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A parametric similarity measure between picture fuzzy sets and its applications in multi-attribute decision-making

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Abstract

Picture fuzzy set is an extension of intuitionistic fuzzy set, which can deal with inconsistent and uncertain information more accurately. Similarity measure, as an important mathematical tool to evaluate the degree of similarity between picture fuzzy sets, has been widely used to deal with multi-attribute decision-making problems. But there are unreasonable and counter-intuitive cases due to a few undesirable properties. In order to handle these unreasonable cases, this paper proposes a parametric similarity measure based on three parameters m_1, m_2 and m_3, in which decision makers with different decision styles can obtain the appropriate similarity measure by adjusting parameters m_1, m_2 and m_3. Moreover, we analyze some existing similarity measures from the perspective of mathematics and show that the proposed similarity measure is effective by numerical examples. In the end, we use the proposed similarity measure to solve the problems of multi-attribute decision-making. Through the comparison and analysis, we find that the proposed similarity measure is more effective than some existing similarity measures between picture fuzzy sets.

Keywords: Picture fuzzy set, similarity measure, multi-attribute decision-making.

1 Introduction

In the process of solving multi-attribute decision-making problems, due to the complexity of decision environment, decision makers often encounter some uncertain information, which is used to characterize decision alternatives. Hence, the treatment of uncertain information will affect the decision results. In order to deal with uncertain information, the theory of fuzzy sets was proposed by Zadeh [32]. Many scholars have proved that fuzzy sets can deal with vague information in practical application. However, it can not separately express the negative and rejected information. Aiming at this issue, various extensions of the fuzzy set have been given. Zadeh [33] proposed the theory of interval-valued fuzzy sets. Atanassov [2] extended the fuzzy set to the intuitionistic fuzzy set. Atanassov and Gargov [3] proved that intuitionistic fuzzy sets and interval-valued fuzzy sets are equipotent. Due to the ability of intuitionistic fuzzy sets to deal with vagueness, it has been widely used in various fields such as decision-making [22, 24, 31], pattern recognition [1, 5, 6, 15, 16, 17, 20, 23] and medical diagnosis [19, 21].

Although intuitionistic fuzzy sets have shown great potential in practical application, their application scope is still limited due to the ability to express more complex fuzzy information. For example [27], in the election of the village director, the voting results can be divided into four categories: vote for, abstain from voting, vote against and refuse to vote. Abstaining from voting means that the ballot is left blank, rejecting both voting for and voting against the candidate, but still casting the ballot. Refusing to vote means either an invalid ballot or bypassing the vote. In this situation, the intuitionistic fuzzy set fails to attain any satisfactory result. In order to solve these problems, Cuong [8] proposed picture fuzzy set (PFS for short), which is formed by the positive membership function, the neutral membership function, the negative membership function and the refusal membership function. Compared to the intuitionistic fuzzy

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set, the PFS divides the hesitancy membership function into two parts, i.e., the refusal membership function and the neutral membership function, which can easily handle uncertain and incomplete information more accurately in real situations such as human opinions involving four responses of the types: yes, abstain, no and refusal. Because of this advantage, many academics have started to study the picture fuzzy set theory. Up to now, many outstanding contributions have been made in the research of PFSs, which can be found in [7, 9, 10, 11, 12].

As an important mathematical tool for pattern recognition, medical diagnosis and decision-making, the similarity measure between PFSs has been attracted the attention of many scholars. Wei [28] proposed cosine similarity measures and cotangent similarity measures based on cosine function and cotangent function respectively, and applied them to strategic decision-making. Wei and Gao [30] developed Dice similarity measures and the generalized Dice similarity measures between PFSs and indicated that the Dice similarity measures are the special cases of the generalized Dice similarity measures in some parameter values. Then they applied the generalized Dice similarity measures to build material recognition. Dinh and Thao [12] proposed some similarity measures and applied them to multi-attribute decision-making. Singh and Mishra [26] pointed out that Dinh and Thao's measures failed to consider the refusal membership function, then they proposed several distance measures and similarity measures. Based on the proposed similarity measures, they developed a clustering algorithm and applied it to assess flood disaster risk in South region of India. Luo and Zhang [18] pointed out that the formula of some existing similarity measures is a simple calculation by additive operation, maximizing operation or minimizing operation, which leads to some unreasonable results, and then they constructed a similarity measure based on the three constituent functions.

From the work listed above, we find that the research on picture fuzzy similarity measure has achieved fruitful results, but some of them cannot fully solve the problem of multi-attribute decision-making (see Example 5.1-5.3), and face some obstacles such as

• it cannot provide the best choice for decision makers in a certain situation when solving the problem of decision-making, since it offers the same degree of similarity for the different decision alternatives.

• it cannot produce reasonable reliable decision outcomes for decision makers, as it does not satisfy the axioms of similarity measure and does not fully contain more fuzzy information.

• it does not provide decision results due to the existence of 'division by zero' problem.

Therefore, to improve the identification ability of the similarity measure and overcome the defects of current similarity measures, it is very necessary to propose a new similarity measure. This paper first studies obstacles of some existing similarity measures (see Section 3), and then attempts to construct a parametric picture fuzzy similarity measure by the following idea: considering that the evaluation value of decision-making experts on decision-making alternatives affects the decision-making results in the decision-making process. Due to the complexity of the decision-making environment and the limitations of human cognition, the decision-making experts who refuse to vote for the decision-making alternative (i.e. picture fuzzy rejection information) will change their attitude with the change of cognition and decision-making environment, that is, some tend to vote for it, some tend to vote against it, some still insist on refusing to vote, and others tend to abstain from voting. In this paper, the parameters m_1, m_2, m_3 are introduced to describe the change process of rejection information, and then a new picture fuzzy information is generated. After that, a new similarity measure between picture fuzzy sets is established. For this similarity measure, decision makers with different decision styles can find the appropriate similarity measure by selecting the appropriate parameters m_1, m_2, m_3. Moreover, the rationality and effectiveness of the proposed similarity measure are illustrated by numerical examples and multi-attribute decision-making problems.

The rest of this paper is organized as follows. In Section 2, some preliminaries related to picture fuzzy sets and similarity measures are reviewed. Section 3 comprehensive analyzes some existing similarity measures from the perspective of mathematics. In Section 4, a parametric similarity measure between PFSs and its explanation are given. Moreover, two numerical examples are established to illustrate that the proposed similarity measure is reasonable. Section 5 explains that the proposed similarity measure is effective and flexible by multi-attribute decision-making problems. The final Section is conclusions.

2 Preliminaries

In this section, some preliminaries related to picture fuzzy sets and similarity measures are reviewed.

Definition 2.1. [8] *Let X be a universe. A picture fuzzy set A on X is an object having the following form:*

$$A = \{\langle x, \mu_A(x), \eta_A(x), \nu_A(x)\rangle | x \in X\},$$

where $\mu_A(x)$ is the positive membership function, $\eta_A(x)$ is the neutral membership function and $\nu_A(x)$ is the negative membership function, they meet the conditions: $\mu_A(x), \eta_A(x), \nu_A(x) \in [0, 1]$, and $\mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1$ for all

$x \in X$. The refusal membership function $\rho_A(x) = 1 - (\mu_A(x) + \eta_A(x) + \nu_A(x))$ for all $x \in X$.

Definition 2.2. [8, 14] *Let $A = \{\langle x, \mu_A(x), \eta_A(x), \nu_A(x)\rangle | x \in X\}$ and $B = \{\langle x, \mu_B(x), \eta_B(x), \nu_B(x)\rangle | x \in X\}$ be any two PFSs on universe X, then*
(1) *$A \subseteq B$ if and only if $\mu_A(x) \le \mu_B(x), \eta_A(x) \ge \eta_B(x), \nu_A(x) \ge \nu_B(x)$, for $x \in X$.*
(2) *$A = B$ if and only if $A \subseteq B$ and $B \subseteq A$, i.e., $\mu_A(x) = \mu_B(x), \eta_A(x) = \eta_B(x), \nu_A(x) = \nu_B(x)$.*
(3) *$A^c = \{\langle x, \nu_A(x), \eta_A(x), \mu_A(x)\rangle | x \in X\}$.*

Definition 2.3. [26] *Let $A = \{\langle x, \mu_A(x), \eta_A(x), \nu_A(x)\rangle | x \in X\}$ and $B = \{\langle x, \mu_B(x), \eta_B(x), \nu_B(x)\rangle | x \in X\}$ be any two PFSs on universe X, then the similarity measure between A and B is defined as $S(A, B)$, which satisfies the following axioms:*
($S1$) *$0 \le S(A, B) \le 1$;*
($S2$) *$S(A, B) = 1$ if and only if $A = B$;*
($S3$) *$S(A, B) = S(B, A)$;*
($S4$) *Let C be any a PFS, if $A \subseteq B \subseteq C$, then $S(A, C) \le S(A, B)$ and $S(A, C) \le S(B, C)$.*

We review some existing similarity measures between PFSs. Let $A = \{\langle x_i, \mu_A(x_i), \eta_A(x_i), \nu_A(x_i)\rangle | x_i \in X\}$ and $B = \{\langle x_i, \mu_B(x_i), \eta_B(x_i), \nu_B(x_i)\rangle | x_i \in X\}$ be two PFSs on $X = \{x_1, x_2, \ldots, x_n\}$, $\rho_A(x_i)$ and $\rho_B(x_i)$ be the refusal degrees of element x_i belonging to PFSs A and B respectively, where $\rho_A(x_i) = 1 - \mu_A(x_i) - \eta_A(x_i) - \nu_A(x_i)$, $\rho_B(x_i) = 1 - \mu_B(x_i) - \eta_B(x_i) - \nu_B(x_i)$. The existing similarity degrees between PFSs A and B are reviewed as follows:

