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Original paper

Connections between commutative rings and some algebras of logic

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Abstract

In this paper using the connections between some subvarieties of residuated lattices, we investigated some properties of the lattice of ideals in commutative and unitary rings. We give new characterizations for commutative rings A in which $Id(A)$ is an MV-algebra, a Heyting algebra or a Boolean algebra and we establish connections between these types of rings. We are very interested in the finite case and we present summarizing statistics. We show that the lattice of ideals in a finite commutative ring of the form $A = \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times ... \times \mathbb{Z}_{k_r}$, where $k_i = p_i^{\alpha_i}$ and p_i a prime number, for all $i \in \{1, 2, ..., r\}$, is a Boolean algebra or an MV-algebra (which is not Boolean). Using this result we generate the binary block codes associated to the lattice of ideals in finite commutative rings and we present a new way to generate all (up to an isomorphism) finite MV-algebras using rings.

Keywords: Commutative ring, ideal, BCK-algebra, residuated lattice, MV-algebra, Boolean algebra, Heyting algebra, Chang property, block codes.

1 Introduction

Residuated lattices were introduced by Dilworth and Ward, through the papers [11, 25]. The study of residuated lattices is originated in 1930 in the context of theory of rings, with the study of ring ideals. It is known that the lattice of ideals of a commutative and unitary ring is a residuated lattice. Based on this result, many researchers ([3, 5, 8, 22, 23]) have been interested in this construction.

In this paper, using the connections between some subvarieties of residuated lattices, we investigated properties of the lattice of ideals in commutative and unitary rings. We are very interested in the finite case. An argument for the importance of this case comes from computational consideration.

In general, a solution that is computational tractable consists in considering algebras with a reasonable small number of elements. This makes us study in this paper, finite residuated lattices and finite rings.

In whole this paper, by a finite commutative unitary ring, we will understand a finite ring of the form $A = \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times ... \times \mathbb{Z}_{k_r}$, where $k_i = p_i^{\alpha_i}$ and p_i a prime number, for all $i \in \{1, 2, ..., r\}$.

Section 2 contains basic properties and definitions that we use in the paper. In Section 3 we give new characterization for commutative rings for which the lattice of ideals is an MV-algebra (Theorem 3.5, Corollaries 3.7, 3.9 and 3.15), a Heyting algebra (Theorem 3.20) or a Boolean algebra (Corollaries 3.27, 3.28, 3.34 and 3.35). Also, we establish new connections between these rings (Corollaries 3.27 and 3.35).

We introduce the notion of Chang ring, as a commutative unitary ring A which satisfies Chang property: $I + J = (J : (J : I))$, for every $I, J \in Id(A)$ and we show that any finite commutative ring has Chang property, that is, its lattice of ideals is a Boolean algebra or an MV-algebra (which is not Boolean), see Corollary 3.33. Also, we prove that if A is a commutative ring, then $Id(A)$ is a Boolean algebra if and only if A is a Von Neumann regular ring satisfying Chang property, or equivalent, if and only if, A is a Von Neumann regular ring in which $Ann(Ann(I)) = I$, for every

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$I \in Id(A)$, see Corollaries 3.28 and 3.35. Moreover, if A is finite, then $Id(A)$ is a Boolean algebra if and only if A is a Von Neumann regular ring, see Corollary 3.28.

One of recent applications of residuated lattices is given by Coding Theory. In Section 4, using the results obtained in Section 3, we describe the binary block codes associated to the lattice of ideals in a finite commutative ring with n ideals. In particular, we construct these binary block codes for $n = 4, 6, 8$.

In Section 5, we present a new way to generate all (up to an isomorphism) finite MV-algebras using commutative rings and we present summarizing statistics.

2 Preliminaries

Let A be a commutative unitary ring.

The set $Id\,(A)$ denotes the set of all ideals of the ring A. We denote by $< x >$ the ideal of A generated by $x \in A$. Let $I, J \in Id\,(A)$. The following sets are also ideals in the ring A:

$$I + J =< I \cup J >= \{i + j, i \in I, j \in J\}, \text{ the sum of two ideals = the ideal generated by } I \cup J;$$

$$I \otimes J = \{\sum_{i=1}^{n} f_i g_i, f_i \in I, g_i \in J\}, \text{ the product of two ideals;}$$

$$(I : J) = \{x \in A, x \cdot J \subseteq I\}, \text{ the quotient of two ideals;}$$

$$Ann\,(I) = (\mathbf{0} : I)\,, \text{ the annihilator of the ideal } I,$$

where $\mathbf{0} =< 0 >$.

Remark 2.1. [7] *Let A be a commutative unitary ring and $I, J, K \in Id\,(A)$. Then the following hold:*
1) $I \otimes J \subseteq I + J$;
2) $Ann\,(I) = \{x \in A, x \cdot I = 0\} = \{x \in A, x \cdot i = 0,$ *for all* $i \in I\}$;
3) $\bigcup_{x \in A, x \neq 0} Ann(< x >) =$ *the set of all zero divisors of the ring* A;
4) $Ann\,(\mathbf{0}) = A$ *and* $Ann\,(A) = \mathbf{0}$;
5) $Ann\,(I + J) = Ann\,(I) \cap Ann\,(J)$;
6) $(A : I) = A$, $(I : A) = I$ *and* $(I : I) = A$.
7) $I \subseteq (I : J)$;
8) $(I : J) \otimes J \subseteq I$;
9) $I \subseteq J \Leftrightarrow (J : I) = A$;
10) $((I : J) : K) = (I : (J \otimes K)) = ((I : K) : J)$;
11) $(K : (I + J)) = (K : I) \cap (K : J)$.

Definition 2.2. [7] *Let A be a commutative unitary ring and $I, J \in Id\,(A)$. The ideals I and J are called coprime if $I + J = A$, that means there are $i \in I$, $j \in J$ such that $i + j = 1$.*

Remark 2.3. 1) *For a commutative ring A, if $I, J \in Id\,(A)$ are coprime ideals, then $I \otimes J = I \cap J$, see* [7], *Proposition 2.19.*

2) *If an ideal I is coprime with $Ann\,(I)$, therefore $I \cap Ann\,(I) = \{0\}$. Indeed, since $I \cap Ann\,(I) = I \otimes Ann\,(I)$, if $x \in I \cap Ann\,(I)$, we have $x = \sum_{i=1}^{n} f_i g_i$, $f_i \in I, g_i \in Ann\,(I)$, therefore $x = 0$.*

Definition 2.4. [2] *The partially ordered set $(\mathcal{L}, \leq)$ is a lattice if for each two elements $x, y \in \mathcal{L}$ their supremum and infimum elements exist, denoted by $sup\{x, y\} = x \vee y$ and $inf\{x, y\} = x \wedge y$.*

The lattice $(\mathcal{L}, \leq)$ is a distributive lattice if for each elements $x, y, z \in \mathcal{L}$ we have the following relation:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

A lattice $(\mathcal{L}, \leq)$ is a bounded lattice if there are the elements 0 and 1, the least element in $\mathcal{L}$, respectively, the greatest element in $\mathcal{L}$. With the above notations, for a lattice $(\mathcal{L}, \leq)$ an element $x \in \mathcal{L}$ has a complement if there is an element $y \in \mathcal{L}$ satisfying the following relations:

$$x \vee y = 1 \text{ and } x \wedge y = 0.$$

In this situation, the element x is called complemented. A complement of an element is not unique, but, if $(\mathcal{L}, \leq)$ is distributive, then each element has at most a complement.

The lattice $(\mathcal{L}, \leq)$ is a complemented lattice if it is a bounded lattice and each element $x \in \mathcal{L}$ has a complement.

Definition 2.5. [2] *An algebra* $(\mathcal{B}, \vee, \wedge, ', 0, 1)$ *is called a Boolean algebra if* $(\mathcal{B}, \vee, \wedge, 0, 1)$ *is a bounded distributive lattice and a complemented lattice in which* $b \vee b' = 1$ *and* $b \wedge b' = 0$, *for all elements* $b \in \mathcal{B}$.

Definition 2.6. [18] *A BCK-algebra is a structure* $(X, \leq, \rightarrow, 1)$ *where* $(X, \leq)$ *is a poset with a greatest element* 1 *and* $\rightarrow$ *is a binary operation on* X *such that:*
$BCK1)$ $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$;
$BCK2)$ $x \leq (x \rightarrow y) \rightarrow y$;
$BCK3)$ $x \leq y$ *iff* $x \rightarrow y = 1$, *for every* $x, y, z \in X$.

Obviously, by a bounded BCK-algebra we mean a BCK-algebra with a least element 0.
For other details regarding BCK-algebras, the readers are referred to [1, 20].
Fundamental examples of BCK-algebras come from algebras of logic:

Definition 2.7. [11, 25] *A* residuated lattice *is an algebra* $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ *equipped with an order* $\leq$ *satisfying the following axioms:*
$LR1)$ $(L, \wedge, \vee, 0, 1)$ *is a bounded lattice;*
$LR2)$ $(L, \odot, 1)$ *is a commutative ordered monoid;*
$LR3)$ $\odot$ *and* $\rightarrow$ *form an adjoint pair, i.e.,* $z \leq x \rightarrow y$ *iff* $x \odot z \leq y$, *for all* $x, y, z \in L$.

It is obviously that if $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a residuated lattice, then $(L, \rightarrow, 0, 1)$ is a bounded BCK-algebra.

Remark 2.8. [24] *Let* $(\mathcal{B}, \wedge, \vee, ', 0, 1)$ *be a* Boolean algebra. *If we define for every* $x, y \in \mathcal{B}, x \odot y = x \wedge y$ *and* $x \rightarrow y = x' \vee y$, *then* $(\mathcal{B}, \wedge, \vee, \odot, \rightarrow, 0, 1)$ *becomes a residuated lattice.*

Proposition 2.9. [2, 10] *Let* $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ *be a residuated lattice. The following assertions are equivalent:*
(i) $x^2 = x$ *and* $x^{**} = x$, *for every* $x \in L$;
(ii) $x^2 = x$ *for every* $x \in L$ *and* [*for* $y \in L$, $y^* = 0 \Rightarrow y = 1$];
(iii) $x \vee x^* = 1$, *for every* $x \in L$.

In a residuated lattice $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ we consider the identities:

$$(div) \qquad x \odot (x \rightarrow y) = x \wedge y \qquad (divisibility);$$
$$(DN) \qquad x^{**} = x \qquad (double\ negation\ condition).$$

Definition 2.10. [18, 24] *The residuated lattice* L *is called:*
(i) divisible *if* L *verifies* (div);
(ii) involutive *or* Girard monoid *if* L *verifies* (DN).

Definition 2.11. [9, 21] *An abelian monoid* $(M, \oplus, 0)$ *is called an MV-algebra if we have an unary operation* * *on* M *such that:*
$MV1)$ $(x^*)^* = x$;
$MV2)$ $x \oplus 0^* = 0^*$;
$MV3)$ $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$, *for all* $x, y \in M$.

Usually, we denote an MV-algebra by $(M, \oplus, ^*, 0)$. In an MV-algebra M, the constant element 0^* is denoted with 1, that means $1 = 0^*$, and the following auxiliary operations are also defined:

$$x \odot y = (x^* \oplus y^*)^* \text{ and } x \rightarrow y = x^* \oplus y,$$

for every $x, y \in M$, see [21]. Also, in [10], for an MV-algebra $(M, \oplus, ^*, 0)$ and $x, y \in M$ is defined the order relation:

$$x \leq y \text{ if and only if } x^* \oplus y = 1.$$

We recall that, the natural order $\leq$ determines on M a bounded distributive lattice structure in which for $x, y \in M$, the join $x \vee y$ and the meet $x \wedge y$ are given by:

$$\begin{aligned} x \vee y &= (x \odot y^*) \oplus y = (y \odot x^*) \oplus x \qquad \text{and} \\ x \wedge y &= (x^* \vee y^*)^* = x \odot (x^* \oplus y) = y \odot (y^* \oplus x). \end{aligned}$$

MV-algebras are particular cases of residuated lattices. If $(M, \oplus, ^*, 0)$ is an MV-algebra, then using the auxiliary operations defined above, $(M, \wedge, \vee, \odot, \rightarrow, 0, 1)$ becomes a residuated lattice.

