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Quintuple intuitionistic fuzzy implications reasoning algorithms and application

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Abstract

Atanassov's intuitionistic fuzzy sets have succussed in the application of decision making, data mining, artificial intelligence, image processing, and so on. In these applications, intuitionistic fuzzy reasoning plays a crucial role. To improve the quality of intuitionistic fuzzy reasoning, this paper presents a quintuple intuitionistic fuzzy implication principle (QIIP) to resolve intuitionistic fuzzy modus ponens (IFMP) and intuitionistic fuzzy modus tollens (IFMT) problems. The QIIP algorithms of IFMP and IFMT problems for intuitionistic R-implication, S-implication, and several fuzzy implications are represented. Moreover, we investigate the recovery property and continuity of QIIP algorithms for IFMP and IFMT. Finally, an application example for medical diagnosis is implemented to illustrate our proposed approaches.

Keywords: Intuitionistic fuzzy set, intuitionistic fuzzy implications, fuzzy reasoning, recovery property, continuity.

1 Introduction

During the last 38 years, medical diagnosis has been addressed with much attention by decision theory researchers. Many studies were investigated for medical diagnosis by utilizing different technologies and are widely used in the field of medical image processing, and segmentation techniques [27, 26]. Nonetheless, the fuzzy sets (FS) theory [25] presented by Zadeh in 1965 has a significant effect on decision theory. To handle ambiguous information more accurately, the FSs had been extended to intuitionistic fuzzy sets (IFSs) by Atanassov [3] in 1986. Because the IFS has a more powerful capability to express the imprecise membership grades, it has been widely studied and applied in the fields of decision-making, data mining, artificial intelligence, image processing, pattern recognition, and so on [17, 1, 11, 23, 28]. These applications conversely motivate people's interest in intuitionistic fuzzy reasoning.

As we all know that intuitionistic fuzzy reasoning depends completely on both intuitionistic fuzzy logic connectives and inference rules. Since they can be utilized to formalize "if...then" rule, intuitionistic fuzzy implications play an important role in intuitionistic fuzzy reasoning. And then, a lot of intuitionistic fuzzy implications have been investigated by many researchers [12, 10, 13, 5, 20, 22]. It is well known that intuitionistic fuzzy modus ponens (IFMP) and intuitionistic fuzzy modus tollens (IFMT) generalized from modus ponens (MP) and modus tollens (MT) in binary logic can be intuitively shown as follows:

Premise 1: IF x is $\tilde{A}$ THEN y is $\tilde{B}$	Premise 1: IF x is $\tilde{A}$ THEN y is $\tilde{B}$
Premise 2: x is $\tilde{A}'$	Premise 2: y is $\tilde{B}'$
Conclusion: y is $\tilde{B}'$,	Conclusion: x is $\tilde{A}'$.

Where $\tilde{A}$, $\tilde{A}'$ and $\tilde{B}$, $\tilde{B}'$ are intuitionistic fuzzy sets on U and V, respectively.

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To obtain an approximation $\tilde{B}'(\tilde{A}')$ of $\tilde{B}(\tilde{A})$, some extensive and intensive studies are done by many scholars. The compositional rule of inference (CRI) method in the setting of intuitionistic fuzzy sets is proposed by Cornelis et al.[12]. Zheng et al. present the intuitionistic fuzzy reasoning version of the Triple I method (TIP)[29, 30]. Jin et al. introduce the quintuple implication principle (QIP) based on interval-valued intuitionistic fuzzy sets (IVIFSs) [16]. However, either the CRI method or TIP method may make the solutions of IFMP and IFMT problems sometimes useless or misleading as shown in the following example.

Example 1.1. *Let the universes be* $U = \{x_1, x_2, x_3, x_4\}$ *and* $V = \{y_1, y_2, y_3\}$. *Suppose that* $\tilde{A} = \{\langle x_1, 1, 0\rangle, \langle x_2, 0.2, 0.7\rangle, \langle x_3, 0, 0.8\rangle, \langle x_4, 0, \rangle\}$, $\tilde{B} = \{\langle y_1, 0, 1\rangle, \langle y_2, 0.1, 0.9\rangle, \langle y_3, 0, 1\rangle\}$, $\tilde{A}' = \{\langle x_1, 0, 1\rangle, \langle x_2, 0.2, 0.7\rangle, \langle x_3, 1, 0\rangle, \langle x_4, 0.6, 0.3\rangle\}$. *Our task is to compute the output* $\tilde{B}'$.

We use the intuitionistic fuzzy implication generated by the representable Łukasiewicz intuitionistic fuzzy t-norm (that is, $\mathcal{T}(x, y) = ((x_1 + y_1 - 1) \vee 0, (x_2 + y_2) \wedge 1)$ [13]) to interpret "if ... then". By the CRI and TIP methods, we have $\tilde{B}' = \{\langle y_1, 1, 0\rangle, \langle y_2, 1, 0\rangle, \langle y_3, 1, 0\rangle\}$. This result is not inconsistent with human thinking.

Based on that discussed above, this paper mainly studies the quintuple intuitionistic fuzzy implication principle (QIIP) for IFMP and IFMT problems to overcome this shortcoming. The remainder of this paper is organized as follows. Section 2 reviews some basic definitions and properties related to this paper. In Section 3, we give the expression of QIIP for intuitionistic fuzzy R-implication, S-implication, and several intuitionistic fuzzy implications from Ref.[5]. Section 4 investigates the recovery and continuity of the QIIP algorithms of IFMP and IFMT problems. In Section 5, we utilize an application example for medical diagnosis to illustrate the practicability of our proposed approach. The last section concludes this paper.

2 Preliminaries

In this section, we briefly review some well-known concepts and properties employed in the rest of this paper.

