Iranian Journal of Fuzzy Systems

Volume 19, Number 6, (2022), pp. 141-160

Original paper

University of Sistan and Baluchestan

# A parametric similarity measure for extended picture fuzzy sets and its application in pattern recognition

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**Abstract**

This article advances the idea of extended picture fuzzy set (E-PFS), which is especially an augmentation of generalised spherical fuzzy set (GSFS) by releasing the restricted selection of $p$ in the description of GSFSs. Moreover, by the use of triangular conorm term in the description of E-PFS, it indeed widens the scope of E-PFS not only compared to picture fuzzy set (PFS) and spherical fuzzy set (SFS), but also to GSFS. In the sequel, a given fundamental theorem concerning E-PFS depicts its more ability in comparison with the special types to deal with the ambiguity and uncertainty. Further, we propose a parametric E-PFS similarity measure which plays a critical role in information theory. In order for revealing the advantages and authenticity of E-PFS similarity measure, we exhibit its applicability in multiple criteria decision making entitling the recognition of building material, the recognition of patterns, and the selection process of mega project(s) in developing countries. Furthermore, through the experimental studies, we demonstrate that E-PFS is able to handle uncertain information in real-life decision procedures with no extra parameter, and it has a prominent role in decision making whenever the concepts of PFS, SFS and GSFS do not make sense.

*Keywords:* Extended picture fuzzy set (E-PFS), picture fuzzy set (PFS), spherical fuzzy set (SFS), generalized spherical fuzzy set (GSFS), similarity measure.

## 1 Introduction

Decision making is an essential activity in different fields of science and technology [8]-[10]. The first attempt regarding the decision-making processes under uncertainty has been made by Zadeh [35] in which the main role is played by the concept of fuzzy set (FS). Although FS theory enables us to model the meaning of natural language expressions, it is restricted only to model the expressions by the membership degree. To overcome such a barrier, the concept of FS was developed to that of intuitionistic fuzzy set (IFS) [6] for modelling the uncertain information supported by the non-membership degree. In the sequel, the concept of picture fuzzy set (PFS) [7] was also introduced to consider more capability of handling uncertain information in the sense of membership, abstinence, and non-membership degrees. With respect to some limitations of PFSs in practical cases, a fruitful concept of spherical fuzzy set (SFS) [17], and further, the extended form of SFS concept, namely, hereafter generalized spherical fuzzy set (GSFS). In other contributions the concept of GSFS is also called T-spherical fuzzy set (T-SFS) [15, 18, 19, 20, 22].

However, if we incorporate the aforementioned concepts into the same framework, we observe that

- An element of FS is described by an expert's opinion in terms of membership and non-membership degrees whose sum is one;
- An element of IFS is described in terms of membership and non-membership degrees whose sum is less than one;

Corresponding Author: B. Farhadinia
Received: November 2021; Revised: June 2022; Accepted: August 2022.
https://doi.org/10.22111/IJFS.2022.7217

- An element of PFS is described by an expert's opinion in terms of membership, abstinence, and non-membership degrees with their own sum being less than one;
- An element of SFS is described by membership, abstinence, and non-membership degrees whose sum of their square forms is less than one;
- An element of GSFS (equivalently, T-SFS) is described in terms of membership, abstinence, and non-membership degrees with their own "$p$"-power sum being less than one. If $p = 1$, then GSFS becomes PFS; while if $p = 2$, then GSFS is reduced to SFS.

In recent years, many scholars have been devoted for applying PFSs to multiple criteria decision-making problems. Wei [31] presented some PFS operations, and Wei developed accordingly the operator of picture 2-tuple Bonferroni mean. By introducing PFS operational laws, Wang et al. [28] defined a number of picture fuzzy geometric operators together with investigating their basic properties. Wang et al. [27] established a projection-based VIKOR technique with picture fuzzy information in order to evaluating the risk parameter of construction project.

Besides the concept of PFS, the SFS has received a lot of attention in the literature. Ashraf et al. [5] investigated the algebraic form of aggregation operators by the help of algebraic T-norm and T-conorm. On the basis of Dombi norms, Ashraf el al. [4] represented a series of SFS Dombi operators for aggregating the criteria information in order to select the best alternative. Jin et al. [16] applied the logarithmic operational laws to define the logarithmic aggregation operators. Zeng et al. [36] presented the spherical fuzzy rough set technique, and moreover, they discussed the application of spherical fuzzy rough set in decision making by emphasising on the TOPSIS technique.

Apart from the PFS applications, Mahmood et al. [18] first developed a number of spherical fuzzy relations along with aggregation operators for T-SFSs, and then discussed SFS and T-SFS practical applications in medical-diagnostic-based decision-making. Garg et al. [15] defined a variety of operational laws for T-SFSs, and then studied the aggregation operators of T-spherical fuzzy weighted geometric interaction averaging, T-spherical fuzzy ordered weighted geometric interaction averaging, and T-spherical fuzzy hybrid geometric interaction averaging. By representing a decision making process based on T-SFSs, Quek et al. [20] developed some generalized T-spherical fuzzy weighted aggregation operators on neutrosophic sets. Munir et al. [19] introduced a variety of T-spherical fuzzy Einstein hybrid aggregation operators, and then, they showed an application of these aggregation operators in multi-attribute decision making problems.

Besides the above-mentioned concepts, there exists another extension of SFS, called complex spherical fuzzy set (CSFS), in which the range of degrees is given in the complex plane within unit disk [1, 3]. This concept is however left out of scope of this research.

Up to now, the above concepts have been receiving more and more attention in decision making researches in the last decades. Thong and Son [23] proposed a hybrid model by merging picture fuzzy clustering and intuitionistic fuzzy recommender system, which is implemented in medical diagnosis process. Wei [29] dealt with multiple attribute decision making by taking a picture fuzzy cross-entropy measure into account. Wei [30] presented a class of cosine- and cotangent-based similarity measures for PFSs, and then employed them in an strategic decision making problem.

Ganie and Singh [13] developed a picture fuzzy similarity measure by considering the operation on the functions of membership, non-membership, neutrality, and refusal together with the upper bound of membership function of two PFSs. Their method releases the using of distance measure or the association between functions of membership, non-membership and neutrality. Ganie et al. [14] proposed a class of picture fuzzy correlation coefficients, and showed their advantages over some existing methods in pattern recognition, medical diagnosis and clustering. Akram et al. [2] redesigned the shortest path Dijkstra-based algorithm to tackle situations in which the parameters of networks may be uncertain. To do this, they allowed that the parameters are to be in the form of special PFS. Wei and Gao [33] generalized the topic of PFS similarity measure to dice form for applying that in the problem of building material recognition.

Ullah et al. [25] proposed a set of SFS and T-SFS similarity measures including cosine-based, grey-based and set theoretic-based similarity measures which are all subjected to a well-known problem of building material recognition. Rafiq et al. [21] investigated a large number of SFS similarity measures, including cosine-based, weighted cosine-based, cotangent-based and weighted cotangent-based similarity measures.

By considering the concept of GSFS, Ullah et al. [24] introduced a class of correlation coefficients, and then applied them to clustering analysis. Garg et al. [15] improved some interactive aggregation operators to introduce a technique of dealing with GSFS-based multi-attribute group decision making. Munir et al. [19] investigated and discussed a variety of GSFS-based Einstein hybrid aggregation operators in decision-making with generalized spherical fuzzy information.

From the above contributions, we find that the concepts of PFS, SFS and GSFS have been widely implemented in different decision making situations. Moreover, in the case where the generalized version of PFS and SFS, i.e., GSFS, is considered, we are obliged to answer the question what the best option for $p$ is? For instance, in the case where an expert provides the information in terms of membership degree as 0.97, abstinence degree as 0.93, and non-membership degree as 0.13, then it is seen that neither $0.97 + 0.93 + 0.13 \leq 1$ nor $(0.97)^2 + (0.93)^2 + (0.13)^2 \leq 1$. Furthermore, $(0.97)^p + (0.93)^p + (0.13)^p \leq 1$ holds true for any integer $p \geq 15$, which restricts the option of selecting $p$ in GSFS theory. This shows that the concepts of PFSs, SFSs and GSFSs are not able to describe the uncertain information properly. In order to manage such a case, and moreover to release the restriction of $p$ selection in GSFS theory, we develop here the concept of GSFS to the extended picture fuzzy set (E-PFS) in which any triangular conorm operation of membership, abstinence, and non-membership degrees is equal or less than 1. Indeed, the fruitful and new concept of E-PFS is more general than the concept of GSFS for handling uncertain information in real-life decision procedures without needing to worry about an extra parameter like that of GSFS.

The aim of this research is briefly addressed as the following issues:

1. We represent the concept of E-PFS, which not only encompasses the concept of PFS and SFS, but also contains the general form of SFS, called GSFS;

2. We present the fundamental principles of E-PFSs together with defining a number of set and algebraic operations on E-PFSs;

3. We propose a parametric form of similarity measure for E-PFSs, and then, some practical problems are examined using the proposed E-PFS similarity measure.

In this study, we first represent the innovative concept of E-PFS, and provide a brief review of some preliminaries in Section 2. Then, a parametric form of similarity measure of E-PFSs is presented in Sections 3. Section 4 is devoted to the application of proposed E-PFS similarity measure in decision making, and comparison analyses are described in the current section. This contribution concludes in Section 5.

## 2 Fundamental principles

In what follows, we review briefly a number of fundamental concepts which are essential to understand better the subject matter.

**Definition 2.1.** [7] *Consider the referential set $X$, the set*

$$A_{pfs} = \{\langle x, \mu_{A_{pfs}}(x), \eta_{A_{pfs}}(x), \nu_{A_{pfs}}(x)\rangle : x \in X\},$$

*describes a picture fuzzy set (PFS) on $X$ in which $\mu_{A_{pfs}}$, $\eta_{A_{pfs}}$ and $\nu_{A_{pfs}}$ indicate respectively the degree of membership, abstinence, and non-membership functions of $A_{pfs}$. Further, it holds that*

$$0 \leq \mu_{A_{pfs}}(x) + \eta_{A_{pfs}}(x) + \nu_{A_{pfs}}(x) \leq 1,$$

*where $\mu_{A_{pfs}}(x), \eta_{A_{pfs}}(x), \nu_{A_{pfs}}(x) \in [0,1]$ for any $x \in X$.*

It seems that the concept of PFS $A_{pfs} = \{\langle x, \mu_{A_{pfs}}(x), \eta_{A_{pfs}}(x), \nu_{A_{pfs}}(x)\rangle : x \in X\}$ should enable us to deal with human opinion in an efficient manner, but sometimes, we observe that $\mu_{A_{pfs}}(x) + \eta_{A_{pfs}}(x) + \nu_{A_{pfs}}(x) \not\leq 1$. For instance, for $\mu_{A_{pfs}}(x) = 0.7$, $\eta_{A_{pfs}}(x) = 0.3$, and $\nu_{A_{pfs}}(x) = 0.5$, we observe that $\mu_{A_{pfs}}(x) + \eta_{A_{pfs}}(x) + \nu_{A_{pfs}}(x) = 0.7 + 0.3 + 0.5 \not\leq 1$, while $\mu^2_{A_{pfs}}(x) + \eta^2_{A_{pfs}}(x) + \nu^2_{A_{pfs}}(x) = (0.7)^2 + (0.3)^2 + (0.5)^2 \leq 1$. In this case, it was of great value to enlarge the structure of PFS such that the grades of satisfaction $\mu_{A_{pfs}}$, abstinence $\eta_{A_{pfs}}$, and dissatisfaction $\nu_{A_{pfs}}$ satisfy $\mu^2_{A_{pfs}}(x) + \eta^2_{A_{pfs}}(x) + \nu^2_{A_{pfs}}(x) \leq 1$. This was the main motivation of introducing the concept of spherical fuzzy set.

**Definition 2.2.** [17] *Consider the referential set $X$. the set*

$$A_{sfs} = \{\langle x, \mu_{A_{sfs}}(x), \eta_{A_{sfs}}(x), \nu_{A_{sfs}}(x)\rangle : x \in X\},$$

*describes a spherical fuzzy set (SFS) on $X$ in which $\mu_{A_{sfs}}$, $\eta_{A_{sfs}}$ and $\nu_{A_{sfs}}$ indicate respectively the degree of membership, abstinence, and non-membership functions of $A_{SFS}$. Further, it holds that*

$$0 \leq \mu^2_{A_{sfs}}(x) + \eta^2_{A_{sfs}}(x) + \nu^2_{A_{sfs}}(x) \leq 1,$$

*where $\mu_{A_{sfs}}(x), \eta_{A_{sfs}}(x), \nu_{A_{sfs}}(x) \in [0,1]$ for any $x \in X$.*

By the way, there exit still some situations in which we are required to consider $0 \leq \mu^p_{A_{sfs}}(x)+\eta^p_{A_{sfs}}(x)+\nu^p_{A_{sfs}}(x) \leq 1$ (for positive integer $p$), meanwhile, the condition $\mu^2_{A_{sfs}}(x)+\eta^2_{A_{sfs}}(x)+\nu^2_{A_{sfs}}(x) \leq 1$ does not hold true. In such a case, it appears the need of defining the concept of generalized spherical fuzzy set.