Moreover, if $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a residuated lattice and for $x, y \in L$ we define $x \oplus y = x^* \rightarrow y$ (equivalent with $x \oplus y = (x^* \odot y^*)^*$), then $(L, \oplus, ^*, 0)$ is an MV algebra iff

$$(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x,$$

for every $x, y \in L$, see [24]. In fact, MV-algebras are commutative bounded BCK-algebras, see [18, 20].

Definition 2.12. [10] *An algebra* $(W, \rightarrow, ^*, 1)$ *of type* $(2, 1, 0)$ *is called a Wajsberg algebra if the axioms are verified:*
$W1)$ $1 \rightarrow x = x$;
$W2)$ $(x \rightarrow y) \rightarrow [(y \rightarrow z) \rightarrow (x \rightarrow z)] = 1$;
$W3)$ $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$;
$W4)$ $(x^* \rightarrow y^*) \rightarrow (y \rightarrow x) = 1$, *for every* $x, y, z \in W$.

We remark that if $(W, \rightarrow, ^*, 1)$ is a Wajsberg algebra and if on W is defined the following binary relation

$$x \leq y \text{ if and only if } x \rightarrow y = 1,$$

then, this relation is an order relation, called *the natural order relation on* W (see [14]).

Remark 2.13. [10], Lemma 4.2.2 and Theorem 4.2.5
i) If $(W, \rightarrow, ^*, 1)$ *is a Wajsberg algebra, defining*

$$x \odot y = (x \rightarrow y^*)^* \quad and \quad x \oplus y = x^* \rightarrow y,$$

for all $x, y \in W$, *we obtain that* $(W, \oplus, \odot, ^*, 0, 1)$ *is an MV-algebra.*
ii) If $(M, \oplus, \odot, ^*, 0, 1)$ *is an MV-algebra, defining on* M *the operation*

$$x \rightarrow y = x^* \oplus y,$$

it results that $(X, \rightarrow, ^*, 1)$ *is a Wajsberg algebra.*

Remark 2.14. [18] *A residuated lattice* $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ *with Chang condition:*

$$(C): \; x \vee y = (x \rightarrow y) \rightarrow y,$$

for every $x, y \in L$, *is an equivalent definition of Wajsberg algebra.*

3 Some remarks regarding the lattice of ideals in a commutative ring

It is known that if A is a commutative unitary ring, then $(Id(A), \cap, +, \otimes \rightarrow, 0 = \{0\}, 1 = A)$ is a residuated lattice in which the order relation is $\subseteq$ and $I \rightarrow J = (J : I)$, for every $I, J \in Id(A)$, see [23].

Thus, since $\otimes$ and $\rightarrow$ form and adjoint pair, LR3) becomes

$$I \otimes J \subseteq K \text{ iff } I \subseteq (K : J), \text{ for every } I, J, K \in Id(A).$$

Proposition 3.1. *Let* A *be a commutative unitary ring. Then* $(Id(A), \subseteq, \rightarrow, 1 = A)$ *is a bounded BCK-algebra in which* $I \rightarrow J = (J : I)$, *for every* $I, J \in Id(A)$.

Proof. To prove BCK1), using Remark 2.1(8), since $(I : J) \otimes J \subseteq I$ we deduce that $(I : J) \otimes J \otimes (K : I) \subseteq I \otimes (K : I) \subseteq K$. Since $(Id(A), \cap, +, \otimes \rightarrow, 0 = \{0\}, 1 = A)$ is a residuated lattice we deduce that $(I : J) \otimes (K : I) \subseteq (K : J)$ and finally, $(I : J) \subseteq ((K : J) : (K : I))$. Now, using LR3) and Remark 2.1 (8) we have $(J : I) \otimes I \subseteq J$ iff $I \subseteq (J : (J : I))$, so BCK2) holds. The condition BCK3) follows from Remark 2.1 (9). $\square$

Proposition 3.2. *Let* A *be a commutative unitary ring and* $I, J \in Id(A)$. *Then:*
(i) $(J : (J : (J : I))) = (J : I)$;
(ii) $(I : J) \subseteq (Ann(J) : Ann(I))$;
(iii) $I \otimes Ann(I) = \mathbf{0}$;
(iv) $I \otimes J = \mathbf{0}$ *iff* $I \subseteq Ann(J)$;
(v) $Ann(I \otimes J) = (Ann(J) : I) = (Ann(I) : J)$;
(vi) I *and* J *coprime ideals implies* I^n *and* J^n *coprime, for every* $n \geq 1$.

Proof. Results from $i) - v)$ can be obtained by straightforward calculation.

(vi). Suppose that I and J coprime ideals. Then $I + J = A$. We recall that if $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a residuated lattice, then $x \vee y = 1$ implies $x^n \vee y^n = 1$, for every $n \geq 1$, see [6]. Since, $(Id(A), \cap, +, \otimes \rightarrow, 0 = \{0\}, 1 = A)$ is a residuated lattice, we deduce that I^n and J^n coprime ideals, for every $n \geq 1$. □

It is known that, a commutative ring in which the lattice of ideals (augmented with the ideal product) is isomorphic to an MV-algebra, is a direct sum of local Artinian chain rings with unit, see [3].

In the following, using the connections between some subvarieties of residuated lattices, we give new characterizations for commutative and unitary rings for which the lattice of ideals is an MV-algebra.

Using Remark 2.14 and Proposition 3.1 we can give the following definition:

Definition 3.3. *A commutative unitary ring A has Chang property if:*

$$I + J = (J : (J : I)),$$

for every $I, J \in Id(A)$.

Example 3.4. *If we consider the commutative ring $(\mathbb{Z}_4, +, \cdot)$, then $Id(\mathbb{Z}_4) = \{\{\widehat{0}\}, \{\widehat{0}, \widehat{2}\}, \mathbb{Z}_4\}$. We remark that*

$+$	$\{\widehat{0}\}$	$\{\widehat{0}, \widehat{2}\}$	$\mathbb{Z}_4$
$\{\widehat{0}\}$	$\{\widehat{0}\}$	$\{\widehat{0}, \widehat{2}\}$	$\mathbb{Z}_4$
$\{\widehat{0}, \widehat{2}\}$	$\{\widehat{0}, \widehat{2}\}$	$\mathbb{Z}_4$	$\mathbb{Z}_4$
$\mathbb{Z}_4$	$\mathbb{Z}_4$	$\mathbb{Z}_4$	$\mathbb{Z}_4$

and $I + J = (J : (J : I))$, for every $I, J \in Id(\mathbb{Z}_4)$, so, the ring $(\mathbb{Z}_4, +, \cdot)$ has Chang property.

Since, MV-algebras are commutative bounded BCK algebras, i.e., bounded BCK-algebras that satisfy the additional condition $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$, see [10] and Wajsberg algebras are termwise equivalent with MV-algebras (see Remark 2.13), we conclude that:

Theorem 3.5. *A commutative ring has Chang property if and only if its lattice of ideals is an MV-algebra.*

Proposition 3.6. *Let $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ be a residuated lattice. The following assertions are equivalent:*

(i) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$, *for every* $x, y \in L$;

(ii) $x \vee y = (x \rightarrow y) \rightarrow y$, *for every* $x, y \in L$;

(iii) $x^{**} = x$, $x \wedge y = x \odot (x \rightarrow y)$ *and* $(x \rightarrow y) \vee (y \rightarrow x) = 1$, *for every* $x, y \in L$;

(iv) $((x \rightarrow y) \rightarrow y) \rightarrow x = y \rightarrow x$, *for every* $x, y \in L$;

(v) *For* $x, y \in L$, $x \leq y$ *implies* $(y \rightarrow x) \rightarrow x \leq y$.

Proof. $(i) \Leftrightarrow (ii)$. Follows by [18] and [24].

$(i) \Leftrightarrow (iii)$. We recall that a BL-algebra is a residuated lattice that verifies divisibility and prelinearity conditions: $x \wedge y = x \odot (x \rightarrow y)$ and $(x \rightarrow y) \vee (y \rightarrow x) = 1$, for every $x, y \in L$. MV-algebras are BL-algebras which satisfies double negation condition: $x^{**} = x$, for every $x \in L$, see [24].

$(i) \Rightarrow (iv)$. Let $x, y \in L$. Using (i), L is an MV-algebra, so, $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x = x \vee y$. Then $((x \rightarrow y) \rightarrow y) \rightarrow x = (x \vee y) \rightarrow x = (x \rightarrow x) \wedge (y \rightarrow x) = y \rightarrow x$.

$(vi) \Rightarrow (i)$. By hypothesis, $y \rightarrow x \leq ((x \rightarrow y) \rightarrow y) \rightarrow x$, so, $[(x \rightarrow y) \rightarrow y] \odot (y \rightarrow x) \leq x$ and $(x \rightarrow y) \rightarrow y \leq (y \rightarrow x) \rightarrow x$. Similarly, $(y \rightarrow x) \rightarrow x \leq (x \rightarrow y) \rightarrow y$, thus, $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$, *for every* $x, y \in L$.

$(i) \Rightarrow (v)$. Obviously.

$(v) \Rightarrow (i)$. Let $x, y \in L$. From $x \leq (x \rightarrow y) \rightarrow y$, we deduce that $((x \rightarrow y) \rightarrow y) \rightarrow x) \rightarrow x \leq (x \rightarrow y) \rightarrow y$, that is, $(((x \rightarrow y) \rightarrow y) \rightarrow x) \rightarrow x = (x \rightarrow y) \rightarrow y$. From $y \leq (x \rightarrow y) \rightarrow y$ we deduce successively $((x \rightarrow y) \rightarrow y) \rightarrow x \leq y \rightarrow x$, $(y \rightarrow x) \rightarrow x \leq (((x \rightarrow y) \rightarrow y) \rightarrow x) \rightarrow x = (x \rightarrow y) \rightarrow y$. So $(y \rightarrow x) \rightarrow x \leq (x \rightarrow y) \rightarrow y$ and similarly $(x \rightarrow y) \rightarrow y \leq (y \rightarrow x) \rightarrow x$. We conclude that $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$. □

Using Theorem 3.5 and Proposition 3.6 we deduce that:

Corollary 3.7. *Let A be a commutative unitary ring. The following assertions are equivalent:*

(i) *A has Chang property;*

(ii) $(Id(A), \cap, +, Ann, 0 = \{0\}, 1 = A)$ *is an MV-algebra;*

(iii) $(I : (I : J)) = (J : (J : I))$, *for every* $I, J \in Id(A)$;

(iv) $Ann(Ann(I)) = I$, $I \cap J = (J : I) \otimes I = (I : J) \otimes J$ *and* $(I : J) + (J : I) = A$, *for every* $I, J \in Id(A)$;

(v) $(I : (J : (J : I))) = (I : J)$, *for every* $I, J \in Id(A)$;

(vi) *For* $I, J \in Id(A)$, $I \subseteq J$ *implies* $(I : (I : J)) \subseteq J$.

Using some connections between divisible and involutive properties in residuated lattices we give a new characterization for commutative rings with Chang property:

Proposition 3.8. *Let $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ be a residuated lattice. Then L is divisible satisfying (DN) condition if and only if L is an MV-algebra.*

Proof. Obviously, an MV-algebra is an involutive divisible residuated lattice.

Conversely, if L is involutive and divisible, $x = x^{**}$ and $x \vee y = (x \vee y)^{**} = (x^* \wedge y^*)^* = [\ x^* \odot (x^* \rightarrow y^*)]^* = (x^* \rightarrow y^*) \rightarrow x^{**} = (x^* \rightarrow y^*) \rightarrow x = (y \rightarrow x) \rightarrow x$, for every $x, y \in L$. Thus, L is an MV-algebra. □

Using Proposition 3.8 we deduce that:

Corollary 3.9. *Let A be a commutative unitary ring. The following assertions are equivalent:*
(i) A has Chang property;
(ii) $I \cap J = (J : I) \otimes I$ and $Ann(Ann(I)) = I$, for every $I, J \in Id(A)$.