Definition 2.1. [3] *An intuitionistic fuzzy set on the nonempty universe* U *is defined by* $\tilde{A} = \{\langle x, \tilde{A}_t(x), \tilde{A}_f(x)\rangle | x \in U\}$, *where* $\tilde{A}_t : U \to [0, 1], \tilde{A}_f : U \to [0, 1]$ *with the condition* $0 \leq \tilde{A}_t(x) + \tilde{A}_f(x) \leq 1$ *for all* $x \in U$. $\tilde{A}_t(x)$ *and* $\tilde{A}_f(x)$ *denote a membership function and a non-membership function of* x *to* $\tilde{A}$, *respectively.*

It is convenient and unambiguous to rewritten an intuitionistic fuzzy set $\tilde{A}$ on U as $\tilde{A}(x) = (x_1, x_2)$, where $(x_1, x_2) \in [0, 1]^2$, $x_1 + x_2 \leq 1, \forall x \in U$. Let $L^I = \{(x_1, x_2) \in [0, 1]^2 | x_1 + x_2 \leq 1\}$. It is very clear that the truth value of intuitionistic fuzzy set varies on the set L^I. The set of all intuitionistic fuzzy sets on U is denoted by $IFS(U)$ in this paper. For further usage, we also denote $x = (x_1, x_2) \in L^I$. Furthermore, the first and second projection mapping pr_1 and pr_2 on L^I are defined as $pr_1 x = x_1$ and $pr_2 x = x_2$ for all $x \in L^I$.

Definition 2.2. [4] *An order relation* $\preceq$ *on* L^I *is defined as follows:*

$$(x_1, x_2) \preceq (y_1, y_2) \Longleftrightarrow x_1 \leq y_1,\ x_2 \geq y_2.$$

It is easy to verify that the $(L^I, \preceq)$ is a bounded lattice which $(1, 0)$ and $(0, 1)$ are the largest and the smallest elements of L^I, denoted by $1_{L^I} = (1, 0)$ and $0_{L^I} = (0, 1)$, respectively. For any nonempty $A \subseteq L^I$, $\sup A = (\sup\{x_1 | (x_1, x_2) \in A\}, \inf\{x_2 | (x_1, x_2) \in A\})$ and $\inf A = (\inf\{x_1 | (x_1, x_2) \in A\}, \sup\{x_2 | (x_1, x_2) \in A\})$ hold. As well-known, the meet operator $\wedge$ and the join operator $\vee$ linked to the ordering $\preceq$ can be defined as follows:

$$(x_1, x_2) \wedge (y_1, y_2) = (x_1 \wedge y_1, x_2 \vee y_2).$$

$$(x_1, x_2) \vee (y_1, y_2) = (x_1 \vee y_1, x_2 \wedge y_2).$$

We can therefore verify that the algebraic structure $(L^I, \wedge, \vee, 0_{L^I}, 1_{L^I})$ is a complete, bounded and distributive lattice. Moreover, it is dense, that is, if $x \succ y$ ($x \succeq y$ and $x \neq y$), then there exists $z \in L^I$ such that $x \succ z \succ y$.

Definition 2.3. [8] *A function* $\mathcal{N} : L^I \to L^I$ *is called an intuitionistic fuzzy negation if*
(N1) $\mathcal{N}(0_{L^I}) = 1_{L^I},\ \mathcal{N}(1_{L^I}) = 0_{L^I}$,
(N2) $\mathcal{N}(x) \succeq \mathcal{N}(y)$ *if* $x \preceq y,\ \forall\ x, y \in L^I$.
Further, an intuitionistic fuzzy negation $\mathcal{N}$ *is strict if it satisfies the following properties:*
(N3) $\mathcal{N}$ *is continuous,*
(N4) $\mathcal{N}(x) \succ \mathcal{N}(y)$ *if* $x \prec y$.
An intuitionistic fuzzy negation is strong if it is involutive, i.e.,
(N5) $\mathcal{N}(\mathcal{N}(x)) = x,\ \forall\ x \in L^I$.

Example 2.4. [8] *The mapping $\mathcal{N}_s$ defined by $\mathcal{N}_s(x_1, x_2) = (x_2, x_1)$, for all $(x_1, x_2) \in L^I$, will be called the standard negation.*

Remark 2.5. *Notice that $(L^I, \vee, \wedge, \mathcal{N}, 0_{L^I}, 1_{L^I})$ is a soft algebra when $\mathcal{N}$ is involutive, that is, $(L^I, \vee, \wedge)$ is a bounded and distributive lattice and keeps De Morgan identities.*

Definition 2.6. [13] *An operation $\mathcal{T} : L^I \times L^I \to L^I$ is called an intuitionistic fuzzy triangular norm, or intuitionistic fuzzy t-norm for short, if the following conditions hold for all $x, y, z \in L^I$:*
(T1) $\mathcal{T}(x, y) = \mathcal{T}(y, x)$,
(T2) $\mathcal{T}(\mathcal{T}(x, y), z) = \mathcal{T}(x, \mathcal{T}(y, z))$,
(T3) $x \preceq y$ *implies* $\mathcal{T}(x, z) \preceq \mathcal{T}(y, z)$,
(T4) $\mathcal{T}(x, 1_{L^I}) = x$.

Example 2.7. [13] *The following is a continuous intuitionistic fuzzy t-norm $\mathcal{T}$*

$$\mathcal{T}(x, y) = ((x_1 + y_1 - x_2y_2 - 1) \vee 0, (x_2 + y_2) \wedge 1), \quad \forall x,\, y \in L^I.$$

Definition 2.8. [13] *An operation $\mathcal{S} : L^I \times L^I \to L^I$ is called an intuitionistic fuzzy triangular conorm, or intuitionistic fuzzy t-conorm for short, which satisfies: $(T1) - (T3)$ and $\mathcal{S}(x, 0_{L^I}) = x$ for any $x \in L^I$.*

Definition 2.9. [13] *An intuitionistic fuzzy t-norm $\mathcal{T}$ is called t-representable iff there exist a t-norm T and a t-conorm S such that, for all $x, y \in L^I$*

$$\mathcal{T}(x, y) = (T(x_1, y_1), S(x_2, y_2)).$$

Definition 2.10. [13] *An intuitionistic fuzzy t-conorm $\mathcal{S}$ is called t-representable iff there exist a t-norm T and a t-conorm S such that, for all $x, y \in L^I$*

$$\mathcal{S}(x, y) = (S(x_1, y_1), T(x_2, y_2)).$$

Remark 2.11. *Notice that not all intuitionistic t-norms (t-conorms) are t-representable [13].*

Definition 2.12. [12] *An intuitionistic fuzzy implication $\mathcal{I}$ is a function on L^I which satisfies:*
(A1) *Non-increasing in the first variable, i.e., $\mathcal{I}(x, z) \preceq \mathcal{I}(y, z)$ if $x \succeq y$,*
(A2) *Non-decreasing in the second variable, i.e., $\mathcal{I}(x, y) \preceq \mathcal{I}(x, z)$ if $y \preceq z$, $\forall\ x, y, z \in L^I$,*
(A3) $\mathcal{I}(0_{L^I}, 0_{L^I}) = 1_{L^I}$,
(A4) $\mathcal{I}(1_{L^I}, 1_{L^I}) = 1_{L^I}$,
(A5) $\mathcal{I}(1_{L^I}, 0_{L^I}) = 0_{L^I}$.