Wei's similarity measures [28]:

$$S_{g_1}(A, B) = \frac{1}{n}\sum_{i=1}^{n} \frac{\mu_A(x_i)\mu_B(x_i) + \eta_A(x_i)\eta_B(x_i) + \nu_A(x_i)\nu_B(x_i)}{\sqrt{\mu_A^2(x_i) + \eta_A^2(x_i) + \nu_A^2(x_i)}\sqrt{\mu_B^2(x_i) + \eta_B^2(x_i) + \nu_B^2(x_i)}}, \tag{1}$$

$$S_{g_2}(A, B) = \frac{1}{n}\sum_{i=1}^{n} \cos\frac{\pi}{2}(|\mu_A(x_i) - \mu_B(x_i)| \vee |\eta_A(x_i) - \eta_B(x_i)| \vee |\nu_A(x_i) - \nu_B(x_i)| \vee |\rho_A(x_i) - \rho_B(x_i)|), \tag{2}$$

$$S_{g_3}(A, B) = \frac{1}{n}\sum_{i=1}^{n} \cos\frac{\pi}{4}(|\mu_A(x_i) - \mu_B(x_i)| + |\eta_A(x_i) - \eta_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\rho_A(x_i) - \rho_B(x_i)|), \tag{3}$$

$$S_{g_4}(A, B) = \frac{1}{n}\sum_{i=1}^{n} \cot[\frac{\pi}{4} + \frac{\pi}{4}(|\mu_A(x_i) - \mu_B(x_i)| \vee |\eta_A(x_i) - \eta_B(x_i)| \vee |\nu_A(x_i) - \nu_B(x_i)|)], \tag{4}$$

$$S_{g_5}(A, B) = \frac{1}{n}\sum_{i=1}^{n} \cot[\frac{\pi}{4} + \frac{\pi}{4}(|\mu_A(x_i) - \mu_B(x_i)| \vee |\eta_A(x_i) - \eta_B(x_i)| \vee |\nu_A(x_i) - \nu_B(x_i)| \vee |\rho_A(x_i) - \rho_B(x_i)|)], \tag{5}$$

$$S_{g_6}(A, B) = \frac{1}{n}\sum_{i=1}^{n} \frac{\mu_A(x_i)\mu_B(x_i) + \eta_A(x_i)\eta_B(x_i) + \nu_A(x_i)\nu_B(x_i) + \rho_A(x_i)\rho_B(x_i)}{\sqrt{\mu_A^2(x_i) + \eta_A^2(x_i) + \nu_A^2(x_i) + \rho_A^2(x_i)}\sqrt{\mu_B^2(x_i) + \eta_B^2(x_i) + \nu_B^2(x_i) + \rho_B^2(x_i)}}, \tag{6}$$

$$S_{g_7}(A, B) = \frac{1}{n}\sum_{i=1}^{n} \cos\frac{\pi}{2}(|\mu_A(x_i) - \mu_B(x_i)| \vee |\eta_A(x_i) - \eta_B(x_i)| \vee |\nu_A(x_i) - \nu_B(x_i)|), \tag{7}$$

Dinh and Thao's similarity measures [12]:

$$S_h(A, B) = 1 - \frac{1}{3n}\sum_{i=1}^{n}[|\mu_A(x_i) - \mu_B(x_i)| + |\eta_A(x_i) - \eta_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|], \tag{8}$$

$$S_e(A, B) = 1 - \sqrt{\frac{1}{3n}\sum_{i=1}^{n}[(\mu_A(x_i) - \mu_B(x_i))^2 + (\eta_A(x_i) - \eta_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2]}. \tag{9}$$

Singh and Mishra's similarity measures [26]:

$$S_{p_1}(A,B)=1-\frac{1}{4n}\sum_{i=1}^{n}(|\mu_A(x_i)-\mu_B(x_i)|+|\eta_A(x_i)-\eta_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|+|\rho_A(x_i)-\rho_B(x_i)|), \tag{10}$$

$$S_{p_2}(A,B)=1-\sqrt{\frac{1}{4n}\sum_{i=1}^{n}(\mu_A(x_i)-\mu_B(x_i))^2+(\eta_A(x_i)-\eta_B(x_i))^2+(\nu_A(x_i)-\nu_B(x_i))^2+(\rho_A(x_i)-\rho_B(x_i))^2}, \tag{11}$$

$$S_{p_3}(A,B)=1-\frac{1}{4n}\sum_{i=1}^{n}(|\mu_A(x_i)-\mu_B(x_i)|\vee|\eta_A(x_i)-\eta_B(x_i)|\vee|\nu_A(x_i)-\nu_B(x_i)|\vee|\rho_A(x_i)-\rho_B(x_i)|), \tag{12}$$

$$S_{p_4}(A,B)=1-\sqrt{\frac{1}{4n}\sum_{i=1}^{n}(\mu_A(x_i)-\mu_B(x_i))^2\vee(\eta_A(x_i)-\eta_B(x_i))^2\vee(\nu_A(x_i)-\nu_B(x_i))^2\vee(\rho_A(x_i)-\rho_B(x_i))^2}, \tag{13}$$

$$S_{p_5}(A,B)=\frac{1}{4n}\sum_{i=1}^{n}\frac{|\mu_A(x_i)-\mu_B(x_i)|\wedge|\eta_A(x_i)-\eta_B(x_i)|\wedge|\nu_A(x_i)-\nu_B(x_i)|\wedge|\rho_A(x_i)-\rho_B(x_i)|}{|\mu_A(x_i)-\mu_B(x_i)|\vee|\eta_A(x_i)-\eta_B(x_i)|\vee|\nu_A(x_i)-\nu_B(x_i)|\vee|\rho_A(x_i)-\rho_B(x_i)|}. \tag{14}$$

Wei and Gao's similarity measures [30]:

$$S_{w_1}(A,B)=\frac{1}{n}\sum_{i=1}^{n}\frac{2(\mu_A(x_i)\mu_B(x_i)+\eta_A(x_i)\eta_B(x_i)+\nu_A(x_i)\nu_B(x_i)+\rho_A(x_i)\rho_B(x_i))}{(\mu_A^2(x_i)+\eta_A^2(x_i)+\nu_A^2(x_i)+\rho_A^2(x_i))+(\mu_B^2(x_i)+\eta_B^2(x_i)+\nu_B^2(x_i)+\rho_B^2(x_i))}, \tag{15}$$

$$S_{w_2}(A,B)=\frac{\sum_{i=1}^{n}2(\mu_A(x_i)\mu_B(x_i)+\eta_A(x_i)\eta_B(x_i)+\nu_A(x_i)\nu_B(x_i)+\rho_A(x_i)\rho_B(x_i))}{\sum_{i=1}^{n}(\mu_A^2(x_i)+\eta_A^2(x_i)+\nu_A^2(x_i)+\rho_A^2(x_i))+\sum_{i=1}^{n}(\mu_B^2(x_i)+\eta_B^2(x_i)+\nu_B^2(x_i)+\rho_B^2(x_i))}. \tag{16}$$

Luo and Zhang's similarity measure [18]:

$$\begin{aligned}S_l(A,B)=&\frac{1}{3n}\sum_{i=1}^{n}(2\sqrt{\mu_A(x_i)\mu_B(x_i)}+2\sqrt{\eta_A(x_i)\eta_B(x_i)}+2\sqrt{\nu_A(x_i)\nu_B(x_i)}\\&+\sqrt{(1-\eta_A(x_i)-\nu_A(x_i))(1-\eta_B(x_i)-\nu_B(x_i))}+\sqrt{(1-\mu_A(x_i)-\nu_A(x_i))(1-\mu_B(x_i)-\nu_B(x_i))}\\&+\sqrt{(1-\mu_A(x_i)-\eta_A(x_i))(1-\mu_B(x_i)-\eta_B(x_i))}).\end{aligned} \tag{17}$$

3 An analysis of some existing picture fuzzy similarity measures

As a mathematical tool to calculate the similarity degree between objects, the similarity measure has been used to solve the problems of decision-making, medical diagnosis and pattern recognition. Although many similarity measures between PFSs have been proposed, they can produce unreasonable and counter-intuitive results in practical application, which can bring great trouble to practical users. In this section, we comprehensive analyze some existing similarity measures from the perspective of mathematics (see Table 1).

The axiom (S2) is one of the most basic axioms of picture similarity measures. By analyzing Table 1, we can easily find that the similarity measures S_{g_1}, S_{g_6}, S_{p_5}, S_{w_1} and S_{w_2} do not satisfy this axiom. The detailed discussion is as follows:

(1) Let $A=\{\langle x_i,\mu_A(x_i),\eta_A(x_i),\nu_A(x_i)\rangle|x_i\in X\}$ and $B=\{\langle x_i,\mu_B(x_i),\eta_B(x_i),\nu_B(x_i)\rangle|x_i\in X\}$ be any two PFSs on $X=\{x_1,x_2,\ldots,x_n\}$. For the similarity measure S_{g_1}, there are two cases in which S_{g_1} does not satisfy the axiom (S2) "$S(A,B)=1$ implies $A=B$," as shown below:

(i) if $\mu_A(x_i)=\eta_A(x_i)=\nu_A(x_i)\neq\mu_B(x_i)=\eta_B(x_i)=\nu_B(x_i)$, i.e., $A\neq B$, based on Eq.(1), we have

$$S_{g_1}(A,B)=\frac{1}{n}\sum_{i=1}^{n}\frac{\mu_A(x_i)\mu_B(x_i)+\eta_A(x_i)\eta_B(x_i)+\nu_A(x_i)\nu_B(x_i)}{\sqrt{\mu_A^2(x_i)+\eta_A^2(x_i)+\nu_A^2(x_i)}\sqrt{\mu_B^2(x_i)+\eta_B^2(x_i)+\nu_B^2(x_i)}}=\frac{3\mu_A(x_i)\mu_B(x_i)}{\sqrt{3\mu_A^2(x_i)}\sqrt{3\mu_B^2(x_i)}}=1;$$

(ii) if $\mu_A(x_i)=2\mu_B(x_i),\eta_A(x_i)=2\eta_B(x_i)$ and $\nu_A(x_i)=2\nu_B(x_i)$, i.e., $A\neq B$, based on Eq.(1), we can obtain

$$S_{g_1}(A,B)=\frac{1}{n}\sum_{i=1}^{n}\frac{\mu_A(x_i)\mu_B(x_i)+\eta_A(x_i)\eta_B(x_i)+\nu_A(x_i)\nu_B(x_i)}{\sqrt{\mu_A^2(x_i)+\eta_A^2(x_i)+\nu_A^2(x_i)}\sqrt{\mu_B^2(x_i)+\eta_B^2(x_i)+\nu_B^2(x_i)}}$$

Table 1: A comprehensive analysis of some existing similarity measures

S	Existing some drawbacks		
	does not meet the axiom (S2)	the division by zero problem	serious information loss
S_{g_1} [28]	√	√	×
S_{g_2} [28]	×	×	√
S_{g_3} [28]	×	×	×
S_{g_4} [28]	×	×	√
S_{g_5} [28]	×	×	√
S_{g_6} [28]	√	×	×
S_{g_7} [28]	×	×	√
S_h [12]	×	×	×
S_e [12]	×	×	×
S_{p_1} [26]	×	×	×
S_{p_2} [26]	×	×	×
S_{p_3} [26]	×	×	√
S_{p_4} [26]	×	×	√
S_{p_5} [26]	√	√	√
S_{w_1} [30]	√	×	×
S_{w_2} [30]	√	×	×
S_l [18]	×	×	×

$$= \frac{1}{n}\sum_{i=1}^{n} \frac{2\mu_B^2(x_i)+2\eta_B^2(x_i)+2\nu_B^2(x_i)}{\sqrt{4\mu_B^2(x_i)+4\eta_B^2(x_i)+4\nu_B^2(x_i)}\sqrt{\mu_B^2(x_i)+\eta_B^2(x_i)+\nu_B^2(x_i)}} = 1.$$

Obviously, in the above cases, the similarity measure S_{g_1} is invalid.