Proposition 3.10. *Let A be a commutative ring and $I, J \in Id\,(A)$. The following relations are true:*
1) $I + J =< I \cup J > \subseteq (I : (I : J))$, for every $I, J \in Id\,(A)$;
2) If A is a principal ideal domain, then $(I : (I : J)) = (J : (J : I)) = I + J$, for I and J nonzero ideals;
3) If $\mathcal{A}$ is a ring factor of a principal ideal domain D, then $(\mathcal{I} : (\mathcal{I} : \mathcal{J})) = (\mathcal{J} : (\mathcal{J} : \mathcal{I})) = \mathcal{I} + \mathcal{J}$, for $\mathcal{I}$ and $\mathcal{J}$ arbitrary ideals in $\mathcal{A}$.

Proof. 1) Let $x \in I + J$, therefore $x = ai + bj, a, b \in A$ and $i \in I$, $j \in J$. We have that $(I : J) = \{y \in A : y \cdot J \subseteq I\}$. For $y \in (I : J)$ we have $xy = (ai + bj)\,y = aiy + bjy$, with $aiy \in I$ and $bjy \in I$. It results that, $xy \in I$.

2) If A is a principal ideal domain, then let $a, b \in A$ be two nonzero elements and $I =< a >$, $J =< b >$ be the principal ideals generated by a and b. Let $d = gcd\{a, b\}$. We have $d = a\gamma + b\beta$, $a, b \in A$, $a = a_1 d$ and $b = b_1 d$, with $1 = gcd\{a_1, b_1\}$. We will prove that $(I : (I : J)) = (J : (J : I)) =< d >= I + J$. First of all, we remark that $(I : J) = < a_1 >$. Indeed, if $y \in (I : J)$, we have $yb_1 d \in I$, that means $yb_1 d = a_1 dq, q \in A$ and $yb_1 = a_1 q$. Since $1 = gcd\{a_1, b_1\}$, it results $a_1 \mid y$ and $y \in< a_1 >$. If $y \in< a_1 >$, we have $y = a_1 z$ and $yb = a_1 z b_1 d \in I$. By repeating the same proof, since $a_1 = gcd\{a, a_1\}$, we obtain that $(I : (I : J)) =< d >= I + J$. In the same way, the equality $(J : (J : I)) =< d >= I + J$ can be obtained. If $I = \{0\}$, since A is an integral domain, we have that $(\mathbf{0} : (\mathbf{0} : J)) = Ann(Ann(J)) = A$, for every $J \in Id\,(A) \backslash \{0\}$ and $(J : (J : \mathbf{0})) = J$. We deduce that if A is a principal ideal domain, then $Id(A)$ is only a noncommutative BCK-algebra.

3) We apply 2) from above. Let D be a principal ideal domain and $W = \alpha D$, $\alpha \in D$, be an ideal in D. First of all, we remark that if D is an integral domain, its factor ring is not allways an integral domain. For example, $\mathbb{Z}$ and $\mathbb{Z}_n$. Let $\mathcal{A} = D/W$ be the factor ring, which is a principal ring. Let $\mathcal{I}, \mathcal{J}$ be two nonzero ideals of $\mathcal{A}$, $\mathcal{I} = I/W$ and $\mathcal{J} = J/W$, where I and J are nonzero ideals in D, with $W \subseteq I$ and $W \subseteq J$. It results that $I = \beta W$ and $J = \gamma W$, $\beta \mid \alpha$ and $\gamma \mid \alpha$, therefore $\mathcal{I} =< \widehat{\beta} >$ and $\mathcal{J} =< \widehat{\gamma} >$, β, $\gamma \in D$. We will prove that $(\mathcal{I} : (\mathcal{I} : \mathcal{J})) = (\mathcal{J} : (\mathcal{J} : \mathcal{I})) =< \widehat{\delta} >= \mathcal{I} + \mathcal{J}$, where $\delta = gcd\{\beta, \gamma\}$. From the above, we have that $\mathcal{I} + \mathcal{J} \subseteq (\mathcal{I} : (\mathcal{I} : \mathcal{J}))$ and $\mathcal{I} + \mathcal{J} \subseteq (\mathcal{J} : (\mathcal{J} : \mathcal{I}))$. Let $\delta = m\beta + n\gamma, \beta = \delta\beta_1$ and $\gamma = \delta\gamma_1$ with $1 = gcd\{\beta_1, \gamma_1\}$. From 2), we have $(\mathcal{I} : \mathcal{J}) =< \widehat{\beta_1} >$. If $\widehat{y} \in< \widehat{\beta_1} >$, we have $\widehat{y} = \widehat{\beta_1}\widehat{z}$ and $\widehat{y}\widehat{\beta} = \widehat{\beta_1}\widehat{z}\widehat{\gamma_1}\widehat{\delta} \in \mathcal{I}$. By repeating the same proof, since $\beta_1 = gcd\{\beta, \beta_1\}$, we obtain that $(\mathcal{I} : (\mathcal{I} : \mathcal{J})) =< \widehat{\delta} >= \mathcal{I} + \mathcal{J}$. In the same way, the equality $(\mathcal{J} : (\mathcal{J} : \mathcal{I})) =< \widehat{\delta} >= \mathcal{I} + \mathcal{J}$ can be obtained. If $I = \{0\}$ and J is a nonzero ideal in D, the equality is also true. We must prove that $(\mathbf{0} : (\mathbf{0} : \mathcal{J})) = (\mathcal{J} : (\mathcal{J} : \mathbf{0}))$, with $\mathbf{0}$ zero ideal in $\mathcal{A}$, which is equivalent to $Ann\,(Ann\,(\mathcal{J})) = \mathcal{J}$. For $\mathcal{J} =< \widehat{\gamma} >$, we have $Ann\,(\mathcal{J}) =< \widehat{\sigma} >$, with $\alpha = \gamma\sigma$, therefore $Ann\,(Ann\,(\mathcal{J})) =< \widehat{\gamma} >= \mathcal{J}$. □

From Corollary 3.7 and Proposition 3.10 (3) we deduce that:

Theorem 3.11. *A ring factor of a principal ideal domain has Chang property.*

Example 3.12. 1) *The ring of integers $(\mathbb{Z}, +, \cdot)$ does not have Chang property because the relation $(I : (I : J)) = (J : (J : I))$, is not true for $I = \mathbf{0}$. Indeed, since $\mathbb{Z}$ is principal ideal domain, we have $(\mathbf{0} : (\mathbf{0} : J)) = Ann\,(Ann\,(J)) = \mathbb{Z}$ and $(J : (J : \mathbf{0})) = J$, for every $J \in Id\,(\mathbb{Z})$.*

2) *Let K be a field, $K\,[X]$ the polynomial ring and $f \in K\,[X]$. Therefore, from Proposition 3.10, the lattice of ideals of the quotient ring $A = K\,[X]\,/\,(f)$ is an MV-algebra and has Chang property.*

Let $K = \mathbb{R}$ and $f\,(X) = (X + 1)\,(X + 2)\,(X + 3)$. Therefore, $A = \mathbb{R}[X]/\,(f) \simeq \mathbb{R}[X]/\,(X + 1) \times \mathbb{R}[X]/\,(X + 2) \times \mathbb{R}[X]/\,(X + 3)$, from Chinese Remainder Theorem. From here, we have that $A \simeq \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. In this case, $Id\,(A)$ is a Boolean lattice. Indeed $Id\,(A) = \{0,\ 0 \times 0 \times \mathbb{R},\ 0 \times \mathbb{R} \times 0,\ \mathbb{R} \times 0 \times 0,\ 0 \times \mathbb{R} \times \mathbb{R},\ \mathbb{R} \times 0 \times \mathbb{R},\ \mathbb{R} \times \mathbb{R} \times 0,\ \mathbb{R} \times \mathbb{R} \times \mathbb{R}\}$ and is a Boolean algebra.

Let $K=\mathbb{R}$ and $f(X)=(X+1)^2(X+2)$. Therefore, $A=\mathbb{R}[X]/(f)\simeq\mathbb{R}[X]/(X+1)^2\times\mathbb{R}[X]/(X+2)\simeq\mathbb{R}[X]/(X+1)^2\times\mathbb{R}$, from Chinese Remainder Theorem. In this case $Id(A)$ is an MV-algebra. Indeed, $I=\mathbb{R}\times 0$ is an ideal in A, $Ann(I)=0\times\mathbb{R}$. In this case I and $Ann(I)$ are not coprime ideals and $Id(A)$ is not a Boolean algebra is an MV-algebra.

The result can be generalized: *If $\mathbb{K}$ is a field and $f=f_1^{k_1}f_2^{k_2}...f_r^{k_r}$, with f_i irreducible polynomials, we have that $A=\mathbb{K}[X]/(f)\simeq\prod\mathbb{K}[X]/\left(f_i^{k_i}\right)$, therefore the algebra $Id(A)$ is a Boolean algebra or an MV-algebra (which is not Boolean).*

Remark 3.13. *Using Example 3.12 (2) we remark that the condition from Theorem 3.11 is a necessary condition for a ring to have Chang property.*

We recall the fundamental theorem of finite abelian groups:

Theorem 3.14. ([16], Theorem 2.14.1) *Every finite abelian group is a direct product of cyclic groups of prime power order. Moreover, the number of terms in the product and the orders of the cyclic groups are uniquely determined by the group.*

It is known that the lattice of ideals of a commutative unitary ring is a modular lattice (with respect to set inclusion), see [5].

Corollary 3.15. *Any finite commutative unitary ring has Chang property.*

Proof. Let A be a finite commutative unitary ring with $|A|=n=p_1^{\alpha_1}\cdot...\cdot p_r^{\alpha_r}$. We prove that the set of all ideals of the rings A forms an MV-algebra $(Id(A),\vee,\wedge,\odot,\rightarrow,0,1)$ in which the order relation is $\subseteq$, $I\odot J=I\otimes J$, $I^*=Ann(I)$, $I\rightarrow J=(J:I)$, $I\vee J=I+J$, $I\wedge J=I\cap J$, $0=\{0\}$ and $1=A$.

From Theorem 3.14, we know that A has the form $A=\mathbb{Z}_{k_1}\times\mathbb{Z}_{k_2}\times...\times\mathbb{Z}_{k_r}$, where $k_i=p_i^{\alpha_i}$ and p_i a prime number, for all $i\in\{1,2,...,r\}$. It results that $Id(A)=Id(\mathbb{Z}_{k_1})\times Id(\mathbb{Z}_{k_2})\times...\times Id(\mathbb{Z}_{k_r})$. Now, we apply Theorem 3.11.

We recall that in an MV-algebra $(A,\oplus,\odot,^*,0,1)$ we have $x\oplus y=(x^*\odot y^*)^*=x^*\rightarrow y=y^*\rightarrow x$ and $x\rightarrow y=x^*\oplus y=(x\odot y^*)^*$, for every $x,y\in A$, see [10] and [21]. Using these equalities, in MV-algebra $Id(A)$ we obtain:

$$I\oplus J=Ann(Ann(I)\otimes Ann(J))=(J:Ann(I))=(I:Ann(J)),$$

and

$$I\rightarrow J=(J:I)=Ann(I)\oplus J=Ann(I\otimes Ann(J)),$$

for every $I,J\in Id(A)$, since, in this case, $Ann(Ann(I))=I$. □

Remark 3.16. *If A is a finite commutative ring with $|A|=n=p_1^{\alpha_1}\cdot...\cdot p_r^{\alpha_r}$, then the set $Id(A)$ has $\mathcal{N}_A=\prod_{i=1}^{r}(\alpha_i+1)$ elements.*

Remark 3.17. *If the commutative ring A is not finite, then $Id(A)$ is not necessarily an MV-algebra. For example, $Id(\mathbb{Z},+,\cdot)$ is only a noncommutative BCK-algebra, see Example 3.12.*

The following results can be obtained by the straightforward calculations.

Proposition 3.18. *Let A be a finite commutative unitary ring. Then:*

(i) $(Id(A),\cap,+,0=\{0\},1=A)$ is a bounded distributive lattice in which

$$I\vee J=I+J=(I:(I:J))=(J:(J:I)),\ \text{and}\ I\wedge J=I\cap J=(J:I)\otimes I=(I:J)\otimes J,$$

for every $I,J\in Id(A)$.

(ii) $Ann(Ann(I))=I$, for every $I\in Id(A)$;

(iii) $I\oplus Ann(I)=A$, for every $I\in Id(A)$;

(iv) $Ann(I\cap J)=Ann(I)+Ann(J)$, for every $I,J\in Id(A)$.

