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New constructions of lattice-valued quasi-overlap functions

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Abstract

In this paper, we first investigate some properties of lattice-valued quasi-overlap functions, which includes the relation between quasi-overlap functions and three kinds of aggregation functions and the Lipschitz property. Then we provide two new construction methods for lattice-valued quasi-overlap functions. One applies t-conorms and negations, the other involves quasi-overlap functions and implications. Examples are presented and some analytical properties are studied as well.

Keywords: Quasi-overlap functions, bounded lattices, t-conorms, negations, implications.

1 Introduction

The concept of overlap arises to solve many problems when assigning a given element or object to one of the several classes. For instance, since the separation between the classes may be not clear, these classes may be fuzzy in nature or the border which is clear in reality may be not clear enough in our picture, experts may not be sure about how to assign elements into a specific class. In order to be able to check whether a certain classification structure fits reality and is useful for the declared decision making purposes in the same time, overlapping needs to be evaluated and the overlap functions, which was introduced by Bustince et al. in [7] as a binary aggregation operator must fulfill some basic properties to measure the degree of overlapping between two classes.

As a brand of vital aggregation functions, overlap functions on totally ordered sets like the real unit interval [0, 1] play a significant role both in actual applications and theory. In applications, overlap functions are widely used in many practical issues like image processing [8, 27], classifications [17, 18, 34, 36], decision making [9, 19] and fuzzy community detection problems [24].

In theory, for related concept generalized study, Bustince et al. [8] considered several analytical properties including migrativity, homogeneity and Lipschitz property, Gómez et al. [23] generalized the overlap functions to the case of more than 2 variables and introduced the concept of n-dimensional overlap functions, De Miguel et al. [12] replaced the boundary conditions which are written as an equivalence by a sufficient condition and reached the concept of general overlap functions and so on. Besides, there are many researchers involved in numerous aspects of studies referred to overlap functions, for instance, some crucial properties [13, 43, 49], corresponding implications [14, 16, 47], the additive and multiplicative generators pairs [15, 40], distributive equations [33, 41, 42] and interval overlap fuctions [3, 39].

Nowadays, the study of overlap functions and other binary aggregation operators such as t-norms, t-conorms, uninorms, nullnorms, etc. on lattices or partially ordered sets have gained more and more attention. For instance, the study involving t-norms and t-conorms [4, 45, 46], the work involving uninorms [11, 30] and the investigations involving nullnorms [10, 28]. As for the study of overlap functions on lattices, Paiva et al. [35] introduced the notions of lattice-valued overlap and quasi-overlap functions where the continuity is eliminated from the definition, and showed some properties of (quasi-) overlap functions on bounded lattices. Furthermore, as is mentioned in [1], the condition of continuity of traditional overlap functions is required in order to enable them to be applied in image processing. However, for computational purpose its implementation does not require continuity and Paiva et al. [35] provided an example

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to illustrate this point. Meanwhile, the notion of continuity on bounded lattices was still an open problem in their work. Besides, for purpose of investigating the construction methods and properties of overlap functions on complete lattices more conveniently, Qiao [37] gave a concrete form of the continuity of overlap on complete lattices that are simultaneously join-preserving and meet-preserving for each argument. Recently, Wang [48] provided a concrete form of the continuity of overlap functions on bounded lattices and presented two methods to construct new overlap functions on bounded lattices. In addition, Qiao [38] recently provided construction methods for quasi-overlap functions via special homomorphisms and the so-called Galois s-connections. The study of these aggregation functions (including overlap functions) on lattices is more general and practical in applications. For example, in classifications, the classification systems may lack system accuracy due to the uncertainty inherent in the definition of membership functions and the limitation of the homogeneous distribution of the linguistic labels. Besides, many questions in applications involve incomparable elements, and it is reasonable to study lattice-valued sets and lattice-valued functions.

However, only a few construction methods arise after extending lattice-valued quasi-overlap functions. In addition, the existing constructions are all on the basis of quasi-overlap functions. On the other hand, as quasi-overlap functions are a kind of non-associative aggregation functions, there must be more feasible construction methods from mathematical point of view. Thus, we intend to use other common operators to enrich this research topic. In this paper, we provide two new construction methods by means of t-conorms and negations, and quasi-overlap functions and implications. Besides, we study some basic analytical properties of these two constructions.

In addition, we investigate some properties of lattice-valued quasi-overlap functions, which includes the relation between quasi-overlap functions and three kinds of aggregation functions and the Lipschitz property of quasi-overlap functions. In order to study the Lipschitz property on lattice, we first recommend the notions of valuation and metric lattices. And based on these, we are able to continue this topic.

The rest of this paper is organized as follows. After necessary preliminaries, in Section 3, we investigate the lattice-valued quasi-overlap function further and provide some new conclusions. For instance, we study the relation between n-dimensional quasi-overlap functions and conjunctive, disjunctive and averaging aggregations, citing the concept of valuation and studying the Lipschitz property. In Section 4, we present two construction methods. Finally, we end up with the conclusion and future works.

2 Preliminaries

In this section, we recall some basic concepts related to our study and we start with lattice theory.

Definition 2.1. [6] *Let $(P, \leq)$ be a partially ordered set and A be a subset of P. An element $x \in P$ is said to be an upper bound of A if $x \geq y$ for all $y \in A$. An upper bound a of A is called the least upper bound if $a \leq x$ for all upper bounds x of A. The definition of the lower bound and the greatest lower bound can be defined dually.*

Definition 2.2. [22] *Let $(P, \leq)$ be a partially ordered set. If any two elements in P have a greatest lower bound, then P is said to be an* inf *semilattice. And if any two elements in P have a least upper bound, then P is said to be a* sup *semilattice.*

Definition 2.3. [5] *A lattice is a partially ordered set where any two elements of it have a greatest lower bound denoted by $x \wedge y$, and a least upper bound denoted by $x \vee y$. If a lattice $(L, \leq)$ has a top element denoted by 1_L and a bottom element denoted by 0_L, then $(L, \leq)$ is called a bounded lattice. A lattice $(L, \leq)$ is complete if every subset of L has a greatest lower bound and a least upper bound.*

We also denote the bounded lattice $(L, \leq)$ as $(L, \leq, 0_L, 1_L)$. Let $a, b \in L$, the notation $a \parallel b$ are used to denote that a is incomparable with b, i.e. $a \not\leq b$ and $a \not\geq b$. The notation $a \nparallel b$ denotes that a is comparable with b, i.e. $a \leq b$ or $b \leq a$. A partially ordered set $(P, \leq)$ is called a chain if for any $x, y \in P$, then it holds either $x \leq y$ or $y \leq x$.