(2) For the similarity measures S_{g_6}, S_{w_1} and S_{w_2}, let $A = \langle x, 0.25, 0.25, 0.25\rangle$ and $B = \langle x, \mu_B, \eta_B, \nu_B\rangle$ be two PFSs on $X = \{x\}$, if μ_B, η_B and ν_B meet the condition $(\mu_B)^2 + (\eta_B)^2 + (\nu_B)^2 + (\rho_B)^2 = 0.25$, then we can get

$$S_{g_6}(A,B) = \frac{1}{n}\sum_{i=1}^{n} \frac{\mu_A\mu_B+\eta_A\eta_B+\nu_A\nu_B+\rho_A\rho_B}{\sqrt{\mu_A^2+\eta_A^2+\nu_A^2+\rho_A^2}\sqrt{\mu_B^2+\eta_B^2+\nu_B^2+\rho_B^2}} = \frac{0.25}{\sqrt{0.25^2\times 4}\times\sqrt{0.25}} = 1,$$

$$S_{w_1}(A,B) = S_{w_2}(A,B) = \frac{2(\mu_A\mu_B+\eta_A\eta_B+\nu_A\nu_B+\rho_A\rho_B)}{(\mu_A^2+\eta_A^2+\nu_A^2+\rho_A^2)+(\mu_B^2+\eta_B^2+\nu_B^2+\rho_B^2)} = \frac{2\times 0.25}{0.25^2\times 4+0.25} = 1.$$

The values of μ_B, η_B and ν_B with $\rho_B = 0.24$ can be regarded as the points on the intersection line of the quarter ball with radius of 0.1924 and the plane $\mu_B + \eta_B + \nu_B = 0.76$, which can be found in Fig.1.

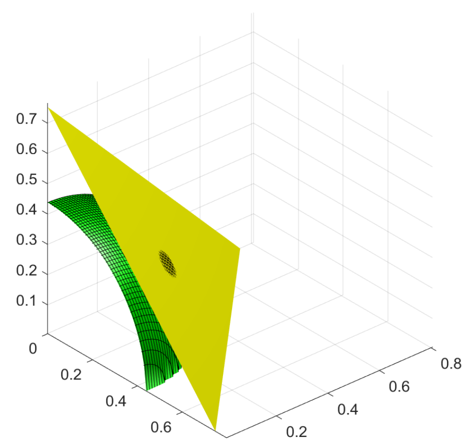

Figure 1: The possible values for μ_B, η_B, ν_B with $\rho_B = 0.24$

Through the above analysis, similarity measures S_{g_6}, S_{w_1} and S_{w_2} do not satisfy the axiom "$S(A,B) = 1$ implies $A = B$", and these similarity measures provide a counter-intuitive result for practical users in this case.

(3) For the similarity measure S_{g_1} and S_{p_5}, when picture fuzzy sets $A = B = \langle x, 0.0, 0.0, 0.0\rangle$ defined on $X = \{x\}$, we have $S_{g_1}(A,B) = S_{p_5}(A,B) = \frac{0}{0}$. In this case, the similarity measure S_{g_1} and S_{p_5} are invalid, i.e., they do not satisfy the axiom (S2)"$A = B$ implies $S(A,B) = 1$", and they have the drawback of "the division by zero problem".

Based on the above analysis, we show that the existing similarity measures S_{g_1}, S_{g_6}, S_{p_5}, S_{w_1} and S_{w_2} do not satisfy the axioms of the picture fuzzy similarity measure.

The ability of the similarity measure to identify the proximity of fuzzy sets is determined by the expression form and the information contained in the expression. The more information the similarity measure pays attention to, the stronger the identification ability. By analysing Table 1, we find that the similarity measure S_{g_2} only considers the difference of positive degree or neutral degree or negative degree or refusal degree between PFSs, which brings a big amount of information losing. For example, let $A = \langle x, 0.1, 0.3, 0.2\rangle$, $B = \langle x, 0.6, 0.1, 0.2\rangle$ be two PFSs. Since $\mid 0.2-0.2 \mid < \mid 0.3-0.1 \mid < \mid (1-0.1-0.3-0.2)-(1-0.6-0.1-0.2) \mid < \mid 0.1-0.6 \mid$, hence, the similarity measure between A and B only considers the difference of the positive degree between A and B by using the similarity measure S_{g_2}. In this case, the similarity measure will cause a lot of information loss in practical application, so that it can not provide more accurate results for practical users. In addition, in this situation, we also find that the similarity measures $S_{g_4}, S_{g_5}, S_{g_7}, S_{p_3}, S_{p_4}$ and S_{p_5} have the same drawback.

4 A parametric similarity measure between picture fuzzy sets

In view of the reasons for the unreasonable results of the above analysis, in this section, we develop a parametric similarity measure between PFSs, in which decision makers with different decision styles can choose the appropriate similarity measure by adjusting the parameters. First of all, we give a parametric picture fuzzy similarity measure by constructing a binary function, and prove that it meets the axiomatic definition of the similarity measure. In this formula, the parameter reflects the influence of picture fuzzy rejection information on similarity measure. After that, we give the corresponding explanation of the parametric similarity measure from the perspective of geometry. Finally, two numerical examples are given to illustrate that the proposed similarity measure is reasonable. In the process of numerical calculation, considering the similarity measures S_{g_1}, S_{g_6}, S_{p_5}, S_{w_1} and S_{w_2} have the drawback that do not satisfy Definition 2.3, we only consider the results that are obtained by the proposed similarity measure and the existing similarity measures $S_{g_2}, S_{g_3}, S_{g_4}, S_{g_5}, S_{g_7}, S_h, S_e, S_{p_1}, S_{p_2}, S_{p_3}, S_{p_4}, S_l$.

4.1 A parametric similarity measure between picture fuzzy sets

Theorem 4.1. *Let* $A = \{\langle x_i, \mu_A(x_i), \eta_A(x_i), \nu_A(x_i)\rangle | x_i \in X\}$ *and* $B = \{\langle x_i, \mu_B(x_i), \eta_B(x_i), \nu_B(x_i)\rangle | x_i \in X\}$ *be any two PFSs on* $X = \{x_1, x_2, \ldots, x_n\}$*, then the function* S_m*:* $PFS(X) \times PFS(X) \to [0,1]$ *defined by*

$$S_m(A,B) = 1 - \Big[\frac{1}{3n}\sum_{i=1}^{n}\big(\Delta^p_{1_{AB}}(x_i) + \Delta^p_{2_{AB}}(x_i) + \Delta^p_{3_{AB}}(x_i)\big)\Big]^{\frac{1}{p}}, \tag{18}$$

is a similarity measure between A *and* B*, where*

$\Delta_{1_{AB}}(x_i) = \frac{1}{m_1+1}|m_1\big(\mu_A(x_i)-\mu_B(x_i)\big) - \big(\eta_A(x_i)-\eta_B(x_i)\big) - \big(\nu_A(x_i)-\nu_B(x_i)\big)|$, $m_1 \in [0,+\infty)$;
$\Delta_{2_{AB}}(x_i) = \frac{1}{2(m_2+1)}|m_2\big(\eta_A(x_i)-\eta_B(x_i)\big) - \big(\mu_A(x_i)-\mu_B(x_i)\big) + \big(\nu_A(x_i)-\nu_B(x_i)\big)|$, $m_2 \in [0,+\infty)$;
$\Delta_{3_{AB}}(x_i) = \frac{1}{2(m_3+1)}|m_3\big(\nu_A(x_i)-\nu_B(x_i)\big) - \big(\mu_A(x_i)-\mu_B(x_i)\big) + \big(\eta_A(x_i)-\eta_B(x_i)\big)|$, $m_3 \in [0,+\infty)$;
$\frac{1}{m_1+1} + \frac{1}{m_2+1} + \frac{1}{m_3+1} \in (0,1]$ *and* p *is any positive integer.*

Proof. In order to prove that Eq. (18) is a similarity measure, we only need to prove Eq. (18) satisfies axioms (S1)-(S4). Let $A = \{\langle x_i, \mu_A(x_i), \eta_A(x_i), \nu_A(x_i)\rangle | x_i \in X\}$, $B = \{\langle x_i, \mu_B(x_i), \eta_B(x_i), \nu_B(x_i)\rangle | x_i \in X\}$ and $C = \{\langle x_i, \mu_C(x_i), \eta_C(x_i), \nu_C(x_i)\rangle | x_i \in X\}$ be any three PFSs on $X = \{x_1, x_2, \ldots, x_n\}$.

(S1) We can write the following equations:

$$\begin{aligned}\Delta_{1_{AB}}(x_i) &= \tfrac{1}{m_1+1}|m_1\big(\mu_A(x_i)-\mu_B(x_i)\big) - \big(\eta_A(x_i)-\eta_B(x_i)\big) - \big(\nu_A(x_i)-\nu_B(x_i)\big)|\\ &= \tfrac{1}{m_1+1}|\big(m_1\mu_A(x_i)-\eta_A(x_i)-\nu_A(x_i)\big) - \big(m_1\mu_B(x_i)-\eta_B(x_i)-\nu_B(x_i)\big)|,\\ \Delta_{2_{AB}}(x_i) &= \tfrac{1}{2(m_2+1)}|m_2\big(\eta_A(x_i)-\eta_B(x_i)\big) - \big(\mu_A(x_i)-\mu_B(x_i)\big) + \big(\nu_A(x_i)-\nu_B(x_i)\big)|\\ &= \tfrac{1}{2(m_2+1)}|\big(m_2\eta_A(x_i)-\mu_A(x_i)+\nu_A(x_i)\big) - \big(m_2\eta_B(x_i)-\mu_B(x_i)+\nu_B(x_i)\big)|,\\ \Delta_{3_{AB}}(x_i) &= \tfrac{1}{2(m_3+1)}|m_3\big(\nu_A(x_i)-\nu_B(x_i)\big) - \big(\mu_A(x_i)-\mu_B(x_i)\big) + \big(\eta_A(x_i)-\eta_B(x_i)\big)|\\ &= \tfrac{1}{2(m_3+1)}|\big(m_3\nu_A(x_i)-\mu_A(x_i)+\eta_A(x_i)\big) - \big(m_3\nu_B(x_i)-\mu_B(x_i)+\eta_B(x_i)\big)|.\end{aligned}$$