By Definition 2.12, we directly obtain the fact that an intuitionistic fuzzy implication satisfies the following properties:
(LB) *Left boundary condition,* $\mathcal{I}(0_{L^I}, y) = 1_{L^I},\ \forall y \in L^I$,
(RB) *Right boundary condition,* $\mathcal{I}(x, 1_{L^I}) = 1_{L^I},\ \forall x \in L^I$.

Definition 2.13. [12] *Let $\mathcal{I}$ be an intuitionistic fuzzy implication function. Then $\mathcal{I}$ can additionally satisfy the following properties:*
(NP) *Left neutrality property, if* $\mathcal{I}(1_{L^I}, y) = y,\ \forall\ y \in L^I$,
(NP) *Identity principle, if* $\mathcal{I}(x, x) = 1_{L^I},\ \forall\ x \in L^I$,
(EP) *Exchange principle, if* $\mathcal{I}(x, \mathcal{I}(y, z)) = \mathcal{I}(y, \mathcal{I}(x, z)),\ \forall\ x, y, z \in L^I$,
(CP(N)) *Law of contraposition with a fuzzy negation if* $\mathcal{I}(x, y) = \mathcal{I}(\mathcal{N}(y), \mathcal{N}(x)),\ \forall\ x, y \in L^I$,
(OP) *Ordering property, if* $\mathcal{I}(x, y) = 1_{L^I} \Longleftrightarrow x \preceq y,\ \forall\ x, y \in L^I$.

Example 2.14. [5] *The following are three intuitionistic fuzzy implications given by, respectively:*

$$\mathcal{I}_1(x,\, y) = ((x_1 \wedge y_1) \vee x_2, x_1 \wedge y_2),$$
$$\mathcal{I}_2(x,\, y) = (x_1y_1 + x_2, x_1y_2),$$
$$\mathcal{I}_3(x,\, y) = (x_2 + y_1 - x_2y_1, x_1y_2).$$

Definition 2.15. [12] *Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm on L^I. The intuitionistic fuzzy R-implication generated by $\mathcal{T}$ is the mapping $\mathcal{I}_\mathcal{T}$ defined as, for all $x,\, y \in L^I$:*

$$\mathcal{I}_\mathcal{T}(x, y) = \sup\{z \in L^I | \mathcal{T}(x,\, z) \preceq y\}.$$

If $\mathcal{T}$ is t-representable, $\mathcal{I}_\mathcal{T}$ is called a representable intuitionistic fuzzy R-implication on L^I.

Example 2.16. [13] *The following is intuitionistic fuzzy R-implication generated by the intuitionistic fuzzy t-norm $\mathcal{T}$ in Example 2.7:*

$$\mathcal{I}_{\mathcal{T}}(x,y) = ((1+x_2-y_2) \wedge ((1+y_1-x_1+x_2((y_2-x_2)\vee 0)) \vee (1+\frac{y_1-x_1}{1+x_2})) \wedge 1, (y_2-x_2)\vee 0).$$

Definition 2.17. [12] *Let $\mathcal{S}$ be an intuitionistic fuzzy t-conorm and $\mathcal{N}$ an involutive intuitionistic fuzzy negation on L^I. The intuitionistic fuzzy S-implication generated by $\mathcal{S}$ and $\mathcal{N}$ is the mapping $\mathcal{I}_{\mathcal{S}}$ defined as, for $x, y \in L^I$:*

$$\mathcal{I}_{\mathcal{S}}(x,y) = \mathcal{S}(\mathcal{N}(x), y).$$

If $\mathcal{S}$ is t-representable, $\mathcal{I}_{\mathcal{S}}$ is called a representable intuitionistic fuzzy S-implication on L^I.

Definition 2.18. [6] *Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm and $\mathcal{I}$ an intuitionistic fuzzy implication. $(\mathcal{T},\mathcal{I})$ is called an intuitionistic fuzzy adjoint pair if $(\mathcal{T},\mathcal{I})$ satisfies the residual principle:*

$$\mathcal{T}(x,y) \preceq z \Longleftrightarrow \mathcal{I}(x,z) \succeq y \quad \forall x, y \in L^I.$$

Remark 2.19. *Notice that $\mathcal{T}$ is an intuitionistic fuzzy left continuous t-norm if $(\mathcal{T},\mathcal{I})$ satisfies the residual principle. However, from intuitionistic fuzzy left-continuity, it cannot be deduced that an intuitionistic fuzzy t-norm satisfies the residuation principle (See Theorem 7.7 and Counterexample 7.1 in [13]).*

3 The quintuple intuitionistic fuzzy implication principle of intuitionistic fuzzy reasoning

In this section, let us first extend the quintuple implication principle [31, 18] in the setting of intuitionistic fuzzy sets.

