (54). Thus u_1 and u_2 are (i)-solutions of (54).

(ii)-solution: Let u be (ii)-GH-differentiable. Then $u' = \big(m_u'; -k_u^{+'}(1-r), -k_u^{-'}(1-r)\big)$. By substituting u' in the equation (54), we obtain

$$u \circledast u' = \big(m_u.m_u', -k_u^-.k_u^{+'}(1-r), -k_u^+.k_u^{-'}(1-r)\big) = \big(m_g; k_g^-(1-r), k_g^+(1-r)\big).$$

Thus, we have

$$m_u.m_u' = m_g, \qquad m_u(0) = u_0, \tag{55}$$

and the system of differential equations with initial conditions

$$-k_u^-.k_u^{+'} = k_g^-, \qquad k_u^-(0) = k_{u0}^-, \tag{56}$$

$$-k_u^+.k_u^{-'} = k_g^+, \qquad k_u^+(0) = k_{u0}^+. \tag{57}$$

Summing the two Eqs. (56) and (57) yields to

$$-(k_u^-.k_u^{+'} + k_u^+.k_u^{-'}) = k_g^- + k_g^+ \Rightarrow \quad -(k_u^-.k_u^+)' = k_g^+ + k_g^-. \tag{58}$$

By integrating both sides of (58), we obtain

$$-k_u^-(x).k_u^+(x) + k_{u0}^- k_{u_0}^+ = \int_{x_0}^x \big(k_g^-(t) + k_g^+(t)\big)dt.$$

Let $f(x) = k_{u0}^- k_{u_0}^+ - \int_{x_0}^x \big(k_g^-(t) + k_g^+(t)\big)dt$. Then we have

$$k_u^-(x) = \frac{f(x)}{k_u^+(x)}. \tag{59}$$

This equality along with (56) leads to the first order differential equation

$$\frac{k_u^{+'}(x)}{k_u^+(x)} = -\frac{k_g^-(x)}{f(x)}, \qquad k_u^-(0) = k_{u0}^-,$$

whose answer is $k_u^+(x) = k_{u0}^- \exp(-\int_{x_0}^x \frac{k_g^-(t)}{f(t)}dt)$. Finally, from (59)

$$k_u^-(x) = \frac{f(x)}{k_u^+(x)} = \frac{f(x)}{k_{u0}^- \exp(-\int_{x_0}^x \frac{k_g^-(t)}{f(t)}dt)} = \frac{f(x)}{k_{u0}^-} \exp\big(\int_{x_0}^x \frac{k_g^-(t)}{f(t)}dt\big).$$

Clearly $k_u^+(x)$ is positive. If $k_u^-(x)$ is also positive, then $u = \big(m_u(x); k_u^-(x)(1-r), k_u^+(x)(1-r)\big) \in \mathbb{R}_{\mathcal{F}}$. Moreover, if $(k_u^+(x))', (k_u^-(x))' < 0$, then u is (ii)-GH-differentiable and since it satisfies problem (54), it is (ii)-solution of (54). Otherwise it doesn't have a (ii)-solution.

Example 5.4. *Consider the following nonlinear fuzzy initial value problem*

$$\begin{cases} u \circledast u' = (-1; 1-r, 1-r), \\ u(0) = (1; 1-r, 1-r). \end{cases} \tag{60}$$

Using the described method, (i)- and (ii)-solutions are $u_{1,2}(x) = \big(\pm\sqrt{1-2x}; \sqrt{1+2x}(1-r), \sqrt{1+2x}(1-r)\big)$ *and* $u_{1,2}(x) = \big(\pm\sqrt{1-2x}, \sqrt{1-2x}(1-r), \sqrt{1-2x}(1-r)\big)$*, respectively. In Figure 3, 1-cut and 0-cut of*

$$u_1(x) = \big(\sqrt{1-2x}; \sqrt{1+2x}(1-r), \sqrt{1+2x}(1-r)\big) \quad \& \quad u_2(x) = \big(\sqrt{1-2x}, \sqrt{1-2x}(1-r), \sqrt{1-2x}(1-r)\big)$$

are plotted.

Figure 3: Solid, dash and dot lines denote upper endpoint of 0-cuts, 1-cuts and lower endpoint of 0-cut, respectively.

6 Conclusion

The main contribution of this paper is obtaining the product rule, i.e. $(fg)' = f'g + fg'$ for fuzzy functions. To do this, first, we have modified the recently introduced MCE-product to take advantage of the shape-preserving property, and called it MMCE-product. This product has interesting advantages, for example, it is distributive and easy to use and it doesn't depend on the signs of multiplied fuzzy numbers. By using these properties, the product rule has been proved by considering all possible cases of (i)- and (ii)-GH-differentiability of involved functions.

Two important applications of fuzzy product rule have been introduced: integration by parts formula and solving a nonlinear differential equation. The integration by parts formula has been obtained before for the case that one of the involving functions was fuzzy and the other was crisp [2]. In the present work, it has been obtained for two fuzzy functions. In the last section of the paper, a nonlinear fuzzy differential equation has been solved which contains multiplication between the unknown fuzzy function and its derivative. Considering the products such as Zadeh extension based product and cross product, such equations are difficult to solve. Because these products depend on the sign of the unknown function u which is unknown before solving. In the present work, we have shown that this equation can be easily solved using MMCE-product and the proposed fuzzy product rule.

Acknowledgement

The authors would like to thank anonymous referees for their careful reading and comments which helped to improve the paper.