We recall that a residuated lattice $(L,\vee,\wedge,\odot,\rightarrow,0,1)$ in which $x^2=x$, for all $x\in L$, is called a *Heyting algebra (or G(RL) algebra or pseudo Boolean algebra)*, see [15, 18, 24]. In [4] was proved that unitary commutative rings for which the semiring of ideals, under ideal sum and ideal product, are Heyting algebras are exactly Von Neumann regular rings, i.e. commutative rings A in which for every element $x\in A$ there exists an element $a\in A$ such that $x=a\cdot x^2$.

In the following, we give new characterizations for commutative rings in which the lattice of ideals is a Heyting algebra.

Proposition 3.19. *Let $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ be a residuated lattice. The following assertions are equivalent:*

(i) *for $x, y, z \in L$, if $x \odot y \leq z$ and $x \leq y$, then $x \leq z$;*

(ii) *for every $z \in L$, $D_z = \{x \in L : z \leq x\}$ is a deductive system of L;*

(iii) *$x^2 \leq y$ implies $x \leq y$, for every $x, y \in L$;*

(iv) *$x \odot y \leq z$ implies $x \odot (x \rightarrow y) \leq z$, for every $x, y, z \in L$;*

(v) *$x = x^2$, for every $x \in L$.*

Proof. $(i) \Rightarrow (ii)$. Since for every $z \in L, z \leq 1$, we have that $1 \in D_z$. If $x, x \rightarrow y \in D_z$, then $z \leq x, x \rightarrow y$. Using (i), $z \leq y$, hence $y \in D_z$, that is, D_z is a deductive system of L.

$(ii) \Rightarrow (i)$. Let $x, y, z \in L$ such that $x \odot y \leq z$ and $x \leq y$. Then $y, y \rightarrow z \in D_x$. Since D_x is a deductive system of L, we obtain $z \in D_x$ so, $x \leq z$.

$(ii) \Rightarrow (iii)$. Let $x, y \in L$ such that $x^2 \leq y$. Then $x \rightarrow y \in D_x$. Since $x, x \rightarrow y \in D_x$, we deduce that $y \in D_x$, that is, $x \leq y$.

$(iii) \Rightarrow (iv)$. Let $x, y, z \in L$ such that $x \odot y \leq z$. Since $y \rightarrow z \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$, we obtain $1 = (x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) \leq x \rightarrow [(x \rightarrow y) \rightarrow (x \rightarrow z)]$, hence $x \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$. Finally, we obtain that $x \leq x \rightarrow [(x \rightarrow y) \rightarrow z]$. By hypothesis, we deduce that $x \leq (x \rightarrow y) \rightarrow z$, hence $x \odot (x \rightarrow y) \leq z$.

$(iv) \Rightarrow (iii)$. Consider $y, z \in L$ such that $y^2 \leq z$. By hypothesis, $y \odot (y \rightarrow y) \leq z$, that is, $y \leq z$.

$(iii) \Rightarrow (i)$. Let $x, y, z \in L$ such that $x \odot y \leq z$ and $x \leq y$. We have $1 = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) \leq (x \rightarrow y) \rightarrow (x \rightarrow (x \rightarrow z))$, hence $x \rightarrow y \leq x \rightarrow (x \rightarrow z)$. Then $x \leq x \rightarrow z$, so, we obtain that $x \leq z$.

$(iii) \Rightarrow (v)$. We have $x^2 \leq x^2$, hence $x \leq x^2$. We deduce that $x = x^2$, for every $x \in L$.

$(v) \Rightarrow (iii)$. Let $x, y \in L$ such that $x^2 \leq y$. Since $x = x^2$, we deduce that $x \leq y$. □

Theorem 3.20. *Let A be a commutative ring. The following conditions are equivalent:*

(i) *A is a Von Neumann regular ring;*

(ii) *$(J : I) \otimes I = I \otimes J = I \cap J$, for every $I, J \in Id(A)$;*

(iii) *for $I, J, K \in Id(A)$, if $I \otimes J \subseteq K$ and $I \subseteq J$, then $I \subseteq K$;*

(iv) *for every $K \in Id(A)$, $D_K = \{I \in Id(A) : K \subseteq I\}$ is a deductive system of the residuated lattice $(Id(A), \cap, +, \otimes \rightarrow, 0 = \{0\}, 1 = A)$;*

(v) *$I \otimes I \subseteq J$ implies $I \subseteq J$, for every $I, J \in Id(A)$;*

(vi) *$I \otimes J \subseteq K$ implies $I \otimes (J : I) \subseteq K$, for every $I, J, K \in Id(A)$.*

Proof. $(i) \Rightarrow (ii)$. Let $I, J \in Id(A)$. Using LR_3 and the properties of residuated lattices (see [6]) we obtain:

$$(J : I) \otimes I \subseteq ((I \otimes J) : (I \otimes I)) \Leftrightarrow (J : I) \otimes I \subseteq ((I \otimes J) : I) \Leftrightarrow$$
$$(J : I) \subseteq (((I \otimes J) : I) : I) = ((I \otimes J) : (I \otimes I)) = ((I \otimes J) : I).$$

We deduce that $(J : I) \otimes I \subseteq I \otimes J$. Since $J \subseteq (J : I)$, then $I \otimes J \subseteq (J : I) \otimes I$, so $I \otimes J = (J : I) \otimes I$. Clearly, $I \otimes J \subseteq I, J$. To prove that $I \otimes J = I \cap J$, let $K \in Id(A)$ such that $K \subseteq I, J$. Then $K = K^2 \subseteq I \otimes J$, that is, $I \otimes J = I \cap J$.

$(ii) \Rightarrow (i)$. For $I = J$ we obtain $I \otimes I = I \cap I = I$, that is, $I \otimes I = I$, for every $I \in Id(A)$.

$(i) \Longleftrightarrow (iii) \Longleftrightarrow (iv) \Longleftrightarrow (v) \Longleftrightarrow (vi)$. Follows by Proposition 3.19. □

Remark 3.21. *Boolean rings are Von Neumann regular rings. Indeed, since in a Boolean ring A, every ideal is idempotent, using Theorem 3.20, we obtain that $Id(A)$ is a Heyting algebra.*

We recall the following result that characterizes unitary rings for which the lattice of ideals is a Boolean algebra:

Theorem 3.22. [5, 8] *For a ring A, the following assertions are equivalent:*

(i) *The lattice $Id(A)$ is a Boolean algebra;*

(ii) *A is isomorphic to a finite direct sum of division rings;*

(iii) *A is isomorphic to a finite direct sum of simple rings;*

(iv) *Ideals form both a Heyting algebra and an ortholattice.*

It is known that a residuated lattice $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a Boolean algebra if and only if $x \vee x^* = 1$, for every $x \in L$, see [15]. We give new necessary and sufficient conditions for rings whose lattice of ideals is a Boolean algebra:

Theorem 3.23. *Let A be a commutative ring and $I \in Id(A)$. The following assertions are equivalent:*

(i) *I and $Ann(I)$ are coprime;*

(ii) *$I \otimes I = I$, $Ann(Ann(I)) = I$, $(Ann(I) : I) + (I : Ann(I)) = A$ and $Ann(I) \otimes (I : Ann(I)) = 0$.*

Proof. $(i) \Rightarrow (ii)$. If $I + Ann(I) = A$, using Remark 2.3 (1), we deduce that $I \otimes I = I \cap I = I$. Moreover, I is a Boolean element in the residuated lattice $(Id(A), \cap, +, \otimes \rightarrow, 0 = \{0\}, 1 = A)$, so $Ann(Ann(I)) = I$ and $I \cap Ann(I) = \mathbf{0}$. Since $Ann(I) \otimes (I : Ann(I)) \subseteq I \cap Ann(I) = \mathbf{0}$ we deduce that $Ann(I) \otimes (I : Ann(I)) = \mathbf{0}$. Also, $Ann(I) \subseteq (Ann(I) : I)$ and $I \subseteq (I : Ann(I))$. Then $A = Ann(I) + I \subseteq (Ann(I) : I) + (I : Ann(I))$, so $(Ann(I) : I) + (I : Ann(I)) = A$.

$(ii) \Rightarrow (i)$. From $I \otimes I = I = I$, we have $(Ann(I) : I) = Ann(I)$. Since $Ann(Ann(I)) = I$, and $(I : Ann(I)) \otimes Ann(I) = \mathbf{0}$, we deduce that $(I : Ann(I)) = I$. Then $Ann(I) + I = (Ann(I) : I) + (I : Ann(I)) = A$. We conclude that I and $Ann(I)$ are coprime. □

Proposition 3.24. *Let A be a commutative unitary ring, $x \in A$, an idempotent element and $I =< x >$ be the ideal generated by x. Therefore, $Ann(I)$ and I are coprime ideals.*

Proof. If $x \neq 1$, since $x + 1 - x = 1$, we have that $1 - x \notin I$. Also, since $(1 - x)x = 0$, we have that $(1 - x)y = 0$, for all $y \in I$, then $1 - x \in Ann(I)$ and $Ann(I)$ and I are coprime. If $x = 1$, then $I = A$ and $Ann(I) = \{0\}$. □

Theorem 3.25. *Let A be a finite commutative ring and $I \in Id(A)$. Then the following assertions are equivalent:*
(i) I and $Ann(I)$ are coprime;
(ii) $I \otimes I = I$.

Proof. $(i) \Rightarrow (ii)$. Obviously, by Theorem 3.23.

$(ii) \Rightarrow (i)$. Follows by Corollary 3.15 and Theorem 3.23, since in an MV-algebra $(L, \oplus, ^*, 0)$, $x^{**} = x$ and $(x \rightarrow y) \vee (y \rightarrow x) = 1$, for every $x, y \in L$. Also, if $x \in L$ such that $x \odot x = x$, we deduce that x is a Boolean element in MV-algebra L, so, x^* is also Boolean. Thus, $x^* \odot x^* = x^*$. Then $x^* \odot (x^* \rightarrow x) = x^* \odot (x^* \rightarrow x^{**}) = x^* \odot (x^* \odot x^*)^* = x^* \odot x^{**} = x^* \odot x = 0$. We deduce that $x^* \odot (x^* \rightarrow x) = 0$. We conclude that in MV-algebra $(Id(A), \oplus, Ann, 0 = \{0\})$, for every $I \in Id(A)$, if $I \otimes I = I$ then $Ann(Ann(I)) = I$, $(Ann(I) : I) + (I : Ann(I)) = A$ and $Ann(I) \otimes (I : Ann(I)) = \mathbf{0}$. □

Remark 3.26. *If I is a non-idempotent ideal in a finite commutative ring A, then I and $Ann(I)$ are not coprime. Indeed, we consider the ring $(\mathbb{Z}_4, +, \cdot)$ and $I = \{\widehat{0}, \widehat{2}\}$. Then $I \otimes I = \{\widehat{0}\}$ and using Example 3.32 (1), I and $Ann(I)$ are not coprime ideals.*

The following results can be obtained by the straightforward calculations.

Corollary 3.27. *Let A be a finite commutative ring. The following conditions are equivalent:*
(i) $(Id(A), \cap, +, Ann, 0 = \{0\}, 1 = A)$ is a Boolean algebra;
(ii) $I \otimes I = I$, for every $I \in Id(A)$;
(iii) I and $Ann(I)$ are coprime, for every $I \in Id(A)$;
(iv) $I \cap Ann(I) = 0$, for every $I \in Id(A)$;
(v) $I \oplus I = I$, for every $I \in Id(A)$.

Moreover, using the characterization of Boolean elements in MV-algebras we obtain that a residuated lattice is a Boolean algebra if and only if it is both Heyting algebra and MV-algebra. Thus, we conclude that:

Corollary 3.28. *Let A be a commutative ring. Then:*
(i) $(Id(A), \cap, +, Ann, 0 = \{0\}, 1 = A)$ is a Boolean algebra if and only if A is a Von Neumann regular ring satisfying Chang property;
(ii) If A is finite, then $(Id(A), \cap, +, Ann, 0 = \{0\}, 1 = A)$ is a Boolean algebra if and only if A is a Von Neumann regular ring.