By $\mu_A(x_i), \eta_A(x_i), \nu_A(x_i), \mu_B(x_i), \eta_B(x_i), \nu_B(x_i) \in [0,1]$ and $\mu_A(x_i)+\eta_A(x_i)+\nu_A(x_i) \le 1$, $\mu_B(x_i)+\eta_B(x_i)+\nu_B(x_i) \le 1$, we have:

$$-1 \le m_1\mu_A(x_i) - \eta_A(x_i) - \nu_A(x_i) \le m_1,$$
$$-m_1 \le -\big(m_1\mu_B(x_i) - \eta_B(x_i) - \nu_B(x_i)\big) \le 1.$$

Hence, $0 \le |\big(m_1\mu_A(x_i)-\eta_A(x_i)-\nu_A(x_i)\big) - \big(m_1\mu_B(x_i)-\eta_B(x_i)-\nu_B(x_i)\big)| \le m_1+1$, i.e., $0 \le \Delta_{1_{AB}}(x_i) \le 1$.
Since

$$-1 \le m_2\eta_A(x_i) + \nu_A(x_i) - \big(1-\eta_A(x_i)-\nu_A(x_i)\big) \le m_2\eta_A(x_i) - \mu_A(x_i) + \nu_A(x_i),$$

$m_2\eta_A(x_i) - \mu_A(x_i) + \nu_A(x_i) \le m_2\eta_A(x_i) + \nu_A(x_i) \le m_2\eta_A(x_i) + (1 - \eta_A(x_i)) = 1 + (m_2 - 1)\eta_A(x_i) \le 1 \vee m_2.$

Then $-1 \le m_2\eta_A(x_i) - \mu_A(x_i) + \nu_A(x_i) \le 1 \vee m_2$.

Similarly, we get the following inequalities:

$-(1 \vee m_2) \le -\big(m_2\eta_B(x_i) - \mu_B(x_i) + \nu_B(x_i)\big) \le 1,$

$-1 \le m_3\nu_A(x_i) - \mu_A(x_i) + \eta_A(x_i) \le 1 \vee m_3,$

$-(1 \vee m_3) \le -\big(m_3\nu_B(x_i) - \mu_B(x_i) + \eta_B(x_i)\big) \le 1,$

then we obtain:

$0 \le |\big(m_2\eta_A(x_i) - \mu_A(x_i) + \nu_A(x_i)\big) - \big(m_2\eta_B(x_i) - \mu_B(x_i) + \nu_B(x_i)\big)| \le 2 \vee (m_2 + 1),$

$0 \le |\big(m_3\nu_A(x_i) - \mu_A(x_i) + \eta_A(x_i)\big) - \big(m_3\nu_B(x_i) - \mu_B(x_i) + \eta_B(x_i)\big)| \le 2 \vee (m_3 + 1).$

It means that:

$0 \le \Delta_{2_{AB}}(x_i) = \frac{1}{2(m_2+1)}|m_2\big(\eta_A(x_i) - \eta_B(x_i)\big) - \big(\mu_A(x_i) - \mu_B(x_i)\big) + \big(\nu_A(x_i) - \nu_B(x_i)\big)| \le \frac{1}{m_2+1} \vee \frac{1}{2} \le 1,$

$0 \le \Delta_{3_{AB}}(x_i) = \frac{1}{2(m_3+1)}|m_3\big(\nu_A(x_i) - \nu_B(x_i)\big) - \big(\mu_A(x_i) - \mu_B(x_i)\big) + \big(\eta_A(x_i) - \eta_B(x_i)\big)| \le \frac{1}{m_3+1} \vee \frac{1}{2} \le 1.$

Finally, we have:

$0 \le 1 - \left[\frac{1}{3n}\sum_{i=1}^{n}\big(\Delta^p_{1_{AB}}(x_i) + \Delta^p_{2_{AB}}(x_i) + \Delta^p_{3_{AB}}(x_i)\big)\right]^{\frac{1}{p}} \le 1.$

Therefore, $0 \le S_m(A, B) \le 1$.

(S2) If $A = B$, then $\mu_A(x_i) = \mu_B(x_i)$, $\eta_A(x_i) = \eta_B(x_i)$ and $\nu_A(x_i) = \nu_B(x_i)$.

Therefore, $\Delta_{1_{AB}}(x_i) = 0$, $\Delta_{2_{AB}}(x_i) = 0$, and $\Delta_{3_{AB}}(x_i) = 0$, i.e., $S_m(A, B) = 1$.

If $S_m(A, B) = 1$, then

$\Delta_{1_{AB}}(x_i) = \frac{1}{m_1+1}|m_1\big(\mu_A(x_i) - \mu_B(x_i)\big) - \big(\eta_A(x_i) - \eta_B(x_i)\big) - \big(\nu_A(x_i) - \nu_B(x_i)\big)| = 0,$

$\Delta_{2_{AB}}(x_i) = \frac{1}{2(m_2+1)}|m_2\big(\eta_A(x_i) - \eta_B(x_i)\big) - \big(\mu_A(x_i) - \mu_B(x_i)\big) + \big(\nu_A(x_i) - \nu_B(x_i)\big)| = 0,$

$\Delta_{3_{AB}}(x_i) = \frac{1}{2(m_3+1)}|m_3\big(\nu_A(x_i) - \nu_B(x_i)\big) - \big(\mu_A(x_i) - \mu_B(x_i)\big) + \big(\eta_A(x_i) - \eta_B(x_i)\big)| = 0.$

By the definition of absolute value, we have:

$m_1\big(\mu_A(x_i) - \mu_B(x_i)\big) - \big(\eta_A(x_i) - \eta_B(x_i)\big) - \big(\nu_A(x_i) - \nu_B(x_i)\big) = 0,$

$m_2\big(\eta_A(x_i) - \eta_B(x_i)\big) - \big(\mu_A(x_i) - \mu_B(x_i)\big) + \big(\nu_A(x_i) - \nu_B(x_i)\big) = 0,$

$m_3\big(\nu_A(x_i) - \nu_B(x_i)\big) - \big(\mu_A(x_i) - \mu_B(x_i)\big) + \big(\eta_A(x_i) - \eta_B(x_i)\big) = 0.$

i.e.,

$$\begin{pmatrix} m_1 & -1 & -1 \\ -1 & m_2 & 1 \\ -1 & 1 & m_3 \end{pmatrix}\begin{pmatrix} \mu_A(x_i) - \mu_B(x_i) \\ \eta_A(x_i) - \eta_B(x_i) \\ \nu_A(x_i) - \nu_B(x_i) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since $\frac{1}{m_1+1} + \frac{1}{m_2+1} + \frac{1}{m_3+1} \in (0, 1]$, then $2 \le m_1m_2m_3 - (m_1 + m_2 + m_3)$.

By the definition of matrix determinant, we can get:

$$\begin{vmatrix} m_1 & -1 & -1 \\ -1 & m_2 & 1 \\ -1 & 1 & m_3 \end{vmatrix} = m_1m_2m_3 + 2 - (m_1 + m_2 + m_3) \ge 4.$$

Therefore, we have

$$\begin{pmatrix} \mu_A(x_i) - \mu_B(x_i) \\ \eta_A(x_i) - \eta_B(x_i) \\ \nu_A(x_i) - \nu_B(x_i) \end{pmatrix} = \begin{pmatrix} m_1 & -1 & -1 \\ -1 & m_2 & 1 \\ -1 & 1 & m_3 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

It means that $\mu_A(x_i) = \mu_B(x_i)$, $\eta_A(x_i) = \eta_B(x_i)$ and $\nu_A(x_i) = \nu_B(x_i)$, i.e., $A = B$.

(S3) Based on the definition of absolute value, we can get the following equations:

$$\begin{aligned}\Delta_{1_{AB}}(x_i) &= \tfrac{1}{m_1+1}|m_1\big(\mu_A(x_i) - \mu_B(x_i)\big) - \big(\eta_A(x_i) - \eta_B(x_i)\big) - \big(\nu_A(x_i) - \nu_B(x_i)\big)| \\ &= \tfrac{1}{m_1+1}|(-1)\big[m_1\big(\mu_B(x_i) - \mu_A(x_i)\big) - \big(\eta_B(x_i) - \eta_A(x_i)\big) - \big(\nu_B(x_i) - \nu_A(x_i)\big)\big]| \\ &= \tfrac{1}{m_1+1}|m_1\big(\mu_B(x_i) - \mu_A(x_i)\big) - \big(\eta_B(x_i) - \eta_A(x_i)\big) - \big(\nu_B(x_i) - \nu_A(x_i)\big)| \\ &= \Delta_{1_{BA}}(x_i).\end{aligned}$$

$$\begin{aligned}\Delta_{2_{AB}}(x_i) &= \tfrac{1}{2(m_2+1)}|m_2\big(\eta_A(x_i) - \eta_B(x_i)\big) - \big(\mu_A(x_i) - \mu_B(x_i)\big) + \big(\nu_A(x_i) - \nu_B(x_i)\big)| \\ &= \tfrac{1}{2(m_2+1)}|(-1)\big[m_2\big(\eta_B(x_i) - \eta_A(x_i)\big) - \big(\mu_B(x_i) - \mu_A(x_i)\big) + \big(\nu_B(x_i) - \nu_A(x_i)\big)\big]| \\ &= \tfrac{1}{2(m_2+1)}|m_2\big(\eta_B(x_i) - \eta_A(x_i)\big) - \big(\mu_B(x_i) - \mu_A(x_i)\big) + \big(\nu_B(x_i) - \nu_A(x_i)\big)| \\ &= \Delta_{2_{BA}}(x_i).\end{aligned}$$

$$\begin{aligned}\Delta_{3_{AB}}(x_i) &= \tfrac{1}{2(m_3+1)}|m_3\big(\nu_A(x_i) - \nu_B(x_i)\big) - \big(\mu_A(x_i) - \mu_B(x_i)\big) + \big(\eta_A(x_i) - \eta_B(x_i)\big)| \\ &= \tfrac{1}{2(m_3+1)}|(-1)\big[m_3\big(\nu_B(x_i) - \nu_A(x_i)\big) - \big(\mu_B(x_i) - \mu_A(x_i)\big) + \big(\eta_B(x_i) - \eta_A(x_i)\big)\big]|\end{aligned}$$

$= \frac{1}{2(m_3+1)}|m_3\big(\nu_B(x_i)-\nu_A(x_i)\big)-\big(\mu_B(x_i)-\mu_A(x_i)\big)+\big(\eta_B(x_i)-\eta_A(x_i)\big)|$
$= \Delta_{3_{BA}}(x_i)$.
Therefore, $S_m(A,B)=S_m(B,A)$.