References

[1] R. Alikhani, F. Bahrami, *Fuzzy partial differential equations under the cross product of fuzzy numbers*, Information Sciences, **494** (2019), 80-99.

[2] A. Armand, T. Allahviranloo, Z. Gouyandeh, *Some fundamental results on fuzzy calculus*, Iranian Journal of Fuzzy Systems, **15**(3) (2018), 27-46.

[3] A. Ban, B. Bede, *Properties of the cross product of fuzzy numbers*, Journal of Fuzzy Mathematics, **14**(3) (2005), 513.

[4] B. Bede, *Mathematics of fuzzy sets and fuzzy logic*, Springer, 2013.

[5] B. Bede, J. Fodor, *Product type operations between fuzzy numbers and their applications in geology*, Acta Polytechnica Hungarica, **3**(1) (2006), 123-139.

[6] B. Bede, S. Gal, *Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations*, Fuzzy Sets and Systems, **151**(3) (2005), 581-599.

[7] B. Bede, I. Rudas, A. Bencsik, *First order linear fuzzy differential equations under generalized differentiability*, Information Sciences, **177**(7) (2007), 1648-1662.

[8] B. Bede, L. Stefanini, *Generalized differentiability of fuzzy-valued functions*, Fuzzy Sets and Systems, **230**(1) (2013), 119-141.

[9] A. M. Bica, D. Fechete, I. Fechete, *Towards the properties of fuzzy multiplication for fuzzy numbers*, Kybernetika, **55**(1) (2019), 44-62.

[10] C. Carlsson, R. Fullér, *On additions of interactive fuzzy numbers*, Proceedings of the Fifth International Symposium of Hungarian Researchers on Computational Intelligence, Budapest, (2004), 227-238.

[11] Y. Chalco-Cano, R. Rodríguez-López, M. D. Jiménez-Gamero, *Characterizations of generalized differentiable fuzzy functions*, Fuzzy Sets and Systems, **295** (2016), 37-56.

[12] Y. Chalco-Cano, A. Rufián-Lizana, H. Román-Flores, M. D. Jiménez-Gamero, *Calculus for interval-valued functions using generalized Hukuhara derivative and applications*, Fuzzy Sets and Systems, **219** (2013), 49-67.

[13] S. S. Chang, L. A. Zadeh, *On fuzzy mapping and control. In fuzzy sets, fuzzy logic, and fuzzy systems: Selected papers by Lotfi A Zadeh*, World Scientific, (1996), 180-184.

[14] D. Dubois, H. Prade, *Operations on fuzzy numbers*, International Journal of Systems Science, **9**(6) (1978), 613-626.

[15] E. Esmi, L. C. Barros, V. F. Wasques, *Some notes on the addition of interactive fuzzy numbers*, International Fuzzy Systems Association World Congress, Springer, Cham, Jun 2019.

[16] R. Fullér, T. Keresztfalvi, *t-norm-based addition of fuzzy intervals*, Fuzzy Sets and Systems, **51**(2) (1992), 155-159.

[17] J. R. Goetschel, W. Voxman, *Elementary fuzzy calculus*, Fuzzy Sets and Systems, **18**(1) (1986), 31-43.

[18] D. H. Hong, *On shape-preserving additions of fuzzy intervals*, Journal of Mathematical Analysis and Applications, **267**(1) (2002), 369-376.

[19] D. H. Hong, S. Y. Hwang, *The convergence of T-product of fuzzy numbers*, Fuzzy Sets and Systems, **85**(3) (1997), 373-378.

[20] M. Ma, M. Friedman, A. Kandel, *A new fuzzy arithmetic*, Fuzzy Sets and Systems, **108**(1) (1999), 83-90.

[21] M. Mizumoto, K. Tanaka, *The four operations of arithmetic on fuzzy numbers*, Computer Control Systems, **7**(5) (1976), 73-81.

[22] H. T. Nguyen, *A note on the extension principle for fuzzy sets*, Journal of Mathematical Analysis and Applications, **64**(2) (1978), 369-380.

[23] L. Stefanini, *A generalization of Hukuhara difference and division for interval and fuzzy arithmetic*, Fuzzy Sets and Systems, **161**(11) (2010), 1564-1584.

[24] V. F. Wasques, E. Esmi, L. C. Barros, P. Sussner, *The generalized fuzzy derivative is interactive*, Information Sciences, **519** (2020), 93-109.

[25] L. A. Zadeh. *Fuzzy sets*, Information and Control, **8**(3) (1965), 338-353.

[26] M. Zeinali, *The existence result of a fuzzy implicit integro-differential equation in semilinear Banach space*, Computational Methods for Differential Equations, **5**(3) (2017), 232-245.

[27] M. Zeinali, G. Eslami, *Uncertainty analysis of temperature distribution in a thermal fin using the concept of fuzzy derivative*, Journal of Mechanical Engineering, **51**(4) (2022), 527-536.

[28] M. Zeinali, S. Shahmorad, *An equivalence lemma for a class of fuzzy implicit integro-differential equations*, Journal of Computational and Applied Mathematics, **327** (2018), 388-399.

[29] M. Zeinali, S. Shahmorad, K. Mirnia, *Fuzzy integro-differential equations: Discrete solution and error estimation*, Iranian Journal of Fuzzy Systems, **10**(1) (2013), 107-122.

[30] D. Zhang, W. Feng, Y. Zhao, J. Qiu, *Global existence of solutions for fuzzy second-order differential equations under generalized H-differentiability*, Computers and Mathematics with Applications, **60**(6) (2010), 1548-1556.