Using Remark 3.21 and Corollary 3.27 we deduce that:

Proposition 3.29. *If A is a finite Boolean ring, then $(Id(A), \cap, +, Ann, 0 = \{0\}, 1 = A)$ is a Boolean algebra.*

Proposition 3.30. *Let $p_1, p_2, ...p_r$, be r prime but not necessarily distinct numbers. We consider the ring $A = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times ... \times \mathbb{Z}_{p_r}$. Therefore, for each $I \in Id(A)$, we have that I and $Ann(I)$ are coprime. The number of ideals in this ring is 2^r.*

Proof. First of all, we remark that if K is a field, then the only ideals are $\{0\}$ and K. Then, $Ann(\{0\}) = K$, $Ann\{K\} = \{0\}$ and it is clear that $Ann(\{0\})$ and K are coprime.

Case 1. We assume that $p_1, p_2, ..., p_r$ are distinct prime numbers. For an arbitrary integer $q, q \in \{1, 2, ..., r\}$, let $v = p_{i_1} p_{i_2} ... p_{i_q}$ and $w = p_{i_{q+1}} p_{i_{q+2}} ... p_{i_r}$, where $\{p_{i_1}, p_{i_2}, ...p_{i_q}, p_{i_{q+1}}, p_{i_{q+2}}, ...p_{i_r}\}$ and $\{p_1, p_2, ..., p_r\}$ are the same set. Let $s = p_1 p_2 ... p_r = p_{i_1} p_{i_2}, ...p_{i_q} p_{i_{q+1}} p_{i_{q+2}} ... p_{i_r}$.

We have that $A=\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}\times...\times\mathbb{Z}_{p_r}\simeq\mathbb{Z}_s=\mathbb{Z}_{vw}$. Let I be an ideal of the ring A. Then I has the form $I\simeq\mathbb{Z}_v\simeq\mathbb{Z}_{p_{i_1}}\times\mathbb{Z}_{p_{i_2}}\times...\times\mathbb{Z}_{p_{i_q}}=\{\widehat{0},\widehat{w},\widehat{2w},\widehat{3w},...,\widehat{(v-1)w}\}$, where $\widehat{x}$ is an element from A. It results that $Ann(I)=\{\widehat{0},\widehat{v},\widehat{2v},\widehat{3v},...,\widehat{(w-1)v}\}$. Since v and w are coprime integers, there are $a,b\in\mathbb{Z}$ such that $1=av+bw$. Therefore, we found the elements $\widehat{av}\in Ann(I)$ and $\widehat{bw}\in I$ such that $\widehat{1}=\widehat{av}+\widehat{bw}$. From here, we get that I and $Ann(I)$ are coprime ideals.

Case 2. The prime numbers $p_1,p_2,...,p_r$ are not distinct. Without losing generality, we suppose that there is a number $s\leq r$ such that $p_1=p_2=...=p_s=p$ and $p_{s+1},p_{s+2},...,p_r$ are distinct prime numbers not equal to p. Let I be an ideal of the ring $A=\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}\times...\times\mathbb{Z}_{p_r}$. Then I can have the following forms:

i) $I\simeq\underbrace{\mathbb{Z}_p\times\mathbb{Z}_p\times...\times\mathbb{Z}_p}_{s-time}$;

ii) $I\simeq\underbrace{\mathbb{Z}_p\times\mathbb{Z}_p\times...\times\mathbb{Z}_p}_{q-time}$, $q<s$;

iii) $I\simeq\underbrace{\mathbb{Z}_p\times\mathbb{Z}_p\times...\times\mathbb{Z}_p}_{q-time}\times\mathbb{Z}_t, q\leq s$ and $t=p_{i_1}p_{i_2}...p_{i_l}$, where $\{p_{i_1},p_{i_2},...,p_{i_l}\}\subseteq\{p_{s+1},p_{s+2},...,p_r\}$ are distinct, $1\leq l\leq r-s$.

If $I\simeq\underbrace{\mathbb{Z}_p\times\mathbb{Z}_p\times...\times\mathbb{Z}_p}_{s-time}$, then the annihilator is $Ann(I)\simeq\mathbb{Z}_{p_{s+1}}\times\mathbb{Z}_{p_{s+2}}\times...\times\mathbb{Z}_{p_r}=\mathbb{Z}_t$, for $l=r-s$. The element $\alpha=\left(\underbrace{\widehat{1},\widehat{1},...,\widehat{1}}_{s-time},\underbrace{\mathbf{0},\mathbf{0},...,\mathbf{0}}_{(r-s)-time}\right)\in I$, where $\widehat{1}$ is the class modulo p and $\mathbf{0}$ are classes in $\mathbb{Z}_{p_{s+1}},\mathbb{Z}_{p_{s+2}},...,\mathbb{Z}_{p_r}$. In the same time, the element $\beta=\left(\underbrace{\widehat{0},\widehat{0},...,\widehat{0}}_{s-time},\underbrace{\mathbf{1},\mathbf{1},...,\mathbf{1}}_{(r-s)-time}\right)\in Ann(I)$, where $\widehat{0}$ is the class modulo p and $\mathbf{1}$ are classes in $\mathbb{Z}_{p_{s+1}},\mathbb{Z}_{p_{s+2}},...,\mathbb{Z}_{p_r}$. With the above notations, we remark that $\alpha+\beta=1$, where $1=\left(\underbrace{\widehat{1},\widehat{1},...,\widehat{1}}_{s-time},\underbrace{\mathbf{1},\mathbf{1},...,\mathbf{1}}_{(r-s)-time}\right)$ is the unit element in A. Therefore I and $Ann(I)$ are coprime.

If $I\simeq\underbrace{\mathbb{Z}_p\times\mathbb{Z}_p\times...\times\mathbb{Z}_p}_{q-time}$, $q<s$, then $Ann(I)\simeq\underbrace{\mathbb{Z}_p\times\mathbb{Z}_p\times...\times\mathbb{Z}_p}_{(s-q)-time}\times\mathbb{Z}_{p_{s+1}}\times\mathbb{Z}_{p_{s+2}}...\times\mathbb{Z}_{p_r}$. The element $\alpha=\left(\underbrace{\widehat{1},\widehat{1},...,\widehat{1}}_{q-time},\underbrace{\widehat{0},\widehat{0},...,\widehat{0}}_{(s-q)-time},\underbrace{\mathbf{0},\mathbf{0},...,\mathbf{0}}_{(r-s)-time}\right)\in I$, where $\widehat{0},\widehat{1}$ are the classes modulo p and $\mathbf{0}$ are classes in $\mathbb{Z}_{p_{s+1}},\mathbb{Z}_{p_{s+2}},...,\mathbb{Z}_{p_r}$. In the same time, the element $\beta=\left(\underbrace{\widehat{0},\widehat{0},...,\widehat{0}}_{q-time},\underbrace{\widehat{1},\widehat{1},...,\widehat{1}}_{(s-q)-time},\underbrace{\mathbf{0},\mathbf{0},...,\mathbf{0}}_{(r-s)-time}\right)\in Ann(I)$, where $\widehat{0},\widehat{1}$ are the classes modulo p and $\mathbf{1}$ are classes in $\mathbb{Z}_{p_{s+1}},\mathbb{Z}_{p_{s+2}},...,\mathbb{Z}_{p_r}$. With the above notations, we remark that $\alpha+\beta=1$, where $1=\left(\underbrace{\widehat{1},\widehat{1},...,\widehat{1}}_{s-time},\underbrace{\mathbf{1},\mathbf{1},...,\mathbf{1}}_{(r-s)-time}\right)$ is the unit element in A. Therefore, I and $Ann(I)$ are coprime.

If $I\simeq\underbrace{\mathbb{Z}_p\times\mathbb{Z}_p\times...\times\mathbb{Z}_p}_{q-time}\times\mathbb{Z}_t, q\leq s$ and $t=p_{i_1}p_{i_2}...p_{i_l}$, where $\{p_{i_1},p_{i_2},...,p_{i_l}\}\subseteq\{p_{s+1},p_{s+2},...,p_r\}$ are distinct, $1\leq l\leq r-s$, then the annihilator is $Ann(I)\simeq\underbrace{\mathbb{Z}_p\times\mathbb{Z}_p\times...\times\mathbb{Z}_p}_{(s-q)-time}\times\mathbb{Z}_u$, where $u=\prod p_j$, with p_j distinct prime numbers $p_j\in\{p_{s+1},p_{s+2},...,p_r\}-\{p_{i_1},p_{i_2},...,p_{i_l}\}$. The element $\alpha=\left(\underbrace{\widehat{1},\widehat{1},...,\widehat{1}}_{q-time},\underbrace{\mathbf{1},\mathbf{1},...,\mathbf{1}}_{l-time},\underbrace{\mathbf{0},\mathbf{0},...,\mathbf{0}}_{(r-q-l)-time}\right)\in I$, where $\widehat{1}$ is the class modulo p and $\mathbf{0},\mathbf{1}$ are classes in $\mathbb{Z}_{p_{q+1}},\mathbb{Z}_{p_{q+2}},...,\mathbb{Z}_{p_r}$. In the same time, the element $\beta=$

$\left(\underbrace{\widehat{0},\widehat{0},...,\widehat{0}}_{q-time},\underbrace{\mathbf{0},\mathbf{0},...,\mathbf{0}}_{l-time},\ \underbrace{\mathbf{1},\mathbf{1},...,\mathbf{1}}_{(r-q-l)-time}\right) \in Ann\,(I)$, where $\widehat{0}$ is the class modulo p and $\mathbf{0},\mathbf{1}$ are classes in $\mathbb{Z}_{p_{q+1}},\mathbb{Z}_{p_{q+2}},...,\mathbb{Z}_{p_r}$.

With the above notations, we remark that $\alpha+\beta=1$, where $1=\left(\underbrace{\widehat{1},\widehat{1},...,\widehat{1}}_{q-time},\ \underbrace{\mathbf{1},\mathbf{1},...,\mathbf{1}}_{(r-q)-time}\right)$ is the unit element in A.

Therefore I and $Ann\,(I)$ are coprime. For the last part of the proposition, it is clear that for the ring A we have $\mathsf{C}_r^0+\mathsf{C}_r^1+\mathsf{C}_r^2+...+\mathsf{C}_r^r=2^r$ ideals. □

Remark 3.31. *Due to the characterization of finite abelian groups, it results that the rings A from the above proposition are the only finite commutative unitary ring with the property that for each $I\in Id\,(A)$ the ideals I and $Ann\,(I)$ are coprime.*

Example 3.32. 1) *Let $A=\mathbb{Z}_4=\mathbb{Z}_{2^2}$. The ideals are $\{\widehat{0}\},\{\widehat{0},\widehat{2}\}$ and $\mathbb{Z}_4$.For $I=\{\widehat{0},\widehat{2}\}$, the annihilator is $Ann\,(I)=\{\widehat{0},\widehat{2}\}$, therefore I and $Ann\,(I)$ are not coprime ideals.*

2) *Let $A=\mathbb{Z}_2\times\mathbb{Z}_2=\{(\widehat{0},\widehat{0}),(\widehat{0},\widehat{1}),(\widehat{1},\widehat{0}),(\widehat{1},\widehat{1})\}$. For $I=\{(\widehat{0},\widehat{0}),(\widehat{0},\widehat{1})\}$, the annihilator is $Ann\,(I)=\{(\widehat{0},\widehat{0}),(\widehat{1},\widehat{0})\}$. Obviously, I and $Ann\,(I)$ are coprime ideals.*

3) *Let $A=\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2=$*
$=\{(\widehat{0},\widehat{0},\widehat{0}),(\widehat{0},\widehat{1},\widehat{0}),(\widehat{1},\widehat{0},\widehat{0}),(\widehat{1},\widehat{1},\widehat{0}),(\widehat{0},\widehat{0},\widehat{1}),(\widehat{1},\widehat{0},\widehat{1}),(\widehat{0},\widehat{1},\widehat{1}),(\widehat{1},\widehat{1},\widehat{1})\}$. Let $I=\{(\widehat{0},\widehat{0},\widehat{0}),(\widehat{0},\widehat{1},\widehat{0})\}$ be an ideal. The annihilator is $Ann\,(I)=\{(\widehat{0},\widehat{0},\widehat{0}),(\widehat{1},\widehat{0},\widehat{0}),(\widehat{0},\widehat{0},\widehat{1}),(\widehat{1},\widehat{0},\widehat{1})\}$. It results that I and $Ann\,(I)$ are coprime ideals and this is true for all ideals of A. We remark that this ring has 8 ideals.