**Definition 2.3.** [22] *Consider the referential set $X$. the set*

$$A_{gsfs} = \{\langle x, \mu_{A_{gsfs}}(x), \eta_{A_{gsfs}}(x), \nu_{A_{gsfs}}(x)\rangle : x \in X\},$$

*describes a generalized spherical fuzzy set (GSFS) on $X$ in which $\mu_{A_{gsfs}}$, $\eta_{A_{gsfs}}$ and $\nu_{A_{gsfs}}$ indicate respectively the degree of membership, abstinence, and non-membership functions of $A_{gsfs}$. Further, it holds that*

$$0 \leq \mu^p_{A_{gsfs}}(x) + \eta^p_{A_{gsfs}}(x) + \nu^p_{A_{gsfs}}(x) \leq 1,$$

*where $\mu_{A_{gsfs}}(x), \eta_{A_{gsfs}}(x), \nu_{A_{gsfs}}(x) \in [0,1]$ for any $x \in X$, and any positive integer $p$ being up to decision makers.*

The concept of GSFS plays role as the most general type among PFSs and SFSs, but very little is known about the pre-determination of parameter $p$. To help overcome this obstacle, we aim here to develop a new concept of extended picture fuzzy set which is not dependent on any value of $p$.

**Definition 2.4.** *Consider the referential set $X$. the set*

$$A_{epfs} = \{\langle x, \mu_{A_{epfs}}(x), \eta_{A_{epfs}}(x), \nu_{A_{epfs}}(x)\rangle : x \in X\},$$

*describes an extended picture fuzzy set (E-PFS) on $X$ in which $\mu_{A_{epfs}}$, $\eta_{A_{epfs}}$ and $\nu_{A_{epfs}}$ indicate respectively the degree of membership, abstinence, and non-membership functions of $A_{epfs}$. Further, it holds that*

$$0 \leq [\mu_{A_{epfs}}(x) \odot \eta_{A_{epfs}}(x)] \odot \nu_{A_{epfs}}(x) \leq 1,$$

*where $\mu_{A_{epfs}}(x), \eta_{A_{epfs}}(x), \nu_{A_{epfs}}(x) \in [0,1]$ for any $x \in X$, and the symbol $\odot$ denotes any triangular conorm operation.*

Generally, a triangular conorm (T-conorm for short) is a binary operation $\odot : [0,1] \times [0,1] \to [0,1]$ which satisfies the following properties (see e.g. [8])

- Axiom 1. $\odot(x,0) = x$ (Boundary condition);
- Axiom 2. For all $x, y, z \in [0,1]$, if $y \leq z$, then $\odot(x,y) \leq \odot(x,z)$ (Monotonicity);
- Axiom 3. For all $x, y \in [0,1]$, $\odot(x,y) = \odot(y,x)$ (Commutativity);
- Axiom 4. For all $x, y, z \in [0,1]$, $\odot(x, \odot(y,z)) = \odot(\odot(x,y), z)$ (Associativity).

The followings are respectively known as Algebraic, Einstein, Hamacher, and Frank T-conorms:

$$\begin{aligned}
&\odot_1(x,y) = x + y - xy;\\
&\odot_2(x,y) = \frac{x+y}{1+xy};\\
&\odot^{\epsilon}_3(x,y) = \frac{x+y-xy-(1-\epsilon)xy}{1-(1-\epsilon)xy}, \quad \epsilon > 0;\\
&\odot^{\epsilon}_4(x,y) = 1 - log_{\epsilon}(1 + \frac{(\epsilon^{1-x}-1)(\epsilon^{1-y}-1)}{\epsilon - 1}), \quad \epsilon > 1.
\end{aligned}$$

Instead of T-conorm, the term *S-norm* is sometimes used synonymously in the literature.
By considering the negation $\aleph(.)$ that is non-increasing on $[0,1]$, and also $\aleph(0) = 1$ and $\aleph(1) = 0$, the corresponding dual form of triangular conorm $\odot$ is expressed by the *triangular norm* (T-norm for short) $\otimes : [0,1] \times [0,1] \to [0,1]$ such that

$$\otimes(x,y) = \aleph(\odot(\aleph(x), \aleph(y))), \tag{1}$$

$$\odot(x,y) = \aleph(\otimes(\aleph(x), \aleph(y))). \tag{2}$$

The duality relationships expressing by (1) and (2) allow one translates many properties of T-conorm into the corresponding properties of T-norm.

In what follows, we present a fundamental theorem which claims any concept of PFS, SFS, and GSFS can be concluded from the concept of E-PFS, and indeed, the concepts PFS, SFS, and GSFS are special cases of E-PFS.

**Proposition 2.5.** *Any PFS, SFS, and GSFS on $X$ is an E-PFS on $X$.*

*Proof.* It is clear that any GSFS $A_{gsfs}$, which is the general form of PFS and SFS, is defined in terms of

$$A_{gsfs} = \{\langle x, \mu_{A_{gsfs}}(x), \eta_{A_{gsfs}}(x), \nu_{A_{gsfs}}(x)\rangle : x \in X\},$$

in which $0 \le \mu^p_{A_{gsfs}}(x) + \eta^p_{A_{gsfs}}(x) + \nu^p_{A_{gsfs}}(x) \le 1$ for any $x \in X$ and any positive integer $p$. If $p = 1$ or $p = 2$, then the GSFS $A_{gsfs}$ is degenerated to the PFS form or the SFS form.
Now, if we set

$$(\mu_{A_{epfs}}(x), \eta_{A_{epfs}}(x), \nu_{A_{epfs}}(x)) = (\mu^p_{A_{gsfs}}(x), \eta^p_{A_{gsfs}}(x), \nu^p_{A_{gsfs}}(x)), \tag{3}$$

then, the latter relation implies that
**Case 1 ($\odot := \odot_1$):**

$$\begin{aligned}
&[\mu_{A_{epfs}}(x) \odot_1 \eta_{A_{epfs}}(x)] \odot_1 \nu_{A_{epfs}}(x)\\
&:= [\mu_{A_{epfs}}(x) + \eta_{A_{epfs}}(x) - \mu_{A_{epfs}}(x)\eta_{A_{epfs}}(x)] \odot_1 \nu_{A_{epfs}}(x)\\
&= [1 - (1 - \mu_{A_{epfs}}(x)) \times (1 - \eta_{A_{epfs}}(x))] \odot_1 \nu_{A_{epfs}}(x)\\
&= [1 - (1 - \mu_{A_{epfs}}(x)) \times (1 - \eta_{A_{epfs}}(x))] + \nu_{A_{epfs}}(x) - [1 - (1 - \mu_{A_{epfs}}(x)) \times (1 - \eta_{A_{epfs}}(x))]\nu_{A_{epfs}}(x)\\
&= [1 - (1 - \mu_{A_{epfs}}(x)) \times (1 - \eta_{A_{epfs}}(x))](1 - \nu_{A_{epfs}}(x)) + \nu_{A_{epfs}}(x)\\
&= 1 - (1 - \mu_{A_{epfs}}(x)) \times (1 - \eta_{A_{epfs}}(x)) \times (1 - \nu_{A_{epfs}}(x)) \qquad\qquad (\textit{by Eq. } (3))\\
&= 1 - (1 - \mu^p_{A_{gsfs}}(x)) \times (1 - \eta^p_{A_{gsfs}}(x)) \times (1 - \nu^p_{A_{gsfs}}(x)) \le 1.
\end{aligned}$$

**Case 2 ($\odot := \odot_2$):**

$$\begin{aligned}
&[\mu_{A_{epfs}}(x) \odot_2 \eta_{A_{epfs}}(x)] \odot_2 \nu_{A_{epfs}}(x) := [\frac{\mu_{A_{epfs}}(x) + \eta_{A_{epfs}}(x)}{1 + \mu_{A_{epfs}}(x)\eta_{A_{epfs}}(x)}] \odot_2 \nu_{A_{epfs}}(x)\\
&= \frac{[\frac{\mu_{A_{epfs}}(x)+\eta_{A_{epfs}}(x)}{1+\mu_{A_{epfs}}(x)\eta_{A_{epfs}}(x)}] + \nu_{A_{epfs}}(x)}{1 + [\frac{\mu_{A_{epfs}}(x)+\eta_{A_{epfs}}(x)}{1+\mu_{A_{epfs}}(x)\eta_{A_{epfs}}(x)}]\nu_{A_{epfs}}(x)}\\
&= \frac{\mu_{A_{epfs}}(x) + \eta_{A_{epfs}}(x) + \nu_{A_{epfs}}(x) + \mu_{A_{epfs}}(x)\eta_{A_{epfs}}(x)\nu_{A_{epfs}}(x)}{1 + \mu_{A_{epfs}}(x)\eta_{A_{epfs}}(x) + \mu_{A_{epfs}}(x)\nu_{A_{epfs}}(x) + \eta_{A_{epfs}}(x)\nu_{A_{epfs}}(x)}\\
&= \frac{\mu_{A_{epfs}}(x) + \eta_{A_{epfs}}(x) + \nu_{A_{epfs}}(x) + \mu_{A_{epfs}}(x)\eta_{A_{epfs}}(x)\nu_{A_{epfs}}(x)}{1 + \mu_{A_{epfs}}(x)\eta_{A_{epfs}}(x) + \mu_{A_{epfs}}(x)\nu_{A_{epfs}}(x) + \eta_{A_{epfs}}(x)\nu_{A_{epfs}}(x)}\\
&\quad + \frac{\pm[\mu_{A_{epfs}}(x)\eta_{A_{epfs}}(x) + \mu_{A_{epfs}}(x)\nu_{A_{epfs}}(x) + \eta_{A_{epfs}}(x)\nu_{A_{epfs}}(x)]}{1 + \mu_{A_{epfs}}(x)\eta_{A_{epfs}}(x) + \mu_{A_{epfs}}(x)\nu_{A_{epfs}}(x) + \eta_{A_{epfs}}(x)\nu_{A_{epfs}}(x)} \qquad\qquad (\textit{by Eq. } (3))\\
&= \frac{1 - (1 - \mu^p_{A_{gsfs}}(x)) \times (1 - \eta^p_{A_{gsfs}}(x)) \times (1 - \nu^p_{A_{gsfs}}(x))}{1 + \mu^p_{A_{gsfs}}(x)\eta^p_{A_{gsfs}}(x) + \mu^p_{A_{gsfs}}(x)\nu^p_{A_{gsfs}}(x) + \eta^p_{A_{gsfs}}(x)\nu^p_{A_{gsfs}}(x)}\\
&\quad + \frac{[\mu^p_{A_{gsfs}}(x)\eta^p_{A_{gsfs}}(x) + \mu^p_{A_{gsfs}}(x)\nu^p_{A_{gsfs}}(x) + \eta^p_{A_{gsfs}}(x)\nu^p_{A_{gsfs}}(x)]}{1 + \mu^p_{A_{gsfs}}(x)\eta^p_{A_{gsfs}}(x) + \mu^p_{A_{gsfs}}(x)\nu^p_{A_{gsfs}}(x) + \eta^p_{A_{gsfs}}(x)\nu^p_{A_{gsfs}}(x)}\\
&\le \frac{1 + [\mu^p_{A_{gsfs}}(x)\eta^p_{A_{gsfs}}(x) + \mu^p_{A_{gsfs}}(x)\nu^p_{A_{gsfs}}(x) + \eta^p_{A_{gsfs}}(x)\nu^p_{A_{gsfs}}(x)]}{1 + \mu^p_{A_{gsfs}}(x)\eta^p_{A_{gsfs}}(x) + \mu^p_{A_{gsfs}}(x)\nu^p_{A_{gsfs}}(x) + \eta^p_{A_{gsfs}}(x)\nu^p_{A_{gsfs}}(x)} = 1.
\end{aligned}$$

**Case 3 ($\odot := \odot_3$):**

$$\begin{aligned}
&[\mu_{A_{epfs}}(x) \odot_3 \eta_{A_{epfs}}(x)] \odot_3 \nu_{A_{epfs}}(x) :=\\
&[\frac{\mu_{A_{epfs}}(x) + \eta_{A_{epfs}}(x) - \mu_{A_{epfs}}(x)\eta_{A_{epfs}}(x) - (1-\epsilon)\mu_{A_{epfs}}(x)\eta_{A_{epfs}}(x)}{1 - (1-\epsilon)\mu_{A_{epfs}}(x)\eta_{A_{epfs}}(x)}] \odot_3 \nu_{A_{epfs}}(x).
\end{aligned}$$

Now, if we consider the equation (3), then, it is easy to find that

$$\mu_{A_{epfs}}(x) + \eta_{A_{epfs}}(x) - \mu_{A_{epfs}}(x)\eta_{A_{epfs}}(x) = \mu^p_{A_{gsfs}}(x) + \eta^p_{A_{gsfs}}(x) - \mu^p_{A_{gsfs}}(x)\eta^p_{A_{gsfs}}(x) \le 1,$$

and therefore,

$$[\mu_{A_{epfs}}(x) \odot_3 \eta_{A_{epfs}}(x)] \odot_3 \nu_{A_{epfs}}(x) \leq [1] \odot_3 \nu_{A_{epfs}}(x).$$

Let us again refer to the equation (3). Then, it follows that

$$[\mu_{A_{epfs}}(x) \odot_3 \eta_{A_{epfs}}(x)] \odot_3 \nu_{A_{epfs}}(x) \leq \frac{1+\nu^p_{A_{gsfs}}(x) - 1\times\nu^p_{A_{gsfs}}(x) - (1-\epsilon)1\times\nu^p_{A_{gsfs}}(x)}{1-(1-\epsilon)1\times\nu^p_{A_{gsfs}}(x)} \leq 1,$$

for any $\epsilon > 0$.
**Case 4 ($\odot := \odot_4$):**

$$\begin{aligned}
&[\mu_{A_{epfs}}(x) \odot_4 \eta_{A_{epfs}}(x)] \odot_4 \nu_{A_{epfs}}(x) := \\
&[1 - log_\epsilon(1 + \frac{(\epsilon^{1-\mu_{A_{epfs}}(x)}-1)(\epsilon^{1-\eta_{A_{epfs}}(x)}-1)}{\epsilon - 1})] \odot_4 \nu_{A_{epfs}}(x) \quad (by\ Eq.\ (3)) \\
&= [1 - log_\epsilon(1 + \frac{(\epsilon^{1-\mu^p_{A_{gsfs}}(x)}-1)(\epsilon^{1-\eta^p_{A_{gsfs}}(x)}-1)}{\epsilon - 1})] \odot_4 \nu_{A_{epfs}}(x).
\end{aligned}$$

The fact that $\epsilon > 1$ gives rise to $\epsilon - 1 > 0$ together with $\epsilon^{1-\mu^p_{A_{gsfs}}(x)} - 1 \geq 0$ and $\epsilon^{1-\eta^p_{A_{gsfs}}(x)} - 1 \geq 0$ for any $\mu^p_{A_{gsfs}}(x), \eta^p_{A_{gsfs}}(x) \leq 1$.