Quintuple intuitionistic implication principle for IFMP $\tilde{B}'(y)$ is the smallest intuitionistic fuzzy subset on V such that the following condition holds for every $x \in U$ and $y \in V$:

$$\mathcal{I}(\mathcal{I}(\tilde{A}(x),\, \tilde{B}(y)), \mathcal{I}(\mathcal{I}(\tilde{A}'(x), \tilde{A}(x)), \mathcal{I}(\tilde{A}'(x), \tilde{B}'(y)))) = 1_{L^I}, \tag{1}$$

Quintuple Implication Principle for IFMT $\tilde{A}'(x)$ is the smallest intuitionistic fuzzy subset on U such that the following condition holds for every $x \in U$ and $y \in V$:

$$\mathcal{I}(\mathcal{I}(\tilde{A}(x),\, \tilde{B}(y)), \mathcal{I}(\mathcal{I}(\tilde{B}(y), \tilde{B}'(y)), \mathcal{I}(\tilde{A}(x), \tilde{A}'(x)))) = 1_{L^I}. \tag{2}$$

Theorem 3.1. *Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm and $\mathcal{I}$ an intuitionistic fuzzy implication. If $(\mathcal{T},\mathcal{I})$ is an intuitionistic fuzzy adjoint pair on L^I, then the QIIP algorithms of IFMP and IFMT problems are given as follows:*

$$\tilde{B}'(y) = \bigvee_{x\in U} \mathcal{T}(\tilde{A}'(x), \mathcal{T}(\mathcal{I}(\tilde{A}'(x), \tilde{A}(x)), \mathcal{I}(\tilde{A}(x), \tilde{B}(y)))), \tag{3}$$

$$\tilde{A}'(x) = \bigvee_{y\in V} \mathcal{T}(\tilde{A}(x), \mathcal{T}(\mathcal{I}(\tilde{A}(x), \tilde{B}(y)), \mathcal{I}(\tilde{B}(y), \tilde{B}'(y)))). \tag{4}$$

Proof. We only consider the QIIP algorithm of IFMP problem. The QIIP algorithm of IFMT problem can be similarly proved. Let $\tilde{C}(y)$ be an intuitionistic fuzzy set on V. Since $(\mathcal{T},\mathcal{I})$ is an intuitionistic fuzzy adjoint pair, $(\mathcal{T},\mathcal{I})$ satisfies the residuation principle. For all $x \in U, y \in V$, we have:

$$\begin{aligned}
&\mathcal{I}(\mathcal{I}(\tilde{A}(x), \tilde{B}(y)), \mathcal{I}(\mathcal{I}(\tilde{A}'(x), \tilde{A}(x)), \mathcal{I}(\tilde{A}'(x), \tilde{C}(y)))) = 1_{L^I}.\\
\Leftrightarrow &\mathcal{T}(\mathcal{I}(\tilde{A}(x), \tilde{B}(y)),\, \mathcal{I}(\tilde{A}'(x), \tilde{A}(x))) \preceq \mathcal{I}(\tilde{A}'(x), \tilde{C}(y)).\\
\Leftrightarrow &\mathcal{T}(\tilde{A}'(x),\, \mathcal{T}(\mathcal{I}(\tilde{A}(x), \tilde{B}(y)),\, \mathcal{I}(\tilde{A}'(x), \tilde{A}(x)))) \preceq \tilde{C}(y).
\end{aligned}$$

Since $\tilde{B}'(y)$ is the smallest intuitionistic fuzzy subset on V defined as in Eq.(3) and $\tilde{B}'(y) \preceq \tilde{C}(y)$. $\tilde{B}'(y)$ takes its maximum, we then have:

$$\tilde{B}'(y) = \min\{\tilde{C}(y) \in V | (\mathcal{I}(\tilde{A}(x), \tilde{B}(y)), \mathcal{I}(\mathcal{I}(\tilde{A}'(x), \tilde{A}(x)), \mathcal{I}(\tilde{A}'(x), \tilde{C}(y)))) = 1_{L^I}\}.$$

For any $x \in U$ and $y \in V$. This implies that $\tilde{B}'(y) = \bigvee_{x\in U} \mathcal{T}(\tilde{A}'(x), \mathcal{T}(\mathcal{I}(\tilde{A}'(x), \tilde{A}(x)), \mathcal{I}(\tilde{A}(x), \tilde{B}(y))))$. □

Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm satisfied by the condition $\sup\limits_{y\in A}\mathcal{T}(x,y)=\mathcal{T}(x,\sup\limits_{y\in A}y)$, where A is any nonempty subset of L^I. For the intuitionistic fuzzy R-implication generated by such an intuitionistic fuzzy t-norm $\mathcal{T}$, we have the following result.

Corollary 3.2. *Let $\mathcal{I}_{\mathcal{T}}$ be an intuitionistic fuzzy R-implication generated by an intuitionistic fuzzy t-norm satisfied the condition $\sup\limits_{y\in A}\mathcal{T}(x,y)=\mathcal{T}(x,\sup\limits_{y\in A}y)$. The QIIP algorithms of IFMP and IFMT problems are given by Eqs.(3) and (4).*

Proof. According to Theorem 7.7 in [13], $(\mathcal{T},\mathcal{I}_{\mathcal{T}})$ becomes an intuitionistic fuzzy adjoint pair in this case. This means the QIIP algorithms of IFMP and IFMT problems are given by Eqs.(3) and (4). □

As that mentioned above, the left-continuity cannot ensure that an intuitionistic fuzzy t-norm satisfies the residuation principle. This means that it is difficult to obtain the expression of QIIP algorithms of IFMP and IFMT problems for common intuitionistic fuzzy R-implications. However, we have the following result for the intuitionistic fuzzy R-implication in Example 2.16.

Theorem 3.3. *Let $\mathcal{I}_{\mathcal{T}}$ be an intuitionistic fuzzy R-implication generated by the intuitionistic fuzzy t-norm $\mathcal{T}$ in Example 2.7 which does not satisfy the residuation principle. Then the QIIP algorithm of IFMP problem is given as follows:*

$$\tilde{B}'(y)=\bigvee_{x\in U}\mathcal{T}(\tilde{A}'(x),\mathcal{T}(\mathcal{I}_{\mathcal{T}}(\tilde{A}'(x),\tilde{A}(x)),\mathcal{I}_{\mathcal{T}}(\tilde{A}(x),\tilde{B}(y)))). \tag{5}$$

Proof. Suppose that $\tilde{A}$, $\tilde{B}$, $\tilde{A}'$ have been fixed. Let $\mathcal{I}_{\mathcal{T}}(\tilde{A},\tilde{B})=(\mathcal{I}_{\mathcal{T}t}(x,\,y),\mathcal{I}_{\mathcal{T}f}(x,\,y))$, it is easy to obtain that:

$$\tilde{B}'_f\preceq(\tilde{A}_f+(\mathcal{I}_{\mathcal{T}f}(\tilde{A},\tilde{B})+\mathcal{I}_{\mathcal{T}f}(\tilde{A}',\tilde{A}))\wedge 1_{L^I})\wedge 1_{L^I}.$$

Next, we consider the following four cases:

(i) $\mathcal{I}_{\mathcal{T}f}(\tilde{A}',\tilde{B}')-\mathcal{I}_{\mathcal{T}f}(\tilde{A}',\tilde{A})\preceq 0_{L^I}$ and $\mathcal{I}_{\mathcal{T}t}(\tilde{A}',\tilde{B}')-\mathcal{I}_{\mathcal{T}t}(\tilde{A}',\tilde{A})\succeq 0_{L^I}$.
In this case, we can obtain $\mathcal{I}_{\mathcal{T}t}(\tilde{A},\tilde{B})\preceq 1_{L^I}$. It is obvious that this does not contact with $\tilde{B}'$.

