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A quadratic optimization problem with bipolar fuzzy relation equation constraints

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Abstract

This paper studies the quadratic programming problem subject to a system of bipolar fuzzy relation equations with the max-product composition. A characterization of structure of its feasible domain is presented using the lower and upper bound vector on its solution set. A sufficient condition is proposed which under the condition, a component of one of its optimal solutions is the corresponding component of either the lower or upper bound vector. Some sufficient conditions are suggested to reveal one of its optimal solutions without resolution of the problem. Furthermore, some sufficient conditions are then given to determine some components from one of its optimal solutions. Based on these conditions, we can simplify the problem and reduce its dimensions. The simplified problem can be reformulated to an 0-1 mixed integer programming problem. Other unknown variables can be found by solving the current problem.

Keywords: Bipolar fuzzy relation equation, Quadratic programming, max-product composition, mixed integer programming problem.

1 Introduction

The system of Fuzzy Relation Equations (FREs) were firstly introduced by Sanchez [29]. Since then, many practical problems were formulated in terms of the system [27, 28, 38, 40, 41, 42]. The problems motivated researchers to study FREs and their related issues. Determining its consistency and solution set is the most important issues. As it is well-known, the consistency is checked by the maximum solution in a polynomial time and its complete solution set is determined by the maximum solution and a finite number of minimal solutions which finding all of them is an NP-hard problem [10]. One of the interesting topics in the area of FREs is the optimization of an objective function subject to a system of FREs. The linear programming problem with FRE constraints was firstly studied by Fang and Li [11]. Some researchers improved their method by some simplification rules [16, 32, 33, 34]. Since most engineering systems and natural phenomena have nonlinear behavior, we cannot formulate them in terms of linear objective functions. Hence, some researchers focussed on the problem with a nonlinear objective function with some metaheuristic algorithms [17, 19, 25] and some exact algorithms were presented for different classes of nonlinear objective functions such as linear fractional [3, 20, 35], geometric [31, 36, 39, 41, 43, 44, 45], and quadratic [1, 2]. Many practical problems in the areas of economics, inventory management, portfolio selection, engineering design, and molecular study can be formulated in terms of the quadratic programming problem. The problem with the max-product fuzzy relation inequality constraints has been studied in [1]. In [1], some sufficient conditions were given to find its optimal solution or some of its optimal components in terms of the maximum solution or the minimal solutions of Fuzzy Relation Inequalities (FRIs) or their components. Also, some simplification procedures were proposed to accelerate its resolution by removing the components having no effect on the solution process. In [2], the problem was converted to a separable programming problem using the properties of square real symmetric indefinite matrices, Cholesky's decomposition, and the least square technique and a closed form was given for its optimal solution. In [1, 2], the authors analyzed the problem in some special cases. Also,

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some simplification rules were proposed to reduce its dimension. Then, they designed an algorithm to solve the problem based on the rules, the branch-and-bound method, and a numerical algorithm. The above FREs and FRIs are increasing with respect to the variables. In some economics applications, we need FRE systems which containing both the decision variable vector and its negation, simultaneously. These systems are called Bipolar FREs (BFREs). Freson et al. [12] firstly studied the linear programming subject to the max-min BFRE constraints. They decomposed the BFREs into several FRE and FRI systems and detected the set of effective elements to optimize the objective function. This process can be very time-consuming and complicated. Li and Liu [22] considered the problem with max-Lukasewiz and proposed an equivalent integer linear programming problem to it using the results in [21]. Aliannezhadi et al. [6] improved their method by some simplification procedures. Liu et al. [24] applied a simple value matrix with some simplification rules to solve the problem. Only two papers [1, 2] have discussed the quadratic programming problem with the max-product fuzzy relation inequality constraints. Due to the nonlinear behavior of natural phenomena and engineering systems, a monomial geometric programming provided to bipolar max-product FRE constraints was presented by Aliannezhadi et al. [7]. They analyzed its feasible domain based on FREs and FRIs and found the effective elements to optimize the problem. The nonlinear programming with bipolar FREs using the max-Lukasiewicz was considered by Zhou et al. [46]. The problem was equivalently converted to an 0-1 mixed integer programming problem and solved by classic integer programming techniques which contain high computational complexity. To increase the computational efficiency, we can consider the nonlinear programming problems in the different classes such as geometric programming, quadratic programming, linear fractional programming, and so on. Among these classes, the geometric programming with BFRE constraint has been investigated in [4, 5]. The quadratic programming problem subject to BFRE system has not been studied up to now. Due to the importance of the quadratic programming and bipolar fuzzy relation equations in the practical problems and theoretical, we are motivated to study the quadratic programming subject to BFREs with the max-product composition operator. The structure of feasible domain in [1, 2] is determined by a unique maximal solution and a finite number of minimal solutions. The structure of feasible domain in this paper is determined by a finite set of maximal and minimal solution pairs. Therefore, the structure of its feasible domain is completely different with the structure of feasible domain of the problem in [1, 2]. Hence, we cannot apply the sufficient conditions in them to simplify and reduce the problem in this paper and the algorithms in [1, 2] cannot be used to find its optimal solution. Five sufficient conditions have been presented to determine one of its optimal solutions without resolution of the problem in terms of Lemmas 1,2, Corollary 3, Theorems 4 and 5 in [1]. The sufficient conditions given in Lemma 1, Corollary 3, Theorem 4, and Theorem 5 from Ref. [1] were presented on the basis of the unique maximal solution of the system of FRIs, which is computed by an explicit formula in polynomial time. In the system of BFREs, the maximal solutions are not necessarily unique, in a general case, and we cannot compute them by the formula. The sufficient conditions expressed in Lemma 2 from Ref. [1] were given based on being positive coefficients of the objective function and the minimal solutions. We will not use this property in any sufficient conditions because the problem of computation of minimal solutions is an NP-hard, in a general case. We express the sufficient conditions in terms of lower and upper bound vector which their computations can be done in polynomial time. Moreover, the model and closed form in [2] were expressed based on the unique maximal solution and the resulted optimal solution was approximate. As we mentioned previously, the maximal solutions of BFRE system are not necessarily unique, in a general case, and the approach cannot apply to solve the proposed problem. Hence, the sufficient conditions and points in [1, 2] cannot apply to determine one of the optimal solutions or solve the proposed problem. Therefore, it is necessary to present some sufficient conditions based on the structure of the objective function and the system of BFRE and simplify the problem. Doing this work, its optimal solution or some its optimal components can be found without its resolution, directly. The simplified problem is converted to an 0-1 mixed integer programming problem and can be solved by classic integer optimization techniques. The proposed method in this paper presents the exact optimal solution. The structure of this paper is organized as follows. In Section 2, the quadratic programming problem with BFRE constraints is formulated and the structure of its feasible domain is discussed. Section 3 presents some sufficient conditions for simplification of the problem. Section 4 designs an algorithm to solve the simplified problem. An application example and some numerical examples are given to illustrate the algorithm and its importance in Section 5. Finally, conclusions and future researches are presented in Section 6.

2 Formulation of the quadratic programming problem with BFRE constraints and its feasible domain structure

Let $A^+ = [a^+_{ij}]$ and $A^- = [a^-_{ij}]$ be two $m \times n$ fuzzy relation matrices with $0 \leq a^+_{ij}, a^-_{ij} \leq 1$ for each $i \in I = \{1, 2, \ldots, m\}$ and $j \in J = \{1, 2, \ldots, n\}$. Also, assume that $b = [b_1, \ldots, b_m]^T \in [0, 1]^m$. Furthermore, $c = [c_1, \ldots, c_n]$ is a vector of cost coefficients and $Q = [q_{ij}]$ is the $n \times n$ matrix of the quadratic form. In this section, we consider the following problem:

$$\min \quad Z(x) = c.x + \frac{1}{2}x^T Qx, \tag{1}$$

$$\text{s.t.} \quad A^+ \circ x \vee A^- \circ \neg x = b, \tag{2}$$

$$x \in [0,1]^n, \tag{3}$$

Its feasible domain is to find the vectors of $x = [x_j]_{n\times 1} \in [0,1]^n$ such that

$$\max_{j\in J} \max\left\{a_{ij}^+.x_j, a_{ij}^-.(1-x_j)\right\} = b_i, \quad \forall i \in I. \tag{4}$$

The system (4) is equivalent to the following two conditions:

(*i*) $\max\left\{a_{ij}^+.x_j, a_{ij}^-.(1-x_j)\right\} \le b_i$ for all $i \in I$ and $j \in J$, and

(*ii*) For each $i \in I$ there exists an index $j_i \in J$ such that $\max\left\{a_{ij_i}^+.x_{j_i}, a_{ij_i}^-.(1-x_{j_i})\right\} = b_i$.

The solution set of system (4) is denoted by $S(A^+, A^-, b) = \{x \in [0,1]^n \mid A^+ \circ x \vee A^- \circ \neg x = b\}$. If $S(A^+, A^-, b) \neq \emptyset$, its solution set is bounded by the lower and upper bound vectors $\check{x} = (\check{x}_1, \dots, \check{x}_n)^T$ and $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n)^T$, respectively, where $\check{x}_j = \max_{i\in I}\{1 - \frac{b_i}{a_{ij}^-} \mid a_{ij}^- > b_i\}$ and $\hat{x}_j = \min_{i\in I}\{\frac{b_i}{a_{ij}^+} \mid a_{ij}^+ > b_i\}$, for each $j \in J$. Also, $\max\emptyset = 0$ and $\min\emptyset = 1$ are defined. If $x \in S(A^+, A^-, b)$, then $\check{x} \le x \le \hat{x}$, but its converse is not necessarily true [4]. We now express two special cases in the following lemmas.