By taking these outcomes into account, we get $log_\epsilon(1 + \frac{(\epsilon^{1-\mu^p_{A_{gsfs}}(x)}-1)(\epsilon^{1-\eta^p_{A_{gsfs}}(x)}-1)}{\epsilon-1}) \geq log_\epsilon 1 = 0$, and

$$\begin{aligned}
&[\mu_{A_{epfs}}(x) \odot_4 \eta_{A_{epfs}}(x)] \odot_4 \nu_{A_{epfs}}(x) = \\
&[1 - log_\epsilon(1 + \frac{(\epsilon^{1-\mu^p_{A_{gsfs}}(x)}-1)(\epsilon^{1-\eta^p_{A_{gsfs}}(x)}-1)}{\epsilon - 1})] \odot_4 \nu_{A_{epfs}}(x) \\
&\leq [1] \odot_4 \nu_{A_{epfs}}(x) = 1 - log_\epsilon(1 + \frac{(\epsilon^{1-1}-1)(\epsilon^{1-\nu^p_{A_{gsfs}}(x)}-1)}{\epsilon - 1}) = 1,
\end{aligned}$$

for any $\epsilon > 1$.

In summarize, the above results demonstrate that the satisfaction of $0 \leq \mu^p_{A_{gsfs}}(x) + \eta^p_{A_{gsfs}}(x) + \nu^p_{A_{gsfs}}(x) \leq 1$ for any $x \in X$ and any positive integer $p$ leads to the fact that $0 \leq [\mu_{A_{epfs}}(x) \odot \eta_{A_{epfs}}(x)] \odot \nu_{A_{epfs}}(x) \leq 1$, that is, the GSFS $A_{gsfs}$ will be understood as the E-PFS $A_{epfs}$. □

**Remark 2.6.** *Since the same treatment strategy can be applied to all types of T-conorm $\odot$, we only consider $\odot := \odot_1$ in the subsequent analysis.*

**Remark 2.7.** *It should be stressed that any E-PFS is not necessarily a GSFS for all positive integer values of $p$. For instance, $(\mu_{A_{epfs}}(x), \eta_{A_{epfs}}(x), \nu_{A_{epfs}}(x)) = (0.93, 0.94, 0.1)$ is not a GSFS for any $p \in [1, 10)$, but it is an E-PFS.*

Before we proceed to the next discussion, we call $(\mu_{A_{epfs}}(x), \eta_{A_{epfs}}(x)), \nu_{A_{epfs}}(x))$ an *extended picture fuzzy number* (E-PFN) for a fixed $x \in X$, which is nothing else than an special case of E-PFS.
Now, we are in a position to propose a number of set and algebraic operations on E-PFNs.

**Theorem 2.8.** *For any E-PFNs $A_{epfn} = (\mu_{A_{epfn}}, \eta_{A_{epfn}}, \nu_{A_{epfn}})$ and $B_{epfn} = (\mu_{B_{epfn}}, \eta_{B_{epfn}}, \nu_{B_{epfn}})$, the following operations are to be carried out:*

$$A^c_{epfn} = (\nu_{A_{epfn}}, \eta_{A_{epfn}}, \mu_{A_{epfn}}); \tag{4}$$
$$A_{epfn} \cap B_{epfn} = (\min\{\mu_{A_{epfn}}, \mu_{B_{epfn}}\}, \max\{\eta_{A_{epfn}}, \nu_{B_{epfn}}\}, \max\{\nu_{A_{epfn}}, \nu_{B_{epfn}}\}), \tag{5}$$
$$A_{epfn} \cup B_{epfn} = (\max\{\mu_{A_{epfn}}, \mu_{B_{epfn}}\}, \min\{\eta_{A_{epfn}}, \nu_{B_{epfn}}\}, \min\{\nu_{A_{epfn}}, \nu_{B_{epfn}}\}), \tag{6}$$
$$A_{epfn} \oplus B_{epfn} = ([1-(1-\mu_{A_{epfn}})(1-\mu_{B_{epfn}})], [\eta_{A_{epfn}}\eta_{B_{epfn}}], [\nu_{A_{epfn}}\nu_{B_{epfn}}]), \tag{7}$$
$$A_{epfn} \otimes B_{epfn} = ([\mu_{A_{epfn}}\mu_{B_{epfn}}], [1-(1-\eta_{A_{epfn}})(1-\eta_{B_{epfn}})], [1-(1-\nu_{A_{epfn}})(1-\nu_{B_{epfn}})]), \tag{8}$$
$$\lambda A_{epfn} = ([1-(1-\mu_{A_{epfn}})^\lambda], [(\eta_{A_{epfn}})^\lambda], [(\nu_{A_{epfn}})^\lambda]), \tag{9}$$
$$A^\lambda_{epfn} = ([(\mu_{A_{epfn}})^\lambda], [1-(1-\eta_{A_{epfn}})^\lambda], [1-(1-\nu_{A_{epfn}})^\lambda]),\ \lambda > 0. \tag{10}$$

*Proof.* It is only sufficient to show that for any operation $*$ it holds $0 \leq [\mu_* \odot_1 \eta_*] \odot_1 \nu_* \leq 1$.

Proof of (4): It is obvious for any E-PFN $A_{epfn} = (\mu_{A_{epfn}}, \eta_{A_{epfn}}, \nu_{A_{epfn}})$, we have $0 \leq [\mu_{A_{epfn}} \odot_1 \eta_{A_{epfn}}] \odot_1 \nu_{A_{epfn}} \leq 1$. By the axiom of associativity of $\odot_1$, we get that $[\mu_{A_{epfn}} \odot_1 \eta_{A_{epfn}}] \odot_1 \nu_{A_{epfn}} = [\nu_{A_{epfn}} \odot_1 \eta_{A_{epfn}}] \odot_1 \mu_{A_{epfn}}$ which leads to the result of $A^c_{epfn} = (\nu_{A_{epfn}}, \eta_{A_{epfn}}, \mu_{A_{epfn}})$ being an E-PFN.

Proof of (5): Since $A_{epfn}$ and $B_{epfn}$ are two E-PFNs, then, we get that

$$\begin{aligned}
0 &\leq [\mu_{A_{epfs}} \odot_1 \eta_{A_{epfs}}] \odot_1 \nu_{A_{epfs}} \\
&:= [\mu_{A_{epfs}} + \eta_{A_{epfs}} - \mu_{A_{epfs}} \eta_{A_{epfs}}] \odot_1 \nu_{A_{epfs}} \\
&= [1 - (1 - \mu_{A_{epfs}}) \times (1 - \eta_{A_{epfs}})] \odot_1 \nu_{A_{epfs}} \\
&= [1 - (1 - \mu_{A_{epfs}}) \times (1 - \eta_{A_{epfs}})] + \nu_{A_{epfs}} - [1 - (1 - \mu_{A_{epfs}}) \times (1 - \eta_{A_{epfs}})] \times \nu_{A_{epfs}} \\
&= 1 - (1 - \mu_{A_{epfs}}) \times (1 - \eta_{A_{epfs}}) \times (1 - \nu_{A_{epfs}}) \\
&\leq 1,
\end{aligned}$$

and

$$\begin{aligned}
0 &\leq [\mu_{B_{epfs}} \odot_1 \eta_{B_{epfs}}] \odot_1 \nu_{B_{epfs}} \\
&:= [\mu_{B_{epfs}} + \eta_{B_{epfs}} - \mu_{B_{epfs}} \eta_{B_{epfs}}] \odot_1 \nu_{B_{epfs}} \\
&= [1 - (1 - \mu_{B_{epfs}}) \times (1 - \eta_{B_{epfs}})] \odot_1 \nu_{B_{epfs}} \\
&= [1 - (1 - \mu_{B_{epfs}}) \times (1 - \eta_{B_{epfs}})] + \nu_{B_{epfs}} - [1 - (1 - \mu_{B_{epfs}}) \times (1 - \eta_{B_{epfs}})] \times \nu_{B_{epfs}} \\
&= 1 - (1 - \mu_{B_{epfs}}) \times (1 - \eta_{B_{epfs}}) \times (1 - \nu_{B_{epfs}}) \\
&\leq 1.
\end{aligned}$$

Now, from the definition of $A_{epfn} \cap B_{epfn}$, we deduce that

$$\begin{aligned}
&[\mu_{A_{epfn} \cap B_{epfn}} \odot_1 \eta_{A_{epfn} \cap B_{epfn}}] \odot_1 \nu_{A_{epfn} \cap B_{epfn}} \\
= \; & 1 - (1 - \mu_{A_{epfn} \cap B_{epfn}}) \times (1 - \eta_{A_{epfn} \cap B_{epfn}}) \times (1 - \nu_{A_{epfn} \cap B_{epfn}}) \\
= \; & 1 - (1 - \min\{\mu_{A_{epfn}}, \mu_{B_{epfn}}\}) \times (1 - \max\{\eta_{A_{epfn}}, \eta_{B_{epfn}}\}) \times (1 - \max\{\nu_{A_{epfn}}, \nu_{B_{epfn}}\}),
\end{aligned}$$

and moreover, the non-negativity property of all the terms $1 - \min\{\mu_{A_{epfn}}, \mu_{B_{epfn}}\}$, $1 - \max\{\eta_{A_{epfn}}, \eta_{B_{epfn}}\}$ and $1 - \max\{\nu_{A_{epfn}}, \nu_{B_{epfn}}\}$ result in

$$0 \leq [\mu_{A_{epfn} \cap B_{epfn}} \odot_1 \eta_{A_{epfn} \cap B_{epfn}}] \odot_1 \nu_{A_{epfn} \cap B_{epfn}} \leq 1.$$

Proof of (6): The proof is much like that of (5).

Proof of (7): Based on the definition of $A_{epfn} \oplus B_{epfn}$, it follows that

$$\begin{aligned}
&[\mu_{A_{epfn} \oplus B_{epfn}} \odot_1 \eta_{A_{epfn} \oplus B_{epfn}}] \odot_1 \nu_{A_{epfn} \oplus B_{epfn}} \\
= \; & 1 - (1 - \mu_{A_{epfn} \oplus B_{epfn}}) \times (1 - \eta_{A_{epfn} \oplus B_{epfn}}) \times (1 - \nu_{A_{epfn} \oplus B_{epfn}}) \\
= \; & 1 - (1 - [1 - (1 - \mu_{A_{epfn}})(1 - \mu_{B_{epfn}})]) \times (1 - \eta_{A_{epfn}} \eta_{B_{epfn}}) \times (1 - \nu_{A_{epfn}} \nu_{B_{epfn}}).
\end{aligned}$$

Again, the non-negativity property of terms $1 - \mu_{A_{epfn}}$, $1 - \mu_{B_{epfn}}$, $1 - \eta_{A_{epfn}} \eta_{B_{epfn}}$ and $1 - \nu_{A_{epfn}} \nu_{B_{epfn}}$ gives rise to

$$0 \leq [\mu_{A_{epfn} \oplus B_{epfn}} \odot_1 \eta_{A_{epfn} \oplus B_{epfn}}] \odot_1 \nu_{A_{epfn} \oplus B_{epfn}} \leq 1.$$

Proof of (8), (9) and (10): The proof is much like that of (7). □

**Definition 2.9.** *For any E-PFNs $A_{epfn} = (\mu_{A_{epfn}}, \eta_{A_{epfn}}, \nu_{A_{epfn}})$ and $B_{epfn} = (\mu_{B_{epfn}}, \eta_{B_{epfn}}, \nu_{B_{epfn}})$, we have*

$$A_{epfn} \subseteq B_{epfn} \quad \text{if and only if} \quad \mu_{A_{epfn}} \leq \mu_{B_{epfn}}\;,\; \eta_{A_{epfn}} \leq \eta_{B_{epfn}} \text{ and } \nu_{A_{epfn}} \geq \nu_{B_{epfn}}. \tag{11}$$

## 3 Similarity measure for E-PFSs

Within this section, a class of existing similarity measures for PFSs, SFSs and GSFSs are reviewed, and then, a new parametric form of similarity measure for E-PFSs is developed.

The following definition provides a base for the properties needed to be verified hold true for a logical similarity measure of PFSs, SFSs, GSFSs, and E-PFSs.