Definition 2.4. [5] *For a bounded lattice $(L, \leq, 0_L, 1_L)$, and $a, b \in L$ with $a \leq b$, a subinterval $[a, b]$ of L is a sublattice of L defined as*

$$[a, b] = \{x \in L \mid a \leq x \leq b\}.$$

Similarly, $]a, b] = \{x \in L \mid a < x \leq b\}$, $[a, b[= \{x \in L \mid a \leq x < b\}$ and $]a, b[= \{x \in L \mid a < x < b\}$.

Now we introduce several related functions on lattices.

Definition 2.5. [32] *Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice and $n \in \mathbb{N}$ be fixed. A mapping $A : L^n \to L$ is called an (n-ary) aggregation function if it is increasing and satisfies the boundary conditions*

$$A(0_L, \ldots, 0_L) = 0_L, \ A(1_L, \ldots, 1_L) = 1_L.$$

Example 2.6. *T-norms and t-conorms* [4]*, uninorms* [30]*, nullnorms* [28]*, t-operators* [44]*, semi-t-operators* [20]*, uni-nullnorms and null-uninorms* [50] *are aggregation functions on bounded lattices.*

Aggregation functions on the unit interval of the real line are usually classified into four different classes by means of their point-wise comparison to the maximum and minimum operators. Here, we only consider three of them, i.e., conjunctive, disjunctive and averaging aggregation functions.

Definition 2.7. [26] *Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice, $n \in \mathbb{N}$ be fixed and $A : L^n \to L$ be an (n-ary) aggregation function. A is called to be conjunctive (resp. disjunctive) if $A(x_1, \ldots, x_n) \leq x_i$ (resp. $A(x_1, \ldots, x_n) \geq x_i$) for all $x_1, \ldots, x_n \in L, i \in \{1, \ldots, n\}$.*

Note that $A : L^n \to L$ is conjunctive (resp. disjunctive) if and only if $A \leq \wedge$ (resp. $A \geq \vee$). Besides, A is called to be averaging if $\wedge \leq A \leq \vee$. In addition, t-norms [4] and semicopulas [31] on bounded lattices must be conjunctive and t-conorms [4] and co-semicopulas [31] on bounded lattices must be disjunctive.

Moreover, an (n-ary) aggregation function on a bounded lattice L is idempotent if for all $x \in L$, $A(x, \ldots, x) = x$. An element $a \in L \setminus \{0_L, 1_L\}$ is a zero divisor of A if there exist some $x_i \in L \setminus \{0_L, 1_L\}$, $i \in \{1, \ldots, n\}$ such that $A(a, x_2, \ldots, x_n) = A(x_1, a, x_3, \ldots, x_n) = \cdots = A(x_1, \ldots, x_{n-1}, a) = 0_L$. And an element $b \in L$ is an absorbing element of A if for all $x_i \in L \setminus \{0_L, 1_L\}$, $i \in \{1, \ldots, n\}$, $A(b, x_2, \ldots, x_n) = A(x_1, b, x_3, \ldots, x_n) = \cdots = A(x_1, \ldots, x_{n-1}, b) = b$.

Definition 2.8. [4] *Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice. A binary function T (resp. S) : $L^2 \to L$ is called a t-norm (resp. t-conorm) if it is commutative, associative, increasing with respect to both variables and has a neutral element 1_L (resp. 0_L).*

Example 2.9. [29] *Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice. T_D and $T_\wedge$ given below are two basic t-norms on bounded lattice L:*

$$T_D(x, y) = \begin{cases} y & \text{if } x = 1_L \\ x & \text{if } y = 1_L \\ 0_L & \text{otherwise,} \end{cases}$$

$$T_\wedge(x, y) = x \wedge y.$$

Dually, S_D and $S_\vee$ given below is a t-conorm on bounded lattice L:

$$S_D(x, y) = \begin{cases} y & \text{if } x = 0_L \\ x & \text{if } y = 0_L \\ 1_L & \text{otherwise,} \end{cases}$$

$$S_\vee(x, y) = x \vee y.$$

Moreover, a t-norm T (resp. t-conorm S) is positive if $T(x, y) = 0_L$ (resp. $S(x, y) = 1_L$) implies either $x = 0_L$ or $y = 0_L$ (resp. either $x = 1_L$ or $y = 1_L$).

Definition 2.10. [25] *Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice. A decreasing function $N : L \to L$ is called a negation if $N(0_L) = 1_L$ and $N(1_L) = 0_L$.*

A negation N on L is said to be non-vanishing if $N(x) = 0_L \Leftrightarrow x = 1_L$. And a negation N on L is said to be non-filling if $N(x) = 1_L \Leftrightarrow x = 0_L$.

The classical (standard) fuzzy negation N_C on the real unit internal [0, 1] is given as $N_C(x) = 1 - x$, for all $x \in [0, 1]$.

Definition 2.11. [2, 21] *Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice. A function $I : L^2 \to L$ is called an implication if it is decreasing in its first variable, increasing in its second one, and fulfills $I(0_L, 0_L) = I(0_L, 1_L) = I(1_L, 1_L) = 1_L$ and $I(1_L, 0_L) = 0_L$.*

In addition, an implication I on L is said to satisfy the *identity principle*, if

$$I(x, x) = 1_L, \quad x \in L. \tag{IP}$$

From the mathematical point of view, the continuity of overlap functions is important in order to solve the problem of fuzziness in real applications. However, for computational purposes its implementation does not require continuity and Paiva et al. [35] provide an example to prove this. Thus, the concept of quasi-overlap functions arises.

Definition 2.12. [35] *Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice. A function $O : L^2 \to L$ is called quasi-overlap whenever it satisfies:*

$(O1)$ *$O(x, y) = O(y, x)$ for all $x, y \in L$;*

$(O2)$ *$O(x, y) = 0_L$ if and only if $x = 0_L$ or $y = 0_L$;*

$(O3)$ *$O(x, y) = 1_L$ if and only if $x = y = 1_L$;*

$(O4)$ *O is increasing in each coordinate.*

Example 2.13. [38]

(1) *Consider $p \in \mathbb{N} \setminus \{0\}$ and the bounded poset $L_0 = \{0, 1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots\}$ with the ordering as real numbers. Then the binary operator $O^{(p)}_{L_0} : L_0 \times L_0 \to L_0$ given, for all $x, y \in L_0$, by*

$$O^{(p)}_{L_0}(x, y) = x^p y^p,$$

is a quasi-overlap function on L_0.