Lemma 2.1. [4] *Suppose that $S(A^+, A^-, b) \neq \emptyset$. (I) If an index $j_0 \in J$ exists such that $\check{x}_{j_0} = \hat{x}_{j_0}$, then $x_{j_0} = \check{x}_{j_0} = \hat{x}_{j_0}$, for all $x \in S(A^+, A^-, b)$. Moreover, let $\overline{I} = \left\{i \in I \mid \max\{a_{ij_0}^+.x_{j_0}, a_{ij_0}^-.(1-x_{j_0})\} = b_i\right\}$ and assume that $\overline{I} \neq \emptyset$. Then the solution sets of system (4) with $\check{x}_j \le x_j \le \hat{x}_j$, for each $j \in J$ is the same to the following system:*

$$\begin{cases} \max\limits_{j\in J-\{j_0\}} \max\left\{a_{ij}^+.x_j, a_{ij}^-.(1-x_j)\right\} = b_i, & \forall i \in I - \overline{I}, \\ \check{x}_j \le x_j \le \hat{x}_j, & \forall j \in J - \{j_0\}; \quad x_{j_0} = \check{x}_{j_0} = \hat{x}_{j_0}, \\ \check{x}_j \text{ and } \hat{x}_j, \text{ for all } j \in J, \text{ are defined based on system (4).} \end{cases}$$

(II) Let $I_0 = \{i \in I \mid b_i = 0\}$. Then the solution set of system (4) with $\check{x}_j \le x_j \le \hat{x}_j$, for each $j \in J$, is the same to the following system:

$$\begin{cases} \max\limits_{j\in J} \max\{a_{ij}^+.x_j, a_{ij}^-.(1-x_j)\} = b_i, & \forall i \in I - I_0, \\ \check{x}_j \le x_j \le \hat{x}_j, & \forall j \in J, \\ \check{x}_j \text{ and } \hat{x}_j, \text{ for all } j \in J, \text{ are defined based on system (4).} \end{cases}$$

Without loss of generality, it is assumed that $\check{x}_j < \hat{x}_j$, for each $j \in J$, and $b_i > 0$, for each $i \in I$, with regard to Lemma 2.1. In the system (4), each equation can be held by $\check{x}_j$ or $\hat{x}_j$. Hence, the characteristic matrix is defined to focus on those values as follows.

Definition 2.2. [4] *(I) Define a characteristic matrix $\tilde{Q} = (\tilde{q}_{ij})_{m\times n}$ for the system of (4) where*

$$\tilde{q}_{ij} = \begin{cases} \{\check{x}_j\}, & \text{if } a_{ij}^-.(1-\check{x}_j) = b_i \neq a_{ij}^+.\hat{x}_j, \\ \{\hat{x}_j\}, & \text{if } a_{ij}^-.(1-\check{x}_j) \neq b_i = a_{ij}^+.\hat{x}_j, \\ \{\check{x}_j, \hat{x}_j\}, & \text{if } a_{ij}^-.(1-\check{x}_j) = b_i = a_{ij}^+.\hat{x}_j, \\ \emptyset, & \text{otherwise,} \end{cases} \tag{5}$$

for each $i \in I$ and $j \in J$.
(II) Define two matrices $Q^+ = (q_{ij}^+)_{m\times n}$ and $Q^- = (q_{ij}^-)_{m\times n}$ where

$$q_{ij}^+ = \begin{cases} 1, & \text{if } \hat{x}_j \in \tilde{q}_{ij}, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad q_{ij}^- = \begin{cases} 1, & \text{if } \check{x}_j \in \tilde{q}_{ij}, \\ 0, & \text{otherwise,} \end{cases} \tag{6}$$

for each $i \in I$ and $j \in J$.
(III) Define the following index sets based on matrices Q^+ and Q^- as follows:

$$I_j^+(x) = \{i \in I \mid x_j = \hat{x}_j \text{ and } q_{ij}^+ = 1\} \text{ and } J_i^+(x) = \{j \in J \mid x_j = \hat{x}_j \text{ and } q_{ij}^+ = 1\}.$$

$$I_j^-(x) = \{i \in I | x_j = \check{x}_j \text{ and } q_{ij}^- = 1\} \text{ and } J_i^-(x) = \{j \in J | x_j = \check{x}_j \text{ and } q_{ij}^- = 1\},$$

for each $i \in I$ and $j \in J$. Moreover, let $I_j(x) = I_j^+(x) \cup I_j^-(x)$, for each $j \in J$. Also, let $I_j^+ = I_j^+(\hat{x})$, $J_i^+ = J_i^+(\hat{x})$, $I_j^- = I_j^-(\check{x})$, and $J_i^- = J_i^-(\check{x})$, for each $i \in I$ and $j \in J$.

The following theorem presents some conditions to check the consistency of system (4). In the theorem, the values of $\hat{x}_j$ and $\check{x}_j$ are labeled with the positive literal y_j and the negative literal $\neg y_j$, respectively, which means that $x_j = \hat{x}_j$ and $x_j = \check{x}_j$ imply $y_j = 1$ and $y_j = 0$, respectively, and vice versa.

Theorem 2.3. [21, 4] *A system of bipolar max-product FREs (4) is consistent if and only if its characteristic boolean formula $C = \bigwedge_{i \in I} C_i$ is well-defined and satisfiable, where $C_i = \bigvee_{j \in J_i^+} y_j \vee \bigvee_{j \in J_i^-} \neg y_j$ and $y_j, \neg y_j \in \{0, 1\}$.*

Throughout this paper, it is assumed that $S(A^+, A^-, b) \neq \emptyset$.

3 Some sufficient conditions for simplification of problem (1)-(3)

In this section, some sufficient conditions are proposed to determine one of the optimal solutions or some its components in problem (1)-(3) without its resolution, directly. At first, we provide a number of sufficient conditions for the objective function to be decreasing or increasing using the components of the lower and upper bound vector of the feasible domain in problem (1)-(3), respectively.

Theorem 3.1. *Suppose that $S(A^+, A^-, b) \neq \emptyset$. Let $\hat{x}$ be the upper bound on the solution set of system (2)-(3) and $x^* = (x_j^*)_{j \in J}$ be an optimal solution of problem (1)-(3). If there exists $k \in J$ such that $-2c_k > \sum_{i=1, i \neq k}^{n} (q_{ik}^+ + q_{ki}^+)\hat{x}_i + 2q_{kk}^+ \hat{x}_k$, then $Z(x)$ is a decreasing function with respect to x_k.*

Proof. Since

$$\begin{aligned}
\sum_{i=1, i \neq k}^{n} (q_{ik}^+ + q_{ki}^+)\hat{x}_i + 2q_{kk}^+.\hat{x}_k &\geq \sum_{i=1, i \neq k}^{n} (q_{ik}^+ + q_{ki}^+)x_i + 2q_{kk}^+.x_k \\
&\geq \sum_{i=1, i \neq k}^{n} (q_{ik}^+ + q_{ki}^+)x_i + 2q_{kk}^+.x_k + \sum_{i=1, i \neq k}^{n} (q_{ik}^- + q_{ki}^-)x_i + 2q_{kk}^-.x_k \\
&= \sum_{i=1, i \neq k}^{n} (q_{ik} + q_{ki})x_i + 2q_{kk}.x_k,
\end{aligned}$$

with regard to the assumption, we have $\frac{\partial Z}{\partial x_k} < 0$ and $Z(x)$ is a decreasing function with respect to x_k. □

Theorem 3.2. *Suppose that $S(A^+, A^-, b) \neq \emptyset$. Let $\hat{x}$ be the upper bound on the solution set of system (2)-(3). If there exists $k \in J$ such that $-2c_k < \sum_{i=1, i \neq k}^{n} (q_{ik}^- + q_{ki}^-)\hat{x}_i + 2q_{kk}^- \hat{x}_k$, then $Z(x)$ is an increasing function with respect to x_k.*

Proof. Since

$$\begin{aligned}
\sum_{i=1, i \neq k}^{n} (q_{ik}^- + q_{ki}^-)\hat{x}_i + 2q_{kk}^-.\hat{x}_k &\leq \sum_{i=1, i \neq k}^{n} (q_{ik}^- + q_{ki}^-)x_i + 2q_{kk}^-.x_k \\
&\leq \sum_{i=1, i \neq k}^{n} (q_{ik}^- + q_{ki}^-)x_i + 2q_{kk}^-.x_k + \sum_{i=1, i \neq k}^{n} (q_{ik}^+ + q_{ki}^+)x_i + 2q_{kk}^+.x_k \\
&= \sum_{i=1, i \neq k}^{n} (q_{ik} + q_{ki})x_i + 2q_{kk}.x_k
\end{aligned}$$

with regard to the assumption, we have $\frac{\partial Z}{\partial x_k} > 0$. Hence, the function $Z(x)$ is an increasing function with respect to x_k. □

Now, we are ready to present some stronger sufficient conditions to be decreasing or increasing the objective function of problem (1)-(3). These points are expressed in two following Theorems.