4) *Let $A=\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\simeq\mathbb{Z}_2\times\mathbb{Z}_6=\{(\widehat{0},\widehat{0},\overline{0}),(\widehat{0},\widehat{1},\overline{0}),(\widehat{1},\widehat{0},\overline{0}),(\widehat{0},\widehat{1},\overline{1}),(\widehat{0},\widehat{1},\overline{2}),(\widehat{1},\widehat{1},\overline{2}),(\widehat{0},\widehat{0},\overline{1}),(\widehat{1},\widehat{1},\overline{0}),(\widehat{1},\widehat{0},\overline{1}),(\widehat{0},\widehat{0},\overline{2}),(\widehat{1},\widehat{0},\overline{2}),(\widehat{1},\widehat{1},\overline{1})\}$.*

For $I=\{(\widehat{0},\widehat{0},\overline{0}),(\widehat{0},\widehat{1},\overline{0})\}$, the annihilator is $Ann\,(I)=\{(\widehat{0},\widehat{0},\overline{0}),(\widehat{1},\widehat{0},\overline{0}),(\widehat{0},\widehat{0},\overline{1}),(\widehat{1},\widehat{0},\overline{1}),(\widehat{0},\widehat{0},\overline{2}),(\widehat{1},\widehat{0},\overline{2})\}$ and is coprime to I.

For $I=\{(\widehat{0},\widehat{0},\overline{0}),(\widehat{0},\widehat{1},\overline{0}),(\widehat{1},\widehat{0},\overline{0}),(\widehat{1},\widehat{1},\overline{0})\}$, the annihilator is $Ann\,(I)=\{(\widehat{0},\widehat{0},\overline{0}),(\widehat{0},\widehat{0},\overline{1}),(\widehat{0},\widehat{0},\overline{2})\}$ and is coprime to I.

For $I=\{(\widehat{0},\widehat{0},\overline{0}),(\widehat{1},\widehat{0},\overline{0})\}$, the annihilator is $Ann\,(I)=\{(\widehat{0},\widehat{0},\overline{0}),(\widehat{0},\widehat{1},\overline{0}),(\widehat{0},\widehat{0},\overline{1}),(\widehat{0},\widehat{1},\overline{1}),(\widehat{0},\widehat{0},\overline{2}),(\widehat{0},\widehat{1},\overline{2})\}$ and is coprime to I, etc.

5) *Let $A=\mathbb{Z}_4\times\mathbb{Z}_3\simeq\mathbb{Z}_{12}$. For $I=\{\widehat{0},\widehat{6}\}$, the annihilator is $Ann\,(I)=\{\widehat{0},\widehat{2},\widehat{4},\widehat{6},\widehat{8},\widehat{10}\}$ and $Ann\,(I)$ is not coprime to I.*

6) *If A is a finite integral domain, then $(Id(A),\cap,+,Ann,0=\{0\},1=A)$ is a Boolean algebra. Indeed, every finite integral domain is a field, so $Id(A)=\{\{0\},A\}\simeq L_2$ is a Boolean algebra.*

7) *If K is a field, then $Id(K)=\{\mathbf{0},K\}$ and I and $Ann(I)$ are coprime for every $I\in Id(K)$.*

Using Corollary 3.15 and Proposition 3.30 we obtain:

Corollary 3.33. *If A is a finite commutative ring with $|A|=n=p_1^{\alpha_1}\cdot...\cdot p_r^{\alpha_r}$, then its set of ideals is an MV-algebra. Of all its representations, only if A is isomorphic to the ring $\underbrace{\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_1}\times...\times\mathbb{Z}_{p_1}}_{\alpha_1-time}\times...\times\underbrace{\mathbb{Z}_{p_r}\times\mathbb{Z}_{p_r}\times...\times\mathbb{Z}_{p_r}}_{\alpha_r-time}$ the lattice of its ideals is a Boolean algebra.*

Using the equivalent characterizations for Boolean elements in residuated lattices (see [6]), we obtain necessary and sufficient conditions for rings in which the lattice of ideals is a Boolean algebra:

Corollary 3.34. *Let A be a commutative unitary ring. The following assertions are equivalent:*

(*i*) *$(Id(A),\cap,+,Ann,0=\{0\},1=A)$ is a Boolean algebra;*
(*ii*) *I and $Ann(I)$ are coprime, for every $I\in Id(A)$;*
(*iii*) *$(I:Ann(I))=I$, for every $I\in Id(A)$;*
(*iv*) *For $J\in Id(A)$, $Ann(J)\subseteq J$ implies $J=A$;*
(*v*) *For $I,J\in Id(A)$, $I\subseteq(J:Ann(J))$ implies $I\subseteq J$;*
(*vi*) *For $I,J,K\in Id(A)$, $I\subseteq(J:Ann(K))$ and $J\subseteq K$; implies $I\subseteq K$;*
(*vii*) *For $I,J\in Id(A)$, $(J:I)\subseteq I$ implies $I=A$.*

Moreover, using some connections between Heyting algebras, MV-algebras and involutive residuated lattices we obtain necessary and sufficient conditions for which the lattice of ideals in commutative unitary rings is a Boolean algebra. The following results can be obtained by the straightforward calculations, using Proposition 2.9:

Corollary 3.35. *Let A be a commutative unitary ring. The following assertions are equivalent:*
(i) *A is a Von Neumann regular ring in which*

$$Ann(Ann(I)) = I,$$

for every $I \in Id(A)$;
(ii) *A is a Von Neumann regular ring that satisfies the condition*

$$Ann(J) = \mathbf{0} \Rightarrow J = A,$$

for $J \in Id(A)$;
(iii) $(Id(A), \cap, +, Ann, 0 = \{0\}, 1 = A)$ *is a Boolean algebra.*

4 Connections with binary block codes

In [13] have been defined binary block codes associated to MV-algebras and Wajsberg algebras and in [12] a classification of these algebras was done.

One of the question which arise is what these codes really represent. In Section 3, we proved that the lattice of ideals of a finite commutative ring is a Boolean algebra or an MV-algebra, see Corollary 3.33.

Using this result, in this section we construct binary block codes associated to the lattice of ideals in a finite commutative ring $A = \{x_1, x_2, ..., x_n\}$ with n elements.

From Remark 3.16, $Id\,(A)$, the set of its ideals of A, has $\mathcal{N}_A$ elements (obviously, $\mathcal{N}_A \leq n$). From Corollary 3.33, $Id\,(A)$ can be a Boolean algebra or an MV-algebra (that is not Boolean) and it can generates a binary block codes. To each ideal I_α from $Id\,(A)$ we associate a codeword α of length $n, \alpha = (\alpha_1, ..., \alpha_n)\,, \alpha_i = 1$ if $x_i \in I_\alpha$ and $\alpha_i = 0$, otherwise.

We denote with $\mathcal{C}_A$ the set of these codewords, which is a binary block code. The obtained block code $\mathcal{C}_A = \{w_1, w_2, ..., w_{\mathcal{N}_A}\}$, with lexicographic order is a totally ordered set. On $\mathcal{C}_A$ we can define a Boolean algebra structure, if $\mathcal{N}_A$ is a power of 2, or an MV-algebra structure, otherwise. This algebra generates a binary block code $\mathcal{C}_2$.

The code $\mathcal{C}_2$ is called the *reduced code* of the code $\mathcal{C}_A$. It is clear that all finite commutative rings A with the set $Id\,(A)$ having the same number of elements, $\mathcal{N}_A$, generate the same reduced binary block code with $\mathcal{N}_A$ codewords of length n.

By using the above definitions from Section 2, we have the following correspondence between the operations of an MV-algebra M :

$$x \odot y = (x^* \oplus y^*)^* \text{ and } x \oplus y = (x^* \odot y^*)^*.$$

In the associated Wajsberg algebra, we have that

$$x \to y = x^* \oplus y = (x \odot y^*)^*.$$

Since, from Corollary 3.33, the associated lattice of ideals in the ring A is a Wajsberg algebra, then the Wajsberg implication is

$$I \to J = Ann\,(I \otimes Ann\,(J)),$$

for every $I, J \in Id(A)$.

It is known that ([12]) if S is a nonempty set and $(W, \to, ^*, 1)$ is a Wajsberg algebra then:

Definition 4.1. *A mapping $f : S \to W$ is called a W-function on S and a map $f_w : S \to \{0, 1\}, w \in W$, such that*

$$f_w\,(x) = 1, \text{ if and only if } w \to f\,(x) = 1, \text{ for every } x \in S,$$

is called a cut function of the map f .
The subset

$$S_w = \{x \in S \ : \ w \to f\,(x) = 1\} \subseteq S,$$

is called a cut subset of the set S.

If $f : S \to W$ is a W-function on S and we define on W the binary relation

$$\forall w_1, w_2 \in W, w_1 \sim w_2 \text{ if and only if } S_{w_1} = S_{w_2},$$

then, this relation is an equivalence relation on W. For every $w \in W$, we denote by $\widetilde{w}$ the equivalence class of w. Now, let A be a finite commutative ring with n ideals, $S = \{1, ..., n\}$ be a nonempty set and $(Id(A), \to, Ann, 1 = A)$ be the Wajsberg algebra of ideals (see Corollary 3.33). Using above notations, to each equivalence class $\widetilde{I}$ (with $I \in Id(A)$), will correspond the codeword $f_I = I_1 I_2 ... I_n$, with $I_i = j$, if and only if $f_I(i) = j, i \in S, j \in \{0, 1\}$.

Example 4.2. *We consider the commutative ring* $A = (\mathbb{Z}_n, +, \cdot)$.

Case 1. $n = 4$**.**

i) For $A = \mathbb{Z}_4$*, the lattice* $Id(\mathbb{Z}_4)$ *has 3 elements,* $Id(\mathbb{Z}_4) = \{\widehat{0}, \{\widehat{0}, \widehat{2}\}, \mathbb{Z}_4\} = \{O, R, E\}$. *Since* $AnnO = E$, $AnnE = O$, $AnnR = R$ *and* $R \otimes R = O$, *so using Corollaries 3.15 and 3.27, the obtained lattice is a Wajsberg (MV) algebra, with the implication given in the below table:*

$\to$	O	R	E
O	E	E	E
R	R	E	E
E	O	R	E

The code $\mathcal{C}_A$ *attached to the algebra* $Id(\mathbb{Z}_4)$ *is* $\mathcal{C}_{\mathbb{Z}_4} = \{0001, 0101, 1111\}$. *Its reduced code is* $\mathcal{C}_2 = \{111, 011, 001\}$.

ii) For $A = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{\left(\widehat{0}, \widehat{0}\right), \left(\widehat{0}, \widehat{1}\right), \left(\widehat{1}, \widehat{0}\right), \left(\widehat{1}, \widehat{1}\right)\}$, *we obtain the lattice*

$Id(\mathbb{Z}_2 \times \mathbb{Z}_2) = \{\left(\widehat{0}, \widehat{0}\right), \{\left(\widehat{0}, \widehat{0}\right), \left(\widehat{0}, \widehat{1}\right)\}, \{\left(\widehat{0}, \widehat{0}\right), \left(\widehat{1}, \widehat{0}\right)\}, \mathbb{Z}_2 \times \mathbb{Z}_2\} = \{O, R, B, E\}$, *which is a Wajsberg (Boolean) algebra since* $I^2 = I$, *for every* $I \in Id(\mathbb{Z}_2 \times \mathbb{Z}_2)$, *see Corollary 3.27. Also,* $AnnO = E, AnnE = O, AnnR = B, AnnB = R$, *thus, the implication table is*

$\to$	O	R	B	E
O	E	E	E	E
R	B	E	B	E
B	R	R	E	E
E	O	R	B	E

The code associated to the lattice $Id(\mathbb{Z}_2 \times \mathbb{Z}_2)$ *is* $\mathcal{C}_{\mathbb{Z}_2 \times \mathbb{Z}_2} = \{1000, 1100, 1010, 1111\}$. *The reduced code is* $\mathcal{C}_2 = \{1111, 0101, 0011, 0001\}$.

Case 2. $n = 6$.