(ii) $\mathcal{I}_{\mathcal{T}f}(\tilde{A}',\tilde{B}')-\mathcal{I}_{\mathcal{T}f}(\tilde{A}',\tilde{A})\preceq 0_{L^I}$ and $\mathcal{I}_{\mathcal{T}t}(\tilde{A}',\tilde{B}')-\mathcal{I}_{\mathcal{T}t}(\tilde{A}',\tilde{A})\preceq 0_{L^I}$.

This case implies $\mathcal{I}_{\mathcal{T}t}(\tilde{A},\tilde{B})\preceq 1_{L^I}+\frac{\mathcal{I}_{\mathcal{T}t}(\tilde{A}',\tilde{B}')-\mathcal{I}_{\mathcal{T}t}(\tilde{A}',\tilde{A})}{\mathcal{I}_f(\tilde{A}',\tilde{A})+1}$.

(iii) $\mathcal{I}_{\mathcal{T}f}(\tilde{A}',\tilde{B}')-\mathcal{I}_{\mathcal{T}f}(\tilde{A}',\tilde{A})\succeq 0_{L^I}$ and $\frac{\mathcal{I}_{\mathcal{T}t}(\tilde{A}',\tilde{B}')-\mathcal{I}_{\mathcal{T}t}(\tilde{A}',\tilde{A})}{\mathcal{I}_{\mathcal{T}f}(\tilde{A}',\tilde{A})+1_{L^I}}\succeq\mathcal{I}_{\mathcal{T}f}(\tilde{A}',\tilde{A})-\mathcal{I}_{\mathcal{T}f}(\tilde{A}',\tilde{B}')$.

In this case, we have: $\mathcal{I}_{\mathcal{T}t}(\tilde{A},\tilde{B})\preceq(1_{L^I}+\mathcal{I}_{\mathcal{T}f}(\tilde{A}',\tilde{A})-\mathcal{I}_{\mathcal{T}f}(\tilde{A}',\tilde{B}'))$.

(iv) $\mathcal{I}_{\mathcal{T}f}(\tilde{A}',\tilde{B}')-\mathcal{I}_{\mathcal{T}f}(\tilde{A}',\tilde{A})\succeq 0_{L^I}$ and $\frac{\mathcal{I}_{\mathcal{T}t}(\tilde{A}',\tilde{B}')-\mathcal{I}_{\mathcal{T}t}(\tilde{A}',\tilde{A})}{\mathcal{I}_{\mathcal{T}f}(\tilde{A}',\tilde{A})+1_{L^I}}\preceq\mathcal{I}_{\mathcal{T}f}(\tilde{A}',\tilde{A})-\mathcal{I}_{\mathcal{T}f}(\tilde{A}',\tilde{B}')$.

This case means $\mathcal{I}_{\mathcal{T}t}(\tilde{A},\tilde{B})\preceq 1_{L^I}+\frac{\mathcal{I}_{\mathcal{T}t}(\tilde{A}',\tilde{B}')-\mathcal{I}_{\mathcal{T}t}(\tilde{A}',\tilde{A})}{\mathcal{I}_f(\tilde{A}',\tilde{A})+1}$.

Combining the above four cases, we then obtain:

$$\mathcal{I}_{\mathcal{T}t}(\tilde{A}',\tilde{B}')\succeq\mathcal{I}_{\mathcal{T}t}(\tilde{A},\tilde{B})-\mathcal{I}_{\mathcal{T}f}(\tilde{A}',\tilde{A})\cdot(1_{L^I}+\mathcal{I}_{\mathcal{T}t}(\tilde{A},\tilde{B}))-1_{L^I}+\mathcal{I}_{\mathcal{T}t}(\tilde{A}',\tilde{A}).$$

For $\mathcal{I}_{\mathcal{T}t}(\tilde{A}',\tilde{B}')$, we can similarly consider the following four cases:

$$\text{cases}=\begin{cases}\mathcal{I}_{\mathcal{T}t}(\tilde{A}',\tilde{B}')=1_{L^I},\\ \tilde{B}'_f-\tilde{A}'_f\succeq 0_{L^I}\quad\text{and}\quad\frac{\tilde{B}'_t-\tilde{A}'_t}{1+\tilde{A}'_f}\succeq\tilde{A}'_f-\tilde{B}'_f,\\ \tilde{B}'_f-\tilde{A}'_f\preceq 0_{L^I}\quad\text{and}\quad\tilde{B}'_t-\tilde{A}'_t\preceq 0_{L^I},\\ \tilde{B}'_f-\tilde{A}'_f\succeq 0_{L^I}\quad\text{and}\quad\frac{\tilde{B}'_t-\tilde{A}'_t}{1+\tilde{A}'_f}\preceq\tilde{A}'_f-\tilde{B}'_f.\end{cases}$$

For all x, $y\in L^I$, we obtain that

$$\tilde{B}'_t\succeq(\tilde{A}'_t+(\mathcal{I}_{\mathcal{T}t}(\tilde{A},\tilde{B})+\mathcal{I}_{\mathcal{T}t}(\tilde{A}',\tilde{A})-\mathcal{I}_{\mathcal{T}f}(\tilde{A}',\tilde{A})\cdot\mathcal{I}_{\mathcal{T}f}(\tilde{A},\tilde{B})-1_{L^I})\vee 0_{L^I}-\tilde{A}'_f\cdot((\mathcal{I}_{\mathcal{T}f}(\tilde{A}',\tilde{A})+\mathcal{I}_{\mathcal{T}f}(\tilde{A},\tilde{B}))\wedge 0_{L^I})-1_{L^I})\vee 0_{L^I}.$$

From Eq.(1) we further have:

$$\tilde{B}'(y) \succeq \mathcal{T}(\tilde{A}'(x),\, \mathcal{T}(\mathcal{I}_{\mathcal{T}}(\tilde{A}'(x), \tilde{A}(x)), \mathcal{I}_{\mathcal{T}}(\tilde{A}(x), \tilde{B}(y)))).$$

Therefore, $\tilde{B}'$ is the smallest intuitionistic fuzzy subset on V such that Eq.(1) holds. □

Nest, we investigate the QIIP algorithms of IFMP and IFMT problems for intuitionistic fuzzy S-implication.