Theorem 3.3. *Suppose that $S(A^+, A^-, b) \neq \emptyset$. Let $\check{x}$ and $\hat{x}$ be the lower and upper bound vector on the solution set of system (2)-(3). If there exists $k \in J$ such that*

$$-2c_k > \sum_{i=1, i\neq k}^{n} (q_{ik}^+ + q_{ki}^+)\hat{x}_i + \sum_{i=1, i\neq k}^{n} (q_{ik}^- + q_{ki}^-)\check{x}_i + 2q_{kk}^+\hat{x}_k + 2q_{kk}^-\check{x}_k,$$

where $q_{ij}^+ = max\{0, q_{ij}\}$ and $q_{ij}^- = min\{0, q_{ij}\}$, for all $i, j \in J$, then $Z(x)$ is a decreasing function with respect to x_k.

Proof. For any $k \in J$, we have $\frac{\partial Z}{\partial x_k} = c_k + \frac{1}{2}(\sum_{i=1, i\neq k}^{n} (q_{ik} + q_{ki})x_i + 2q_{kk}.x_k)$. Since

$$\begin{aligned}
&\sum_{i=1, i\neq k}^{n} (q_{ik}^+ + q_{ki}^+)\hat{x}_i + \sum_{i=1, i\neq k}^{n} (q_{ik}^- + q_{ki}^-)\check{x}_i + 2q_{kk}^+.\hat{x}_k + 2q_{kk}^-.\check{x}_k \\
\geq &\sum_{i=1, i\neq k}^{n} (q_{ik}^+ + q_{ki}^+)x_i + \sum_{i=1, i\neq k}^{n} (q_{ik}^- + q_{ki}^-)x_i + 2q_{kk}^+.x_k + 2q_{kk}^-.x_k \\
= &\sum_{i=1, i\neq k}^{n} [(q_{ik}^- + q_{ik}^+)x_i + (q_{ki}^- + q_{ki}^+)x_i] + 2(q_{kk}^+ + q_{kk}^-).x_k \\
= &\sum_{i=1, i\neq k}^{n} (q_{ik} + q_{ki})x_i + 2q_{kk}.x_k,
\end{aligned}$$

with regard to the assumption, we have $\frac{\partial Z}{\partial x_k} < 0$ or equivalently $Z(x)$ is a decreasing function with respect to x_k. □

Corollary 3.4. *If there exists $k \in J$ such that the conditions of Theorem 3.3 are satisfied then the conditions of Theorem 3.1 are true.*

Proof. Its reason is as follows:

$$\sum_{i=1, i\neq k}^{n} (q_{ik}^+ + q_{ki}^+)\hat{x}_i + 2q_{kk}^+.\hat{x}_k \geq \sum_{i=1, i\neq k}^{n} (q_{ik}^+ + q_{ki}^+)\hat{x}_i + \sum_{i=1, i\neq k}^{n} (q_{ik}^- + q_{ki}^-)\check{x}_i + 2q_{kk}^+.\hat{x}_k + 2q_{kk}^-.\check{x}_k.$$

With regard to the above inequalities, it is possible that for some $k \in J$, the conditions of Theorem 3.1 are satisfied but the conditions of Theorem 3.3 are not held. □

Remark 3.5. *With regard to the structure of conditions of Theorem 3.3, we can first check the conditions. If the conditions were not met then we can examine the conditions of Theorem 3.1.*

Theorem 3.6. *Suppose that $S(A^+, A^-, b) \neq \emptyset$. Let $\check{x}$ and $\hat{x}$ be the lower and upper bound on the solution set of system (2)-(3). If there exists $k \in J$ such that*

$$-2c_k > \sum_{i=1, i\neq k}^{n} (q_{ik}^+ + q_{ki}^+)\check{x}_i + \sum_{i=1, i\neq k}^{n} (q_{ik}^- + q_{ki}^-)\hat{x}_i + 2q_{kk}^+\check{x}_k + 2q_{kk}^+\hat{x}_k,$$

where $q_{ij}^+ = max\{0, q_{ij}\}$ and $q_{ij}^- = min\{0, q_{ij}\}$, for all $i, j \in J$, then $Z(x)$ is an increasing function with respect to x_k.

Proof. Similar to the proof of Theorem 3.1, since

$$\begin{aligned}
&\sum_{i=1, i\neq k}^{n} (q_{ik}^+ + q_{ki}^+)\check{x}_i + \sum_{i=1, i\neq k}^{n} (q_{ik}^- + q_{ki}^-)\hat{x}_i + 2q_{kk}^+.\check{x}_k + 2q_{kk}^-.\hat{x}_k \\
\leq &\sum_{i=1, i\neq k}^{n} (q_{ik}^+ + q_{ki}^+)x_i + \sum_{i=1, i\neq k}^{n} (q_{ik}^- + q_{ki}^-)x_i + 2q_{kk}^+.x_k + 2q_{kk}^-.x_k \\
= &\sum_{i=1, i\neq k}^{n} (q_{ik} + q_{ki})x_i + 2q_{kk}.x_k,
\end{aligned}$$

with regard to the assumption, we have $\frac{\partial Z}{\partial x_k} > 0$. Hence, $Z(x)$ is an increasing function with respect to x_k. □

The following corollary is a direct result of the above discussions.

Corollary 3.7. *(I) If the objective function $Z(x)$ with respect to x_k, for each $k \in J$, is a decreasing function and the upper bound vector $\hat{x}$ is a feasible solution, then $x^* = \hat{x}$.*
(II) If the objective function $Z(x)$ with respect to x_k, for each $k \in J$, is an increasing function and the lower bound vector $\check{x}$ is a feasible solution, then $x^* = \check{x}$.

We intend to study a useful property from problem (1)-(3) in the following theorem.

Theorem 3.8. *Suppose that $S(A^+, A^-, b) \neq \emptyset$. Let $\check{x}$ and $\hat{x}$ be the lower and upper bound vector on the solution set of system (2)-(3). If there exists $k \in J$ such that $Z(x)$ is a decreasing or increasing function with respect to x_k, then there exists an optimal solution $x^* = (x_j^*)_{j \in J}$ for problem (1)-(3) such that $x_k^* = \hat{x}_k$ or $x_k^* = \check{x}_k$, respectively.*

Proof. Assume that x^* is an optimal solution of problem (1)-(3) and there is an index $k \in J$ such that $\check{x}_k < x_k^* < \hat{x}_k$. This vector remains feasible if the component of x_k^* is increased to $\hat{x}_k$ or decreased to $\check{x}_k$. If the objective function $Z(x)$ is decreasing with respect to x_k, then the vector x^{**} is defined as follows:

$$x_j^{**} = \begin{cases} \hat{x}_j & j = k, \\ x_j^*, & j \neq k, \end{cases}$$

for each $j \in J$. The definition of vector x^{**} implies that $x^* \leq x^{**}$ and $x^* \neq x^{**}$. The function Z is decreasing with respect to x_k. So, we have $Z(x^*) \geq Z(x^{**})$. Since vector x^* is an optimal solution for problem (1)-(3), $Z(x^*) > Z(x^{**})$ cannot occur. Hence, we must have $Z(x^{**}) = Z(x^*)$. Therefore, vector x^{**} is also an optimal solution such that $x_k^{**} = \hat{x}_k$.
If the objective function $Z(x)$ is increasing with respect to x_k, then the vector x^{**} is defined as follows:

$$x_j^{**} = \begin{cases} \check{x}_j & j = k, \\ x_j^*, & j \neq k, \end{cases}$$

for each $j \in J$. With regard to the definition of vector x^{**}, we have $x^{**} \leq x^*$. Since the function $Z(x)$ is increasing with respect to x_k, $Z(x^{**}) \leq Z(x^*)$. The expression $Z(x^{**}) < Z(x^*)$ cannot occur because the vector x^* is an optimal solution for problem (1)-(3). So, we must have $Z(x^{**}) = Z(x^*)$. Hence, the vector x^{**} is an optimal solution for problem (1)-(3) such that $x_k^{**} = \check{x}_k$. □

Now, we define the following sets: $\bar{K} = \{j \in J \mid Z(x) \;\; \textit{is increasing with respect to } x_j\}$ and $\underline{K} = \{j \in J \mid Z(x) \;\; \textit{is decreasing with respect to } x_j\}$, where $\bar{K} \bigcap \underline{K} = \emptyset$.
Now, we are ready to present some sufficient conditions which under them, we set $x_j^* = \check{x}_j$, for each $j \in \underline{K}$, and $x_j^* = \hat{x}_j$, for each $j \in \bar{K}$.

Lemma 3.9. *If the following conditions hold:*
1) $\underline{K} \bigcup \bar{K} = J$,
2) $(\bigcup_{j \in \bar{K}} I_j^-) \bigcup (\bigcup_{j \in \underline{K}} I_j^+) = I$,
then the optimal solution $x^* = (x_j^*)_{j \in J}$ of problem (1)-(3) can be obtained as follows:

$$x_j^* = \begin{cases} \check{x}_j, & j \in \bar{K}, \\ \hat{x}_j, & j \in \underline{K}. \end{cases}$$

Proof. The vector x^* is a feasible solution for problem (1)-(3) according to condition 2. Therefore, it is enough to show that $Z(x^*) \leq Z(x)$ for each $x \in S(A^+, A^-, b)$. With regard to the condition 1 and the definitions of sets $\underline{K}$ and $\bar{K}$, for each $x \in S(A^+, A^-, b)$, we have:

$$Z(x) = Z(x_1, x_2, ..., x_n) = Z((x_p)_{p \in \underline{K}}, (x_p)_{p \in \bar{K}}) \geq Z((\hat{x}_p)_{p \in \underline{K}}, (x_p)_{p \in \bar{K}}) \geq Z((\hat{x}_p)_{p \in \underline{K}}, (\check{x}_p)_{p \in \bar{K}}) = Z(x^*).$$

□

The following lemma shows the structure of the optimal solution of problem (1)-(3) where $\underline{K} \bigcup \bar{K} \subset J$ and $\underline{K} \bigcup \bar{K} \neq J$.