(S4) Assume $A\subseteq B\subseteq C$, then $\mu_A(x_i)\le\mu_B(x_i)\le\mu_C(x_i)$, $\eta_C(x_i)\le\eta_B(x_i)\le\eta_A(x_i)$ and $\nu_C(x_i)\le\nu_B(x_i)\le\nu_A(x_i)$. Therefore, we can have:

$$m_1\mu_A(x_i)-\eta_A(x_i)-\nu_A(x_i)\le m_1\mu_B(x_i)-\eta_B(x_i)-\nu_B(x_i)\le m_1\mu_C(x_i)-\eta_C(x_i)-\nu_C(x_i),$$
$$m_2\eta_C(x_i)-\mu_C(x_i)+\nu_C(x_i)\le m_2\eta_B(x_i)-\mu_B(x_i)+\nu_B(x_i)\le m_2\eta_A(x_i)-\mu_A(x_i)+\nu_A(x_i),$$
$$m_3\nu_C(x_i)-\mu_C(x_i)+\eta_C(x_i)\le m_3\nu_B(x_i)-\mu_B(x_i)+\eta_B(x_i)\le m_3\nu_A(x_i)-\mu_A(x_i)+\eta_A(x_i).$$

By the property of inequality, we can obtain:

$$\begin{aligned}
&|\big(m_1\mu_A(x_i)-\eta_A(x_i)-\nu_A(x_i)\big)-\big(m_1\mu_B(x_i)-\eta_B(x_i)-\nu_B(x_i)\big)|\\
\le&|\big(m_1\mu_A(x_i)-\eta_A(x_i)-\nu_A(x_i)\big)-\big(m_1\mu_C(x_i)-\eta_C(x_i)-\nu_C(x_i)\big)|,\\
&|\big(m_2\eta_A(x_i)-\mu_A(x_i)+\nu_A(x_i)\big)-\big(m_2\eta_B(x_i)-\mu_B(x_i)+\nu_B(x_i)\big)|\\
\le&|\big(m_2\eta_A(x_i)-\mu_A(x_i)+\nu_A(x_i)\big)-\big(m_2\eta_C(x_i)-\mu_C(x_i)+\nu_C(x_i)\big)|,\\
&|\big(m_3\nu_A(x_i)-\mu_A(x_i)+\eta_A(x_i)\big)-\big(m_3\nu_B(x_i)-\mu_B(x_i)+\eta_B(x_i)\big)|\\
\le&|\big(m_3\nu_A(x_i)-\mu_A(x_i)+\eta_A(x_i)\big)-\big(m_3\nu_C(x_i)-\mu_C(x_i)+\eta_C(x_i)\big)|.
\end{aligned}$$

i.e.,

$$\Delta_{1_{AB}}(x_i)\le\Delta_{1_{AC}}(x_i),\ \Delta_{2_{AB}}(x_i)\le\Delta_{2_{AC}}(x_i),\ \Delta_{3_{AB}}(x_i)\le\Delta_{3_{AC}}(x_i).$$

Therefore, we have

$$1-\Big[\tfrac{1}{3n}\sum_{i=1}^{n}\big(\Delta^p_{1_{AC}}(x_i)+\Delta^p_{2_{AC}}(x_i)+\Delta^p_{3_{AC}}(x_i)\big)\Big]^{\frac{1}{p}}\le 1-\Big[\tfrac{1}{3n}\sum_{i=1}^{n}\big(\Delta^p_{1_{AB}}(x_i)+\Delta^p_{2_{AB}}(x_i)+\Delta^p_{3_{AB}}(x_i)\big)\Big]^{\frac{1}{p}}.$$

It means that $S_m(A,C)\le S_m(A,B)$.
Similarly, we have $S_m(A,C)\le S_m(B,C)$. □

Example 4.2. (1) *When $m_1=0, m_2=m_3=+\infty$, Eq.(18) can be written as:*

$$S_1(A,B)=1-\Big[\frac{1}{3n}\sum_{i=1}^{n}\Big(|\big(\eta_A(x_i)-\eta_B(x_i)\big)+\big(\nu_A(x_i)-\nu_B(x_i)\big)|^p+\frac{|\eta_A(x_i)-\eta_B(x_i)|^p}{2^p}+\frac{|\nu_A(x_i)-\nu_B(x_i)|^p}{2^p}\Big)\Big]^{\frac{1}{p}} \quad (19)$$

(2) *When $m_1=m_2=+\infty, m_3=0$, Eq.(18) can be written as:*

$$S_2(A,B)=1-\Big[\frac{1}{3n}\sum_{i=1}^{n}\Big(|\mu_A(x_i)-\mu_B(x_i)|^p+\frac{|\eta_A(x_i)-\eta_B(x_i)|^p}{2^p}+\frac{|\big(\eta_A(x_i)-\eta_B(x_i)\big)-\big(\mu_A(x_i)-\mu_B(x_i)\big)|^p}{2^p}\Big)\Big]^{\frac{1}{p}} \quad (20)$$

Theorem 4.3. *For any two PFSs $A=\{\langle x_i,\mu_A(x_i),\eta_A(x_i),\nu_A(x_i)\rangle|x_i\in X\}$ and $B=\{\langle x_i,\mu_B(x_i),\eta_B(x_i),\nu_B(x_i)\rangle|x_i\in X\}$ on $X=\{x_1,x_2,\dots,x_n\}$, $\omega_i\in[0,1]$ and $\sum_{i=1}^{n}\omega_i=1$, the function S_ω: $PFS(X)\times PFS(X)\to[0,1]$ defined by*

$$S_\omega(A,B)=1-\Big[\frac{1}{3}\sum_{i=1}^{n}\omega_i\big(\Delta^p_{1_{AB}}(x_i)+\Delta^p_{2_{AB}}(x_i)+\Delta^p_{3_{AB}}(x_i)\big)\Big]^{\frac{1}{p}}, \quad (21)$$

is a weighted similarity measure between A and B.

Proof. The proof is similar to Theorem 4.1. □

4.2 Geometric interpretation of the parametric picture fuzzy similarity measure

Let $A=\langle x,\mu_A,\eta_A,\nu_A\rangle$ and $B=\langle x,\mu_B,\eta_B,\nu_B\rangle$ be two PFSs on $X=\{x\}$. μ_A can be equal to any value in $[\mu_A,\mu_A+\rho_A]$, η_A can be equal to any value in $[\eta_A,\eta_A+\rho_A]$, ν_A can be equal to any value in $[\nu_A,\nu_A+\rho_A]$. The possible values for the picture fuzzy set A are illustrated in Fig. 2.

In Fig.2, $A'=\langle x,\mu'_A,\eta'_A,\nu'_A\rangle=\langle x,\mu_A+\frac{\rho_A}{m_1+1},\eta_A+\frac{\rho_A}{m_2+1},\nu_A+\frac{\rho_A}{m_3+1}\rangle$ ($m_1,m_2,m_3\in[0,+\infty)$, $\frac{1}{m_1+1}+\frac{1}{m_2+1}+\frac{1}{m_3+1}\in(0,1]$) can represent any point in the tetrahedron $M-NAQ$. It is easy to find that the point A' coincides with the point A when $m_1=m_2=m_3=+\infty$; the point A' coincides with the point $Q=\langle x,\mu_A,\eta_A+\rho_A,\nu_A\rangle$ when $m_1=m_3=+\infty$ and $m_2=0$; the point A' coincides with the point $N=\langle x,\mu_A+\rho_A,\eta_A,\nu_A\rangle$ when $m_1=0$ and $m_2=m_3=+\infty$; the point A' coincides with the point $M=\langle x,\mu_A,\eta_A,\nu_A+\rho_A\rangle$ when $m_1=m_2=+\infty$ and $m_3=0$. Moreover, we can obtain a point with the most concentrated information, which is a centroid point of the tetrahedron $M-NAQ$ when

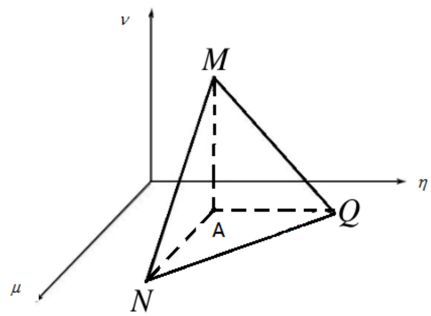

Figure 2: The possible values for the picture fuzzy set $A = \langle x, \mu_A, \eta_A, \nu_A \rangle$ on $X = \{x\}$

$m_1 = m_2 = m_3 = 3$.
Since $\rho_A = 1 - \mu_A - \eta_A - \nu_A$, we can get the following equality:
$A' = \langle x, \frac{(m_1+1)\mu_A+\rho_A}{m_1+1}, \frac{(m_2+1)\eta_A+\rho_A}{m_2+1}, \frac{(m_3+1)\nu_A+\rho_A}{m_3+1} \rangle = \langle x, \frac{1+m_1\mu_A-\eta_A-\nu_A}{m_1+1}, \frac{1+m_2\eta_A-\mu_A-\nu_A}{m_2+1}, \frac{1+m_3\nu_A-\mu_A-\eta_A}{m_3+1} \rangle$.
Similarly, we can get the point B' as follows:
$B' = \langle x, \frac{(m_1+1)\mu_B+\rho_B}{m_1+1}, \frac{(m_2+1)\eta_B+\rho_B}{m_2+1}, \frac{(m_3+1)\nu_B+\rho_B}{m_3+1} \rangle = \langle x, \frac{1+m_1\mu_B-\eta_B-\nu_B}{m_1+1}, \frac{1+m_2\eta_B-\mu_B-\nu_B}{m_2+1}, \frac{1+m_3\nu_B-\mu_B-\eta_B}{m_3+1} \rangle$.
The similarity measure between A' and B' is defined by:
$S(A', B') = 1 - [\frac{1}{3}(\Delta_1^p + \Delta_2^p + \Delta_3^p)]^{\frac{1}{p}}$,
where $\Delta_1 = \frac{1}{m_1+1}|m_1(\mu_A - \mu_B) - (\eta_A - \eta_B) - (\nu_A - \nu_B)|$,
$\Delta_2 = \frac{1}{2(m_2+1)}|m_2(\eta_A - \eta_B) - (\mu_A - \mu_B) + (\nu_A - \nu_B)|$,
$\Delta_3 = \frac{1}{2(m_3+1)}|m_3(\nu_A - \nu_B) - (\mu_A - \mu_B) + (\eta_A - \eta_B)|$.

It is easy to see that m_1, m_2 and m_3 are three different parameters adjusting the effect of refusal membership function in the computation. If m_1, m_2 and m_3 are very large, then the effect of refusal membership function can be ignored in the calculation. If m_1, m_2 and m_3 are very small, then the effect of refusal membership function need to be considered in the calculation.