**Definition 3.1.** *A similarity measure between the sets $A_*$ and $B_*$ is a mapping $\mathcal{S} : *(X) \times *(X) \to [0,1]$ which satisfies the following axiomatic requirements:*

**(S1)** $\mathcal{S}(A_*, B_*) \in [0,1]$;

**(S2)** $\mathcal{S}(A_*, B_*) = 1$ *if and only if* $A_* = B_*$;

**(S3)** $\mathcal{S}(A_*, B_*) = \mathcal{S}(B_*, A_*)$;

**(S4)** $\mathcal{S}(A_*, C_*) \le \mathcal{S}(A_*, B_*)$ *and* $\mathcal{S}(A_*, C_*) \le \mathcal{S}(B_*, C_*)$ *if* $A_* \subseteq B_* \subseteq C_*$;

**(S5)** $\mathcal{S}(A_*, A_*^c) = 0$ *if* $A_*$ *is a crisp set.*

*The sub-index $*$ could be devoted to any concept of PFS, SFS, GSFS and E-PFS.*

## 3.1 Similarity measure of PFSs, SFSs and GSFSs

Here, we are going to review in brief the earlier similarity measures of PFSs, SFSs and GSFSs.

For any PFSs $A_{pfs} = \{\langle x_i, \mu_{A_{pfs}}(x_i), \eta_{A_{pfs}}(x_i), \nu_{A_{pfs}}(x_i)\rangle : x_i \in X\}$ and $B_{pfs} = \{\langle x_i, \mu_{B_{pfs}}(x_i), \eta_{B_{pfs}}(x_i), \nu_{B_{pfs}}(x_i)\rangle : x_i \in X\}$, Wei [32] implemented the bellow similarity measures:

$$\mathcal{S}^{W1}_{pfs}(A_{pfs}, B_{pfs}) = \sum_{i=1}^{n} \omega_i \frac{\mu_{A_{pfs}}(x_i)\mu_{B_{pfs}}(x_i) + \eta_{A_{pfs}}(x_i)\eta_{B_{pfs}}(x_i) + \nu_{A_{pfs}}(x_i)\nu_{B_{pfs}}(x_i)}{(\mu^2_{A_{pfs}}(x_i) + \eta^2_{A_{pfs}}(x_i) + \nu^2_{A_{pfs}}(x_i))^{\frac{1}{2}} \times (\mu^2_{B_{pfs}}(x_i) + \eta^2_{B_{pfs}}(x_i) + \nu^2_{B_{pfs}}(x_i))^{\frac{1}{2}}}; \tag{12}$$

$$\mathcal{S}^{W2}_{pfs}(A_{pfs}, B_{pfs}) = \sum_{i=1}^{n} \omega_i \frac{\mu_{A_{pfs}}(x_i)\mu_{B_{pfs}}(x_i) + \eta_{A_{pfs}}(x_i)\eta_{B_{pfs}}(x_i) + \nu_{A_{pfs}}(x_i)\nu_{B_{pfs}}(x_i)}{\max\{\mu^2_{A_{pfs}}(x_i) + \eta^2_{A_{pfs}}(x_i) + \nu^2_{A_{pfs}}(x_i), \mu^2_{B_{pfs}}(x_i) + \eta^2_{B_{pfs}}(x_i) + \nu^2_{B_{pfs}}(x_i)\}}; \tag{13}$$

$$\mathcal{S}^{W3}_{pfs}(A_{pfs}, B_{pfs}) = \frac{1}{3}\sum_{i=1}^{n} \omega_i \Big(\frac{\underline{\Delta^{\mu}_i} + \overline{\Delta^{\mu}_i}}{\Delta^{\mu}_i + \overline{\Delta^{\mu}_i}} + \frac{\underline{\Delta^{\eta}_i} + \overline{\Delta^{\eta}_i}}{\Delta^{\eta}_i + \overline{\Delta^{\eta}_i}} + \frac{\underline{\Delta^{\nu}_i} + \overline{\Delta^{\nu}_i}}{\Delta^{\nu}_i + \overline{\Delta^{\nu}_i}}\Big), \tag{14}$$

where for any $* = \mu, \eta$ and $\nu$, it is symbolized that $\Delta^*_i = |*_{A_{pfs}}(x_i) - *_{B_{pfs}}(x_i)|$, $\underline{\Delta^*_i} = \min_i\{|*_{A_{pfs}}(x_i) - *_{B_{pfs}}(x_i)|\}$ and $\overline{\Delta^*_i} = \max_i\{|*_{A_{pfs}}(x_i) - *_{B_{pfs}}(x_i)|\}$. Moreover $\omega = (\omega_1, \omega_2, ..., \omega_n)$ denotes the weight vector of alternatives $x_i$'s with the conditions $\omega_i \in [0,1]$ and $\sum_{i=1}^{n} \omega_i = 1$.

For any SFSs $A_{sfs} = \{\langle x_i, \mu_{A_{sfs}}(x_i), \eta_{A_{sfs}}(x_i), \nu_{A_{sfs}}(x_i)\rangle : x_i \in X\}$ and $B_{sfs} = \{\langle x_i, \mu_{B_{sfs}}(x_i), \eta_{B_{sfs}}(x_i), \nu_{B_{sfs}}(x_i)\rangle : x_i \in X\}$, Khan et al. [17] provided the following similarity measures:

$$\mathcal{S}^{R1}_{sfs}(A_{sfs}, B_{sfs}) = \sum_{i=1}^{n} \omega_i \frac{\mu^2_{A_{sfs}}(x_i)\mu^2_{B_{sfs}}(x_i) + \eta^2_{A_{sfs}}(x_i)\eta^2_{B_{sfs}}(x_i) + \nu^2_{A_{sfs}}(x_i)\nu^2_{B_{sfs}}(x_i)}{(\mu^4_{A_{sfs}}(x_i) + \eta^4_{A_{sfs}}(x_i) + \nu^4_{A_{sfs}}(x_i))^{\frac{1}{2}} \times (\mu^4_{B_{sfs}}(x_i) + \eta^4_{B_{sfs}}(x_i) + \nu^4_{B_{sfs}}(x_i))^{\frac{1}{2}}}; \tag{15}$$

$$\mathcal{S}^{R2}_{sfs}(A_{sfs}, B_{sfs}) = \sum_{i=1}^{n} \omega_i \cos\Big[\frac{\pi}{2}(\max\{|\mu^2_{A_{sfs}}(x_i) - \mu^2_{B_{sfs}}(x_i)|, |\eta^2_{A_{sfs}}(x_i) - \eta^2_{B_{sfs}}(x_i)|, |\nu^2_{A_{sfs}}(x_i) - \nu^2_{B_{sfs}}(x_i)|\})\Big]; \tag{16}$$

$$\mathcal{S}^{R3}_{sfs}(A_{sfs}, B_{sfs}) = \sum_{i=1}^{n} \omega_i \cos\Big[\frac{\pi}{4}(|\mu^2_{A_{sfs}}(x_i) - \mu^2_{B_{sfs}}(x_i)| + |\eta^2_{A_{sfs}}(x_i) - \eta^2_{B_{sfs}}(x_i)| + |\nu^2_{A_{sfs}}(x_i) - \nu^2_{B_{sfs}}(x_i)|)\Big]; \tag{17}$$

$$\mathcal{S}^{R4}_{sfs}(A_{sfs}, B_{sfs}) = \sum_{i=1}^{n} \omega_i \cot\Big[\frac{\pi}{4} + \frac{\pi}{4}(\max\{|\mu^2_{A_{sfs}}(x_i) - \mu^2_{B_{sfs}}(x_i)|, |\eta^2_{A_{sfs}}(x_i) - \eta^2_{B_{sfs}}(x_i)|, |\nu^2_{A_{sfs}}(x_i) - \nu^2_{B_{sfs}}(x_i)|\})\Big]; \tag{18}$$

$$\mathcal{S}^{R5}_{sfs}(A_{sfs}, B_{sfs}) = \sum_{i=1}^{n} \omega_i \cot\Big[\frac{\pi}{4} + \frac{\pi}{8}(|\mu^2_{A_{sfs}}(x_i) - \mu^2_{B_{sfs}}(x_i)| + |\eta^2_{A_{sfs}}(x_i) - \eta^2_{B_{sfs}}(x_i)| + |\nu^2_{A_{sfs}}(x_i) - \nu^2_{B_{sfs}}(x_i)|)\Big]; \tag{19}$$

$$\mathcal{S}_{sfs}^{R6}(A_{sfs}, B_{sfs}) = \sum_{i=1}^{n} \omega_i \cos[\frac{\pi}{2}(\max\{|\mu_{A_{sfs}}^2(x_i) - \mu_{B_{sfs}}^2(x_i)|, |\eta_{A_{sfs}}^2(x_i) - \eta_{B_{sfs}}^2(x_i)|, |\nu_{A_{sfs}}^2(x_i) - \nu_{B_{sfs}}^2(x_i)|, |\mu_{A_{sfs}}^2(x_i) - \mu_{B_{sfs}}^2(x_i) + \eta_{A_{sfs}}^2(x_i) - \eta_{B_{sfs}}^2(x_i) + \nu_{A_{sfs}}^2(x_i) - \nu_{B_{sfs}}^2(x_i)|\})]; \quad (20)$$

$$\mathcal{S}_{sfs}^{R7}(A_{sfs}, B_{sfs}) = \sum_{i=1}^{n} \omega_i \cos[\frac{\pi}{4}(|\mu_{A_{sfs}}^2(x_i) - \mu_{B_{sfs}}^2(x_i)| + |\eta_{A_{sfs}}^2(x_i) - \eta_{B_{sfs}}^2(x_i)| + |\nu_{A_{sfs}}^2(x_i) - \nu_{B_{sfs}}^2(x_i)| + |\mu_{A_{sfs}}^2(x_i) - \mu_{B_{sfs}}^2(x_i) + \eta_{A_{sfs}}^2(x_i) - \eta_{B_{sfs}}^2(x_i) + \nu_{A_{sfs}}^2(x_i) - \nu_{B_{sfs}}^2(x_i)|)]; \quad (21)$$

$$\mathcal{S}_{sfs}^{R8}(A_{sfs}, B_{sfs}) = \sum_{i=1}^{n} \omega_i \cot[\frac{\pi}{4} + \frac{\pi}{4}(\max\{|\mu_{A_{sfs}}^2(x_i) - \mu_{B_{sfs}}^2(x_i)|, |\eta_{A_{sfs}}^2(x_i) - \eta_{B_{sfs}}^2(x_i)|, |\nu_{A_{sfs}}^2(x_i) - \nu_{B_{sfs}}^2(x_i)|, |\mu_{A_{sfs}}^2(x_i) - \mu_{B_{sfs}}^2(x_i) + \eta_{A_{sfs}}^2(x_i) - \eta_{B_{sfs}}^2(x_i) + \nu_{A_{sfs}}^2(x_i) - \nu_{B_{sfs}}^2(x_i)|\})]; \quad (22)$$

$$\mathcal{S}_{sfs}^{R9}(A_{sfs}, B_{sfs}) = \sum_{i=1}^{n} \omega_i \cot[\frac{\pi}{4} + \frac{\pi}{8}(|\mu_{A_{sfs}}^2(x_i) - \mu_{B_{sfs}}^2(x_i)| + |\eta_{A_{sfs}}^2(x_i) - \eta_{B_{sfs}}^2(x_i)| + |\nu_{A_{sfs}}^2(x_i) - \nu_{B_{sfs}}^2(x_i)| + |\mu_{A_{sfs}}^2(x_i) - \mu_{B_{sfs}}^2(x_i) + \eta_{A_{sfs}}^2(x_i) - \eta_{B_{sfs}}^2(x_i) + \nu_{A_{sfs}}^2(x_i) - \nu_{B_{sfs}}^2(x_i)|)], \quad (23)$$

$$\mathcal{S}_{sfs}^{K}(A_{sfs}, B_{sfs}) = \frac{\sum_{i=1}^{n} \omega_i(\mu_{A_{sfs}}^2(x_i)\mu_{B_{sfs}}^2(x_i) + \eta_{A_{sfs}}^2(x_i)\eta_{B_{sfs}}^2(x_i) + \nu_{A_{sfs}}^2(x_i)\nu_{B_{sfs}}^2(x_i))}{\sum_{i=1}^{n}(\max\{\mu_{A_{sfs}}^4(x_i), \mu_{B_{sfs}}^4(x_i)\} + \max\{\eta_{A_{sfs}}^4(x_i), \eta_{B_{sfs}}^4(x_i)\} + \max\{\nu_{A_{sfs}}^4(x_i), \nu_{B_{sfs}}^4(x_i)\})} \backslash (\sum_{i=1}^{n}\omega_i), \quad (24)$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)$, $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \omega_i = 1$.