Lemma 3.10. *If the following conditions hold:*
1) $\underline{K}\bigcup\bar{K}\subset J$, *where* $\underline{K}\bigcup\bar{K}\neq J$
2) $(\bigcup_{j\in\bar{K}}I_j^-)\bigcup(\bigcup_{j\in\underline{K}}I_j^+)=I$,
then there exists an optimal solution $x^*=(x_j^*)_{j\in J}$ *for problem* (1)-(3) *such that for each* $j\in\bar{K}\bigcup\underline{K}$, *we have*

$$x_j^*=\begin{cases}\check{x}_j, & j\in\bar{K},\\ \hat{x}_j, & j\in\underline{K}.\end{cases} \tag{7}$$

Proof. According to condition 2, there exists an optimal solution $x^*=(x_j^*)_{j\in J}$ such that its components, for $j\in\bar{K}\bigcup\underline{K}$, are the same to relation (7) because for each $x\in S(A^+,A^-,b)$, we have

$$\begin{aligned}Z(x)&=Z((x_j)_{j\in\bar{K}},(x_j)_{j\in\underline{K}},(x_j)_{j\in J-(\underline{K}\bigcup\bar{K})})\\&\geq Z((\check{x}_j)_{j\in\bar{K}},(x_j)_{j\in\underline{K}},(x_j)_{j\in J-(\underline{K}\bigcup\bar{K})})\\&\geq Z((\check{x}_j)_{j\in\bar{K}},(\hat{x}_j)_{j\in\underline{K}},(x_j)_{j\in J-(\underline{K}\bigcup\bar{K})}).\end{aligned}$$

The above inequalities hold with regard to the definitions of sets $\bar{K}$ and $\underline{K}$. So, there exists an optimal solution $x^*=(x_j^*)_{j\in J}$ such that its components, for $j\in\underline{K}\bigcup\bar{K}$, are the same to relation (7). □

The following corollary is a direct result of the above theorem.

Corollary 3.11. *Suppose that* $(\bigcup_{j\in\bar{K}}I_j^-)\bigcup(\bigcup_{j\in\underline{K}}I_j^+)=I$. *Then the problem* (1)-(3) *has an optimal solution* $x^*=(x_j^*)_{j\in J}$ *as follows:* $x_j^*=\begin{cases}\check{x}_j, & j\in\bar{K},\\ \hat{x}_j, & j\in\underline{K},\end{cases}$ $\forall j\in\bar{K}\bigcup\underline{K}$. *Moreover, the problem* (1)-(3) *is simplified as follows:*

$$\begin{aligned}\min\quad & Z(x),\\ \text{s.t.}\quad & x\in S(A^+,A^-,b).\end{aligned}$$

where $x=(x_j)_{j\in J'\bigcup K}$, $J'=J-(\bar{K}\bigcup\underline{K})$, *and* $K=\bar{K}\bigcup\underline{K}$.

Proof. Since $(\bigcup_{j\in\bar{K}}I_j^-)\bigcup(\bigcup_{j\in\underline{K}}I_j^+)=I$, the system (2)-(3) is obviously feasible. With regard to the definitions of sets $\bar{K}$ and $\underline{K}$, we can easily set for $j\in\bar{K}$, the optimal component $x_j^*=\check{x}_j$ and for $j\in\underline{K}$, the optimal component $x_j^*=\hat{x}_j$. By setting the values x_j^*, for $j\in\bar{K}\bigcup\underline{K}$, the simplified problem is obtained. This completes the proof. □

The following lemma presents some sufficient conditions to determine some the optimal components of problem (1)-(3) based on the set $\bar{K}$.

Lemma 3.12. *Suppose that* $S(A^+,A^-,b)\neq\emptyset$. *If* $\bigcup_{j\in\bar{K}_1}I_j^+\subseteq\bigcup_{j\in\bar{K}_1}I_j^-$, *where* $\bar{K}_1\subseteq\bar{K}$, *then there exists an optimal solution for problem* (1)-(3) *such that for each* $j\in\bar{K}_1$, *we have* $x_j^*=\check{x}_j$.

Proof. Since $\bigcup_{j\in\bar{K}_1}I_j^+\subseteq\bigcup_{j\in\bar{K}_1}I_j^-$, all the constraints of $i\in\bigcup_{j\in\bar{K}_1}I_j^-$ as well as all the constraints of $i\in\bigcup_{j\in\bar{K}_1}I_j^+$ are satisfied by setting $x_j^*=\check{x}_j$, for each $j\in\bar{K}_1$. On the other hand, for each $j\in\bar{K}_1$, $x_j^*=\check{x}_j$, implies that the objective function $Z(x)$ is minimized with respect to the variables x_j according to the definition of set $\bar{K}$. □

The next lemma proposes the sufficient conditions based on the set $\underline{K}$.

Lemma 3.13. *Suppose that* $S(A^+,A^-,b)\neq\emptyset$. *If* $\bigcup_{j\in\underline{K}_1}I_j^-\subseteq\bigcup_{j\in\underline{K}_1}I_j^+$, *where* $\underline{K}_1\subseteq\underline{K}$, *then there exists an optimal solution* $x^*=(x_j^*)_{j\in J}$ *for problem* (1)-(3) *such that* $x_j^*=\hat{x}_j$ *for each* $j\in\underline{K}_1$.

Proof. Similar to the proof of Lemma 3.12, since $\bigcup_{j\in\underline{K}_1}I_j^-\subseteq\bigcup_{j\in\underline{K}_1}I_j^+$, all the constraints of $i\in\bigcup_{j\in\underline{K}_1}I_j^+$ as well as all the constraints of $i\in\bigcup_{j\in\underline{K}_1}I_j^-$ are satisfied when we let $x_j^*=\hat{x}_j$, for each $j\in\underline{K}_1$. Furthermore, $x_j^*=\hat{x}_j$, for each $j\in\underline{K}_1$, minimizes the objective function $Z(x)$ with respect to the variables x_j according to the definition of set $\underline{K}$. □

Some sufficient conditions are given to detect one of the optimal components of problem (1)-(3) in the following lemma.

Lemma 3.14. *Suppose that $S(A^+, A^-, b) \neq \emptyset$. If there exists $i_0 \in I$ and $j_0 \in \bar{K}$ such that the following conditions are satisfied:*
1. $q^+_{i_0 j_0} = 0$,
2. $I^+_{j_0} - I^-_{j_0} \subseteq I^+_j$, *for each* $j \in J^+_{i_0}$,
3. $I^+_{j_0} - I^-_{j_0} \subseteq I^-_j$, *for each* $j \in J^-_{i_0}$,
then $x^*_{j_0} = \check{x}_{j_0}$.

Proof. If $q^-_{i_0 j_0} = 0$, then similar to Rule 3 in [6], we can show that $x^*_{j_0} = \check{x}_{j_0}$, for $j_0 \in \bar{K}$. Otherwise, $q^-_{i_0 j_0} = 1$. Then $j_0 \in J^-_{i_0}$. Then condition 3 implies that $I^+_{j_0} - I^-_{j_0} \subseteq I^-_{j_0}$. Hence, we have $I^+_{j_0} \subseteq I^-_{j_0}$. Therefore, it is concluded that $x^*_{j_0} = \check{x}_{j_0}$. □

Lemma 3.15. *Suppose that $S(A^+, A^-, b) \neq \emptyset$. If there exists $i_0 \in I$ and $j_0 \in \underline{K}$ such that the following conditions are satisfied:*
1. $q^-_{i_0 j_0} = 0$,
2. $I^-_{j_0} - I^+_{j_0} \subseteq I^+_j$, *for each* $j \in J^+_{i_0}$,
3. $I^-_{j_0} - I^+_{j_0} \subseteq I^-_j$, *for each* $j \in J^-_{i_0}$,
then $x^*_{j_0} = \hat{x}_{j_0}$.

Proof. The proof is similar to the proof of Lemma 3.14. □

Now, we are ready to design an algorithm to solve problem (1)-(3) based on the above points and concepts in the next section.

4 An algorithm for solving problem (1)-(3)

In this section, we will focuss on the resolution of problem (1)-(3). First of all, we simplify the problem using the sufficient conditions mentioned in Section 3. If all its optimal variables are found, then the problem has completely been solved. Otherwise, we will solve the simplified problem by converting it to an 0-1 mixed integer quadratic programming problem. The current problem is solved by the classic integer optimization techniques. Suppose that after simplifying the problem (1)-(3), the optimal values of variables x^*_j, for each $j \in P$, have been determined where P is a subset of set J and the optimal values of variables x_j, for each $j \in J - P$, have been remained. Now, we should solve the following problem:

$$\min \quad Z(x), \tag{8}$$
$$\text{s.t.} \quad x \in S(A^+, A^-, b), \tag{9}$$

where $x = (x_j)_{j \in P \bigcup (J-P)}$. With regard to Theorem 1 in [46], the problem (8)-(9) can equivalently be converted to an 0-1 mixed integer programming as follows:

$$\min \quad Z(x), \tag{10}$$
$$\text{s.t.} \quad V.U^+ + \check{x} \leq x \leq -V.U^- + \hat{x}, \tag{11}$$
$$Q^+.U^+ + Q^-.U^- \geq e, \tag{12}$$
$$U^+ + U^- \leq e, \tag{13}$$
$$U^+, U^- \in \{0,1\}^n, \tag{14}$$
$$x \in [0,1]^n. \tag{15}$$

where $V = diag(\hat{x} - \check{x})$ and vector e is the vector of all ones. With regard to the above points, an algorithm is designed to solve the problem (1)-(3).