In practical application, there are more than one feature in the discourse of universe, such as $X = \{x_1, x_2, \dots, x_n\}$, then the similarity measure between A and B on X is Eq.(18).

In the process of decision-making, the decision maker's decision style has a great impact on the decision results. For decision makers with different decision styles, they can find the appropriate similarity measure by adjusting the parameters m_1, m_2 and m_3. For example, for the decision makers with risky decision style and conservative decision style, they can choose the similarity measure with $m_1 = 0, m_2 = m_3 = +\infty$ and $m_1 = m_2 = +\infty, m_3 = 0$ respectively, i.e., Eq.(19) and Eq.(20). For the decision makers of defensive style, the similarity measure with $m_1 = m_2 = m_3 = 2$ is suitable.

4.3 Numerical comparisons

Example 4.4. *Let $A = \langle x, 0.10, 0.30, 0.20 \rangle$, $B = \langle x, 0.60, 0.10, 0.20 \rangle$ and $C = \langle x, 0.60, 0.00, 0.00 \rangle$ be three different PFSs on $X = \{x\}$. From the point of ratings, it is an obvious fact that the PFS A is more similar to B than the C, that is to say, $S(A, B) > S(A, C)$.*

To prove the correctness of this view for our proposed similarity measure S_m and the existing ones namely $S_{g_2}, S_{g_3}, S_{g_4}$, S_{g_5}, $S_{g_7}, S_h, S_e, S_{p_1}, S_{p_2}, S_{p_3}, S_{p_4}$, S_l, we use each similarity measure on the preseted date set. Table 2 shows their relative results.

Through the analysis and comparison of Table 2, we can easily find that the similarity measures $S_{g_2}, S_{g_3}, S_{g_4}$, S_{g_5}, $S_{g_7}, S_e, S_{p_1}, S_{p_2}, S_{p_3}, S_{p_4}$ think A has the same similarity with B and C, which is irrelevant; the proposed similarity measure S_m and the existing measures S_h, S_l fully comply with the fact of $S(A, B) > S(A, C)$. Therefore, they are rational.

Example 4.5. *Suppose that $A = \langle x, 1, 0, 0 \rangle$, $B = \langle x, 0, 0.5, 0 \rangle$ and $C = \langle x, 0, 0, 0.5 \rangle$ are three PFSs on $X = \{x\}$. When we use the 100 person voting model to explain A, B and C, then $A = \langle x, 1, 0, 0 \rangle$ means that 100 people vote for it, $B = \langle x, 0, 0.5, 0 \rangle$ represents that 50 people remain neutral, in other words, the vote cast is invalid, $C = \langle x, 0, 0, 0.5 \rangle$*

Table 2: The results of Example 4.4

	$S(A,B)$	$S(A,C)$	Relation
S_{g_2} [28]	0.7071	0.7071	$S(A,B) = S(A,C)$
S_{g_3} [28]	0.7071	0.7071	$S(A,B) = S(A,C)$
S_{g_4} [28]	0.4144	0.4144	$S(A,B) = S(A,C)$
S_{g_5} [28]	0.4144	0.4144	$S(A,B) = S(A,C)$
S_{g_7} [28]	0.7071	0.7071	$S(A,B) = S(A,C)$
S_h [12]	0.7667	0.6667	$S(A,B) > S(A,C)$
S_e [12]	0.6891	0.6891	$S(A,B) = S(A,C)$
S_{p_1} [26]	0.7500	0.7500	$S(A,B) = S(A,C)$
S_{p_2} [26]	0.6918	0.6918	$S(A,B) = S(A,C)$
S_{p_3} [26]	0.8750	0.8750	$S(A,B) = S(A,C)$
S_{p_4} [26]	0.7500	0.7500	$S(A,B) = S(A,C)$
S_l [18]	0.8754	0.7387	$S(A,B) > S(A,C)$
$S_m(p=1, m_1=m_2=m_3=3)$	0.7833	0.7083	$S(A,B) > S(A,C)$

signifies that 50 people vote against it. In this case, we intuitively believe that the PFS A is more similar to B than the C, in other words, $S(A,B) > S(A,C)$.

To justify the fact for our proposed similarity measure S_m and the existing ones namely $S_{g_2}, S_{g_3}, S_{g_4}, S_{g_5}, S_{g_7}, S_h, S_e, S_{p_1}, S_{p_2}, S_{p_3}, S_{p_4}, S_l$, we use each similarity measure on the given date set. Table 3 shows their relative results. From this table, we can observe that the similarity measures $S_{g_2}, S_{g_3}, S_{g_4}, S_{g_5}, S_{g_7}, S_h, S_e, S_{p_1}, S_{p_2}, S_{p_3}, S_{p_4} S_l$ think A has the same similarity with B and C. The proposed similarity measure is in line with human intuition.

Table 3: The results of Example 4.5

	$S(A,B)$	$S(A,C)$	Relation
S_{g_2} [28]	0.0000	0.0000	$S(A,B) = S(A,C)$
S_{g_3} [28]	0.0000	0.0000	$S(A,B) = S(A,C)$
S_{g_4} [28]	0.0000	0.0000	$S(A,B) = S(A,C)$
S_{g_5} [28]	0.0000	0.0000	$S(A,B) = S(A,C)$
S_{g_7} [28]	0.0000	0.0000	$S(A,B) = S(A,C)$
S_h [12]	0.5000	0.5000	$S(A,B) = S(A,C)$
S_e [12]	0.3545	0.3545	$S(A,B) = S(A,C)$
S_{p_1} [26]	0.5000	0.5000	$S(A,B) = S(A,C)$
S_{p_2} [26]	0.3876	0.3876	$S(A,B) = S(A,C)$
S_{p_3} [26]	0.7500	0.7500	$S(A,B) = S(A,C)$
S_{p_4} [26]	0.5000	0.5000	$S(A,B) = S(A,C)$
S_l [18]	0.2357	0.2357	$S(A,B) = S(A,C)$
$S_m(p=2, m_1=1, m_2=2, m_3=3)$	0.5139	0.5092	$S(A,B) > S(A,C)$

5 Algorithm and applications

Generally speaking, the reasonable of similarity measure between picture fuzzy sets is tested by practical investigations. In order to demonstrate this, in this section, we test the rationality of the proposed similarity measure through multi-attribute decision-making problems, which shows that the proposed similarity measure is reasonable and in line with human cognition.

In the following, we only consider the results that are obtained by the proposed similarity measure S_m and the existing similarity measures $S_{g_2}, S_{g_3}, S_{g_4}, S_{g_5}, S_{g_7}, S_h, S_e, S_{p_1}, S_{p_2}, S_{p_3}, S_{p_4}, S_l$.

5.1 Algorithm for multi-attribute decision-making

Let $X = \{x_1, x_2, \ldots, x_m\}$ be a set of attributes, there are n alternatives $A_i = \{A_{ij}\} = \{\langle x_j, \mu_{A_i}(x_j), \eta_{A_i}(x_j), \nu_{A_i}(x_j)\rangle \mid x_j \in X\}$, where $\mu_{A_i}(x_j), \eta_{A_i}(x_j), \nu_{A_i}(x_j) \in [0,1], \mu_{A_i}(x_j) + \eta_{A_i}(x_j) + \nu_{A_i}(x_j) \leq 1$, $\mu_{A_i}(x_j)$ is a positive membership function, which is used to describe the degree that alternative A_i satisfies the criteria x_j ($i = \{1, 2, \ldots, n\}$, $j = \{1, 2, \ldots, m\}$). $\eta_{A_i}(x_j)$ is a neutral membership function, which is used to describe the degree that alternative A_i dose not satisfies the criteria x_j. $\nu_{A_i}(x_j)$ indicates the degree that alternative A_i dose not fulfill the criteria x_j. Which is the best alternative? The decision-making steps are as follows:

Step 1. Normalize decision alternatives.

In multi-attribute decision-making problems, attributes can often be divided into two categories: cost type and benefit type. The cost attribute need to be changed into benefit attribute by the following formula in the decision-making process. If all attributes are beneficial, there is no need for transformation.

$$\bar{A}_{ij} = \begin{cases} A_{ij}, & for\ benefit\ attribute\ x_j, \\ A_{ij}^c, & for\ cost\ attribute\ x_j, \end{cases} \tag{22}$$

where $A_{ij}^c = \langle \nu_{A_i}(x_j), \eta_{A_i}(x_j), \mu_{A_i}(x_j)\rangle$, $i = \{1, 2, \ldots, n\}, j = \{1, 2, \ldots, m\}$.

Based on the transformation formula above, the alternative A_i can be rewritten as $A_i = \{\bar{A}_{ij}\}$.

Step 2. Calculate the similarity measure $S(A_i, A)$ $(i = 1, 2, 3, \ldots, n)$, where $A = \{\overbrace{\langle 1, 0, 0\rangle, \langle 1, 0, 0\rangle, \ldots, \langle 1, 0, 0\rangle}^{m \text{ times}}\}$ is an optimal alternative with m attributes.

Step 3. Choose the maximum one $S(A_{i_0}, A)$ from $S(A_i, A)$ $(i = 1, 2, 3, \ldots, n)$, i.e., $S(A_{i_0}, A) = \max_{1\le i\le n}\{S(A_i, A)\}$. Then alternative A_{i_0} is the best choice according to the principle of the maximum of similarity measures.

Step 4. Calculate the degree of confidence Doc, $Doc^{(i_0)} = \sum_{i=1,i\neq i_0}^{n} \mid S(A_{i_0}, A) - S(A_i, A) \mid$ ([15]). If the $Doc^{(i_0)}$ is larger, the result of similarity measure is more credible.

In the following examples, we take $p = 2, m_1 = m_2 = m_3 = 3$ for the similarity measure S_m.

5.2 Applications for multi-attribute decision-making

Example 5.1. *There are three alternatives A_1, A_2, A_3 with four different attributes x_1, x_2, x_3, x_4, which are described by PFSs respectively, as shown in Table 4. The weight of x_j $(1 \le j \le 4)$ are 0.1, 0.5, 0.3 and 0.1 respectively.*

Table 4: Three alternatives with four different attributes in Example 5.1

	x_1	x_2	x_3	x_4
A_1	$\langle x_1, 0.5, 0.1, 0.2\rangle$	$\langle x_2, 0.5, 0.3, 0.1\rangle$	$\langle x_3, 0.5, 0.1, 0.3\rangle$	$\langle x_4, 0.2, 0.3, 0.4\rangle$
A_2	$\langle x_1, 0.5, 0.1, 0.2\rangle$	$\langle x_2, 0.5, 0.3, 0.1\rangle$	$\langle x_3, 0.5, 0.1, 0.3\rangle$	$\langle x_4, 0.2, 0.3, 0.0\rangle$
A_3	$\langle x_1, 0.5, 0.1, 0.2\rangle$	$\langle x_2, 0.5, 0.3, 0.1\rangle$	$\langle x_3, 0.5, 0.1, 0.3\rangle$	$\langle x_4, 0.2, 0.0, 0.4\rangle$

Table 5: The results of similarity measures and decision results in Example 5.1.