(2) *The binary operator $O : [0, \infty]^{op} \times [0, \infty]^{op} \to [0, \infty]^{op}$ given, for all $x, y \in [0, \infty]^{op}$, by*

$$O(x, y) = (x + y) \times (x \wedge_{[0,\infty]^{op}} y),$$

is a quasi-overlap function on the bounded poset $[0, \infty]^{op}$, where $[0, \infty]^{op}$ denotes the extended interval of all non-negative real numbers with the opposite ordering as real numbers and $+$ and $\times$ are the usual addition and multiplication on real numbers extended to cope with infinity such that $x + \infty = \infty$ for every $x \in [0, \infty]^{op}$ and $x \times \infty = \infty$ for every $x \in]0, \infty]^{op}$ and $0 \times \infty = 0$.

If the condition of continuity is added into the notion above, we can reach the concept of overlap functions on L. There are several constructions of continuous functions on lattices, readers can refer to [35, 37, 48] for interest.

Next, we recall the migrativity of binary operations on lattices.

Definition 2.14. [35] *For an aggregation operation A, a bounded lattice L and fix an $\alpha \in L$. A binary operation $F : L^2 \to L$ is (α, A)-migrative if, for any $x, y \in L$, it satisfies*

$$F(A(\alpha, x), y) = F(x, A(\alpha, y)).$$

If F is (α, A)-migrative for all $\alpha \in L$, then it is A-migrative.

3 Further investigation of lattice-valued quasi-overlap functions

In this section, we focus on some new conclusions about quasi-overlap functions on lattices. Let us denote the set of all quasi-overlap functions on a lattice $(L, \leq, 0_L, 1_L)$ by LO. Then we can reach the following theorem.

Definition 3.1. *Define the ordering $\leq_{LO}$ on the set LO for $O_1, O_2 \in LO$ as $O_1 \leq_{LO} O_2$ if and only if $O_1(x, y) \leq O_2(x, y)$ for all $x, y \in L$. In addition, let $(O_1 \wedge_{LO} O_2)(x, y) = O_1(x, y) \wedge O_2(x, y)$ and $(O_1 \vee_{LO} O_2)(x, y) = O_1(x, y) \vee O_2(x, y)$ for all $x, y \in L$.*

Theorem 3.2. *Let $(L, \leq, 0, 1)$ be a bounded lattice with $x \wedge y > 0_L$ and $x \vee y < 1_L$ for all $x, y \in L \setminus \{0_L, 1_L\}$. Then $(LO, \leq_{LO})$ is a lattice.*

Proof. First, we prove that $(LO, \leq_{LO})$ is an inf semilattice, i.e., $(O_1 \wedge O_2) \in LO$ for any $O_1, O_2 \in LO$. It can be obtained that $0_L < (O_1 \wedge_{LO} O_2)(x, y) = O_1(x, y) \wedge O_2(x, y) < 1_L$ for all $(x, y) \in (L \setminus \{0_L\})^2 \setminus (\{1_L\})^2$. This is because for any $(x, y) \in (L \setminus \{0_L\})^2 \setminus (\{1_L\})^2$, $0_L < O_1(x, y) < 1_L$ and $0_L < O_2(x, y) < 1_L$. It is easy to check that $O_1 \wedge_{LO} O_2$ satisfies the conditions (O1-O4), thus $(O_1 \wedge_{LO} O_2) \in LO$.

Similarly, we can prove that $(LO, \leq_{LO})$ is also a sup semilattice, hence $(LO, \leq_{LO})$ is a lattice. □

3.1 Conjunctive, disjunctive and averaging aggregations

Taking into account of the classification of aggregation functions, now we will try to seek the relations between these three classes that mentioned in Definition 2.7 and quasi-overlap functions on a bounded lattice. Notice that aggregation functions are usually presented by n-ary forms, where $n \in \mathbb{N}$, thus we extend the notion of quasi-overlap function into n-dimensional case first.

Definition 3.3. *Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice and $n \in \mathbb{N}$. A function $O : L^n \to L$ is called an n-dimensional quasi-overlap function if and only if it satisfies the following conditions:*

($NO1$) *O is symmetric.*

($NO2$) *$O(x_1, \ldots, x_n) = 0_L$ if and only if $x_i = 0_L$ for some $i \in \{1, \ldots, n\}$.*

($NO3$) *$O(x_1, \ldots, x_n) = 1_L$ if and only if $x_i = 1_L$ for all $i \in \{1, \ldots, n\}$.*

($NO4$) *O is increasing in each coordinate.*

Example 3.4. (1) *Consider $p \in \mathbb{N} \setminus \{0\}$, $n \in \mathbb{N}$ and the bounded poset $L_0 = \{0, 1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots\}$ with the ordering as real numbers. Then the n-dimensional operator $O^{(p)}_{L_0} : (L_0)^n \to L_0$ given, for all $x_1, x_2, \ldots, x_n \in L_0$, by*

$$O^{(p)}_{L_0}(x_1, x_2, \ldots, x_n) = {x_1}^p {x_2}^p \cdots {x_n}^p,$$

is an n-dimensional quasi-overlap function on L_0.

(2) *The n-dimensional operator $O : L^n \to L$ given, for all $x_1, x_2, \ldots, x_n \in L$, by*

$$O(x_1, x_2, \ldots, x_n) = x_1 \wedge x_2 \wedge \cdots \wedge x_n,$$

is an n-dimensional quasi-overlap function on any bounded lattice L with $a \wedge b > 0_L$ for all $a, b \in L \setminus \{0_L\}$.

In this paper, we write $\boldsymbol{x} = (x_1, \ldots, x_n) \in L^n$.