Algorithm 4.1. *An algorithm for solving the problem* (1)-(3).
Step 1. *Calculate the vectors of the lower and upper bound $\check{x}$ and $\hat{x}$ using mentioned relations in Section 2 and put*

$P := \emptyset$.
Step 2. *If $b_i > 0$ for each $i \in I$ and $\check{x}_j < \hat{x}_j$ for each $j \in J$, then go to Step 3. Otherwise, apply Lemma 2.1.*
Step 3. *Compute two characteristic matrices Q^+ and Q^- and the index sets I^+_j, I^-_j, J^+_i, and J^-_i using Definition 2.2.*

Step 4. *Check the consistency of the bipolar system in* (2)-(3) *by Theorem 2.3. If it is inconsistent, then stop!*
Step 5. *Check the following simplifications:*

5.1. *If the function of* $Z(x)$ *with respect to* x_k*, for each* $k \in J$*, is a decreasing function and vector* $\hat{x}$ *is a feasible solution, then according to Corollary 3.7, set* $x^* = \hat{x}$ *and Stop!*

5.2. *If function* $Z(x)$ *with respect to* x_k*, for each* $k \in J$*, is an increasing function and vector* $\check{x}$ *is a feasible solution, then according to Corollary 3.7, set* $x^* = \check{x}$ *and Stop!*

5.3. *Create two sets* $\underline{K}$ *and* $\bar{K}$ *according to the relations in Section 3 such that* $\underline{K} \bigcap \bar{K} = \emptyset$*. If the conditions: (i)* $\underline{K} \bigcup \bar{K} = J$ *and (ii)* $(\bigcup_{j\in\bar{K}} I_j^-) \bigcup (\bigcup_{j\in\underline{K}} I_j^+) = I$ *are satisfied, then according to Lemma 3.9, the optimal solution* $x^* = (x_j^*)_{j\in J}$ *is as follows:* $x_j^* = \check{x}_j$*, for* $j \in \bar{K}$*, and* $x_j^* = \hat{x}_j$*, for* $j \in \underline{K}$*. Stop!*

5.4. *If the conditions* $\underline{K} \bigcup \bar{K} \subset J$ *and* $(\bigcup_{j\in\bar{K}} I_j^-) \bigcup (\bigcup_{j\in\underline{K}} I_j^+) = I$ *are satisfied then according to Lemma 3.10, there exists an optimal solution* $x^* = (x_j^*)_{j\in J}$ *for problem* (1)-(3) *where* $x_j^* = \check{x}_j$*, for* $j \in \bar{K}$*, and* $x_j^* = \hat{x}_j$*, for* $j \in \underline{K}$*. Remove row(s)* $i \in I_j^-$*, for each* $j \in \bar{K}$*,* $i \in I_j^+$*, for each* $j \in \underline{K}$*, and columns* j*, where* $j \in \underline{K} \bigcup \bar{K}$*, from the matrices* Q^+ *and* Q^-*. Let* $P := P \bigcup (\underline{K} \bigcup \bar{K})$*. Update sets* I *and* J*. Also, update sets* $I_j^+, I_j^-, J_i^+, J_i^-$*, for each* $i \in I$ *and* $j \in J$*, and sets* $\underline{K}$ *and* $\bar{K}$*.*

5.5. *If* $(\bigcup_{j\in\bar{K}_1} I_j^+) \subseteq (\bigcup_{j\in\bar{K}_1} I_j^-)$*, for one set* $\bar{K}_1 (\subseteq \bar{K})$*, then according to Lemma 3.12, set* $x_j^* = \check{x}_j$*, for each* $j \in \bar{K}_1$*. Remove row(s)* $i \in I_j^-$*, for each* $j \in \bar{K}_1$*, columns* j *where* $j \in \bar{K}_1$ *from the matrices* Q^+ *and* Q^-*. Let* $P := P \bigcup \bar{K}_1$*. Update sets* I *and* J*. Also, update sets* $I_j^+, I_j^-, J_i^+, J_i^-$*, for each* $i \in I$ *and* $j \in J$*, and sets* $\underline{K}$ *and* $\bar{K}$*.*

5.6. *If* $(\bigcup_{j\in\underline{K}_1} I_j^-) \subseteq (\bigcup_{j\in\underline{K}_1} I_j^+)$*, for one set* $\underline{K}_1 (\subseteq \underline{K})$*, then according to Lemma 3.13, set* $x_j^* = \hat{x}_j$*, for each* $j \in \underline{K}_1$*. Remove row(s)* $i \in I_j^+$*, for each* $j \in \underline{K}_1$*, columns* j *where* $j \in \underline{K}_1$ *from the matrices* Q^+ *and* Q^-*. Let* $P := P \bigcup \underline{K}_1$*. Update sets* I *and* J*. Also, update sets* $I_j^+, I_j^-, J_i^+, J_i^-$*, for each* $i \in I$ *and* $j \in J$*, and sets* $\underline{K}$ *and* $\bar{K}$*.*

5.7. *If there exists* $i_0 \in I$ *and* $j_0 \in \bar{K}$ *such that (1)* $q^+_{i_0 j_0} = 0$*, (2)* $I^+_{j_0} - I^-_{j_0} \subseteq I_j^+$*, for each* $j \in J^+_{i_0}$*, and (3)* $I^+_{j_0} - I^-_{j_0} \subseteq I_j^-$*, for each* $j \in J^-_{i_0}$*, then according to Lemma 3.14, set* $x^*_{j_0} = \check{x}_{j_0}$*. Remove row(s)* $i \in I^-_{j_0}$ *and column* j_0 *from the matrices* Q^+ *and* Q^-*. Let* $P := P \bigcup \{j_0\}$*. Update sets* I *and* J*. Also, update sets* $I_j^+, I_j^-, J_i^+, J_i^-$*, for each* $i \in I$ *and* $j \in J$*, and sets* $\underline{K}$ *and* $\bar{K}$*.*

5.8. *If there exists* $i_0 \in I$ *and* $j_0 \in \underline{K}$ *such that (1)* $q^-_{i_0 j_0} = 0$*, (2)* $I^-_{j_0} - I^+_{j_0} \subseteq I_j^+$*, for each* $j \in J^+_{i_0}$*, and (3)* $I^-_{j_0} - I^+_{j_0} \subseteq I_j^-$*, for each* $j \in J^-_{i_0}$*, then according to Lemma 3.15, set* $x^*_{j_0} = \hat{x}_{j_0}$*. Remove row(s)* $i \in I^+_{j_0}$ *and column* j_0 *from the matrices* Q^+ *and* Q^-*. Let* $P := P \bigcup \{j_0\}$*. Update sets* I *and* J*. Also, update sets* I_j^+, I_j^-, J_i^+*, and* J_i^-*, for each* $i \in I$ *and* $j \in J$*, and sets* $\underline{K}$ *and* $\bar{K}$*.*

It is necessary to remind that if the problem is reduced by one of rules 5.4-5.8, then we should again check the rules 5.4-5.8 for the reduced problem. If the conditions are satisfied, then we can apply their results for further simplifications.
Step 6. *If optimal values of all variables have been determined, then Stop!*
Step 7. *Create the following problem to find other unknown variables:*

$$\min \quad Z(x) \tag{16}$$
$$\text{s.t.} \quad V.U^+ + \check{x} \le x \le -V.U^- + \hat{x}, \tag{17}$$
$$Q^+.U^+ + Q^-.U^- \ge e, \tag{18}$$
$$U^+ + U^- \le e, \tag{19}$$
$$U^+, U^- \in \{0,1\}^n, \tag{20}$$
$$x \in [0,1]^n. \tag{21}$$

where $x = (x_j)_{j\in P \bigcup (J-P)}$*. Apply one of the algorithms of the mixed integer programming to solve this problem.*
Step 8. *End.*

Some examples are presented to illustrate the importance of the problem (1)-(3) and Algorithm 4.1 in the next section. The resolution process of the problem is explained step-by-step in the next section.

5 Application background of the bipolar fuzzy relation equation programming and numerical examples

This section is divided to two subsections. The first subsection explains the application background of the bipolar fuzzy relation equation programming (1)-(3) according to Refs. [12, 37, 24]. The second subsection presents some numerical examples to illustrate Algorithm 4.1 as step-by-step.