For any GSFSs $A_{gsfs} = \{\langle x_i, \mu_{A_{gsfs}}(x_i), \eta_{A_{gsfs}}(x_i), \nu_{A_{gsfs}}(x_i)\rangle : x_i \in X\}$ and $B_{gsfs} = \{\langle x_i, \mu_{B_{gsfs}}(x_i), \eta_{B_{gsfs}}(x_i), \nu_{B_{gsfs}}(x_i)\rangle : x_i \in X\}$, Wu et al. [34] proposed the next similarity measures on the basis of cosine and cotangent functions:

$$\mathcal{S}_{gsfs}^{WCS1}(A_{gsfs}, B_{gsfs}) = \sum_{i=1}^{n} \omega_i \cos[\frac{\pi}{2}(\max\{|\mu_{A_{gsfs}}^p(x_i) - \mu_{B_{gsfs}}^p(x_i)|, |\eta_{A_{gsfs}}^p(x_i) - \eta_{B_{gsfs}}^p(x_i)|, |\nu_{A_{gsfs}}^p(x_i) - \nu_{B_{gsfs}}^p(x_i)|\})]; \quad (25)$$

$$\mathcal{S}_{gsfs}^{WCS2}(A_{gsfs}, B_{gsfs}) = \sum_{i=1}^{n} \omega_i \cos[\frac{\pi}{4}(|\mu_{A_{gsfs}}^p(x_i) - \mu_{B_{gsfs}}^p(x_i)| + |\eta_{A_{gsfs}}^p(x_i) - \eta_{B_{gsfs}}^p(x_i)| + |\nu_{A_{gsfs}}^p(x_i) - \nu_{B_{gsfs}}^p(x_i)|)]; \quad (26)$$

$$\mathcal{S}_{gsfs}^{WCS3}(A_{gsfs}, B_{gsfs}) = \sum_{i=1}^{n} \omega_i \cos[\frac{\pi}{2}(\max\{|\mu_{A_{gsfs}}^p(x_i) - \mu_{B_{gsfs}}^p(x_i)|, |\eta_{A_{gsfs}}^p(x_i) - \eta_{B_{gsfs}}^p(x_i)|, |\nu_{A_{gsfs}}^p(x_i) - \nu_{B_{gsfs}}^p(x_i)|, |\mu_{A_{gsfs}}^p(x_i) - \mu_{B_{gsfs}}^p(x_i) + \eta_{A_{gsfs}}^p(x_i) - \eta_{B_{gsfs}}^p(x_i) + \nu_{A_{gsfs}}^p(x_i) - \nu_{B_{gsfs}}^p(x_i)|\})]; \quad (27)$$

$$\mathcal{S}_{gsfs}^{WCS4}(A_{gsfs}, B_{gsfs}) = \sum_{i=1}^{n} \omega_i \cos[\frac{\pi}{4}(|\mu_{A_{gsfs}}^p(x_i) - \mu_{B_{gsfs}}^p(x_i)| + |\eta_{A_{gsfs}}^p(x_i) - \eta_{B_{gsfs}}^p(x_i)| + |\nu_{A_{gsfs}}^p(x_i) - \nu_{B_{gsfs}}^p(x_i)| + |\mu_{A_{gsfs}}^p(x_i) - \mu_{B_{gsfs}}^p(x_i) + \eta_{A_{gsfs}}^p(x_i) - \eta_{B_{gsfs}}^p(x_i) + \nu_{A_{gsfs}}^p(x_i) - \nu_{B_{gsfs}}^p(x_i)|)]; \quad (28)$$

$$\mathcal{S}_{gsfs}^{WCT1}(A_{gsfs}, B_{gsfs}) = \sum_{i=1}^{n} \omega_i \cot[\frac{\pi}{4} + \frac{\pi}{4}(\max\{|\mu_{A_{gsfs}}^p(x_i) - \mu_{B_{gsfs}}^p(x_i)|, |\eta_{A_{gsfs}}^p(x_i) - \eta_{B_{gsfs}}^p(x_i)|, |\nu_{A_{gsfs}}^p(x_i) - \nu_{B_{gsfs}}^p(x_i)|\})]; \quad (29)$$

$$\mathcal{S}_{gsfs}^{WCT2}(A_{gsfs}, B_{gsfs}) =$$
$$\sum_{i=1}^{n} \omega_i \cot[\frac{\pi}{4} + \frac{\pi}{8}(|\mu_{A_{gsfs}}^p(x_i) - \mu_{B_{gsfs}}^p(x_i)| + |\eta_{A_{gsfs}}^p(x_i) - \eta_{B_{gsfs}}^p(x_i)| + |\nu_{A_{gsfs}}^p(x_i) - \nu_{B_{gsfs}}^p(x_i)|)]; \quad (30)$$
$$\mathcal{S}_{gsfs}^{WCT3}(A_{gsfs}, B_{gsfs}) =$$
$$\sum_{i=1}^{n} \omega_i \cot[\frac{\pi}{4} + \frac{\pi}{4}(\max\{|\mu_{A_{gsfs}}^p(x_i) - \mu_{B_{gsfs}}^p(x_i)|, |\eta_{A_{gsfs}}^p(x_i) - \eta_{B_{gsfs}}^p(x_i)|, |\nu_{A_{gsfs}}^p(x_i) - \nu_{B_{gsfs}}^p(x_i)|,$$
$$|\mu_{A_{gsfs}}^p(x_i) - \mu_{B_{gsfs}}^p(x_i) + \eta_{A_{gsfs}}^p(x_i) - \eta_{B_{gsfs}}^p(x_i) + \nu_{A_{gsfs}}^p(x_i) - \nu_{B_{gsfs}}^p(x_i)|\})]; \quad (31)$$
$$\mathcal{S}_{gsfs}^{WCT4}(A_{gsfs}, B_{gsfs}) =$$
$$\sum_{i=1}^{n} \omega_i \cot[\frac{\pi}{4} + \frac{\pi}{8}(|\mu_{A_{gsfs}}^p(x_i) - \mu_{B_{gsfs}}^p(x_i)| + |\eta_{A_{gsfs}}^p(x_i) - \eta_{B_{gsfs}}^p(x_i)| + |\nu_{A_{gsfs}}^p(x_i) - \nu_{B_{gsfs}}^p(x_i)|$$
$$+|\mu_{A_{gsfs}}^p(x_i) - \mu_{B_{gsfs}}^p(x_i) + \eta_{A_{gsfs}}^p(x_i) - \eta_{B_{gsfs}}^p(x_i) + \nu_{A_{gsfs}}^p(x_i) - \nu_{B_{gsfs}}^p(x_i)|)], \quad (32)$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)$, $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \omega_i = 1$, and moreover, $p$ is a positive integer.

## 3.2 Similarity measure of E-PFSs

Now, we are in a position to introduce a new similarity measure of E-PFSs which satisfies all the axioms **(S1)-(S5)** given in Definition 3.1.

**Theorem 3.2.** *Let* $A_{epfs} = \{\langle x_i, \mu_{A_{epfs}}(x_i), \eta_{A_{epfs}}(x_i), \nu_{A_{epfs}}(x_i)\rangle : x_i \in X\}$ *and*

$$B_{epfs} = \{\langle x_i, \mu_{B_{epfs}}(x_i), \eta_{B_{epfs}}(x_i), \nu_{B_{epfs}}(x_i)\rangle : x_i \in X\},$$

*be two E-PFSs. The mapping* $\mathcal{S}_{epfs} : E - PFS(X) \times E - PFS(X) \to [0, 1]$ *given by*

$$\mathcal{S}_{epfs}(A_{epfs}, B_{epfs}) =$$
$$\frac{\sum_{i=1}^{n}(k_1 \min\{\mu_{A_{epfs}}(x_i), \mu_{B_{epfs}}(x_i)\} + k_2 \min\{\eta_{A_{epfs}}(x_i), \eta_{B_{epfs}}(x_i)\} + k_3 \min\{1 - \nu_{A_{epfs}}(x_i), 1 - \nu_{B_{epfs}}(x_i)\})}{\sum_{i=1}^{n}(k_1 \max\{\mu_{A_{epfs}}(x_i), \mu_{B_{epfs}}(x_i)\} + k_2 \max\{\eta_{A_{epfs}}(x_i), \eta_{B_{epfs}}(x_i)\} + k_3 \max\{1 - \nu_{A_{epfs}}(x_i), 1 - \nu_{B_{epfs}}(x_i)\})}, \quad (33)$$

*satisfies all the properties* ***(S1)-(S5)*** *given in Definition 3.1.*
*Note that the decision maker's preferences* $k_1, k_2$ *and* $k_3$ *satisfy the condition* $\sum_{l=1}^{3} k_l = 1$ *for any* $k_1, k_2, k_3 \geq 0$.

*Proof.* The proof of properties **(S1)** and **(S3)** is obvious.

Proof of **(S2)**: As follows from definition of $\mathcal{S}_{epfs}$, we conclude that

$$\mathcal{S}_{epfs}(A_{epfs}, B_{epfs}) = 1,$$

if and only if

$$\frac{\sum_{i=1}^{n}(k_1 \min\{\mu_{A_{epfs}}(x_i), \mu_{B_{epfs}}(x_i)\} + k_2 \min\{\eta_{A_{epfs}}(x_i), \eta_{B_{epfs}}(x_i)\} + k_3 \min\{1 - \nu_{A_{epfs}}(x_i), 1 - \nu_{B_{epfs}}(x_i)\})}{\sum_{i=1}^{n}(k_1 \max\{\mu_{A_{epfs}}(x_i), \mu_{B_{epfs}}(x_i)\} + k_2 \max\{\eta_{A_{epfs}}(x_i), \eta_{B_{epfs}}(x_i)\} + k_3 \max\{1 - \nu_{A_{epfs}}(x_i), 1 - \nu_{B_{epfs}}(x_i)\})} = 1,$$

if and only if

$$\min\{\mu_{A_{epfs}}(x_i), \mu_{B_{epfs}}(x_i)\} = \max\{\mu_{A_{epfs}}(x_i), \mu_{B_{epfs}}(x_i)\},$$
$$\min\{\eta_{A_{epfs}}(x_i), \eta_{B_{epfs}}(x_i)\} = \max\{\eta_{A_{epfs}}(x_i), \eta_{B_{epfs}}(x_i)\}, \text{ and}$$
$$\min\{1 - \nu_{A_{epfs}}(x_i), 1 - \nu_{B_{epfs}}(x_i)\} = \max\{1 - \nu_{A_{epfs}}(x_i), 1 - \nu_{B_{epfs}}(x_i)\},$$

for any $x_i \in X$.
These findings give rise to $A_{epfs} = B_{epfs}$, and in a compact form

$$\mathcal{S}_{epfs}(A_{epfs}, B_{epfs}) = 1, \quad \text{if and only if} \quad A_{epfs} = B_{epfs}.$$

Proof of **(S4)**: If it holds that $A_{epfs} \subseteq B_{epfs} \subseteq C_{epfs}$, then from Definition 2.9, we deduce the following results:

$$\mu_{A_{epfn}}(x_i) \leq \mu_{B_{epfn}}(x_i) \leq \mu_{C_{epfn}}(x_i), \tag{34}$$

$$\eta_{A_{epfn}}(x_i) \leq \eta_{B_{epfn}}(x_i) \leq \eta_{C_{epfn}}(x_i), \tag{35}$$

$$1-\nu_{A_{epfn}}(x_i) \leq 1-\nu_{B_{epfn}}(x_i) \leq 1-\nu_{C_{epfn}}(x_i), \tag{36}$$

for any $x_i \in X$.
The monotonicity conditions of (34)-(36) ensure that

$$\mu_{A_{epfn}}(x_i) = \min\{\mu_{A_{epfn}}(x_i), \mu_{C_{epfn}}(x_i)\}, \quad \mu_{C_{epfn}}(x_i) = \max\{\mu_{A_{epfn}}(x_i), \mu_{C_{epfn}}(x_i)\}; \tag{37}$$

$$\eta_{A_{epfn}}(x_i) = \min\{\eta_{A_{epfn}}(x_i), \eta_{C_{epfn}}(x_i)\}, \quad \eta_{C_{epfn}}(x_i) = \max\{\eta_{A_{epfn}}(x_i), \eta_{C_{epfn}}(x_i)\}; \tag{38}$$

$$1-\nu_{A_{epfn}}(x_i) = \min\{1-\nu_{A_{epfn}}(x_i), 1-\nu_{C_{epfn}}(x_i)\}, \quad 1-\nu_{C_{epfn}}(x_i) = \max\{1-\nu_{A_{epfn}}(x_i), 1-\nu_{C_{epfn}}(x_i)\}; \tag{39}$$

and

$$\mu_{A_{epfn}}(x_i) = \min\{\mu_{A_{epfn}}(x_i), \mu_{B_{epfn}}(x_i)\}, \quad \mu_{B_{epfn}}(x_i) = \max\{\mu_{A_{epfn}}(x_i), \mu_{B_{epfn}}(x_i)\}; \tag{40}$$

$$\eta_{A_{epfn}}(x_i) = \min\{\eta_{A_{epfn}}(x_i), \eta_{B_{epfn}}(x_i)\}, \quad \eta_{B_{epfn}}(x_i) = \max\{\eta_{A_{epfn}}(x_i), \eta_{B_{epfn}}(x_i)\}; \tag{41}$$

$$1-\nu_{A_{epfn}}(x_i) = \min\{1-\nu_{A_{epfn}}(x_i), 1-\nu_{B_{epfn}}(x_i)\}, \quad 1-\nu_{B_{epfn}}(x_i) = \max\{1-\nu_{A_{epfn}}(x_i), 1-\nu_{B_{epfn}}(x_i)\}. \tag{42}$$

Once again from (34)-(36), we have

$$\frac{\sum_{i=1}^{n}[k_1\mu_{A_{epfn}}(x_i) + k_2\eta_{A_{epfn}}(x_i) + k_3(1-\nu_{A_{epfn}}(x_i))]}{\sum_{i=1}^{n}[k_1\mu_{C_{epfn}}(x_i) + k_2\eta_{C_{epfn}}(x_i) + k_3(1-\nu_{C_{epfn}}(x_i))]} \leq \frac{\sum_{i=1}^{n}[k_1\mu_{A_{epfn}}(x_i) + k_2\eta_{A_{epfn}}(x_i) + k_3(1-\nu_{A_{epfn}}(x_i))]}{\sum_{i=1}^{n}[k_1\mu_{B_{epfn}}(x_i) + k_2\eta_{B_{epfn}}(x_i) + k_3(1-\nu_{B_{epfn}}(x_i))]},$$

for any parameters $k_1, k_2$ and $k_3$ satisfying $\sum_{l=1}^{3} k_l = 1$ for any $k_1, k_2, k_3 \geq 0$.