Proposition 3.5. *Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice with at least three elements and $A_n : L^n \to L$ be an aggregation function.*

1. *If A_n is disjunctive, then A_n is not an n-dimensional quasi-overlap functions on L.*
2. *If A_n is conjunctive, then A_n is an n-dimensional quasi-overlap functions on L if and only if it is symmetric and has no zero divisors.*
3. *If A_n is averaging and has no zero divisors, then A_n is an n-dimensional quasi-overlap functions on L if and only if it is symmetric and has zero as absorbing element and satisfies $A_n(x, 1_L, \ldots, 1_L) \neq 1_L$ for any $x \neq 1_L$.*

Proof. 1. If A_n is disjunctive, then for any $x_i \in L$, $i \in \{1, \ldots, n\}$, $A_n(\boldsymbol{x}) \geq \vee_{i=1}^n x_i$, which implies $A_n(a, 0_L, \ldots, 0_L) \geq a$ for any $a \in L$ and $a > 0_L$. Thus the condition (NO2) of n-dimensional quasi-overlap functions is not satisfied.

2. The necessity is obvious. Then we need to prove the sufficiency. If A_n is conjunctive, symmetric and has no zero divisors, then A_n satisfies (NO1) and (NO4) directly. Hence it only remains to prove (NO2) and (NO3).

(NO2): If $A_n(\boldsymbol{x}) = 0_L$, notice that A_n has no zero divisors, then we know $x_i = 0_L$ for some $i \in \{1, \ldots, n\}$. Conversely, if $x_i = 0_L$ for some $i \in \{1, \ldots, n\}$, then $\wedge_{i=1}^n x_i = 0_L$. From $A_n(\boldsymbol{x}) \leq \wedge_{i=1}^n x_i = 0_L$ we know $A_n(\boldsymbol{x}) = 0_L$.

(NO3): If $A_n(\boldsymbol{x}) = 1_L$, since A_n is conjunctive, then $1_L = A_n(\boldsymbol{x}) \leq \wedge_{i=1}^n x_i \leq 1_L$, thus $\wedge_{i=1}^n x_i = 1_L$, which implies $x_i = 1_L$ for all $i \in \{1, \ldots, n\}$. Conversely, if $x_i = 1_L$ for all $i \in \{1, \ldots, n\}$, then $A_n(\boldsymbol{x}) = 1_L$ naturally.

3. First we prove the necessity. Assume A_n be an n-dimensional quasi-overlap function on L and A_n is averaging. Then A_n is symmetric directly. From the condition (NO2), if $x_i = 0_L$ for some $i \in \{1, \ldots, n\}$, then $A_n(\boldsymbol{x}) = 0_L$, thus we have A_n has zero as absorbing element. Besides, from the condition (NO3), we know that $A_n(x, 1_L, \ldots, 1_L) \neq 1_L$ for any $x \neq 1_L$. Next, we turn to the sufficiency. Similarly, we only need to check the condition (NO2) and (NO3).

(NO2): Suppose averaging aggregation functions do not have zero divisors, thus $A_n(\boldsymbol{x}) = 0_L$ implies $x_i = 0_L$ for some $i \in \{1, \ldots, n\}$. Conversely, if $x_i = 0_L$ for some $i \in \{1, \ldots, n\}$, then $A_n(\boldsymbol{x}) = 0_L$ since zero is absorbing element.

(NO3): If $A_n(\boldsymbol{x}) = 1_L$ then from the monotonicity of A_n, we have $A_n(x_1, 1_L, \ldots, 1_L) = 1_L$ for any $x_1 \in L$, and then $x_1 = 1_L$ by assumption. Conversely, if $x_i = 1_L$ for all $i \in \{1, \ldots, n\}$, then $A_n(\boldsymbol{x}) = 1_L$ immediately.

□

3.2 Lipschitz property

Lipschitz property is a useful properties that has been studied in relation to aggregation functions. In order to generalize the definition of Lipschitz property to a lattice, we need to recommend the following concepts that will help.

Definition 3.6. [5] *A valuation on a lattice L is a real-valued function $v(x)$ on L which satisfies*

$$v(x) + v(y) = v(x \vee y) + v(x \wedge y),$$

for all $x, y \in L$.

A valuation is increasing if and only if $x \geq y$ implies $v(x) \geq v(y)$ and positive if and only if $x > y$ implies $v(x) > v(y)$.

Example 3.7. [5]

(1) *In any real finite-dimensional vector space $\mathbb{R}^n$, lattice-ordered by letting $(x_1, \cdots, x_n) \leq (y_1, \cdots, y_n)$ mean that $x_k \leq y_k$ for all $k \in \{1, \cdots, n\}$, any linear functional $c(x) = c_1x_1 + \cdots + c_nx_n$ is a valuation, where $c_i \in \mathbb{R}$ for $i \in \{1, 2, \ldots, n\}$.*

(2) *For the lattice $N_5 = \{a, b, c, d, e\}$ given in Fig. 1, the equations $v(a) = v(b) = v(c) = \frac{1}{2}$, $v(e) = 1$ and $v(d) = 0$ give a nontrivial increasing valuation on N_5.*

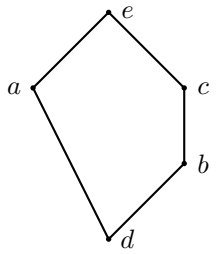


Figure 1: The lattice N_5

Definition 3.8. [5] *In any lattice L with an increasing valuation, the distance function on L with respect to v is defined as follows:*

$$d(x, y) = v(x \vee y) - v(x \wedge y),$$

for all $x, y \in L$.

A lattice with increasing valuation is called a pseudo-metric (or quasi-metric) lattice. And lattices with a positive valuation are called metric lattices. Thanks to these concepts, we are able to present the definition of k-Lipschitz property on metric lattices.

Definition 3.9. *Let L be a metric lattice, $O : L^n \to L$ be a mapping, d be a distance function on L with respect to v and let $k > 0$ be a positive real value. Then, the function O is k-Lipschitz if and only if for any $\boldsymbol{x}, \boldsymbol{y} \in L^n$ the inequality*

$$d(O(\boldsymbol{x}), O(\boldsymbol{y})) \leq k(\|\boldsymbol{x} - \boldsymbol{y}\|),$$

holds, where $\|\boldsymbol{x} - \boldsymbol{y}\| = \sum_i d(x_i, y_i)$.