S	The results of similarity measures			Decision-making		
	$S(A_1, A)$	$S(A_2, A)$	$S(A_3, A)$	Ranking	The best alternative	$Doc^{(2)}$
S_{g_2} [28]	0.6673	0.6673	0.6673	$A_1 = A_2 = A_3$	Cannot be determined	\
S_{g_3} [28]	0.6673	0.6673	0.6673	$A_1 = A_2 = A_3$	Cannot be determined	\
S_{g_4} [28]	0.3886	0.3886	0.3886	$A_1 = A_2 = A_3$	Cannot be determined	\
S_{g_5} [28]	0.3886	0.3886	0.3886	$A_1 = A_2 = A_3$	Cannot be determined	\
S_{g_7} [28]	0.6673	0.6673	0.6673	$A_1 = A_2 = A_3$	Cannot be determined	\
S_h [12]	0.6833	0.6943	0.6933	$A_2 > A_3 > A_1$	A_2	0.0120
S_e [12]	0.6353	0.6427	0.6394	$A_2 > A_3 > A_1$	A_2	0.0107
S_{p_1} [26]	0.7350	0.7350	0.7350	$A_1 = A_2 = A_3$	Cannot be determined	\
S_{p_2} [26]	0.6791	0.6760	0.6767	$A_1 > A_3 > A_2$	A_1	\
S_{p_3} [26]	0.8675	0.8675	0.8675	$A_1 = A_2 = A_3$	Cannot be determined	\
S_{p_4} [26]	0.7312	0.7312	0.7312	$A_1 = A_2 = A_3$	Cannot be determined	\
S_l [18]	0.7068	0.7164	0.7143	$A_2 > A_3 > A_1$	A_2	0.0117
S_m	0.6719	0.6839	0.6810	$A_2 > A_3 > A_1$	A_2	0.0149

We want to choose the best alternative from A_1, A_2, A_3. To address it, we use different similarity measures to calculate the degree of similarity between A and A_i $(i = 1, 2, 3)$. The results of the calculation and the decision are presented in Table 5.

Although there are three very similar decision alternatives in this case, based on the principle of maximum similarity measure, the alternative A_2 can be regarded as the best option with the help of the similarity measures S_h, S_e, S_l and S_m. For the similarity measures $S_{g_2}, S_{g_3}, S_{g_4}, S_{g_5}, S_{g_7}, S_{p_1}, S_{p_2}, S_{p_3}, S_{p_4}$, they cannot provide a reasonable decision result as $S(A_1, A) = S(A_2, A) = S(A_3, A)$. Moreover, in order to illustrate the reliability of decision results, we consider the confident measure of some similarity measures and give the results in Table 5 and Fig.3. Through Fig.3, we can easily find that the confident measure of the proposed similarity measure S_m is highest than that of other similarity measures such as S_h, S_e, S_l, which illustrates that the proposed similarity measure is more accurate for decision-making.

Example 5.2. [12, 31] *This example is used to evaluate the university faculty for tenure and promotion. There are six faculty candidates (denoted as $A_1, A_2, A_3, A_4, A_5, A_6$) to be evaluated. According to the evaluation criteria of some universities, the evaluation team needs to evaluate the candidates from three aspects, i.e., teaching (x_1), reach (x_2) and*

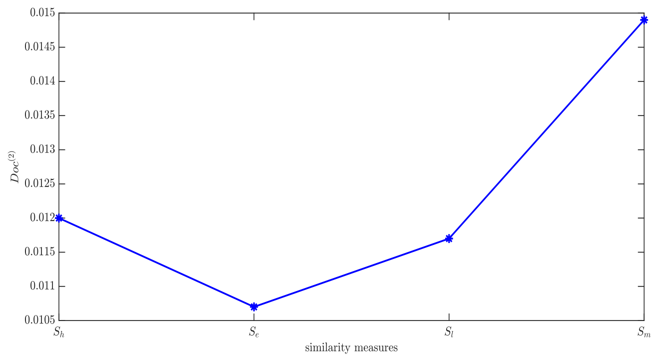

Figure 3: The degree of confidence $Doc^{(2)}$ in Example 5.1

service (x_3), whose weight vector is $w = (0.55, 0.25, 0.20)$. The evaluation results are given in Table 6. In Table 6, we take the faculty candidate A_1 as an example to explain the practice meaning of faculty candidates: the picture fuzzy set $\langle x_1, 0.61, 0.08, 0.11\rangle$ is the evaluation of expert group to "if A_1 the best candidate in view of teaching", which indicates that 61% say yes, 8% have no opinion, 11% say no and 20% do not participate. Which is the best faculty candidate? In Table 7, we summarize the results of different similarity measures and decision results.

Table 6: The evaluation results of six faculty candidates in Example 5.2

	x_1	x_2	x_3
A_1	$\langle x_1, 0.61, 0.08, 0.11\rangle$	$\langle x_2, 0.59, 0.04, 0.26\rangle$	$\langle x_3, 0.41, 0.36, 0.08\rangle$
A_2	$\langle x_1, 0.54, 0.45, 0.01\rangle$	$\langle x_2, 0.50, 0.15, 0.25\rangle$	$\langle x_3, 0.66, 0.12, 0.01\rangle$
A_3	$\langle x_1, 0.72, 0.23, 0.03\rangle$	$\langle x_2, 0.53, 0.05, 0.35\rangle$	$\langle x_3, 0.50, 0.03, 0.08\rangle$
A_4	$\langle x_1, 0.63, 0.30, 0.06\rangle$	$\langle x_2, 0.59, 0.29, 0.08\rangle$	$\langle x_3, 0.58, 0.07, 0.16\rangle$
A_5	$\langle x_1, 0.59, 0.30, 0.09\rangle$	$\langle x_2, 0.41, 0.57, 0.01\rangle$	$\langle x_3, 0.59, 0.19, 0.10\rangle$
A_6	$\langle x_1, 0.73, 0.10, 0.15\rangle$	$\langle x_2, 0.54, 0.37, 0.06\rangle$	$\langle x_3, 0.46, 0.24, 0.06\rangle$

Table 7: The results of similarity measures and decision results in Example 5.2.

S	The results of similarity measures						Decision-making		
	$S(A_1, A)$	$S(A_2, A)$	$S(A_3, A)$	$S(A_4, A)$	$S(A_5, A)$	$S(A_6, A)$	Ranking	The best alternative	$Doc^{(3)}$
S_{g_2} [28]	0.7700	0.7615	0.8240	0.8176	0.7499	0.8211	$A_3 > A_6 > A_4 > A_1 > A_2 > A_5$	A_3	0.1999
S_{g_3} [28]	0.7700	0.7615	0.8240	0.8176	0.7499	0.8211	$A_3 > A_6 > A_4 > A_1 > A_2 > A_5$	A_3	0.1999
S_{g_4} [28]	0.4980	0.4660	0.5424	0.5197	0.4582	0.5436	$A_3 > A_6 > A_4 > A_1 > A_2 > A_5$	A_3	0.2265
S_{g_5} [28]	0.4980	0.4660	0.5424	0.5197	0.4582	0.5436	$A_3 > A_6 > A_4 > A_1 > A_2 > A_5$	A_3	0.2265
S_{g_7} [28]	0.7700	0.7615	0.8240	0.8176	0.7499	0.8211	$A_3 > A_6 > A_4 > A_1 > A_2 > A_5$	A_3	0.1999
S_h [12]	0.7658	0.7250	0.7878	0.7578	0.7092	0.7745	$A_3 > A_6 > A_1 > A_4 > A_2 > A_5$	A_3	0.2067
S_e [12]	0.7113	0.6646	0.7350	0.7214	0.6548	0.7313	$A_3 > A_6 > A_4 > A_1 > A_2 > A_5$	A_3	0.1236
S_{p_1} [26]	0.7825	0.7770	0.8143	0.8050	0.7725	0.8143	$A_3 = A_6 > A_4 > A_1 > A_2 > A_5$	Cannot be determined	\
S_{p_2} [26]	0.7357	0.7047	0.7538	0.7548	0.6997	0.7610	$A_6 > A_4 > A_3 > A_1 > A_2 > A_5$	A_6	\
S_{p_3} [26]	0.8913	0.8885	0.9071	0.9025	0.8863	0.9071	$A_3 = A_6 > A_4 > A_1 > A_2 > A_5$	Cannot be determined	\
S_{p_4} [26]	0.7790	0.7753	0.8074	0.8047	0.7692	0.8055	$A_3 > A_6 > A_4 > A_1 > A_2 > A_5$	A_3	0.1033
S_l [18]	0.7844	0.7571	0.8119	0.7919	0.7441	0.8037	$A_3 > A_6 > A_4 > A_1 > A_2 > A_5$	A_3	0.1783
S_m	0.7380	0.7044	0.7763	0.7463	0.6910	0.7520	$A_3 > A_6 > A_4 > A_1 > A_2 > A_5$	A_3	0.2498

From Table 7, we have $S_m(A_3, A) > S_m(A_6, A) > S_m(A_4, A) > S_m(A_1, A) > S_m(A_2, A) > S_m(A_5, A)$, hence, the best faculty candidate is A_3. It is consistent with the decision result of the similarity measures S_{g_2}, S_{g_3}, S_{g_4}, S_{g_5}, S_{g_7}, S_h, S_e, S_{p_4}, S_l, which overcome the drawback of the similarity measures S_{p_1}, S_{p_3}, where the similarity measures S_{p_1}, S_{p_3} cannot provide the final decision result. Further, the proposed similarity measure has the highest confident measure (see Fig.4).

Moreover, based on Eq.(19), we can get $S_1(A_1, A) = 0.8223, S_1(A_2, A) = 0.7459, S_1(A_3, A) = 0.8219, S_1(A_4, A) = 0.7870, S_1(A_5, A) = 0.7288, S_1(A_6, A) = 0.8043$, according to the principle of the maximum of similarity measures, the best faculty candidate is A_1. For the similarity measure S_2 in Eq.(20), the best faculty candidate is A_3 as $S_2(A_1, A) = 0.6886, S_2(A_2, A) = 0.6428, S_2(A_3, A) = 0.7276, S_2(A_4, A) = 0.6973, S_2(A_5, A) = 0.6263, S_2(A_6, A) = 0.7076$. Therefore, for the decision makers with risky decision style, they can choose A_1 as the best candidate, but for the decision makers with conservative decision style, they can choose A_3.