5.1 Application background of the bipolar fuzzy relation equation programming

Example 5.1. *Consider two suppliers which provide two categories of products. The first supplier provides the products, denoted by $P_1, P_2, ..., P_r$ and the second supplier provides the products, denoted by $P_{r+1}, P_{r+2}, ..., P_n$.*

The suppliers aim to optimize the public awareness and therefore attribute to all the products a degree of appreciation, $x_1, x_2, ..., x_n$. For the j^{th} product, the degree of appreciation x_j is reflected by a real number in the unit interval $[0, 1]$. Correspondingly, its degree of disappreciation is indeed $\bar{x}_j = 1 - x_j$. The degree of appreciation thus acquires a bipolar character on $[0, 1]$, in the sense that values 0.5 means neutrality with respect to appreciation. For example, assume that the products are promoted in m area or markets, denoted by $A_1, A_2, ..., A_m$. If the degree of appreciation of P_j at the i^{th} market A_i is a^+_{ij} ($a^+_{ij} \in [0, 1]$), then the effective public awareness is $a^+_{ij}.x_j$. Analogously, if the degree of disappreciation of P_j at A_i is a^-_{ij} ($a^-_{ij} \in [0, 1]$), then the degree of disappreciation could be represented by $a^-_{ij}.\bar{x}_j$. As pointed out in [12], "in practice, often bad publicity can be seen as a form of publicity too".

Hence, not only a high degree of appreciation is helpful for public awareness, but also a high degree of disappreciation can in some situations be worthwile. Both these two aspects are effective for publicity of the products. The public awareness of product P_j at market A_i could be represented by $max\{a^+_{ij}.x_j, a^-_{ij}.\bar{x}_j\}$.

Assume further that market research has revealed that public awareness at a level of b_i is the best value for the products to be sold at market A_i. For better product sales, the companies hope the public awareness reaches such best value. For arbitrary market A_i, there should exist at least one product, of which the public awareness reaches the best value b_i. Therefore, the above-mentioned conditions could be formulated as:

$$\max_{j\in J} \max\left\{a^+_{ij}.x_j, a^-_{ij}.(1 - x_j)\right\} = b_i, \quad \forall i \in I.$$

The first supplier is responsible for raising the public awareness about the products $P_1, P_2, ..., P_n$. The cost of promoting each product is proportional to its degree of appreciation. The supplier expends cost $c_j.x_j$ to promote the product P_j, for $j = 1, ..., r$, and earns fee $c_j.x_j$ to promote the product P_j, for $j = r + 1, ..., n$, from the second supplier. Therefore, total his costs are as: $Z = \sum_{j=1}^{r} c_j.x_j - \sum_{j=r+1}^{n} c_j.x_j$. If $Z \geq 0$, then he has expended cost Z. Otherwise, he has earned a salary as $|Z|$. The coefficients c_j, for $j = 1, ..., n$, are not fixed but they are normally distributed random variables due to the uncertainty in the markets. The function Z will thus be random variable with mean $\bar{Z} = \sum_{j=1}^{r} \bar{c}_j.x_j - \sum_{j=r+1}^{n} \bar{c}_j.x_j$ and variance $\sigma^2_Z = x^t.V_Z.x$, where $\bar{c}_j$, for $j \in J$, are the means of c_j, for $j \in J$, and $x = (x_j)_{j\in J}$. Also, the matrix $V_Z = [v_{ij}]_{n\times n}$ is the covariance matrix c_j defined as follows:

$$v_{ij} = \begin{cases} var(c_i) & i = j, \\ cov(c_i, c_j) & i \neq j. \end{cases}$$

It is convenient to normalize the value Z by the function of utility of U so that (1) $U = 0$ for $Z = 0$ and (2) $U = 1$ as Z approaches the value $+\infty$. The function U is called the supplier's utility function and it is usually non-decreasing continuous function.

Different curves can be expressed mathematically for U as: $U(Z) = 1 - e^{-Z}$. The density function ϕ_Z is given by the following function:$\phi_Z(Z) = \frac{1}{\sigma_Z\sqrt{2\pi}} exp(-\frac{1}{2}(\frac{Z-\bar{Z}}{\sigma_Z})^2)$. The supplier want to minimize the expected value of the utility by the following relation:
$\int_{-\infty}^{+\infty}(1 - e^{-Z})\phi_Z(Z)dZ = 1 - \frac{1}{\sigma_Z\sqrt{2\pi}} \int_{-\infty}^{+\infty} exp(-Z - \frac{1}{2}(\frac{Z-\bar{Z}}{\sigma_Z})^2)dZ = 1 - exp(-\bar{Z} + \frac{1}{2}\sigma^2_Z)$.
It is equivalent to minimizing $\bar{Z} - (\frac{1}{2})\sigma^2_Z$. Substituting for $\bar{Z}$ and σ^2_Z, we have:

$$\min \quad Z(x) = \bar{c}^T.x - \frac{1}{2}x^T V_Z x, \tag{22}$$

$$\text{s.t.} \quad A^+ \circ x \vee A^- \circ \neg x = b, \tag{23}$$

$$x \in [0, 1]^n, \tag{24}$$

where $\bar{c} = (\bar{c}_j)_{j\in J}$. Now, we apply the data in Refs. [37, 24] for matrices A^+ and A^- with 10×9 dimensions as follows:

$$A^+ = \begin{pmatrix} 0.18 & 0.15 & 0.12 & 0.25 & 0.22 & 0.35 & 0.21 & 0.12 & 0.31 \\ 0.23 & 0.56 & 0.71 & 0.62 & 0.8 & 0.93 & 0.45 & 0.43 & 0.38 \\ 0.75 & 0.9 & 0.76 & 0.32 & 0.95 & 0.61 & 0.49 & 0.64 & 0.68 \\ 0.43 & 0.56 & 0.72 & 0.57 & 0.81 & 0.19 & 0.8 & 0.38 & 0.47 \\ 0.7 & 0.72 & 0.45 & 0.54 & 0.7 & 0.9 & 0.34 & 0.46 & 0.63 \\ 0.65 & 0.82 & 0.72 & 0.61 & 0.53 & 0.78 & 0.82 & 0.62 & 0.72 \\ 0.42 & 0.43 & 0.58 & 0.7 & 0.67 & 0.8 & 0.33 & 0.45 & 0.26 \\ 0.82 & 0.61 & 0.67 & 0.65 & 0.8 & 0.63 & 0.54 & 0.76 & 0.42 \\ 0.35 & 0.68 & 0.43 & 0.76 & 0.64 & 0.55 & 0.45 & 0.25 & 0.8 \\ 0.45 & 0.46 & 0.48 & 0.36 & 0.7 & 0.45 & 0.52 & 0.32 & 0.77 \end{pmatrix},$$

$$A^- = \begin{pmatrix} 0.23 & 0.2 & 0.17 & 0.3 & 0.27 & 0.4 & 0.26 & 0.17 & 0.36 \\ 0.13 & 0.46 & 0.61 & 0.52 & 0.7 & 0.83 & 0.35 & 0.33 & 0.28 \\ 0.85 & 0.98 & 0.86 & 0.42 & 1 & 0.71 & 0.59 & 0.74 & 0.78 \\ 0.28 & 0.41 & 0.57 & 0.96 & 0.66 & 0.04 & 0.65 & 0.23 & 0.32 \\ 0.8 & 0.8 & 0.55 & 0.64 & 0.8 & 1 & 0.44 & 0.56 & 0.73 \\ 0.57 & 1 & 0.64 & 0.53 & 0.45 & 0.7 & 0.74 & 0.54 & 0.64 \\ 0.54 & 0.55 & 0.7 & 0.82 & 0.79 & 0.92 & 0.45 & 0.57 & 0.38 \\ 0.74 & 0.53 & 0.59 & 0.57 & 0.72 & 0.55 & 0.46 & 0.68 & 0.34 \\ 0.41 & 0.74 & 0.49 & 0.82 & 0.7 & 0.61 & 0.51 & 0.31 & 0.86 \\ 0.58 & 0.59 & 0.61 & 0.49 & 0.83 & 0.58 & 0.65 & 0.45 & 0.9 \end{pmatrix}, \; b = \begin{pmatrix} 0.19 \\ 0.5 \\ 0.54 \\ 0.56 \\ 0.51 \\ 0.57 \\ 0.46 \\ 0.64 \\ 0.46 \\ 0.45 \end{pmatrix}, \; \textit{and } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{pmatrix}.$$

Here, the random variables c_j, for $j \in J$, are independent and $r = 5$. Hence, $cov(c_i, c_j) = 0$, for each i, j, with $i \neq j$. The vector $\bar{c} = [\bar{c}_j]_{9\times 1}$ and the matrix $V_Z = [v_{ij}]_{9\times 9}$ are as follows:

$$\bar{c} = \begin{pmatrix} \bar{c}_1 \\ \bar{c}_2 \\ \bar{c}_3 \\ \bar{c}_4 \\ \bar{c}_5 \\ \bar{c}_6 \\ \bar{c}_7 \\ \bar{c}_8 \\ \bar{c}_9 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 7 \\ 1.5 \\ -6 \\ -5 \\ -2.3 \\ -0.5 \end{pmatrix}, \; \textit{and } V_Z = \begin{pmatrix} 0.64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 \end{pmatrix}.$$