Theorem 3.10. *Let $(L, \leq, 0_L, 1_L)$ be a bounded metric lattice with $v : L \to [0, 1]$ be a positive valuation with $v(0_L) = 0$ and $v(1_L) = 1$, $d : L^2 \to [0, 1]$ be a distance function on L with respect to v and $k \geq 1$ be a positive real value. Then, for any k-Lipschitz n-dimensional quasi-overlap function $O : L^n \to L$, the following inequality holds:*

$$max\{\sum_i kv(x_i) - nk + 1, 0\} \leq v(O(\boldsymbol{x})) \leq min\{1, kv(x_1), \ldots, kv(x_n)\},$$

for any $\boldsymbol{x} \in L^n$.

Proof. Suppose that $v(O(\boldsymbol{x})) > min\{1, kv(x_1), \ldots, v(x_n)\}$ for some $\boldsymbol{x} \in L^n$. Since $v(O(\boldsymbol{x})) \leq 1$, we have $min\{1, kv(x_1), \ldots, kv(x_n)\} = kv(x_{i_0})$, for some $i_0 \in \{1, \ldots, n\}$, i.e., $kv(x_{i_0}) < v(O(\boldsymbol{x}))$. However, take $\boldsymbol{x} = (1_L, \ldots, 1_L, x_{i_0}, 1_L, \ldots, 1_L)$ and $\boldsymbol{y} = (1_L, \ldots, 1_L, 0_L, 1_L, \ldots, 1_L)$ with a single zero at the i_0-th position. Then $v(O(\boldsymbol{x})) = v(O(\boldsymbol{x}) \vee O(\boldsymbol{y})) - v(O(\boldsymbol{x}) \wedge O(\boldsymbol{y})) = d(O(\boldsymbol{x}), O(\boldsymbol{y})) \leq kv(x_{i_0})$, which leads to a contradiction.

Take $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (1_L, 1_L, \ldots, 1_L)$. Then $v(O(\boldsymbol{x}) \vee O(\boldsymbol{y})) - v(O(\boldsymbol{x}) \wedge O(\boldsymbol{y})) = d(O(\boldsymbol{x}), O(\boldsymbol{y})) \leq k(\sum_i (v(1_L) - v(x_i)))$. By $v(O(1_L, 1_L, \ldots, 1_L)) = v(1_L) = 1$ we get $1 - v(O(\boldsymbol{x})) \leq k(\sum_i (1 - v(x_i)))$, i.e., $v(O(\boldsymbol{x})) \geq \sum_i kv(x_i) - kn + 1$. On the other hand $v(O(\boldsymbol{x})) \geq 0$, so $v(O(\boldsymbol{x})) \geq \max\{0, \sum_i kv(x_i) - kn + 1\}$. □

And the following corollary is straightforward.

Corollary 3.11. *Let $(L, \leq, 0_L, 1_L)$ be a bounded metric lattice with $v : L \to [0, 1]$ be a positive valuation with $v(0_L) = 0$ and $v(1_L) = 1$, $d : L^2 \to [0, 1]$ be a distance function on L with respect to v. Then, for any 1-Lipschitz n-dimensional quasi-overlap function $O : L^n \to L$, the following inequality holds:*

$$v(O(\boldsymbol{x})) \leq min\{v(x_1), \ldots, v(x_n)\},$$

for any $\boldsymbol{x} \in L^n$.

4 Two construction methods for lattice-valued quasi-overlap functions

In this section, we investigate two new construction methods for quasi-overlap functions on L. At first, we make use of t-conorms and negations. And then, we construct quasi-overlap functions on the basis of quasi-overlap functions and implications. Besides, these two methods are both on the basis of two binary operators.

4.1 Construction via t-conorms and negations

In this subsection, we intend to present an alternative construction of quasi-overlap functions on a bounded lattice with the help of t-conorms and negations.

Theorem 4.1. *Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice, $S_1, S_2 : L^2 \to L$ be two t-conorms with S_1 is positive and $N : L \to L$ be a non-vanishing and non-filling negation. Then the mapping $O : L^2 \to L$ defined by*

$$O(x, y) = S_1(f(x, y), N(S_2(f(x, y), g(x, y)))),$$

where $f, g : L^2 \to L$ are commutative functions, is a quasi-overlap function if and only if

(1) *$f(x, y) = 0_L$, if and only if $g(x, y) = 1_L$, if and only if $x = 0_L$ or $y = 0_L$;*

(2) *$f(x, y) = 1_L$ if and only if $x = y = 1_L$;*

(3) *$S_1(f(x, y), N(S_2(f(x, y), g(x, y))))$ is increasing.*

Proof. For necessity, we suppose $O(x, y) = S_1(f(x, y), N(S_2(f(x, y), g(x, y))))$ is a quasi-overlap function on L, $(x, y) \in L^2$. Then condition (3) is immediately obtained from the monotonicity of O.

If $O(x, y) = 0_L$, i.e., $S_1(f(x, y), N(S_2(f(x, y), g(x, y)))) = 0_L$, then this coincides with $f(x, y) = N(S_2(f(x, y), g(x, y))) = 0_L$. Since N is non-vanishing, then we have $O(x, y) = 0_L \Leftrightarrow f(x, y) = 0_L$ and $S_2(f(x, y), g(x, y)) = 1_L$. Besides, $f(x, y) = 0_L$ and $S_2(f(x, y), g(x, y)) = 1_L$ are equivalent to $f(x, y) = 0_L$ and $S_2(0_L, g(x, y)) = g(x, y) = 1_L$. Hence, $O(x, y) = 0_L \Leftrightarrow f(x, y) = 0_L$ and $g(x, y) = 1_L$. Then f and g must fulfill condition (1).

If $O(x, y) = 1_L$, i.e., $S_1(f(x, y), N(S_2(f(x, y), g(x, y)))) = 1_L$, then, since S_1 is positive, this coincides with $f(x, y) = 1_L$ or $N(S_2(f(x, y), g(x, y))) = 1_L$. Since N is non-filling, then we have $N(S_2(f(x, y), g(x, y))) = 1_L \Leftrightarrow S_2(f(x, y), g(x, y)) = 0_L$. Thus, $O(x, y) = 1_L \Leftrightarrow f(x, y) = 1_L$ or $S_2(f(x, y), g(x, y)) = 0_L$. If $O(x, y) = 1_L \Leftrightarrow f(x, y) = 1_L$, then we have $f(x, y) = 1_L$ if and only if $x = y = 1_L$ since O is a quasi-overlap function. If $O(x, y) = 1_L \Leftrightarrow S_2(f(x, y), g(x, y)) = 0_L$, then we have $O(x, y) = 1_L \Leftrightarrow f(x, y) = g(x, y) = 0_L \Leftrightarrow x = y = 1_L$. However, this contradicts with condition (1). Hence, $O(x, y) = 1_L \Leftrightarrow f(x, y) = 1_L \Leftrightarrow x = y = 1_L$, i.e., condition (2) must be fulfilled.