For $A = \mathbb{Z}_6$, *the lattice* $Id(\mathbb{Z}_6)$ *has 4 elements,* $Id(\mathbb{Z}_6) = \{\widehat{0}, \{\widehat{0}, \widehat{3}\}, \{\widehat{0}, \widehat{2}, \widehat{4}\}, \mathbb{Z}_6\} = \{O, R, B, E\}$. *Since* $AnnO = E$, $AnnE = O$, $AnnR = B$, $AnnB = R$ *and* $I^{\,2} = I$, *for every* $I \in Id(\mathbb{Z}_6)$, *the obtained lattice is a Wajsberg (Boolean) algebra, with the implication table*

$\to$	O	R	B	E
O	E	E	E	E
R	B	E	B	E
B	R	R	E	E
E	O	R	B	E

The code attached to the lattice $Id(\mathbb{Z}_6)$ *is* $\mathcal{C}_{\mathbb{Z}_6} = \{000001, 001001, 010101, 111111\}$ *and the reduced code is* $\mathcal{C}_2 = \{1111, 0101, 0011, 0001\}$, *the same as in the Case 1, ii).*

The case $A = \mathbb{Z}_2 \times \mathbb{Z}_3 = \{\left(\widehat{0}, \overline{0}\right), \left(\widehat{0}, \overline{1}\right), \left(\widehat{0}, \overline{2}\right), \left(\widehat{1}, \overline{0}\right), \left(\widehat{1}, \overline{1}\right), \left(\widehat{1}, \overline{2}\right)\}$ *is similar with the above case, since* $\mathbb{Z}_2 \times \mathbb{Z}_3 \simeq \mathbb{Z}_6$.

Case 3. $n = 8$.

i) For $A = \mathbb{Z}_8$, *the lattice* $Id(\mathbb{Z}_8)$ *has 4 elements,* $Id(\mathbb{Z}_8) = \{\widehat{0}, \{\widehat{0}, \widehat{4}\}, \{\widehat{0}, \widehat{2}, \widehat{4}, \widehat{6}\}, \mathbb{Z}_8\} = \{O, R, B, E\}$. *Since* $AnnO = E$, $AnnE = O$, $AnnR = B$, $AnnB = R$ *and* $R \otimes R = O$, $B \otimes B = R$, *using Corollary 3.27, the obtained lattice is a Wajsberg (MV) algebra, with the implication table given in the below table:*

$\to$	O	R	B	E
O	E	E	E	E
R	B	E	E	E
B	R	B	E	E
E	O	R	B	E

The code attached to the lattice $Id(\mathbb{Z}_8)$ *is* $\mathcal{C}_{\mathbb{Z}_8} = \{00000001, 00010001, 01010101, 11111111\}$ *and the reduced code is* $\mathcal{C}_2 = \{1111, 0111, 0011, 0001\}$.

ii) For $A = \mathbb{Z}_2 \times \mathbb{Z}_4 = \{(\widehat{0},\overline{0}), (\widehat{0},\overline{1}), (\widehat{0},\overline{2}), (\widehat{0},\overline{3}), (\widehat{1},\overline{0}), (\widehat{1},\overline{1}), (\widehat{1},\overline{2}), (\widehat{1},\overline{3})\}$, *the lattice of ideals is*
$Id(\mathbb{Z}_2 \times \mathbb{Z}_4)=\{(\widehat{0},\overline{0}),\{(\widehat{0},\overline{0}),(\widehat{0},\overline{1}),(\widehat{0},\overline{2}),(\widehat{0},\overline{3})\}$,
$\{(\widehat{0},\overline{0}),(\widehat{1},\overline{0}),(\widehat{0},\overline{2}),(\widehat{1},\overline{2})\},\{(\widehat{0},\overline{0}),(\widehat{0},\overline{2})\}, \{(\widehat{0},\overline{0}),(\widehat{1},\overline{0})\}, \mathbb{Z}_2 \times \mathbb{Z}_4\} = \{O, B, D, R, C, E\}$. *Since* $AnnO = E$, $AnnE = O$, $AnnR = D$, $AnnB = C$, $AnnD = R$, $AnnC = B$ *and* $R \otimes R = O$, *using Corollary 3.27, the obtained lattice is a Wajsberg (MV) algebra, with the implication table given below:*

$\rightarrow$	O	R	B	C	D	E
O	E	E	E	E	E	E
R	D	E	E	D	E	E
B	C	D	E	C	D	E
C	B	B	B	E	E	E
D	R	B	B	D	E	E
E	O	R	B	C	D	E

The code attached to the lattice $Id(\mathbb{Z}_2 \times \mathbb{Z}_4)$ *is* $\mathcal{C}_{\mathbb{Z}_2\times\mathbb{Z}_4} = \{00000001, 00000101, 00001111, 00010001, 01010101, 11111111\}$ *and the reduced code is* $\mathcal{C}_2 = \{111111, 011011, 001001, 000111, 000011, 00000001\}$.

iii) For $A = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(\widehat{0},\widehat{0},\widehat{0}), (\widehat{0},\widehat{0},\widehat{1}), (\widehat{0},\widehat{1},\widehat{0}), (\widehat{0},\widehat{1},\widehat{1}), (\widehat{1},\widehat{0},\widehat{0}), (\widehat{1},\widehat{0},\widehat{1}), (\widehat{1},\widehat{1},\widehat{0}), (\widehat{1},\widehat{1},\widehat{1})\}$, *we obtain the lattice* $Id(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = \{(\widehat{0},\widehat{0},\widehat{0}), \{(\widehat{0},\widehat{0},\widehat{0}),(\widehat{0},\widehat{0},\widehat{1})\}, \{(\widehat{0},\widehat{0},\widehat{0}),(\widehat{0},\widehat{1},\widehat{0})\}, \{(\widehat{0},\widehat{0},\widehat{0}),(\widehat{1},\widehat{0},\widehat{0})\}$, $\{(\widehat{0},\widehat{0},\widehat{0}),(\widehat{0},\widehat{0},\widehat{1}),(\widehat{0},\widehat{1},\widehat{0}),(\widehat{0},\widehat{1},\widehat{1})\}, \{(\widehat{0},\widehat{0},\widehat{0}),(\widehat{0},\widehat{0},\widehat{1}),(\widehat{1},\widehat{0},\widehat{0}),(\widehat{1},\widehat{0},\widehat{1})\}, \{(\widehat{0},\widehat{0},\widehat{0}),(\widehat{0},\widehat{1},\widehat{0}),(\widehat{1},\widehat{0},\widehat{0}),(\widehat{1},\widehat{1},\widehat{0})\}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\} = \{O, X, Y, T, Z, U, V, E\}$, *which is a Wajsberg (Boolean) algebra. Since* $AnnO = E$, $AnnE = O$, $AnnX = V$, $AnnY = U$, $AnnZ = T$, $AnnT = Z$, $AnnU = Y$, $AnnV = X$, *the implication is given in the below table:*

$\rightarrow$	O	X	Y	Z	T	U	V	E
O	E	E	E	E	E	E	E	E
X	V	E	V	E	V	E	V	E
Y	U	U	E	E	U	U	E	E
Z	T	U	V	E	T	U	V	E
T	Z	Z	Z	Z	E	E	E	E
U	Y	Z	Y	Z	V	E	V	E
V	X	X	Z	Z	U	U	E	E
E	O	X	Y	Z	T	U	V	E

The code attached to the lattice $Id(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ *is* $\mathcal{C}_{\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2} = \{00000001, 00000011, 00000101, 00010001, 00001111, 00110011, 01010101, 11111111\}$, *which is similar to its reduced code* $\mathcal{C}_2$.

Definition 4.3. [19] *Let* $\mathcal{C}$ *be a code. The Hamming distance* $d(c_1, c_2)$, *between two codewords* c_1, c_2 *of the same length, is the number of positions in which the corresponding symbols are different. The minimum Hamming distance of the code* $\mathcal{C}$, *denoted* d_H, *is*

$$d_H = min\{d(c_1, c_2), c_1, c_2 \in \mathcal{C}, c_1 \neq c_2\}.$$

The code $\mathcal{C}$ *is considered to be* k*-error detecting if and only if* d_H *between any two of its codewords is at least* $k+1$. *The code* $\mathcal{C}$ *is* k*-errors correcting if and only if* d_H *between any two of its codewords is at least* $2k+1$. *Therefore, if* $d_H \geq 2$, *the code* $\mathcal{C}$ *is an error detecting code. If* $d_H \geq 3$, *the code* $\mathcal{C}$ *is an error correcting code.*

Remark 4.4. *The codes generated by a Wajsberg (MV) algebra is not error detecting, nor error correcting codes, since* d_H *is always* 1.

Proposition 4.5. *Let* A *be a finite commutative unitary ring,* $A = \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times ... \times \mathbb{Z}_{k_r}$, *where* $k_i = p_i^{\alpha_i}$, p_i *a prime number, for all* $i \in \{1, 2, ..., r\}$.

(i) *If* $p_i \geq 3$, *for all* $i \in \{1, 2, ..., r\}$, *then the attached code* $\mathcal{C}_A$ *is an error detecting code.*

(ii) *If* $p_i \geq 5$, *for all* $i \in \{1, 2, ..., r\}$, *then the attached code* $\mathcal{C}_A$ *is an error correcting code.*

Proof. If $p_i \geq 3$, for all $i \in \{1, 2, ..., r\}$, we have that d_H is minimum 2. If $p_i \geq 5$, for all $i \in \{1, 2, ..., r\}$, then d_H is minimum 3. ☐

Remark 4.6. *Let W be a Wajsberg (MV or Boolean) algebra with $\mathcal{N}_A = \prod_{i=1}^{r} (\alpha_i + 1)$ elements and $\mathcal{C}_W$ its attached binary block code. There is a finite unitary commutative ring $A = \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times ... \times \mathbb{Z}_{k_r}$, where $k_i = p_i^{\alpha_i}$, p_i a prime number, for all $i \in \{1, 2, ..., r\}$, with lattice of ideals $Id(A)$ having $\mathcal{N}_A$ elements and its attached binary block code $\mathcal{C}_A$ such that $\mathcal{C}_W$ is the reduced code of the code $\mathcal{C}_A$.*

5 Generation of finite MV-algebras using finite commutative rings

In this section, using the results obtained in Section 3, we present a way to generate finite MV-algebras using finite commutative rings. We recall that an MV-algebra is finite if and only if it is isomorphic to a finite product of totally ordered MV-algebras, see [17]. Also, it is known that, on a finite totally ordered set there is only one way to define an MV-algebra, see [14]. Now, let $n \geq 2$ be a natural number. If we consider the decomposition of n in factors greater than 1, then this decomposition is not unique. We denote by $\pi(n)$ the number of all such decompositions. We conclude that for every natural number $n \geq 2$ there are $\pi(n)$ non-isomorphic MV-algebras with n elements which are obtained as a finite product of totally ordered MV-algebras. Furthermore, there is only one MV-algebra with n elements which is a chain. We deduce that:

Proposition 5.1. *For every natural number $n \geq 2$, there are $\pi(n) + 1$ non-isomorphic MV-algebras with n elements.*

Example 5.2. *For $n = 8$, we have $8 = 2 \cdot 4 = 2 \cdot 2 \cdot 2$. Thus, we have two types (up to an isomorphism) of MV-algebras with 8 elements obtained as a finite product of MV chains. In addition, we have only one MV-algebra with 8 elements which is a chain.Therefore, there are three types of MV-algebras (up to an isomorphism) with 8 elements.*

Table 1 present a summary for the number of MV-algebras and Boolean algebras with $n \leq 8$ elements:

Table 1

	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
MV-algebras	1	1	2	1	2	1	3
Boolean algebras	1	–	1	–	–	–	1

An interesting thing is the relationship between the number of MV-algebras with $n \geq 2$ elements and the number of MV-chains with n elements (only one) or the number of MV-algebras which are Boolean algebras (only one if n is a power of 2). Using Corollary 3.33, if A is a finite commutative ring, its lattice of ideals is a Wajsberg (MV) algebra or a Wajsberg (Boolean) algebra.