Lemma 3.4. *Let $\mathcal{I}_{\mathcal{S}}$ be an intuitionistic fuzzy S-implication generated by a positive intuitionistic fuzzy t-conorm. Then $\mathcal{I}_{\mathcal{S}}(x, y) = 1_{L^I}$ if $x = 0_{L^I}$ or $y = 1_{L^I}$.*

Proof. Since $\mathcal{I}_{\mathcal{S}}$ is an intuitionistic fuzzy S-implication induced by a positive t-conorm, we have $\mathcal{I}_{\mathcal{S}}(x, y) = \mathcal{S}(\mathcal{N}(x),\, y)$. Which implies:

$$\mathcal{I}_{\mathcal{S}}(x, 1_{L^I}) = \mathcal{S}(\mathcal{N}(x),\, 1_{L^I}) = 1_{L^I},$$
$$\mathcal{I}_{\mathcal{S}}(0_{L^I}, y) = \mathcal{S}(\mathcal{N}(0_{L^I}),\, y) = 1_{L^I}.$$

Therefore, we have $\mathcal{I}_{\mathcal{S}}(x, y) = 1_{L^I}$ if $x = 0_{L^I}$ or $y = 1_{L^I}$. □

Lemma 3.5. *Let $\mathcal{I}_{\mathcal{S}}$ be an intuitionistic fuzzy S-implication generated by a positive intuitionistic fuzzy t-conorm. Then the QIIP algorithm of IFMP problem is given as follows:*

$$\tilde{B}'(y) = \begin{cases} 1_{L^I}, & \tilde{A}', \tilde{A}, \tilde{B} \neq 0_{L^I}, \\ 0_{L^I}, & \textit{otherwise.} \end{cases}$$

Based on the argument above, we will assume that the intuitionistic fuzzy S-implication $\mathcal{I}_{\mathcal{S}}$ generated by a non-positive intuitionistic fuzzy t-conorm in this paper.

Lemma 3.6. *Let $\mathcal{S}$ be an intuitionistic fuzzy t-conorm. The mapping $\mathcal{N}_{\mathcal{S}} : L^I \to L^I$ defined as*

$$\mathcal{N}_{\mathcal{S}}(x) = \inf\{y \in L^I | \mathcal{S}(x, y) = 1_{L^I}\} \textit{ for } x \in L^I.$$

is an intuitionistic fuzzy negation.

Proof. This can be directly verified according to Definition 2.3. □

Theorem 3.7. *Let $\mathcal{I}_{\mathcal{S}}$ be the intuitionistic fuzzy S-implication induced by an inf-morphism intuitionistic fuzzy t-conorm $\mathcal{S}$. Then the QIIP algorithms of IFMP and IFMT problems for the intuitionistic fuzzy S-implication $\mathcal{I}_{\mathcal{S}}$ are given by the following:*

$$\tilde{B}'(y) = \bigvee_{x \in U} \mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(\tilde{A}'(x)), \mathcal{S}(\mathcal{S}(\mathcal{T}(\tilde{A}'(x), \mathcal{N}_{\mathcal{S}}(\tilde{A}(x))), \mathcal{T}(\tilde{A}(x), \mathcal{N}_{\mathcal{S}}(\tilde{B}(y)))))), \tag{6}$$

$$\tilde{A}'(x) = \bigvee_{y \in V} \mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(\tilde{A}(x)), \mathcal{S}(\mathcal{S}(\mathcal{T}(\tilde{A}(x), \mathcal{N}_{\mathcal{S}}(\tilde{B}(y))), \mathcal{T}(\tilde{B}(y), \mathcal{N}_{\mathcal{S}}(\tilde{B}'(y)))))). \tag{7}$$

Proof. We only consider the case of IFMP. The case of IFMT can be similarly proved. Let $\tilde{B}'$ be defined as Eq.(6). It can easily be verified that

$$\mathcal{I}_{\mathcal{S}}(\mathcal{I}_{\mathcal{S}}(\tilde{A}(x), \tilde{B}(y)), \mathcal{I}_{\mathcal{S}}(\mathcal{I}_{\mathcal{S}}(\tilde{A}'(x), \tilde{A}(x)), \mathcal{I}_{\mathcal{S}}(\tilde{A}'(x), \tilde{B}'(y)))) = 1_{L^I}.$$

holds for all $x \in U$ and $y \in V$.

Suppose that $\tilde{C}(y)$ is an arbitrary intuitionistic fuzzy subset on V such that:

$$\mathcal{I}_{\mathcal{S}}(\mathcal{I}_{\mathcal{S}}(\tilde{A}(x), \tilde{B}(y)), \mathcal{I}_{\mathcal{S}}(\mathcal{I}_{\mathcal{S}}(\tilde{A}'(x), \tilde{A}(x)), \mathcal{I}_{\mathcal{S}}(\tilde{A}'(x), \tilde{C}(y)))) = 1_{L^I}.$$

We can obtain:

$$\begin{aligned} \tilde{C}(y) \succeq & \mathcal{S}(\tilde{A}'(x), \mathcal{S}(\mathcal{I}_{\mathcal{S}}(\tilde{A}(x), \tilde{B}(y)), \mathcal{I}_{\mathcal{S}}(\tilde{A}'(x), \tilde{A}(x)))) \\ \succeq & \mathcal{S}(\tilde{A}'(x), \mathcal{S}(\mathcal{S}(\mathcal{N}_{\mathcal{S}}(\tilde{A}(x)), \tilde{B}(y)), \mathcal{S}(\mathcal{N}_{\mathcal{S}}(\tilde{A}'(x)), \tilde{A}(x)))) \\ \succeq & \mathcal{S}(\tilde{A}'(x), \mathcal{S}(\mathcal{T}(\tilde{A}(x), \mathcal{N}_{\mathcal{S}}\tilde{B}(y)), \mathcal{T}(\tilde{A}'(x), \mathcal{N}_{\mathcal{S}}(\tilde{A}(x))))) \\ \succeq & \mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(\tilde{A}'(x)), \mathcal{S}(\mathcal{S}(\mathcal{T}(\tilde{A}'(x), \mathcal{N}_{\mathcal{S}}(\tilde{A}(x))), \mathcal{T}(\tilde{A}(x), \mathcal{N}_{\mathcal{S}}(\tilde{B}(y)))))). \end{aligned}$$