For sufficiency, it is easy to check that O satisfies the condition (O1) and (O4), then we only need to check the remaining conditions.

(O2): Let $x, y \in L$. If $O(x, y) = S_1(f(x, y), N(S_2(f(x, y), g(x, y)))) = 0_L$, therefore, $f(x, y) = N(S_2(f(x, y), g(x, y))) = 0_L$. This is equivalent to $f(x, y) = 0_L$ since $f(x, y) = 0_L$ if and only if $g(x, y) = 1_L$. From condition (1), $f(x, y) = 0_L$ if and only if $x = 0_L$ or $y = 0_L$.

(O3): For any $x, y \in L$, $O(x,y) = 1_L$, i.e., $S_1(f(x,y), N(S_2(f(x,y), g(x,y)))) = 1_L$, there are two possible cases:

Case 1. $f(x,y) = 1_L$. From the hypothesis, $O(x,y) = f(x,y) = 1_L$ if and only if $x = y = 1_L$;

Case 2. $f(x,y) \neq 1_L$ and $S_2(f(x,y), g(x,y)) = 0_L$. Hence, we have $f(x,y) = g(x,y) = 0_L$. However, it cannot be satisfied from the condition (1).

In the end, O satisfies all the four conditions and it is a quasi-overlap function on L. □

To have a better comprehension, we now provide several examples.

Example 4.2.

1. *Let $(L, \leq, 0_L, 1_L)$ be a chain, $N : L \to L$ be a non-vanishing and non-filling negation and denote a t-conorm $S : L^2 \to L$ by, for all $x, y \in L$,*
$$S(x,y) = x \vee y.$$
Take $f(x,y) = x \wedge y$, $g(x,y) = N(f(x,y))$ and $S_1 = S_2 = S$, thus, by simple computation, we have
$$O(x,y) = f(x,y) \vee N(f(x,y) \vee N(f(x,y))).$$
Notice that L is a chain, which means that any two elements on L are comparable, and the presentation of O can be written as follows:
$$O(x,y) = \begin{cases} f(x,y) \vee N(N(f(x,y))), & f(x,y) < N(f(x,y)) \\ f(x,y), & f(x,y) \geq N(f(x,y)). \end{cases}$$
It is easy to check that O satisfies all the conditions (O1)-(O4), and it is a quasi-overlap function on L.

2. *Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice with $x \vee y < 1_L$ for all $x, y \in L \setminus \{1_L\}$. For any $a \in L \setminus \{0_L, 1_L\}$, we denote a t-conorm $S : L^2 \to L$ by, for all $x, y \in L$,*
$$S(x,y) = \begin{cases} x, & y = 0_L \\ y, & x = 0_L \\ a \vee x \vee y, & otherwise. \end{cases}$$
Take $f(x,y) = G(x,y)$, where $G(x,y)$ is a quasi-overlap function on L, $g(x,y) = N(f(x,y))$, $S_1 = S_2 = S$ and N is defined as follows:
$$N(x) = \begin{cases} 1, & x = 0_L \\ 0, & x = 1_L \\ a, & otherwise. \end{cases}$$
Thus, by a simple computation, we have
$$O(x,y) = \begin{cases} 1_L, & x = y = 1_L \\ 0_L, & x = 0_L \text{ or } y = 0_L \\ a \vee G(x,y), & otherwise. \end{cases}$$
It is easy to check that O satisfies all the conditions (O1)-(O4), and it is a quasi-overlap function on L.

Although the condition (3) in Theorem 4.1 is necessary and sufficient, it may be not handy in practical use. So we try to exchange this condition and provide the following corollary.

Corollary 4.3. *Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice, $S_1, S_2 : L^2 \to L$ be two t-conorms with S_1 is positive and $N : L \to L$ be a non-vanishing and non-filling negation. Then the mapping $O : L^2 \to L$ defined by*
$$O(x,y) = S_1(f(x,y), N(S_2(f(x,y), g(x,y)))),$$
where $f, g : L^2 \to L$ are commutative functions, is a quasi-overlap function if

(1) *$f(x,y) = 0_L$, if and only if $g(x,y) = 1_L$, if and only if $x = 0_L$ or $y = 0_L$;*

(2) *$f(x,y) = 1_L$ if and only if $x = y = 1_L$;*

(3) *f is increasing and $S_2(f(x,y),g(x,y))$ is decreasing on $L^2 \setminus \{1_L\}^2$.*

Proof. Notice that O satisfies conditions (O1)-(O3), which has been proven by Theorem 4.1. Thus we only need to prove that O is increasing. From condition (3), it is easy to obtain that $O(x,y)$ is increasing on $L^2 \setminus \{1_L\}^2$. In fact, $O(1_L,1_L) = S_1(f(1_L,1_L),N(S_2(f(1_L,1_L),g(1_L,1_L)))) = S_1(1_L,N(S_2(1_L,g(1_L,1_L)))) = S_1(1_L,0_L) = 1_L$. Then $1_L = O(1_L,1_L) \geq O(x,y)$ for all $x,y \in L$, thus O is increasing on the whole lattice L. To sum up, O is a quasi-overlap function on L. □

Example 4.4. *Under the conditions of Corollary 4.3, let $L = \{0_L, a, b, c, d, e, f, 1_L\}$ be a bounded lattice with the order given in Fig. 2, $S_1 = S_2 = \vee$, the increasing function f is defined as follows:*

$$f(x,y) = \begin{cases} 1_L, & x = y = 1_L \\ 0_L, & x = 0_L \text{ or } y = 0_L \\ f, & \text{otherwise}, \end{cases}$$

and g and N are given in Table 1 and 2, respectively.