Table 2 present a basic summary for the structure of the lattice of ideals in a finite and commutative ring A with $2 \leq n \leq 10$ elements:

Table 2

$\|A\| =$ **n**	A **is isomorphic to one of the rings:**	$Id(A)$ **is:**
$n=2$	Z_2	Boolean algebra
$n=3$	Z_3	Boolean algebra
$n=4$	Z_4	MV-algebra
	$Z_2 \times Z_2$	Boolean algebra
$n=5$	Z_5	Boolean algebra
$n=6$	$Z_6 \simeq Z_3 \times Z_2$	Boolean algebra
$n=7$	Z_7	Boolean algebra
$n=8$	Z_8	MV-algebra
	$Z_4 \times Z_2$	MV-algebra
	$Z_2 \times Z_2 \times Z_2$	Boolean algebra
$n=9$	Z_9	MV-algebra
	$Z_3 \times Z_3$	Boolean algebra
$n=10$	$Z_{10} \simeq Z_5 \times Z_2$	Boolean algebra

One interesting thing is that for any finite commutative ring, just for one representation of this ring, the lattice of ideals is a Boolean algebra. In all other cases, this lattice is an MV-algebra which is not a Boolean algebra. Thus, in order to generate all MV-algebras with $n \geq 2$ elements it is suffices to find finite commutative rings with n ideals.

Remark 5.3. *All finite MV-algebras (up to an isomorphism) with $n \geq 2$ elements correspond to finite commutative rings A in which $|Id(A)| = n$.*

Lemma 5.4. *If $p \geq 2$ is a prime number and $k \geq 1$ is a natural number, then $(Id(Z_{p^k}), \cap, +, \otimes, \rightarrow, 0 = \{0\}, 1 = Z_{p^k})$ is the only MV-chain (up to an isomorphism) with $k+1$ elements.*

Proof. Obviously, the ring $(Z_{p^k}, +, \cdot)$ has $k+1$ ideals: $I_0 = \{0\}$, $I_1 = \widehat{p^{k-1}}Z_{p^k}$, ..., $I_{k-2} = \widehat{p^2}Z_{p^k}$, $I_{k-1} = \widehat{p}Z_{p^k}$, $I_k = Z_{p^k}$. Since $I_0 \subseteq I_1 \subseteq I_2 \subseteq ... \subseteq I_k$, using Corollary 3.33, $Id(Z_{p^k})$ is an MV-chain. Moreover, for every $i, j \in \{0, ..., k\}$ we have $I_i \rightarrow I_j = Z_{p^k}$ if $i \leq j$ and I_{k-i+j} otherwise. Also, $I_i^* = Ann(I_i) = I_{k-i}$ for every $i \in \{0, ..., k\}$. Moreover, for every $i, j \in \{0, ..., k\}$,

$$I_i \oplus I_j = Ann(I_i) \rightarrow I_j = I_{k-i} \rightarrow I_j = Z_{p^k} \text{ if } k \leq i+j \text{ and } I_{i+j} \text{ otherwise.}$$

□

Example 5.5. *We generate all finite MV-algebras M with $n = 6 = 2 \cdot 3$ elements (up to an isomorphism). Thus, we have $\pi(6)+1 = 2$ types of non-isomorphic MV-algebras: one MV-algebra is an MV-chain (for example, $Id(Z_{32})$ which has 6 ideals, since $32 = 2^5$, see Lemma 5.4) and one MV-algebra is a product of MV-chains (for example, $Id(Z_2 \times Z_4)$ which has $2 \cdot 3 = 6$ ideals).*

Case 1. *In $(\mathbb{Z}_{32}, +, \cdot)$ the lattice of ideals has 6 elements:*
$Id(\mathbb{Z}_{32})$=$\{I_0 = \{\widehat{0}\}$, $I_1 = \widehat{16}\mathbb{Z}_{32} = \{\widehat{0}, \widehat{16}\}$, $I_2 = \widehat{8}\mathbb{Z}_{32} = \{\widehat{0}, \widehat{8}, \widehat{16}, \widehat{24}\}$, $I_3 = \widehat{4}\mathbb{Z}_{32} = \{\widehat{0}, \widehat{4}, \widehat{8}, ..., \widehat{28}\}$, $I_4 = \widehat{2}\mathbb{Z}_{32} = \{\widehat{0}, \widehat{2}, \widehat{4}, ..., \widehat{30}\}$, , $I_5 = \mathbb{Z}_{32}\}$. Using Corollary 3.27, since $I_1^2 = I_0$, the obtained lattice is an MV chain $(M, \oplus, ^*, O)$ with $M = \{O, B, D, R, C, E\}$ in which $O \leq B \leq D \leq R \leq C \leq E$ and the addition table is given below:

$\oplus$	O	R	B	C	D	E
O	O	R	B	C	D	E
R	R	E	C	E	E	E
B	B	C	D	E	R	E
C	C	E	E	E	E	E
D	D	E	R	E	C	E
E	E	E	E	E	E	E

Using Lemma 5.4, in this MV-algebra, $O^* = E$, $E^* = O$, $R^* = D$, $B^* = C$, $D^* = R$ and $C^* = B$.

Case 2. *In $\mathbb{Z}_2 \times \mathbb{Z}_4$ the lattice of ideals has 6 elements, see Example 4.2, Case 3 (ii).*
Using Corollary 3.33, we obtain an MV algebra $(M, \oplus, ^*, O)$ with $M = \{O, B, D, R, C, E\}$ in which $O \leq R, C \leq D \leq E, O \leq R \leq B \leq E$ and C, R respective D, B are incomparable.

The addition table is given below:

$\oplus$	O	R	B	C	D	E
O	O	R	B	C	D	E
R	R	B	B	D	E	E
B	B	B	B	E	E	E
C	C	D	E	C	D	E
D	D	E	E	D	E	E
E	E	E	E	E	E	E

Since $^* = Ann$ in this $MV-algebra$,

$$O^* = E,\ E^* = O,\ R^* = D,\ B^* = C,\ D^* = R \text{ and } C^* = B.$$

In the following, in **Table 3,** using Corollaries 3.27, 3.33 and Lemma 5.4, we sum briefly describe a way to generate finite MV-algebras with $2 \leq n \leq 8$ elements.

In Table 3, the first column contains the number n of elements of the finite MV-algebra M.

The second column corresponds to the number of MV-algebras with n elements.

The third column contains the rings A which generates these MV-algebras and the subvariety of MV algebras to which the lattice of ideals $Id(A)$ belongs. In this column, $p \geq 2$ is a prime number.

Table 3:

$\vert M\vert = \mathbf{n}$	**Number of MV-algebras**	**Rings which generates MV-algebras**
$n = 2$	1	$\mathbb{Z}_p$ (Boole chain)
$n = 3$	1	$\mathbb{Z}_{p^2}$ (MV chain)
$n = 4$	2	$\mathbb{Z}_{p^3}$ (MV chain) and $\mathbb{Z}_p \times \mathbb{Z}_p$ (Boole)
$n = 5$	1	$\mathbb{Z}_{p^4}$ (MV chain)
$n = 6$	2	$\mathbb{Z}_{p^5}$ (MV chain) and $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$ (MV)
$n = 7$	1	$\mathbb{Z}_{p^6}$ (MV chain)
$n = 8$	3	$\mathbb{Z}_{p^7}$ (MV chain) and $\mathbb{Z}_p \times \mathbb{Z}_{p^3}$ (MV) and $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ (Boole)

6 Conclusions

MV-algebras are algebraic structures corresponding to Łukasiewicz $\infty-$ valued propositional logic. In this paper using the connections between some subvarieties of residuated lattices, we present a way to generate all (up to an isomorphism) finite MV-algebras using rings. For this, we investigated some properties of the lattice of ideals in commutative rings and we show that, for finite rings of the form $A = \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times ... \times \mathbb{Z}_{k_r}$, where $k_i = p_i^{\alpha_i}$ and p_i a prime number, for all $i \in \{1, 2, ..., r\}$, this lattice is a Boolean algebra or an MV-algebra (which is not Boolean). As a further research, we intend to continue this study by extended it to other classes of logical algebras. Also we will investigate the codes attached to such structures, since we can have here two directions of research. First direction, by studying of some logical algebras, we will try to find when their attached codes are good and performing. Logical algebras can be a way to define codes. We are looking for those structures which give us codes with good parameters. For the first time we find such a structure, attached to a logical algebra, on which we can find codes with Hamming distance greater that 3 (see Proposition 4.5). The second direction is the reverse of the first direction. The study of the codes can give us new properties and applications of logical algebras.

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References

[1] H. A. S. Abujabal, M. Aslam, A. B. Thaheem, *A representation of bounded commutative BCK-algebras*, International Journal of Mathematics and Mathematical Sciences, **19**(4) (1996), 733-736.

[2] R. Balbes, P. Dwinger, *Distributive lattices*, Columbia, Missouri: University of Missouri Press. XIII, 1974.

[3] L. P. Belluce, A. Di Nola, *Commutative rings whose ideals form an MV-algebra*, Mathematical Logic Quarterly, **55**(5) (2009), 468-486.

[4] L. P. Belluce, A. Di Nola, E. Marchioni, *Rings and Gödel algebras*, Algebra University, **64**(1-2) (2010), 103-116.

[5] R. L. Blair, *Ideal lattices and the structure of rings*, Transactions of the AMS - American Mathematical Society, **75** (1953), 136-153.

[6] K. Blount, C. Tsinakis, *The structure of residuated lattices*, International Journal of Algebra and Computation, **13**(4) (2003), 437-461.

[7] D. Busneag, D. Piciu, *Lectii de algebra*, Ed. Universitaria, Craiova, 2002.

[8] I. Chajda, H. Länger, *Commutative rings whose ideal lattices are complemented*, Asian-European Journal of Mathematics, **3** (2019), DOI:10.1142/S1793557119500396.

[9] C. C. Chang, *Algebraic analysis of many-valued logic*, Transactions of the AMS - American Mathematical Society, **88** (1958), 467-490.

[10] R. Cignoli, I. M. L. D'Ottaviano, D. Mundici, *Algebraic foundations of many-valued reasoning*, Trends in Logic-Studia Logica Library 7, Dordrecht: Kluwer Acad. Publ, **2000**, DOI:10.1007/978-94-015-9480-6.

[11] R. P. Dilworth, *Abstract residuation over lattices*, Bulletin of the AMS - American Mathematical Society, **44** (1938), 262-268.

[12] C. Flaut, S. Hoskova-Mayerova, A. Borumand Saeid, R. Vasile, *Wajsberg algebras of order* $n(n \leq 9)$, Neural Computing and Applications, **32** (2020), 13301-13312.

[13] C. Flaut, R. Vasile, *Wajsberg algebras arising from binary block codes*, Soft Computing, **24** (2020), 6047-6058.

[14] J. M. Font, A. J. Rodriguez, A. Torrens, *Wajsberg algebras*, Stochastica, **8**(1) (1984), 5-30.

[15] B. Van Gasse, G. Deschrijver, C. Cornelis, E. Kerre, *Filters of residuated lattices and triangle algebras*, Information Sciences, **180**(16) (2010), 3006-3020.

[16] I. N. Hernstein, *Topics in algebra*, 2end edition, John Wiley and Son, New York, 1975.

[17] U. Höhle, S. E. Rodabaugh, *Mathematics of fuzzy sets: Logic, topology and measure theory*, Springer, Berlin, 1999.

[18] A. Iorgulescu, *Algebras of logic as BCK algebras*, A.S.E., Bucharest, 2009.

[19] J. H. Lint, *Introduction to coding theory*, third edition, Graduate Texts in Mathematics, **86**, Springer Verlag, Berlin, 1999.

[20] J. Meng, Y. B. Jun, *BCK-algebras*, Kyung Moon Sa Co. Seoul, Korea, 1994.

[21] D. Mundici, *MV-algebras-a short tutorial*, Department of Mathematics Ulisse Dini, University of Florence, 2007.

[22] D. Piciu, *Algebras of fuzzy logic*, Editura Universitaria, Craiova, 2007.

[23] S. V. Tchoffo Foka, M. Tonga, *Rings and residuated lattices whose fuzzy ideals form a Boolean algebra*, Soft Computing, **26** (2022), 535-539.

[24] E. Turunen, *Mathematics behind fuzzy logic*, Physica-Verlag Heidelberg, 1999.

[25] M. Ward, R. P. Dilworth, *Residuated lattices*, Transactions of the AMS - American Mathematical Society, **45** (1939), 335-354.