By the use of (37)-(42) together with the latter relation, the next result is immediate

$$\begin{aligned}
&\frac{\sum_{i=1}^{n}[k_1\min\{\mu_{A_{epfn}}(x_i), \mu_{C_{epfn}}(x_i)\} + k_2\min\{\eta_{A_{epfn}}(x_i), \eta_{C_{epfn}}(x_i)\} + k_3\min\{1-\nu_{A_{epfn}}(x_i), 1-\nu_{C_{epfn}}(x_i)\}]}{\sum_{i=1}^{n}[k_1\max\{\mu_{A_{epfn}}(x_i), \mu_{C_{epfn}}(x_i)\} + k_2\max\{\eta_{A_{epfn}}(x_i), \eta_{C_{epfn}}(x_i)\} + k_3\max\{1-\nu_{A_{epfn}}(x_i), 1-\nu_{C_{epfn}}(x_i)\}]} \\
&\leq \frac{\sum_{i=1}^{n}[k_1\min\{\mu_{A_{epfn}}(x_i), \mu_{B_{epfn}}(x_i)\} + k_2\min\{\eta_{A_{epfn}}(x_i), \eta_{B_{epfn}}(x_i)\} + k_3\min\{1-\nu_{A_{epfn}}(x_i), 1-\nu_{B_{epfn}}(x_i)\}]}{\sum_{i=1}^{n}[k_1\max\{\mu_{A_{epfn}}(x_i), \mu_{B_{epfn}}(x_i)\} + k_2\max\{\eta_{A_{epfn}}(x_i), \eta_{B_{epfn}}(x_i)\} + k_3\max\{1-\nu_{A_{epfn}}(x_i), 1-\nu_{B_{epfn}}(x_i)\}]},
\end{aligned}$$

that is,

$$\mathcal{S}_{epfs}(A_{epfs}, C_{epfs}) \leq \mathcal{S}_{epfs}(A_{epfs}, B_{epfs}).$$

By a similar reasoning given above, we conclude that

$$\mathcal{S}_{epfs}(A_{epfs}, C_{epfs}) \leq \mathcal{S}_{epfs}(B_{epfs}, C_{epfs}).$$

Proof of **(S5)**: We assume that $A_{epfs}$ is a crisp set, that is, $A_{epfs} = \{\langle x, 1, 0, 0\rangle\}$ (or $A_{epfs} = \{\langle x, 0, 0, 1\rangle\}$). In this regard, the complement set $A^c_{epfs}$ is defined as $A^c_{epfs} = \{\langle x, 0, 0, 1\rangle\}$ (or $A^c_{epfs} = \{\langle x, 1, 0, 0\rangle\}$). Hence,

$$\begin{aligned}
&\mathcal{S}_{epfs}(A_{epfs}, A^c_{epfs}) = \\
&\frac{\sum_{i=1}^{n}(k_1\min\{\mu_{A_{epfs}}(x_i), \mu_{A^c_{epfs}}(x_i)\} + k_2\min\{\eta_{A_{epfs}}(x_i), \eta_{A^c_{epfs}}(x_i)\} + k_3\min\{1-\nu_{A_{epfs}}(x_i), 1-\nu_{A^c_{epfs}}(x_i)\})}{\sum_{i=1}^{n}(k_1\max\{\mu_{A_{epfs}}(x_i), \mu_{A^c_{epfs}}(x_i)\} + k_2\max\{\eta_{A_{epfs}}(x_i), \eta_{A^c_{epfs}}(x_i)\} + k_3\max\{1-\nu_{A_{epfs}}(x_i), 1-\nu_{A^c_{epfs}}(x_i)\})}, \\
&= \frac{\sum_{i=1}^{n}(k_1\min\{1,0\} + k_2\min\{0,0\} + k_3\min\{1-0, 1-1\})}{\sum_{i=1}^{n}(k_1\max\{1,0\} + k_2\max\{0,0\} + k_3\max\{1-0, 1-1\})} = 0.
\end{aligned}$$

□

# 4 Applications of E-PFSs

Because of widely applications of similarity measure in multiple criteria decision making, we are going here to apply the proposed similarity measure for E-PFSs to the filed of building material recognition [25], pattern recognition problem [17], and the selection process of mega projects in developing countries [17].

## 4.1 Recognizing building materials

In the following part, we assume that $M_i$ $(i = 1, 2, ..., m)$ are $m$ alternatives, and $x_j$ $(j = 1, 2, ..., n)$ are $n$ criteria.

**Similarity-based algorithm for recognizing building materials**

- Input: We construct the decision matrix.
- Output: Given the weights $w_j$ $(j = 1, 2, ..., n)$ of criteria, we characterize the ideal alternative $M$ using the principle of maximum degree of similarity between $M$ and $M_i$ $(i = 1, 2, ..., m)$ as:

$$i^* = \arg \max_{1 \le i \le m} \{\mathcal{S}(M_i, M)\}. \tag{43}$$

According to *Input* step of *similarity-based algorithm for recognizing building materials*, Wei's [32] and Wu et al.'s [34] evaluation of building materials of $M_1$: sealant, $M_2$: floor varnish, $M_3$: wall paint and $M_4$: polyvinyl chloride flooring using the criteria $X = \{x_1, x_2, ..., x_7\}$ will be in the form of PFSs given in Table 1. Here, the weight vector of criteria is considered to be $w = (w_1, w_2, ..., w_7) = (0.12, 0.15, 0.09, 0.16, 0.20, 0.10, 0.18)$.

By performing *Output* step of *similarity-based algorithm for recognizing building materials*, the ranking order of building materials can be derived in accordance with the degree of similarity between $M_i$'s and the unknown material $M$ as listed in Table 2.

Table 1. The data corresponding to building materials.

| Material | $M_1$ | $M_2$ | $M_2$ | $M_4$ | $M$ |
|---|---|---|---|---|---|
| $x_1$ | (0.17, 0.53, 0.13) | (0.51, 0.24, 0.21) | (0.31, 0.39, 0.25) | (1.00, 0.00, 0.00) | (0.91, 0.03, 0.05) |
| $x_2$ | (0.10, 0.81, 0.05) | (0.62, 0.12, 0.07) | (0.60, 0.26, 0.11) | (1.00, 0.00, 0.00) | (0.78, 0.12, 0.07) |
| $x_3$ | (0.53, 0.33, 0.09) | (1.00, 0.00, 0.00) | (0.91, 0.03, 0.02) | (0.85, 0.09, 0.05) | (0.90, 0.05, 0.02) |
| $x_4$ | (0.89, 0.08, 0.03) | (0.13, 0.64, 0.21) | (0.07, 0.09, 0.05) | (0.74, 0.16, 0.10) | (0.68, 0.08, 0.21) |
| $x_5$ | (0.42, 0.35, 0.18) | (0.03, 0.82, 0.13) | (0.04, 0.85, 0.10) | (0.02, 0.89, 0.05) | (0.05, 0.87, 0.06) |
| $x_6$ | (0.08, 0.89, 0.02) | (0.73, 0.15, 0.08) | (0.68, 0.26, 0.06) | (0.08, 0.84, 0.06) | (0.13, 0.75, 0.09) |
| $x_7$ | (0.33, 0.51, 0.12) | (0.52, 0.31, 0.16) | (0.15, 0.76, 0.07) | (0.16, 0.71, 0.05) | (0.15, 0.73, 0.08) |

Table 2. Similarity measures between building materials $M_i$ $(i = 1, 2, 3, 4)$ and the unknown material $M$.

| Similarity measure | $\mathcal{S}(M_1, M)$ | $\mathcal{S}(M_2, M)$ | $\mathcal{S}(M_3, M)$ | $\mathcal{S}(M_4, M)$ |
|---|---|---|---|---|
| Wei [32]'s measures | | | | |
| $\mathcal{S}_{pfs}^{W1}$ | 0.7160 | 0.7630 | 0.8580 | **0.9940** |
| $\mathcal{S}_{pfs}^{W2}$ | 0.5560 | 0.6570 | 0.6930 | **0.9200** |
| $\mathcal{S}_{pfs}^{W3}$ | 0.6600 | 0.7620 | 0.8300 | **0.9010** |
| Wu et al. [34]'s measures | | | | |
| $\mathcal{S}_{epfs}^{WCS1}$ | 0.7485 | 0.8332 | 0.8410 | **0.9858** |
| $\mathcal{S}_{epfs}^{WCS2}$ | 0.7665 | 0.8363 | 0.8849 | **0.9864** |
| $\mathcal{S}_{epfs}^{WCS3}$ | 0.7485 | 0.8332 | 0.8079 | **0.9857** |
| $\mathcal{S}_{epfs}^{WCS4}$ | 0.7464 | 0.8265 | 0.8056 | **0.9836** |
| $\mathcal{S}_{epfs}^{WCT1}$ | 0.5115 | 0.6125 | 0.6865 | **0.8772** |
| $\mathcal{S}_{epfs}^{WCT2}$ | 0.5267 | 0.6297 | 0.7274 | **0.8769** |
| $\mathcal{S}_{epfs}^{WCT3}$ | 0.515 | 0.6125 | 0.6665 | **0.8746** |
| $\mathcal{S}_{epfs}^{WCT4}$ | 0.5102 | 0.6064 | 0.6652 | **0.8600** |
| Proposed measure | | | | |
| $\mathcal{S}_{epfs}$ (for $k_1 = \frac{1}{8}, k_2 = \frac{2}{8}, k_3 = \frac{5}{8}$) | 0.8803 | 0.9547 | 0.9336 | **0.9675** |
| $\mathcal{S}_{epfs}$ (for $k_1 = \frac{5}{8}, k_2 = \frac{2}{8}, k_3 = \frac{1}{8}$) | 0.6317 | 0.8572 | 0.8164 | **0.9270** |

With reference to the results of Table 2, we find that Wei [32]'s and Wu et al. [34]'s similarity measures return the pair of $(M_4, M)$ as the maximum similarity value, while, the proposed similarity measure is sensitive to the weight of satisfaction grade $\mu$, abstinence grade $\eta$, and dissatisfaction grade $\nu$, and the impact of such a sensitivity can be seen in the last two rows.

For further investigation, we consider the above recognition of building materials once again, but now by a rather different data of first row in Table 1 being replaced by new E-PFSs. In such a case, all the above-mention similarity measures are not able to return any logical result which demonstrates the proposed similarity measure has more feasibility and effectiveness than the others described above.

Table 3. The E-PFS data corresponding to building materials.

| Material | $M_1$ | $M_2$ | $M_2$ | $M_4$ | $M$ |
|---|---|---|---|---|---|
| $x_1$ | **(0.97, 0.93, 0.13)** | **(0.91, 0.94, 0.81)** | **(0.91, 0.99, 0.25)** | (1.00, 0.00, 0.00) | **(091, 0.93, 0.85)** |
| $x_2$ | (0.10, 0.81, 0.05) | (0.62, 0.12, 0.07) | (0.60, 0.26, 0.11) | (1.00, 0.00, 0.00) | (0.78, 0.12, 0.07) |
| $x_3$ | (0.53, 0.33, 0.09) | (1.00, 0.00, 0.00) | (0.91, 0.03, 0.02) | (0.85, 0.09, 0.05) | (0.90, 0.05, 0.02) |
| $x_4$ | (0.89, 0.08, 0.03) | (0.13, 0.64, 0.21) | (0.07, 0.09, 0.05) | (0.74, 0.16, 0.10) | (0.68, 0.08, 0.21) |
| $x_5$ | (0.42, 0.35, 0.18) | (0.03, 0.82, 0.13) | (0.04, 0.85, 0.10) | (0.02, 0.89, 0.05) | (0.05, 0.87, 0.06) |
| $x_6$ | (0.08, 0.89, 0.02) | (0.73, 0.15, 0.08) | (0.68, 0.26, 0.06) | (0.08, 0.84, 0.06) | (0.13, 0.75, 0.09) |
| $x_7$ | (0.33, 0.51, 0.12) | (0.52, 0.31, 0.16) | (0.15, 0.76, 0.07) | (0.16, 0.71, 0.05) | (0.15, 0.73, 0.08) |

Table 4. Similarity measures between building materials $M_i$ $(i = 1, 2, 3, 4)$ and unknown material $M$ in the case where some of data are E-PSFs.

| Similarity measure | $\mathcal{S}(M_1, M)$ | $\mathcal{S}(M_2, M)$ | $\mathcal{S}(M_3, M)$ | $\mathcal{S}(M_4, M)$ |
|---|---|---|---|---|
| Wei [32]'s measures | | | | |
| $\mathcal{S}_{pfs}^{W1}$ | * | * | * | * |
| $\mathcal{S}_{pfs}^{W2}$ | * | * | * | * |
| $\mathcal{S}_{pfs}^{W3}$ | * | * | * | * |
| Wu et al. [34]'s measures | | | | |
| $\mathcal{S}_{epfs}^{WCS1}$ | * | * | * | * |
| $\mathcal{S}_{epfs}^{WCS2}$ | * | * | * | * |
| $\mathcal{S}_{epfs}^{WCS3}$ | * | * | * | * |
| $\mathcal{S}_{epfs}^{WCS4}$ | * | * | * | * |
| $\mathcal{S}_{epfs}^{WCT1}$ | * | * | * | * |
| $\mathcal{S}_{epfs}^{WCT2}$ | * | * | * | * |
| $\mathcal{S}_{epfs}^{WCT3}$ | * | * | * | * |
| $\mathcal{S}_{epfs}^{WCT4}$ | * | * | * | * |
| Proposed measure | | | | |
| $\mathcal{S}_{epfs}$ (for $k_1 = \frac{1}{8}, k_2 = \frac{2}{8}, k_3 = \frac{5}{8}$) | 0.8503 | **0.9849** | 0.9173 | 0.8563 |
| $\mathcal{S}_{epfs}$ (for $k_1 = \frac{5}{8}, k_2 = \frac{2}{8}, k_3 = \frac{1}{8}$) | 0.7487 | **0.9473** | 0.9254 | 0.8608 |

## 4.2 Recognizing patterns

Here, we are going to provide the decision maker with an algorithm whose output exhibits the most similar pattern to an input pattern from a set of reference patterns.