Since $\tilde{B}'(y)$ is the smallest intuitionistic fuzzy subset on V, we have $\tilde{B}'(y) \preceq \tilde{C}(y)$. □

Further, $\mathcal{N}_{\mathcal{S}} = \mathcal{N}$ holds if $\mathcal{I}_{\mathcal{S}}$ satisfies (OP). For an involutive intuitionistic fuzzy negation $\mathcal{N}$, let $\mathcal{T}$ be the the $\mathcal{N}$-dual of $\mathcal{S}$. We have the following result.

Corollary 3.8. *Let $\mathcal{I}_{\mathcal{S}}$ be the intuitionistic fuzzy S-implication induced by an inf-morphism intuitionistic fuzzy t-conorm $\mathcal{S}$ and involutive intuitionistic fuzzy negation $\mathcal{N}$. If $\mathcal{I}_{\mathcal{S}}$ satisfies (OP), then the QIIP algorithms of IFMP and IFMT problems for the intuitionistic fuzzy S-implication $\mathcal{I}_{\mathcal{S}}$ are given as follows:*

$$\tilde{B}'(y) = \bigvee_{x\in U} \mathcal{T}(\tilde{A}'(x), \mathcal{T}(\mathcal{S}(\mathcal{N}_{\mathcal{S}}(\tilde{A}'(x)), \tilde{A}(x)), \mathcal{S}(\mathcal{N}_{\mathcal{S}}(\tilde{A}(x)), \tilde{B}(y)))), \tag{8}$$

$$\tilde{A}'(x) = \bigvee_{y\in V} \mathcal{T}(\tilde{A}(x), \mathcal{T}(\mathcal{S}(\mathcal{N}_{\mathcal{S}}(\tilde{A}(x)), \tilde{B}(y)), \mathcal{S}(\mathcal{N}_{\mathcal{S}}(\tilde{B}(y)), \tilde{B}'(y)))). \tag{9}$$

Theorem 3.9. *For intuitionistic fuzzy implications $\mathcal{I}_1, \mathcal{I}_2$ and $\mathcal{I}_3$, the QIIP algorithm of IFMP problem $\tilde{B}'_i(y)(i = 1, 2, 3)$ can be respectively expressed as follows:*

$$\tilde{B}'_{1t}(y) = \bigvee_{x\in U} (\tilde{A}'_{1t}(x) \vee \tilde{A}'_{1f}(x)) \wedge (\mathcal{I}_{1t}(\tilde{A}'(x), \tilde{A}(x)) \vee \mathcal{I}_{1f}(\tilde{A}'(x), \tilde{A}(x))) \wedge \mathcal{I}_{1t}(\tilde{A}(x), \tilde{B}(y)), \tag{10}$$

$$\tilde{B}'_{1f}(y) = \bigwedge_{x\in U} \tilde{A}'_t(x) \wedge \mathcal{I}_{1t}(\tilde{A}'(x), \tilde{A}(x)) \wedge \mathcal{I}_{1f}(\tilde{A}(x), \tilde{B}(y)). \tag{11}$$

$$\tilde{B}'_{2t}(y) = \bigvee_{x\in U} \frac{\mathcal{I}_{2t}(\tilde{A}(x), \tilde{B}(y)) - \mathcal{I}_{2f}(\tilde{A}'(x), \tilde{A}(x)) - \tilde{A}'_{2f}(x) \cdot \mathcal{I}_{2t}(\tilde{A}'(x), \tilde{A}(x))}{\tilde{A}'_{2t}(x) \cdot \mathcal{I}_{2t}(\tilde{A}'(x), \tilde{A}(x))}, \tag{12}$$

$$\tilde{B}'_{2f}(y) = \bigwedge_{x\in U} \frac{\mathcal{I}_{2f}(\tilde{A}(x), \tilde{B}(y))}{\tilde{A}'_{2t}(x) \cdot \mathcal{I}_{2t}(\tilde{A}'(x), \tilde{A}(x))}. \tag{13}$$

$$\tilde{B}'_{3t}(y) = \bigvee_{x\in U} \frac{\mathcal{I}_{3t}(\tilde{A}(x), \tilde{B}(y)) + \mathcal{I}_{3f}(\tilde{A}'(x), \tilde{A}(x)) - \tilde{A}'_{3f}(x)}{\mathcal{I}_{3f}(\tilde{A}'(x), \tilde{A}(x))) \cdot \tilde{A}'_{3f}(x)}, \tag{14}$$

$$\tilde{B}'_{3f}(y) = \bigwedge_{x\in U} \frac{\mathcal{I}_{3f}(\tilde{A}(x), \tilde{B}(y))}{\tilde{A}'_{3t}(x) \cdot \mathcal{I}_{3t}(\tilde{A}'(x), \tilde{A}(x))}. \tag{15}$$

Proof. Suppose that $\tilde{A}(x)$, $\tilde{B}(y)$, $\tilde{A}'(x)$ have been fixed, and $\mathcal{I}_1(x, y)$ satisfies (OP), we can obtain

$$\mathcal{I}_1(\tilde{A}(x), \tilde{B}(y)) \preceq ((\mathcal{I}_{1t}(\tilde{A}'(x), \tilde{A}(x)) \wedge \mathcal{I}_{1t}(\tilde{A}'(x), \tilde{B}'(y))) \vee \mathcal{I}_{1f}(\tilde{A}'(x), \tilde{A}(x)), \mathcal{I}_{1t}(\tilde{A}'(x), \tilde{A}(x)) \wedge \mathcal{I}_{1f}(\tilde{A}'(x), \tilde{B}'(y))).$$