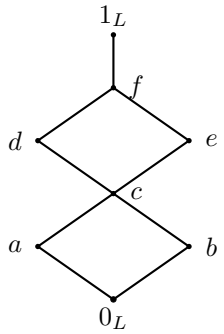


Figure 2: The lattice L in Example 4.4

Table 1: The function g in Example 4.4

$g(x,y)$	0_L	a	b	c	d	e	f	1_L
0_L	1_L	1_L	1_L	1_L	1_L	1_L	1_L	1_L
a	1_L	d	d	d	d	d	d	d
b	1_L	d	d	c	d	d	d	d
c	1_L	d	c	d	d	d	d	d
d	1_L	d	d	d	d	d	d	d
e	1_L	d	d	d	d	d	b	b
f	1_L	d	d	d	d	b	f	0_L
1_L	1_L	d	d	d	d	b	0_L	0_L

Table 2: The negation N in Example 4.4

x	0_L	a	b	c	d	e	f	1_L
$N(x)$	1_L	f	e	e	c	c	a	0_L

Then we have

$$S_2(f(x,y),g(x,y)) = f(x,y) \vee g(x,y) = \begin{cases} 1_L, & x = 0_L \text{ or } y = 0_L \text{ or } x = y = 1_L \\ f, & \text{otherwise}, \end{cases}$$

and it is decreasing on $L^2 \setminus \{1_L\}^2$. Thus, by computation, we can obtain that

$$O(x,y) = f(x,y) \vee N(f(x,y) \vee g(x,y)) = \begin{cases} 1_L, & x = y = 1_L \\ 0_L, & x = 0_L \text{ or } y = 0_L \\ f, & \text{otherwise}, \end{cases}$$

and O is a quasi-overlap function.

As for analytical properties for quasi-overlap functions constructed by Theorem 4.1 or Corollary 4.3, the following propositions are straightforward and we omit the proofs.

Proposition 4.5. *Under the conditions of Theorem 4.1 or Corollary 4.3, if f and g are (α, A)-migrative, then O is (α, A)-migrative.*

Proposition 4.6. *Under the conditions of Theorem 4.1 or Corollary 4.3, if x is the idempotent element of f, g and S_2, and $S_1(x, N(x)) = x$, then x is the idempotent element of O.*

4.2 Construction via quasi-overlap functions and implications

The previous subsection used t-conorms to construct new quasi-overlap function, however, the associativity of t-conorms is not essential. Therefore, it is possible to consider other operators that are not associative. In this subsection, we provide another construction method involving quasi-overlap functions and implications.

Theorem 4.7. *Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice, $O_1 : L^2 \to L$ be a quasi-overlap function and $I : L^2 \to L$ be an implication. Then the mapping $O : L^2 \to L$ defined by*

$$O(x, y) = O_1(f(x, y), I(g(x, y), f(x, y))),$$

where $f, g : L^2 \to L$ are commutative functions, is a quasi-overlap function if

(1) *f is increasing and g is decreasing;*

(2) *$f(x, y) = 0_L$, if and only if $g(x, y) = 1_L$, if and only if $x = 0_L$ or $y = 0_L$;*

(3) *$f(x, y) = 1_L$ if and only if $x = y = 1_L$;*

(4) *$I(x, y) = 0_L$ if and only if $x = 1_L$ and $y = 0_L$.*

Proof. O is obviously commutative. And we only need to verify the remaining conditions.

(O2): Let $x, y \in L$. If $O(x, y) = O_1(f(x, y), I(g(x, y), f(x, y))) = 0_L$, therefore, $f(x, y) = 0_L$ or $I(g(x, y), f(x, y)) = 0_L$. $I(g(x, y), f(x, y)) = 0_L$ is equivalent to $f(x, y) = 0_L$ since $f(x, y) = 0_L$ if and only if $g(x, y) = 1_L$ and $I(x, y) = 0_L$ if and only if $x = 1$ and $y = 0_L$. From condition (2), $f(x, y) = 0_L$ if and only if $x = 0_L$ or $y = 0_L$.

(O3): For any $x, y \in L$, $O(x, y) = 1_L$, i.e., $O_1(f(x, y), I(g(x, y), f(x, y))) = 1_L$, which coincides with $f(x, y) = I(g(x, y), f(x, y)) = 1_L$. Since $f(x, y) = 1_L$ if and only if $x = y = 1_L$ and $I(x, 1_L) = 1_L$, then we have $O(x, y) = 1_L$ if and only if $x = y = 1_L$.

(O4): Take $x, y, z \in L$ with $x \leq y$. Then $I(g(x, z), f(x, z)) \leq I(g(y, z), f(x, z)) \leq I(g(y, z), f(y, z))$ and thus $O(x, z) = O_1(f(x, z), I(g(x, z), f(x, z))) \leq O_1(f(y, z), I(g(y, z), f(y, z))) = O(y, z)$.

at the end, O satisfies all the four conditions and it is a quasi-overlap function on L. □

Example 4.8. 1. *In Theorem 4.7, let $L = [0, 1]$, $f(x, y) = xy$, $g(x, y) = 1 - xy$, $O_1(x, y) = min\{x, y\}$ and*

$$I(x, y) = I_{LK}(x, y) \begin{cases} 1, & x \leq y \\ 1 - x + y, & otherwise. \end{cases}$$

Then by computation, we have $O(x, y) = xy$, which is a quasi-overlap function.

2. *Under the conditions of Theorem 4.7, let $L = \{0_L, x_1, x_2, \ldots, x_8, 1_L\}$ be a bounded lattice with the order given in Fig. 3, $f(x, y) = O_1(x, y) = x \wedge y$ and g and I are given in Table 3 and 4, respectively. Then by computation, we have the function O in Table 5:*

From Tab 5, the function O satisfies all the four conditions and is a quasi-overlap functions on L.