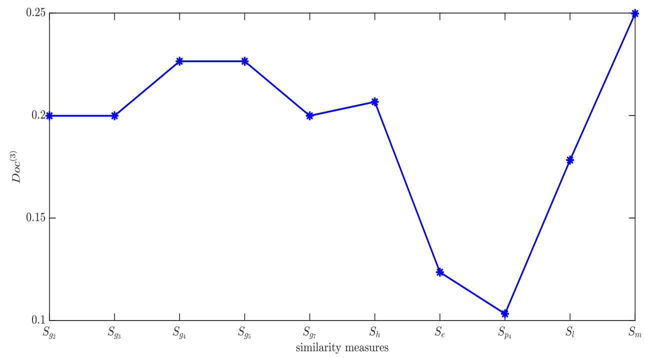

Figure 4: The degree of confidence $Doc^{(3)}$ in Example 5.2

Example 5.3. [4, 25, 29] *An organization wants to select the best enterprise resource planning system from five potential systems (represent as A_1, A_2, A_3, A_4 and A_5, respectively). The selection criteria includes the system's function and technology (x_1), the system's strategic fitness (x_2), the vendor's ability (x_3), and the vendor's reputation (x_4), whose weight vector is $w = (0.2, 0.1, 0.3, 0.4)$. Table 8 gives the evaluation results of five potential systems. In Table 9, we summarize the results of different similarity measures and decision results.*

Table 8: The evaluation results of five potential systems in Example 5.3

	x_1	x_2	x_3	x_4
A_1	$\langle x_1, 0.53, 0.33, 0.09\rangle$	$\langle x_2, 0.89, 0.08, 0.03\rangle$	$\langle x_3, 0.42, 0.35, 0.18\rangle$	$\langle x_4, 0.08, 0.89, 0.02\rangle$
A_2	$\langle x_1, 0.73, 0.12, 0.08\rangle$	$\langle x_2, 0.13, 0.64, 0.21\rangle$	$\langle x_3, 0.03, 0.82, 0.13\rangle$	$\langle x_4, 0.73, 0.15, 0.08\rangle$
A_3	$\langle x_1, 0.91, 0.03, 0.02\rangle$	$\langle x_2, 0.07, 0.09, 0.05\rangle$	$\langle x_3, 0.04, 0.85, 0.10\rangle$	$\langle x_4, 0.68, 0.26, 0.06\rangle$
A_4	$\langle x_1, 0.85, 0.09, 0.05\rangle$	$\langle x_2, 0.74, 0.16, 0.10\rangle$	$\langle x_3, 0.02, 0.89, 0.05\rangle$	$\langle x_4, 0.08, 0.84, 0.06\rangle$
A_5	$\langle x_1, 0.90, 0.05, 0.02\rangle$	$\langle x_2, 0.68, 0.08, 0.21\rangle$	$\langle x_3, 0.05, 0.87, 0.06\rangle$	$\langle x_4, 0.13, 0.75, 0.09\rangle$

Table 9: The results of similarity measures and decision results in Example 5.3.

S	The results of similarity measures					Decision-making		
	$S(A_1, A)$	$S(A_2, A)$	$S(A_3, A)$	$S(A_4, A)$	$S(A_5, A)$	Ranking	The best alternative	$Doc^{(3)}$
S_{g_2} [28]	0.4804	0.5813	0.5783	0.3458	0.3898	$A_2 > A_3 > A_1 > A_5 > A_4$	A_2	\
S_{g_3} [28]	0.4804	0.5813	0.5783	0.3458	0.3898	$A_2 > A_3 > A_1 > A_5 > A_4$	A_2	\
S_{g_4} [28]	0.3004	0.4047	0.4250	0.2532	0.2827	$A_3 > A_2 > A_1 > A_5 > A_4$	A_3	0.4590
S_{g_5} [28]	0.3004	0.4047	0.4250	0.2532	0.2827	$A_3 > A_2 > A_1 > A_5 > A_4$	A_3	0.4590
S_{g_7} [28]	0.4804	0.5813	0.5783	0.3458	0.3898	$A_2 > A_3 > A_1 > A_5 > A_4$	A_2	\
S_h [12]	0.5783	0.6527	0.6787	0.5287	0.5523	$A_3 > A_2 > A_1 > A_5 > A_4$	A_3	0.4028
S_e [12]	0.4607	0.5276	0.5327	0.3767	0.4090	$A_3 > A_2 > A_1 > A_5 > A_4$	A_3	0.3568
S_{p_1} [26]	0.6765	0.7300	0.7365	0.6410	0.6575	$A_3 > A_2 > A_1 > A_5 > A_4$	A_3	0.2410
S_{p_2} [26]	0.5326	0.5764	0.5903	0.4600	0.4880	$A_3 > A_2 > A_1 > A_5 > A_4$	A_3	0.3042
S_{p_3} [26]	0.8383	0.8650	0.8683	0.8205	0.8288	$A_3 > A_2 > A_1 > A_5 > A_4$	A_3	0.1206
S_{p_4} [26]	0.6518	0.6816	0.6831	0.6006	0.6173	$A_3 > A_2 > A_1 > A_5 > A_4$	A_3	0.1217
S_l [18]	0.5558	0.6123	0.6329	0.4411	0.4945	$A_3 > A_2 > A_1 > A_5 > A_4$	A_3	0.4279
S_m	0.4945	0.5838	0.6159	0.4348	0.4633	$A_3 > A_2 > A_1 > A_5 > A_4$	A_3	0.4872

From Table 9, we can see that the decision result of the proposed similarity measure S_m coincides with the decision results of the similarity measures S_{g_1}, S_{g_4}, S_{g_5}, S_{g_6}, S_{g_8}, S_h, S_e, $S_{p_1}, S_{p_2}, S_{p_3}, S_{p_4}$, S_{p_5}, S_l, i.e, the best enterprise resource planning system is A_3. Further, by analyzing Fig. 5, we easily find that the proposed similarity measure has the higher confident measure.

Moreover, we can have $S_1(A_1, A) = 0.5723, S_1(A_2, A) = 0.6174, S_1(A_3, A) = 0.6457, S_1(A_4, A) = 0.5061, S_1(A_5, A) = 0.5286$ by using the Eq.(19) and $S_2(A_1, A) = 0.4214$, $S_2(A_2, A) = 0.4875$, $S_2(A_3, A) = 0.4924$, $S_2(A_4, A) = 0.3296$, $S_2(A_5, A) = 0.3636$ by using the Eq.(20). According to the principle of the maximum of similarity measures, we get the enterprise resource planning system is A_3, which is not only the same as the above decision result, but also consistent with the decision results obtained by aggregation operators in [4, 25, 29]. Therefore, for the decision makers with risky decision style and conservative decision style, they can choose A_3 as the best enterprise resource planning system in this example.

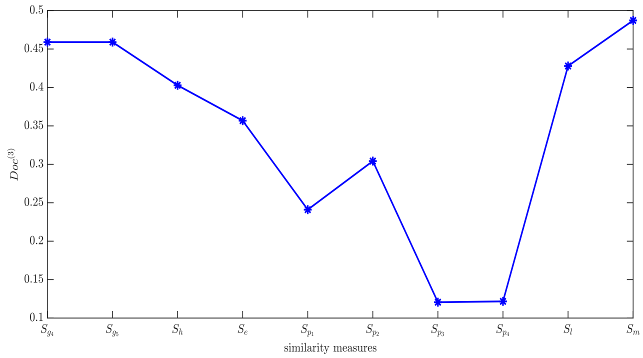

Figure 5: The degree of confidence $Doc^{(3)}$ in Example 5.3

5.3 Analysis and discussion

Through the analysis of the numerical experiment and multi-attribute decision-making problems, the effectiveness and flexibility of the proposed similarity measure are illustrated. There are some advantages that cannot be ignored when compared with some existing similarity measures $S_{g_1}, S_{g_2}, S_{g_4}, S_{g_5}, S_{g_6}, S_{g_7}, S_{g_8}, S_h, S_e, S_{p_1}, S_{p_2}, S_{p_3}, S_{p_4}, S_{p_5}$, S_{w_1}, S_{w_2}, S_l and S_m.

(1) With the help of the existing similarity measures, decision makers only pay attention to the ranking of decision-making alternatives, but they ignore the influence of decision maker's style and decision environment on decision-making results. The similarity measure proposed in this paper increases its flexibility by introducing parameters m_1, m_2, m_3. In this case, decision makers can look for the appropriate parameters m_1, m_2, m_3 to obtain the reasonable similarity measure, which is in line with the current decision maker style and decision environment.

(2) The proposed similarity measure satisfies the axiom (S2) of the similarity measure, which makes the proposed similarity measure avoid the situation of counter intuition"$A \neq B$ implies $S(A, B) = 1$". In addition, the proposed similarity measure increases the identification ability by containing a lot of fuzzy information and overcomes the defects of some existing similarity measures S_{g_1}, S_{g_2}, S_{g_4}, S_{g_5}, S_{g_7}, S_{p_3}, S_{p_4} and S_{p_5}.

(3) The proposed similarity measure can provide reasonable reliable decision outcomes for decision makers. The proposed similarity measure not only has the highest degree of credibility, but also can solve the decision-making problems that the existing similarity measures cannot solve and obtain reasonable decision results.

Therefore, the proposed similarity measure is reasonable and flexible.

6 Conclusions

Although several similarity measures between PFSs have been proposed, most of them have provided counter-intuitive results. In this paper, we analyze and discuss the causes of counter-intuitive results, and then we develop a new parametric similarity measure between PFSs, which can overcome the drawbacks of some existing similarity measures between PFSs. Through the comprehensive analysis and comparison between the proposed similarity measure and the existing similarity measures in Table 1 - 3, we illustrate that the proposed similarity measure is reasonable and in line with human cognition. From the decision results in Examples 5.1-5.3, we find that the proposed similarity measure gets correct decision results. In other words, the proposed similarity measure is the most reasonable similarity measure than some existing similarity measures between PFSs for multi-attribute decision making. It can not only better deal with the decision-making problems, but also has the highest confident measure. In addition, decision makers with different decision styles can choose the appropriate similarity measure by adjusting the parameters m_1, m_2, m_3 of the proposed similarity measure in a wide variety of decision making situations. In the future, we will study the relationship among similarity measure, distance and entropy between PFSs. Inspired by the idea of [13, 34], we will look for the conversion formula from crisp data to PFS data and explore new methods of multi-attribute decision-making to solve multi-attribute decision-making problem with real data.

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