**Similarity-based algorithm for recognizing patterns**

We assume that $P_1$ and $P_2$ are two known patterns and $P$ is the unknown pattern.

- Question: Which pattern does $P$ belongs to?
- Answer: The solution is determined by the following two steps:

**Step 1.** We compute the similarity measures $\mathcal{S}(P_1, P)$ and $\mathcal{S}(P_2, P)$.

**Step 2.** We assign $P$ to the class of a pattern having the larger value of similarity measure in Step 1.

In order to have a better understanding of proposed similarity measure performance, we adopt here a variety of counter examples which are concerned with SFSs, and investigated by Khan et al. [17] in the field of pattern recognition.

We suppose that $P_1$ and $P_2$ are two known patterns and $P$ is the unknown pattern which are represented in the form of SFSs on $X = \{x_1, x_2, x_3\}$ in Table 5. The aim here is to assign the unknown pattern to a specific class of known patterns which is returned by *similarity-based algorithm for recognizing patterns* depending on how the similarity measures are ordered.

*Step 1.* The similarity values of $\mathcal{S}(P_1, P)$ and $\mathcal{S}(P_2, P)$ have been reported in Table 6.

Table 5. The data corresponding to known patterns.

| | Known patterns $P_1$ | $P_2$ | Unknown pattern $P$ |
|---|---|---|---|
| $x_1$ | (0.7, 0.5, 0.3) | (0.5, 0.7, 0.3) | (0.5, 0.4, 0.4) |
| $x_2$ | (0.4, 0.3, 0.5) | (0.3, 0.4, 0.5) | (0.4, 0.6, 0.5) |
| $x_3$ | (0.6, 0.4, 0.3) | (0.4, 0.6, 0.3) | (0.6, 0.7, 0.3) |

Table 6. Similarity measures between known patterns $P_i$ $(i = 1, 2)$ and unknown pattern $P$ corresponding to Table 5.

| Similarity measure | $\mathcal{S}(P_1, P)$ | $\mathcal{S}(P_2, P)$ |
|---|---|---|
| Rafiq et al. [21]'s measures | | |
| $\mathcal{S}_{sfs}^{R1}$ | 0.8813 | 0.9101 |
| $\mathcal{S}_{sfs}^{R2}$ | 0.9033 | 0.9236 |
| $\mathcal{S}_{sfs}^{R3}$ | **0.9104** | **0.9104** |
| $\mathcal{S}_{sfs}^{R4}$ | 0.6354 | 0.6780 |
| $\mathcal{S}_{sfs}^{R5}$ | **0.7679** | **0.7679** |
| $\mathcal{S}_{sfs}^{R6}$ | 0.8993 | 0.8829 |
| $\mathcal{S}_{sfs}^{R7}$ | **0.8829** | **0.8829** |
| $\mathcal{S}_{sfs}^{R8}$ | 0.6278 | 0.6025 |
| $\mathcal{S}_{sfs}^{R9}$ | **0.6025** | **0.6025** |
| Khan et al. [17]'s measure | | |
| $\mathcal{S}_{sfs}^{K}$ | 0.5559 | 0.5758 |
| Proposed measure | | |
| $\mathcal{S}_{epfs}$ (for $k_1 = \frac{1}{8}, k_2 = \frac{2}{8}, k_3 = \frac{5}{8}$) | 0.8581 | 0.8667 |
| $\mathcal{S}_{epfs}$ (for $k_1 = \frac{5}{8}, k_2 = \frac{2}{8}, k_3 = \frac{1}{8}$) | 0.8214 | 0.7910 |

*Step 2.* As can be observed from the data in Table 6, the following issues are addressed:

- Some similarity measures are not able to distinguish the difference of similarity degree of both known patterns $P_i$ $(i = 1, 2)$ and the unknown pattern $P$ (see the results of $\mathcal{S}_{sfs}^{R3}$, $\mathcal{S}_{sfs}^{R5}$, $\mathcal{S}_{sfs}^{R7}$ and $\mathcal{S}_{sfs}^{R9}$);
- Some similarity measures reflect less similarity for $(P_1, P)$ compared to $(P_2, P)$ (see the results of $\mathcal{S}_{sfs}^{R1}$, $\mathcal{S}_{sfs}^{R2}$, $\mathcal{S}_{sfs}^{R4}$ and $\mathcal{S}_{sfs}^{K}$), and
- Some similarity measures reflect greater similarity for $(P_1, P)$ compared to $(P_2, P)$ (see the results of $\mathcal{S}_{sfs}^{R6}$ and $\mathcal{S}_{sfs}^{R8}$).

These findings implicate that the existing similarity measures are not enough sensitive to the influencing factors: $k_1$ (the weight of satisfaction grade), $k_2$ (the weight of abstinence grade), and $k_3$ (the weight of dissatisfaction grade).

Table 7. The data corresponding to known patterns.

| | Known patterns $P_1$ | $P_2$ | Unknown pattern $P$ |
|---|---|---|---|
| $x_1$ | (0.5, 0.7, 0.3) | (0.6, 0.7, 0.3) | (0.6, 0.4, 0.3) |
| $x_2$ | (0.3, 0.8, 0.4) | (0.8, 0.4, 0.4) | (0.5, 0.5, 0.4) |
| $x_3$ | (0.6, 0.3, 0.1) | (0.6, 0.4, 0.2) | (0.7, 0.4, 0.2) |

The remaining tables provide other counter examples for the existing SFS similarity measures compared to the proposed one in pattern recognition. By the results of Tables 7-12, it is brought out that the existing SFS similarity measures cannot properly classify the unknown pattern while the proposed E-PSF similarity measure is able to logically classify the unknown pattern. This shows that the parametric form of E-PSF similarity measure is more applicable than the other existing measures in a pattern recognition problem.

Table 8. Similarity measures between known patterns $P_i$ $(i = 1, 2)$ and unknown pattern $P$ corresponding to Table 7.

| Similarity measure | $\mathcal{S}(P_1, P)$ | $\mathcal{S}(P_2, P)$ |
|---|---|---|
| Rafiq et al. [21]'s measures | | |
| $\mathcal{S}_{sfs}^{R1}$ | 0.8617 | 0.9070 |
| $\mathcal{S}_{sfs}^{R2}$ | **0.8887** | **0.8887** |
| $\mathcal{S}_{sfs}^{R3}$ | 0.8419 | 0.8916 |
| $\mathcal{S}_{sfs}^{R4}$ | **0.6381** | **0.6381** |
| $\mathcal{S}_{sfs}^{R5}$ | 0.7256 | 0.7839 |
| $\mathcal{S}_{sfs}^{R6}$ | 0.8741 | 0.8887 |
| $\mathcal{S}_{sfs}^{R7}$ | 0.8741 | 0.8887 |
| $\mathcal{S}_{sfs}^{R8}$ | 0.5971 | 0.6381 |
| $\mathcal{S}_{sfs}^{R9}$ | 0.5971 | 0.6381 |
| Khan et al. [17]'s measure | | |
| $\mathcal{S}_{sfs}^{K}$ | 0.5038 | 0.5646 |
| Proposed measure | | |
| $\mathcal{S}_{epfs}$ (for $k_1 = \frac{1}{8}, k_2 = \frac{2}{8}, k_3 = \frac{5}{8}$) | 0.8614 | 0.9241 |
| $\mathcal{S}_{epfs}$ (for $k_1 = \frac{5}{8}, k_2 = \frac{2}{8}, k_3 = \frac{1}{8}$) | 0.7667 | 0.8228 |

Table 9. The data corresponding to known patterns.

| | Known patterns | | Unknown pattern |
|---|---|---|---|
| | $P_1$ | $P_2$ | $P$ |
| $x_1$ | (0.5, 0.7, 0.3) | (0.6, 0.7, 0.3) | (0.6, 0.4, 0.3) |
| $x_2$ | (0.3, 0.8, 0.4) | (0.8, 0.4, 0.4) | (0.5, 0.5, 0.4) |
| $x_3$ | (0.5, 0.6, 0.4) | (0.6, 0.5, 0.4) | (0.6, 0.5, 0.2) |

Table 10. Similarity measures between known patterns $P_i$ $(i = 1, 2)$ and unknown pattern $P$ corresponding to Table 9.

| Similarity measure | $\mathcal{S}(P_1, P)$ | $\mathcal{S}(P_2, P)$ |
|---|---|---|
| Rafiq et al. [21]'s measures | | |
| $\mathcal{S}_{sfs}^{R1}$ | 0.8322 | 0.8977 |
| $\mathcal{S}_{sfs}^{R2}$ | **0.8897** | **0.8897** |
| $\mathcal{S}_{sfs}^{R3}$ | 0.8170 | 0.8926 |
| $\mathcal{S}_{sfs}^{R4}$ | 0.6425 | 0.6425 |
| $\mathcal{S}_{sfs}^{R5}$ | 0.7020 | 0.7863 |
| $\mathcal{S}_{sfs}^{R6}$ | **0.8897** | **0.8897** |
| $\mathcal{S}_{sfs}^{R7}$ | 0.8741 | 0.8897 |
| $\mathcal{S}_{sfs}^{R8}$ | **0.6425** | **0.6425** |
| $\mathcal{S}_{sfs}^{R9}$ | 0.5971 | 0.6425 |
| Khan et al. [17]'s measure | | |
| $\mathcal{S}_{sfs}^{K}$ | 0.4921 | 0.5856 |
| Proposed measure | | |
| $\mathcal{S}_{epfs}$ (for $k_1 = \frac{1}{8}, k_2 = \frac{2}{8}, k_3 = \frac{5}{8}$) | 0.8293 | 0.8679 |
| $\mathcal{S}_{epfs}$ (for $k_1 = \frac{5}{8}, k_2 = \frac{2}{8}, k_3 = \frac{1}{8}$) | 0.7568 | 0.8387 |

Table 11. The data corresponding to known patterns.

| | Known patterns | | Unknown pattern |
|---|---|---|---|
| | $P_1$ | $P_2$ | $P$ |
| $x_1$ | (0.5, 0.7, 0.3) | (0.6, 0.7, 0.3) | (0.6, 0.4, 0.3) |
| $x_2$ | (0.3, 0.8, 0.4) | (0.8, 0.4, 0.4) | (0.5, 0.5, 0.4) |
| $x_3$ | (0.5, 0.6, 0.4) | (0.6, 0.5, 0.4) | (0.5, 0.5, 0.2) |

Table 12. Similarity measures between known patterns $P_i$ $(i = 1, 2)$ and unknown pattern $P$ corresponding to Table 11.

| Similarity measure | $\mathcal{S}(P_1, P)$ | $\mathcal{S}(P_2, P)$ |
|---|---|---|
| Rafiq et al. [21]'s measures | | |
| $\mathcal{S}_{sfs}^{R1}$ | 0.8487 | 0.8945 |
| $\mathcal{S}_{sfs}^{R2}$ | **0.8897** | **0.8897** |
| $\mathcal{S}_{sfs}^{R3}$ | 0.8419 | 0.8770 |
| $\mathcal{S}_{sfs}^{R4}$ | **0.6425** | **0.6425** |
| $\mathcal{S}_{sfs}^{R5}$ | 0.7256 | 0.7610 |
| $\mathcal{S}_{sfs}^{R6}$ | **0.8741** | **0.8741** |
| $\mathcal{S}_{sfs}^{R7}$ | **0.8741** | **0.8741** |
| $\mathcal{S}_{sfs}^{R8}$ | **0.5971** | **0.5971** |
| $\mathcal{S}_{sfs}^{R9}$ | **0.5971** | **0.5971** |
| Khan et al. [17]'s measure | | |
| $\mathcal{S}_{sfs}^{K}$ | 0.4971 | 0.5493 |
| Proposed measure | | |
| $\mathcal{S}_{epfs}$ (for $k_1 = \frac{1}{8}, k_2 = \frac{2}{8}, k_3 = \frac{5}{8}$) | 0.8344 | 0.8616 |
| $\mathcal{S}_{epfs}$ (for $k_1 = \frac{5}{8}, k_2 = \frac{2}{8}, k_3 = \frac{1}{8}$) | 0.7832 | 0.8065 |

Once again, we re-consider the above-mentioned problem of pattern recognition by a rather different data in the first row of Table 13 which is given in the form of E-PFS. In this case, all the above-mention SFS similarity measures are not able to return any logical result which demonstrates the proposed E-PFS similarity measure is more effective than the others (see Table 14).