With regard to the above data, the problem is reformulated as follows:

$$\min \quad 2x_1 + 4x_2 + 3x_3 + 7x_4 + 1.5x_5 - 6x_6 - 5x_7 - 2.3x_8 - 0.5x_9 - 0.32x_1^2 \tag{25}$$

$$- x_2^2 - 0.8x_3^2 - 1.5x_4^2 - 0.1x_5^2 - 1.2x_6^2 - 1.7x_7^2 - 0.6x_8^2 - 0.1x_9^2, \tag{26}$$

$$\text{s.t.} \quad A^+ \circ x \vee A^- \circ \neg x = b, \tag{27}$$

$$x \in [0,1]^9, \tag{28}$$

we now apply Algorithm 1 to solve the problem. In computations, the numbers are rounded to two decimal places.
Step 1. *The vectors of lower and upper bound $\check{x}$ and $\hat{x}$ are as follows:*
$\check{x} = (0.36, 0.45, 0.37, 0.44, 0.46, 0.52, 0.31, 0.27, 0.5)^T$ *and* $\hat{x} = (0.6, 0.7, 0.6, 0.57, 0.54, 0.54, 0.7, 0.84, 0.58)^T$, *and* $P := \emptyset$.
Step 2. *In this example, $b_i > 0$, for each $i \in I = \{1, 2, ..., 10\}$, and $\check{x}_j < \hat{x}_j$, for each $j \in J = \{1, 2, ..., 9\}$. Hence, we go to Step 3.*
Step 3. *Two characteristic matrices Q^+ and Q^- are as follows:*

$$Q^+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \textit{and} \quad Q^- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

The index sets I_j^+, I_j^-, J_i^+, and J_i^-, are as follows:

$I_1^+ = \emptyset, I_2^+ = \{6\}, I_3^+ = I_4^+ = I_5^+ = \emptyset, I_6^+ = \{1,2\}, I_7^+ = \{4,6\}, I_8^+ = \{3,8\}, I_9^+ = \{9,10\}$, $I_1^- = \{3,5\}, I_2^- = \{3\}, I_3^- = \{3\}, I_4^- = \{7,9\}, I_5^- = \{3,10\}, I_6^- = \{1\}, I_7^- = \{10\}, I_8^- = \{3\}$, *and* $I_9^- = \{10\}$.
Step 4. *The characteristic Boolean formula corresponding to this example is as:* $C = \bigwedge_{i=1}^{10} C_i$, *where* $C_1 = (\neg y_6) \bigvee y_6$, $C_2 = y_6$, $C_3 = y_8 \bigvee(\neg y_1) \bigvee(\neg y_2) \bigvee(\neg y_3) \bigvee(\neg y_5) \bigvee(\neg y_8)$, $C_4 = y_7$, $C_5 = (\neg y_1)$, $C_6 = y_7 \bigvee y_2$,$C_7 = (\neg y_4)$, $C_8 = y_8$, $C_9 = y_9 \bigvee(\neg y_4)$, *and* $C_{10} = y_9 \bigvee(\neg y_5) \bigvee(\neg y_7) \bigvee(\neg y_9)$. *After simplifying* C, *we have* $C = \bigwedge_{i=1}^{10} C_i = y_6 \bigwedge y_7 \bigwedge(\neg y_1) \bigwedge (\neg y_4) \bigwedge y_8$ *is satisfiable if* $y_6 = y_7 = y_8 = 1$ *and* $y_1 = y_4 = 0$ *are assigned. Hence, the system is consistent.*
Step 5. *The conditions of Substeps 5.1 and 5.2 are not satisfied for this example. Consider two sets* $\underline{K}$ *and* $\overline{K}$ *as follows:* $\underline{K} = \{6,7,8,9\}$ *and* $\overline{K} = \{1,2,3,4,5\}$. *The objective function can be rewritten as follows:*
$Z(x) = (2x_1 - 0.32x_1^2) + (4x_2 - x_2^2) + (3x_3 - 0.8x_3^2) + (7x_4 - 1.5x_4^2) + (1.5x_5 - 0.1x_5^2) + (-6x_6 - 1.2x_6^2) + (-5x_7 - 1.7x_7^2) + (-2.3x_8 - 0.6x_8^2) + (-0.5x_9 - 0.1x_9^2)$.
The function $Z(x)$ *is increasing with respect to* x_j, *for each* $j \in \overline{K}$, *and decreasing with respect to* x_j, *for each* $j \in \underline{K}$, *on* $[0,1]^9$. *Also,* $\underline{K} \bigcap \overline{K} = \emptyset$. *Furthermore, the conditions of Substep 5.3 are satisfied: (i)* $\underline{K} \bigcup \overline{K} = J$ *and* $(\bigcup_{j\in\overline{K}} I_j^-) \bigcup (\bigcup_{j\in\underline{K}} I_j^+) = (I_1^- \bigcup I_2^- \bigcup I_3^- \bigcup I_4^- \bigcup I_5^-) \bigcup (I_6^+ \bigcup I_7^+ \bigcup I_8^+ \bigcup I_9^+) = (\{3,5,7,9,10\} \bigcup \{1,2,3,4,6,8,9,10\}) = \{1,2,...,10\} = I$. *Then according to Lemma 3.9, the optimal solution* $x^* = (x_j^*)_{j\in J}$ *is as follows:* $x^* = (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*, x_7^*, x_8^*, x_9^*)^T = (\check{x}_1, \check{x}_2, \check{x}_3, \check{x}_4, \check{x}_5, \hat{x}_6, \hat{x}_7, \hat{x}_8, \hat{x}_9)^T = (0.36, 0.45, 0.37, 0.44, 0.46, 0.54, 0.7, 0.84, 0.58)^T$ *and* $Z^* = -3.86$. *The algorithm ends.*

Other managerial applications can be seen in the areas of BitTorrent-like Peer-to-Peer file sharing system, wireless communication, supply chain, and modelling the behaviour of a motor. Due to the uncertainty existence of the probability type, we are interested in optimizing the expected value of the objective function. It leads to the problem of quadratic programming problem with consraints in terms of bipolar fuzzy relation equations. Hence, we need to solve the problems using Algorithm 1.
The sensitivity analysis of model (1)-(3) can be performed on its traditional equivalent quadratic model (16)-(21) based on the proposed methods in [15, 14, 8, 26, 18, 13, 23, 9, 30].

5.2 Numerical examples

Example 5.2. *Consider the following problem.*

$$\min \quad Z(x) = (-x_1^2 - 2x_1) + (-x_2^2) + (2x_3^2 + x_3) + (5x_4) \tag{29}$$
$$\text{s.t.} \quad x \in S(A^+, A^-, b) = \{x \in [0,1]^4 \mid A^+ \circ x \vee A^- \circ \neg x = b\}, \tag{30}$$

where $A^+ = \begin{pmatrix} 0.18 & 0.13 & 0.21 & 0.05 \\ 0.08 & 0.4 & 0.13 & 0.2 \\ 0.84 & 0.6 & 0.35 & 0.7 \\ 0.15 & 0.3 & 0.17 & 0.2 \\ 0.35 & 0.2 & 0.24 & 0.21 \\ 0.18 & 0.07 & 0.3 & 0.27 \end{pmatrix}$, $A^- = \begin{pmatrix} 0.07 & 0.3 & 0.02 & 0.42 \\ 0.6 & 0.12 & 0.08 & 0.11 \\ 0.5 & 0.4 & 0.34 & 0.25 \\ 0.11 & 0.04 & 0.11 & 0.15 \\ 0.26 & 0.6 & 0.21 & 0.84 \\ 0.13 & 0.14 & 1 & 0.21 \end{pmatrix}$, $b = (0.21, 0.24, 0.63, 0.18, 0.42, 0.3)^T$, *and* $x = (x_1, x_2, x_3, x_4)^T$. *This problem is solved by Algorithm 1.*
Step 1. *The lower and upper bound vectors of* $\check{x}$ *and* $\hat{x}$ *are as follows:*
$\check{x} = (0.6, 0.3, 0.7, 0.5)^T$ *and* $\hat{x} = (0.75, 0.6, 1, 0.9)^T$.
Step 2. *In this example,* $b_i > 0$, *for each* $i \in I = \{1, ..., 6\}$, *and* $\check{x}_j < \hat{x}_j$, *for each* $j \in J = \{1, ..., 4\}$.
Step 3. *The characteristic matrices* Q^+ *and* Q^- *are as follows:*

$$Q^+ = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad Q^- = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The index sets I_j^+, I_j^-, J_i^+, *and* J_i^-, *for each* $i \in I$ *and* $j \in J$, *are as follows:*
$I_1^+ = \{3\}, I_2^+ = \{2,4\}, I_3^+ = \{1,6\}, I_4^+ = \{3,4\}, I_1^- = \{2\}, I_2^- = \{1,5\}, I_3^- = \{6\}, I_4^- = \{1,5\}, J_1^+ = \{3\}, J_2^+ = \{2\}, J_3^+ = \{1,4\}, J_4^+ = \{2,4\}, J_5^+ = \emptyset, J_6^+ = \{3\}, J_1^- = \{2,4\}, J_2^- = \{1\}, J_3^- = J_4^- = \emptyset, J_5^- = \{2,4\}$, *and* $J_6^- = \{3\}$.
Step 4. *The characteristic Boolean formula corresponding to this example is as:* $C = \bigwedge_{i=1}^{6} C_i$, *where* $C_1 = y_3 \bigvee(\neg y_2) \bigvee(\neg y_4)$, $C_2 = y_2 \bigvee(\neg y_1)$, $C_3 = y_1 \bigvee y_4$, $C_4 = y_2 \bigvee y_4$, $C_5 = (\neg y_2) \bigvee(\neg y_4)$, *and* $C_6 = y_3 \bigvee(\neg y_3)$. *The clause* C_6 *can be removed.*