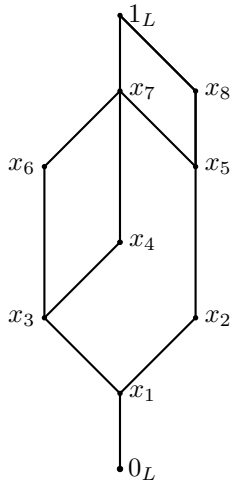


Figure 3: The lattice L in Example 4.8

Table 3: The function g in Example 4.8

$g(x,y)$	0_L	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	1_L
0_L	1_L	1_L	1_L	1_L	1_L	1_L	1_L	1_L	1_L	1_L
x_1	1_L	x_8	x_8	x_8	x_8	x_5	x_5	x_5	x_2	0_L
x_2	1_L	x_8	x_8	x_5	x_5	x_5	x_5	x_2	x_1	0_L
x_3	1_L	x_8	x_5	x_5	x_5	x_5	x_2	x_2	x_1	0_L
x_4	1_L	x_8	x_5	x_5	x_5	x_5	x_2	x_2	x_1	0_L
x_5	1_L	x_5	x_5	x_5	x_5	x_5	x_2	x_2	x_1	0_L
x_6	1_L	x_5	x_5	x_2	x_2	x_2	x_2	x_2	x_1	0_L
x_7	1_L	x_5	x_2	x_2	x_2	x_2	x_2	x_2	x_1	0_L
x_8	1_L	x_2	x_1	x_1	x_1	x_1	x_1	x_1	x_1	0_L
1_L	1_L	0_L	0_L	0_L	0_L	0_L	0_L	0_L	0_L	0_L

Table 4: The function I in Example 4.8

$I(x,y)$	0_L	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	1_L
0_L	1_L	1_L	1_L	1_L	1_L	1_L	1_L	1_L	1_L	1_L
x_1	x_7	x_7	x_7	x_7	x_7	1_L	1_L	1_L	1_L	1_L
x_2	x_7	x_7	x_7	x_7	x_7	x_7	x_7	1_L	1_L	1_L
x_3	x_4	x_7	x_7	x_7	x_7	x_7	x_7	1_L	1_L	1_L
x_4	x_4	x_4	x_4	x_7	x_7	x_7	x_7	1_L	1_L	1_L
x_5	x_4	x_4	x_4	x_7	x_7	x_7	x_7	1_L	1_L	1_L
x_6	x_3	x_3	x_3	x_7	x_7	x_7	x_7	1_L	1_L	1_L
x_7	x_1	x_1	x_1	x_2	x_2	x_5	x_5	x_8	1_L	1_L
x_8	x_1	x_1	x_1	x_2	x_2	x_5	x_5	x_8	x_8	1_L
1_L	0_L	x_1	x_1	x_2	x_2	x_5	x_5	x_8	x_8	1_L

Remark 4.9. *In fact, the values of $g(0_L,\cdot)$ in Tab. 3 do not affect the resulting function O. For example, if we take $g(0_L,0_L)=x_8$, $g(x_k,0_L)=x_8$, $k\in\{1,2,\ldots,6\}$, $g(x_7,0_L)=x_5$, $g(x_8,0_L)=x_2$ and $g(0_L,1_L)=0_L$ while other values still remain in Tab 3, then the function O defined by $O(x,y)=O_1(f(x,y),I(g(x,y),f(x,y)))$ still remain the structure given by Tab 5. In other words, the restriction of function g can be deleted, i.e., g can be an arbitrary decreasing commutative function.*

Besides, the restriction of implication I can be weakened as well. So we give the following corollary.

Corollary 4.10. *Let $(L,\leq,0_L,1_L)$ be a bounded lattice, $O_1:L^2\to L$ be a quasi-overlap function and $I:L^2\to L$ be an implication. Then the mapping $O:L^2\to L$ defined by*

$$O(x,y)=O_1(f(x,y),I(g(x,y),f(x,y))),$$

Table 5: The function O in Example 4.8

$O(x,y)$	0_L	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	1
0_L	0_L	0_L	0_L	0_L	0_L	0_L	0_L	0_L	0_L	0_L
x_1	0_L	x_1	x_1	x_1	x_1	x_1	x_1	x_1	x_1	x_1
x_2	0_L	x_1	x_1	x_1	x_1	x_1	x_1	x_2	x_2	x_2
x_3	0_L	x_1	x_1	x_3	x_3	x_1	x_3	x_3	x_1	x_3
x_4	0_L	x_1	x_1	x_3	x_4	x_1	x_3	x_4	x_1	x_4
x_5	0_L	x_1	x_1	x_1	x_1	x_5	x_1	x_5	x_5	x_5
x_6	0_L	x_1	x_1	x_3	x_3	x_1	x_6	x_6	x_1	x_6
x_7	0_L	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_5	x_7
x_8	0_L	x_1	x_2	x_1	x_1	x_5	x_1	x_5	x_8	x_8
1_L	0_L	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	1_L

where $f, g : L^2 \to L$ are commutative functions, is a quasi-overlap function if

(1) *f is increasing and g is decreasing;*

(2) *$f(x,y) = 0_L$ if and only if $x = 0_L$ or $y = 0_L$;*

(3) *$f(x,y) = 1_L$ if and only if $x = y = 1_L$;*

(4) *If $I(x,y) = 0_L$, then $y = 0_L$.*

Proof. O is obviously commutative. Further, in view of Theorem 4.7, mapping O satisfies conditions (O3) and (O4), hence we only need to verify the remaining condition (O2).

(O2): Let $x, y \in L$. If $O(x,y) = O_1(f(x,y), I(g(x,y), f(x,y))) = 0_L$, therefore, there are two possible cases.

Case 1. $f(x,y) = 0_L$. In this case, $O(x,y) = 0_L$ if and only if $x = 0_L$ or $y = 0_L$.

Case 2. $f(x,y) \neq 0_L$ and $I(g(x,y), f(x,y)) = 0_L$. From item (4), we have $f(x,y) = 0_L$ in this case, which is a contradiction.

Finally, O satisfies all of four conditions and hence it is a quasi-overlap function on L. □

As for analytical properties, the following proposition is straightforward and we omit the proof.

Proposition 4.11. *Under the conditions of Theorem 4.7 or Corollary 4.10, the following items hold:*

(*i*) *If f and g are (α, A)-migrative, then O is (α, A)-migrative.*

(*ii*) *If x is the idempotent element of f and g, I satisfies (IP) and 1 is the neutral element of O_1, then x is the idempotent element of O.*

5 Conclusion

We mainly devoted this paper to provide two new construction methods for quasi-overlap functions on bounded lattices. At first, we investigated some properties of lattice-valued quasi-overlap functions, which includes the relation between quasi-overlap functions and three kinds of aggregation functions and the Lipschitz property. Then we provided two new construction methods for lattice-valued quasi-overlap functions. One applies t-conorms and negations, the other involves quasi-overlap functions and implications. Examples were presented and some analytical properties were studied as well.

In the future, we aim to continue studying not only overlap functions but also other aggregation functions on lattices. Besides, on the basis of the valuation on lattices, we are able to study quasi-copula on lattices.

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