Table 13. The data corresponding to known patterns.

| | Known patterns | | Unknown pattern |
|---|---|---|---|
| | $P_1$ | $P_2$ | $P$ |
| $x_1$ | **(0.7, 0.7, 0.3)** | (0.6, 0.7, 0.3) | (0.6, 0.4, 0.3) |
| $x_2$ | (0.3, 0.8, 0.4) | (0.8, 0.4, 0.4) | (0.5, 0.5, 0.4) |
| $x_3$ | (0.5, 0.6, 0.4) | (0.6, 0.5, 0.4) | (0.5, 0.5, 0.2) |

Table 14. Similarity measures between known patterns $P_i$ $(i = 1, 2)$ and unknown pattern $P$ corresponding to Table 13. '$*$' indicates that this similarity measure does not mean.

| Similarity measure | $\mathcal{S}(P_1, P)$ | $\mathcal{S}(P_2, P)$ |
|---|---|---|
| Rafiq et al. [21]'s measures | | |
| $\mathcal{S}_{sfs}^{R1}$ | $*$ | $*$ |
| $\mathcal{S}_{sfs}^{R2}$ | $*$ | $*$ |
| $\mathcal{S}_{sfs}^{R3}$ | $*$ | $*$ |
| $\mathcal{S}_{sfs}^{R4}$ | $*$ | $*$ |
| $\mathcal{S}_{sfs}^{R5}$ | $*$ | $*$ |
| $\mathcal{S}_{sfs}^{R6}$ | $*$ | $*$ |
| $\mathcal{S}_{sfs}^{R7}$ | $*$ | $*$ |
| $\mathcal{S}_{sfs}^{R8}$ | $*$ | $*$ |
| $\mathcal{S}_{sfs}^{R9}$ | $*$ | $*$ |
| Khan et al. [17]'s measure | | |
| $\mathcal{S}_{sfs}^{K}$ | $*$ | $*$ |
| Proposed measure | | |
| $\mathcal{S}_{epfs}$ (for $k_1 = \frac{1}{8}, k_2 = \frac{2}{8}, k_3 = \frac{5}{8}$) | 0.8354 | 0.8616 |
| $\mathcal{S}_{epfs}$ (for $k_1 = \frac{5}{8}, k_2 = \frac{2}{8}, k_3 = \frac{1}{8}$) | 0.7905 | 0.8065 |

## 4.3 Selection of the best mega project

In what follows, we suppose that $MP_i$ $(i = 1, 2, ..., m)$ are $m$ alternatives, and $x_j$ $(j = 1, 2, ..., n)$ are $n$ criteria.

**Similarity-based algorithm for selecting mega project**

- Input: We construct the decision matrix, and do the following steps:

  **Step 1.** Given the weights $w_j$ $(j = 1, 2, ..., n)$ of criteria, we characterize the ideal alternative $IMP$ as follows:

$$(\mu_{IMP_{sfs}}(x_j), \eta_{IMP_{sfs}}(x_j), \nu_{IMP_{sfs}}(x_j)) = \\ (\left\{ \begin{array}{l} \max_{1\le i\le 5}\{\mu_{MP_{i-sfs}}(x_j)\} \\ \min_{1\le i\le 5}\{\mu_{MP_{i-sfs}}(x_j)\} \end{array} \right\}, \left\{ \begin{array}{l} \min_{1\le i\le 5}\{\eta_{MP_{i-sfs}}(x_j)\} \\ \max_{1\le i\le 5}\{\eta_{MP_{i-sfs}}(x_j)\} \end{array} \right\}, \left\{ \begin{array}{l} \min_{1\le i\le 5}\{\nu_{MP_{i-sfs}}(x_j)\} \\ \max_{1\le i\le 5}\{\nu_{MP_{i-sfs}}(x_j)\} \end{array} \right\}), \quad (44)$$

  where the first and second rows stand for the benefit and cost criteria, respectively.

  **Step 2.** We compute the similarity measure of $\mathcal{S}(MP_i, IMP)$, for $i = 1, 2, ..., n$.

- Output: We make ranking of all the alternatives as:

  **Step 3.** By the use of similarity measure $\mathcal{S}$, we conclude that $MP_{i_1} \succ MP_{i_2}$ if $\mathcal{S}(MP_{i_1}, IMP) > \mathcal{S}(MP_{i_2}, IMP)$ for any $i_1, i_2 = 1, 2, ..., n$.

One of the critical problems in under-developing countries is the process of selecting mega projects whose main characteristics are large investment commitment, huge complexity and long-lasting impact on the society, economy and environment.

Following Khan et al. [17], we assume that an specified country is going to invest in starting a mega project whose loan is approved by the world bank. Since that country must refund the investment money within a determined time period, the country considers five projects including $MP_1$: *one million house construction*; $MP_2$: *dam construction*; $MP_3$: *orange metro train*; $MP_4$: *invest in industry*; and $MP_5$: *power sector*.

In order for selecting a priority project, a set of criteria including $x_1$: *long term benefits*; $x_2$: *time*; $x_3$: *impact*; $x_4$: *revenue generated*; $x_5$: *costs*; and $x_6$: *short term benefits* is considered by a group of experts.

Moreover, the weight vector of all criteria $x_j$ $(j = 1, 2, ..., 6)$ is supposed to be $w = (0.12, 0.25, 0.09, 0.16, 0.20, 0.18)$. The group of experts evaluate mega projects $MP_i$ $(i = 1, 2, ..., 5)$ with respect to the criteria $x_j$ $(j = 1, 2, ..., 6)$ in the form of SFS decision matrix given in Table 15. With the use of *similarity-based algorithm for selecting mega project*, we perform:

*Step 1.* The ideal mega project $IMP$ is constructed from given data as:

$$(\mu_{IMP_{sfs}}(x_j), \eta_{IMP_{sfs}}(x_j), \nu_{IMP_{sfs}}(x_j)) = (\max_{1\le i\le 5}\{\mu_{MP_{i-sfs}}(x_j)\}, \min_{1\le i\le 5}\{\eta_{MP_{i-sfs}}(x_j)\}, \min_{1\le i\le 5}\{\nu_{MP_{i-sfs}}(x_j)\}),$$

which is represented in the last column of Table 15.

*Steps 2 and 3.* The ranking orders of mega projects are given in Table 16 in accordance with the similarity measure between each mega project $MP_i$ and the ideal mega project $IMP$.

Table 15. The mega projects $MP_i$ $(i = 1, 2, ..., 5)$ and the ideal mega project $IMP$.

| | $MP_1$ | $MP_2$ | $MP_3$ | $MP_4$ | $MP_5$ | $IMP$ |
|---|---|---|---|---|---|---|
| $x_1$ | (0.53, 0.33, 0.09) | (0.73, 0.12, 0.08) | (0.91, 0.03, 0.02) | (0.85, 0.09, 0.05) | (0.90, 0.05, 0.02) | (0.91, 0.03, 0.02) |
| $x_2$ | (0.89, 0.08, 0.03) | (0.13, 0.64, 0.21) | (0.07, 0.09, 0.05) | (0.74, 0.16, 0.10) | (0.68, 0.08, 0.21) | (0.89, 0.08, 0.03) |
| $x_3$ | (0.42, 0.35, 0.18) | (0.03, 0.82, 0.13) | (0.04, 0.85, 0.10) | (0.02, 0.89, 0.05) | (0.05, 0.87, 0.06) | (0.42, 0.35, 0.05) |
| $x_4$ | (0.08, 0.89, 0.02) | (0.73, 0.15, 0.08) | (0.68, 0.26, 0.06) | (0.08, 0.84, 0.06) | (0.13, 0.75, 0.09) | (0.73, 0.15, 0.02) |
| $x_5$ | (0.33, 0.51, 0.12) | (0.52, 0.31, 0.16) | (0.15, 0.76, 0.07) | (0.16, 0.71, 0.05) | (0.15, 0.73, 0.08) | (0.52, 0.31, 0.05) |
| $x_6$ | (0.17, 0.53, 0.13) | (0.51, 0.24, 0.21) | (0.31, 0.39, 0.25) | (0.81, 0.15, 0.09) | (0.91, 0.03, 0.05) | (0.91, 0.03, 0.05) |

Table 16. Similarity measures between mega projects $MP_i$ $(i = 1, 2, ..., 5)$ and ideal mega project $IMP$ corresponding to Table 15.

| Similarity measure | $S(MP_1, IMP)$ | $S(MP_2, IMP)$ | $S(MP_3, IMP)$ | $S(MP_4, IMP)$ | $S(MP_5, IMP)$ | Ranking |
|---|---|---|---|---|---|---|
| Rafiq et al. [21]'s measures | | | | | | |
| $S_{sfs}^{R1}$ | 0.6122 | 0.7165 | 0.6229 | 0.6857 | 0.6869 | $MP_2 > MP_5 > MP_4 > MP_3 > MP_1$ |
| $S_{sfs}^{R2}$ | 0.7241 | 0.7251 | 0.6330 | 0.8031 | 0.8266 | $MP_5 > MP_4 > MP_2 > MP_1 > MP_3$ |
| $S_{sfs}^{R3}$ | 0.6742 | 0.5168 | 0.4881 | 0.7363 | 0.7362 | $MP_4 > MP_5 > MP_1 > MP_2 > MP_3$ |
| $S_{sfs}^{R4}$ | 0.5925 | 0.5669 | 0.4593 | 0.5720 | 0.6237 | $MP_5 > MP_1 > MP_4 > MP_2 > MP_3$ |
| $S_{sfs}^{R5}$ | 0.6727 | 0.6806 | 0.6290 | 0.6768 | 0.7041 | $MP_5 > MP_2 > MP_4 > MP_1 > MP_3$ |
| $S_{sfs}^{R6}$ | 0.7241 | 0.7251 | 0.6330 | 0.8050 | 0.8266 | $MP_5 > MP_4 > MP_2 > MP_1 > MP_3$ |
| $S_{sfs}^{R7}$ | 0.7231 | 0.7235 | 0.6316 | 0.8043 | 0.8242 | $MP_5 > MP_4 > MP_2 > MP_1 > MP_3$ |
| $S_{sfs}^{R8}$ | 0.5925 | 0.5669 | 0.4593 | 0.5720 | 0.6237 | $MP_5 > MP_1 > MP_4 > MP_2 > MP_3$ |
| $S_{sfs}^{R9}$ | 0.5895 | 0.5657 | 0.4583 | 0.5716 | 0.6218 | $MP_5 > MP_1 > MP_4 > MP_2 > MP_3$ |
| Khan et al. [17]'s measure | | | | | | |
| $S_{sfs}^{K}$ | 0.0650 | 0.0546 | 0.0481 | 0.0798 | 0.0894 | $MP_5 > MP_4 > MP_1 > MP_2 > MP_3$ |
| Proposed measure | | | | | | |
| $S_{epfs}$ (for $k_1 = \frac{1}{8}, k_2 = \frac{2}{8}, k_3 = \frac{5}{8}$) | 0.8240 | 0.7819 | 0.8178 | 0.8410 | 0.8369 | $MP_4 > MP_5 > MP_1 > MP_3 > MP_2$ |
| $S_{epfs}$ (for $k_1 = \frac{5}{8}, k_2 = \frac{2}{8}, k_3 = \frac{1}{8}$) | 0.6030 | 0.6037 | 0.5168 | 0.6459 | 0.6707 | $MP_5 > MP_4 > MP_2 > MP_1 > MP_3$ |

Overall, we find that the new proposed measure for E-PFSs provides a good measure of similarity while reflecting the importance weight of membership, abstinence, and non-membership degrees.

### 4.4 Discussion

We applied the proposed similarity measure for E-PFSs to the filed of building material recognition [25], pattern recognition problem [17], and the selection process of mega projects in under-developed countries [17]. Summarizing the findings of this section leads to the following conclusions:

- With reference to the ranking results given in the even-labelled tables, we found that the outcome of existing GSFS-based similarity measures is coincide with that of E-PFS-based similarity measure, meanwhile, the existing GSFS-based similarity measures are not able to return any logical result for the E-PFS data. This demonstrates that the E-PFS similarity measure has more feasibility and effectiveness than the existing GSFS-based similarity measures.
- The notable superiority of E-PFS similarity measure over other existing ones is due to its outcome ranking. That is, the E-PFS similarity-based ranking order, as indicated in the last two rows of even-labelled tables, is sensitive to the weight of satisfaction grade $\mu$, abstinence grade $\eta$, and dissatisfaction grade $\nu$.
- Unlike the existing decision-making techniques based on PFS, SFS and GSFS, the structure of E-PFS similarity-based technique is more broader and not limited to an extra parameter.

## 5 Conclusions

In this contribution, we first reviewed briefly the definition of PFSs, SFSs and GSFSs, and then their limitations were highlighted. In order for mitigating such limitations, we introduced the new concept of E-PFS which not only encompasses the concept of PFS, but also it includes the concepts of SFS and GSFS. By taking a variety of triangular conorm operations into consideration, we demonstrated that different forms of E-PFSs may be derived. By a fundamental theorem which guarantees any concept of PFS, SFS, and GSFS can be deduced from the concept of E-PFS, we developed the content of this study. In an attempt to increase the conceptual understanding of E-PFS, we proposed a number of set and algebraic operations on E-PFSs. Besides, an innovative and parametric E-PFS similarity measure was proposed in this contribution. Then, we presented a comparative study that is made by applying the existing PFS, SFS and GSFS similarity measures together with E-PFS similarity measure to the problems of recognizing the building materials, recognizing the patterns, and selection of the best mega project. The results from the parametric form of E-PFS similarity measure indicated that this measure is more consistent with human judgements. Moreover, the prominent role of E-PFS in decision making is its apparent in the situations where PFSs, SFSs and GSFSs do not make sense. The direction of future works of this study may be further focused on E-PFS information measures, different forms of E-PFS aggregation operators, etc.

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