The formula $C=\bigwedge_{i=1}^{5} C_i$ is satisfiable if $y_1=y_2=y_3=1$, and $y_4=0$ are assigned. Hence, the system is consistent.
Step 5. *The mentioned conditions in Substeps 5.1-5.5 are checked as follows:*

Substep 5.1. *The example does not satisfy the conditions of this step.*

Substep 5.2. *The example does not satisfy the conditions of this step.*

Substep 5.3. *The sets $\underline{K}$ and $\bar{K}$ are as follows: $\underline{K}=\{1,2\}$ and $\bar{K}=\{3,4\}$, where $\underline{K}\bigcap\bar{K}=\emptyset$. The conditions: (i) $\underline{K}\bigcup\bar{K}=J$ and (ii) $(\bigcup_{j\in\bar{K}} I_j^-)\bigcup(\bigcup_{j\in\underline{K}} I_j^+)=(I_3^-\bigcup I_4^-)\bigcup(I_1^+\bigcup I_2^+)=(\{6\}\bigcup\{1,5\})\bigcup(\{3\}\bigcup\{2,4\})=\{1,2,3,4,5,6\}=I$ are satisfied. According to Substep 5.3, we have: $x_1^*=\hat{x}_1=0.75$, $x_2^*=\hat{x}_2=0.6$, $x_3^*=\check{x}_3=0.7$, and $x_4^*=\check{x}_4=0.5$. Also, the optimal objective value is as: $Z(x^*)=1.7575$. Stop!*

Example 5.3. *Consider the following problem.*

$$\min \quad Z(x)=4x_1+3x_2-4x_1.x_2+2x_1^2+4x_2^2+(-x_3^2-x_3)+(-3x_4) \tag{31}$$
$$\text{s.t.} \quad x\in S(A^+,A^-,b)=\{x\in[0,1]^4 \mid A^+\circ x\vee A^-\circ\neg x=b\}, \tag{32}$$

where $A^+=\begin{pmatrix}0.12&0.1&0.21&0.15\\0.05&0.4&0.21&0.22\\0.6&0.54&0.4&0.7\\0.13&1&0.12&1\\0.22&0.32&0.42&0.32\\0.17&0.01&0.3&0.27\end{pmatrix}$, $A^-=\begin{pmatrix}0.14&0.3&0.1&0.42\\0.6&0.15&0.2&0.16\\0.45&0.55&0.24&0.39\\0.13&0.12&0.09&0.13\\0.15&0.6&0.33&0.84\\0.21&0.24&1&0.28\end{pmatrix}$, $b=(0.21,0.24,0.42,0.6,0.42,0.3)^T$, *and* $x=(x_1,x_2,x_3,x_4)^T$. *This problem is solved by Algorithm 1.*
Step 1. *The lower and upper bound vectors of $\check{x}$ and $\hat{x}$ are as follows:*
$\check{x}=(0.6,0.3,0.7,0.5)^T$ *and* $\hat{x}=(0.7,0.6,1,0.6)^T$.
Step 2. *In this example, $b_i>0$, for each $i\in I=\{1,...,6\}$, and $\check{x}_j<\hat{x}_j$, for each $j\in J=\{1,...,4\}$.*
Step 3. *The characteristic matrices Q^+ and Q^- are as follows:*

$$Q^+=\begin{pmatrix}0&0&1&0\\0&1&0&0\\1&0&0&1\\0&1&0&1\\0&0&1&0\\0&0&1&0\end{pmatrix} \quad \text{and} \quad Q^-=\begin{pmatrix}0&1&0&1\\1&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&1\\0&0&1&0\end{pmatrix}.$$

The index sets I_j^+, I_j^-, J_i^+, and J_i^-, for each $i\in I$ and $j\in J$, are as follows:
$I_1^+=\{3\}, I_2^+=\{2,4\}, I_3^+=\{1,5,6\}, I_4^+=\{3,4\}, I_1^-=\{2\}, I_2^-=\{1,5\}, I_3^-=\{6\}, I_4^-=\{1,5\}, J_1^+=\{3\}, J_2^+=\{2\}, J_3^+=\{1,4\}, J_4^+=\{2,4\}, J_5^+=\{3\}, J_6^+=\{3\}, J_1^-=\{2,4\}, J_2^-=\{1\}, J_3^-=J_4^-=\emptyset, J_5^-=\{2,4\}$, *and* $J_6^-=\{3\}$.
Step 4. *The characteristic Boolean formula corresponding to this example is as: $C=\bigwedge_{i=1}^{6} C_i$, where $C_1=y_3\bigvee(\neg y_2)\bigvee(\neg y_4)$, $C_2=y_2\bigvee(\neg y_1)$, $C_3=y_1\bigvee y_4$, $C_4=y_2\bigvee y_4$, $C_5=(\neg y_2)\bigvee(\neg y_4)$, and $C_6=y_3\bigvee(\neg y_3)$. The clause C_6 can be deleted. The formula $C=\bigwedge_{i=1}^{5} C_i$ is satisfiable if $y_1=y_2=y_3=1$, and $y_4=0$ are assigned. Hence, the system is consistent.*
Step 5. *The mentioned conditions in Substeps 5.1-5.5 are not satisfied. The conditions 5.6 are only satisfied for this example. To check the conditions, let $\underline{K}_1=\{3,4\}$. Then, $\bigcup_{j\in\underline{K}_1} I_j^-=I_3^-\bigcup I_4^-=\{1,5,6\}$ and $\bigcup_{j\in\underline{K}_1} I_j^+=I_3^+\bigcup I_4^+=\{1,3,4,5,6\}$. So, $\bigcup_{j\in\underline{K}_1} I_j^-\subseteq\bigcup_{j\in\underline{K}_1} I_j^+$. According to Substep 5.6, we have $x_3^*=\hat{x}_3=1$ and $x_4^*=\hat{x}_4=0.6$. The updated matrices Q^+ and Q^- are as follows:*

$$Q^+=\begin{pmatrix}0&1\end{pmatrix} \quad \text{and} \quad Q^-=\begin{pmatrix}1&0\end{pmatrix}.$$

Also, $P=\{3,4\}$, $I=\{2\}$, and $J=\{1,2\}$. Moreover, $I_1^+=I_3^+=I_4^+=\emptyset$, $I_2^+=\{2\}$, $J_1^+=\emptyset$, $J_2^+=\{2\}$, $J_3^+=\{1\}$, $J_4^+=\{2\}$, $J_5^+=J_6^+=\emptyset$, $I_1^-=\{2\}$, $I_2^-=I_3^-=I_4^-=\emptyset$, $J_1^-=\{2\}$, $J_2^-=\{1\}$, $J_3^-=J_4^-=\emptyset$, $J_5^-=\{2\}$, and $J_6^-=\emptyset$.
Step 6. *The optimal values of variables x_1^* and x_2^* have remained.*

Step 7. *Create the following problem to find variables x_1^* and x_2^* as follows:*

$$\begin{aligned}
\min \quad & Z(x) = 4x_1 + 3x_2 - 4x_1x_2 + 2x_1^2 + 4x_2^2 - 3.8, && (33)\\
\text{s.t.} \quad & V.U^+ + \check{x} \le x \le -V.U^- + \hat{x}, && (34)\\
& Q^+.U^+ + Q^-.U^- \ge e, && (35)\\
& U^+ + U^- \le e, && (36)\\
& U^+, U^- \in \{0,1\}^4, && (37)\\
& x \in [0,1]^4. && (38)
\end{aligned}$$

where

$$V = \begin{pmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0.1 \end{pmatrix}, \; Q^+ = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; Q^- = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; U^+ = \begin{pmatrix} u_1^+ \\ u_2^+ \\ u_3^+ \\ u_4^+ \end{pmatrix}, \; U^- = \begin{pmatrix} u_1^- \\ u_2^- \\ u_3^- \\ u_4^- \end{pmatrix},$$

$$\check{x} = \begin{pmatrix} 0.6 \\ 0.3 \\ 0.7 \\ 0.5 \end{pmatrix}, \; \hat{x} = \begin{pmatrix} 0.7 \\ 0.6 \\ 1 \\ 0.6 \end{pmatrix}, \; e = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \; \textit{and} \;\; x = \begin{pmatrix} x_1 \\ x_2 \\ 1 \\ 0.6 \end{pmatrix}.$$

The values of optimal variables of x_3^* and x_4^* are as: $x_3^* = 1$ and $x_4^* = 0.6$. With regard to Steps (1)-(7), the optimal vector and the optimal objective function value are as $x^* = (0.6, 0.6, 1, 0.6)^T$ and $Z^* = 1.12$, respectively.

6 Conclusions and future researches

The quadratic programming problem with the max-product BFRE constraint was studied in this paper. A useful property was expressed about the components of its optimal solutions and the lower and upper bound vector. Some sufficient conditions were then proposed to find one of its optimal solutions or some components of its optimal solutions without its resolution. The problem was simplified by the current conditions, if possible. The simplified problem was equivalently converted to an 0-1 mixed integer programming problem and solved by the related classic optimization techniques.

Future researches: Future researches on the problem (1)-(3) are itemized as follows:
1. Determining the structure of the set of all the optimal solutions of problem (1)-(3).
2. Designing an algorithm to find all the optimal solutions of problem (1)-(3).
3. Modelling Peer-to-Peer data transmission network system in the stochastic case using problem (1)-(3).
4. Resolution of the problem when the constraints are bipolar FRIs.
5. Resolution of the problem when the constraints are type-2 fuzzy systems.

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