According to the Definition 2.2, we can obtain the following formula:

$$\mathcal{I}_{1t}(\tilde{A}(x), \tilde{B}(y)) = \mathcal{I}_{1t}(\tilde{A}'(x), \tilde{A}(x)) \wedge (\tilde{A}'_t(x) \wedge \tilde{B}'_t(y)) \vee \tilde{A}'_f(x) \vee \mathcal{I}_{1f}(\tilde{A}'(x), \tilde{A}(x)).$$

This implies that

$$\tilde{B}'_{1t}(y) \succeq (\tilde{A}'_t(x) \vee \tilde{A}'_f(x)) \wedge (\mathcal{I}_{1t}(\tilde{A}'(x), \tilde{A}(x)) \vee \mathcal{I}_{1f}(\tilde{A}'(x), \tilde{A}(x))) \wedge \mathcal{I}_{1t}(\tilde{A}(x), \tilde{B}(y)).$$

On the other hand

$$\mathcal{I}_{1f}(\tilde{A}(x), \tilde{B}(y)) \preceq \tilde{A}'_t(x) \wedge \mathcal{I}_{1t}(\tilde{A}'(x), \tilde{A}(x)) \wedge \mathcal{I}_{1f}(\tilde{A}(x), \tilde{B}(y)).$$

It is obvious that $\tilde{B}'(y)$ can be defined. The proof of $\mathcal{I}_2$ and $\mathcal{I}_3$ are similar to that of $\mathcal{I}_1$. □

4 The recovery and continuity of the quintuple intuitionistic fuzzy implications algorithms

In this section, we will investigate the recovery properties and continuity of the QIIP algorithms for IFMP and IFMT problems obtained above section. Here, we only discuss the recovery and continuity of the QIIP algorithm for IFMP problem. The recovery and continuity of the QIIP algorithm for IFMT problem can be similarly considered.

4.1 The recovery property of quintuple intuitionistic fuzzy implication algorithm for IFMP

Definition 4.1. [24] *A method of FMP probelm is said recoverable if $A' = A$ implies $B' = B$. Similarly, a method of FMT problem is recoverable if $B' = B$ implies $A' = A$.*

Definition 4.2. *An intuitionistic fuzzy set $\tilde{A}$ on U is normal if there exist some $x \in U$ such that $\tilde{A}(x) = 1_{L^I}$.*

Theorem 4.3. *Let $\mathcal{I}$ be an intuitionistic fuzzy implication and $\mathcal{T}$ an intuitionistic fuzzy t-norm $\mathcal{T}$ such that $(\mathcal{T},\mathcal{I})$ satisfies the residual principle. Then the QIIP algorithm of IFMP problem given by Eq.(3) is recoverable (that is, $\tilde{B}' = \tilde{B}$ whenever $\tilde{A}' = \tilde{A}$) if $\tilde{A}$ is normal.*

Proof. Suppose that $\tilde{A}' = \tilde{A}$ and $\tilde{A}$ is normal on U. This means that there exists $x_0 \in X$ such that $\tilde{A}(x_0) = 1_{L^I}$. Following Eq.(3), we obtain

$$\tilde{B}'(y) \preceq \mathcal{T}(\mathcal{T}(1_{L^I}, 1_{L^I}), \mathcal{I}(1_{L^I}, \tilde{B}(y))) = \tilde{B}(y).$$

For any $y \in V$. On the other hand, for any $x \in U,\ y \in V$,

$$\begin{aligned}
\tilde{B}'(y) &= \bigvee_{x\in U} \mathcal{T}(\tilde{A}'(x), \mathcal{T}(\mathcal{I}_{\mathcal{T}}(\tilde{A}'(x), \tilde{A}(x))\mathcal{I}(\tilde{A}(x), \tilde{B}(y)))) \\
&= \bigvee_{x\in U} \mathcal{T}(\mathcal{T}(\tilde{A}'(x), 1_{L^I}), \mathcal{I}(\tilde{A}(x), \tilde{B}(y))) \\
&= \bigvee_{x\in U} \mathcal{T}(\tilde{A}'(x), \mathcal{I}(\tilde{A}(x), \tilde{B}(y))) \\
&\succeq \tilde{B}(y).
\end{aligned}$$

which implies $\tilde{B}(y) = \tilde{B}'(y)$. □

Theorem 4.4. *If $\mathcal{I}_{\mathcal{T}}$ is an intuitionistic fuzzy R-implication generated by $\mathcal{T}$ which satisfies the condition $\sup_{y\in A} \mathcal{T}(x, y) = \mathcal{T}(x, \sup_{y\in A} y)$, then the QIIP algorithm of IFMP problem given by Eq.(5) is recoverable if $\tilde{A}$ is normal.*

Proof. It follows immediately from Theorem 4.3. □

Theorem 4.5. *If $\mathcal{I}_{\mathcal{S}}$ is an intuitionistic fuzzy S-implication generated by an intuitionistic fuzzy t-conorm $\mathcal{S}$ and an intuitionistic fuzzy strong negation $\mathcal{N}$, then the QIIP algorithm of IFMP problem given by Eq.(6) is recoverable if $\tilde{A}'$ is normal.*

Proof. Suppose $\tilde{A}' = \tilde{A}$ and there exists $x_0 \in U$ such that $\tilde{A}(x_0) = 1_{L^I}$. We have

$$\begin{aligned}
\tilde{B}(y) &\succeq \tilde{B}'(y) \\
&= \bigvee_{x\in U} \mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(\tilde{A}'(x)), \mathcal{S}(\mathcal{S}(\mathcal{T}(\tilde{A}'(x), \mathcal{N}_{\mathcal{S}}(\tilde{A}(x))), \mathcal{T}(\tilde{A}(x), \mathcal{N}_{\mathcal{S}}(\tilde{B}(y)))))) \\
&= \bigvee_{x\in U} \mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(\tilde{A}(x_0)), \mathcal{S}(\mathcal{S}(\mathcal{T}(\tilde{A}(x_0), \mathcal{N}_{\mathcal{S}}(\tilde{A}(x_0))), \mathcal{T}(\tilde{A}(x_0), \mathcal{N}_{\mathcal{S}}(\tilde{B}(y)))))) \\
&\succeq \mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(\tilde{A}(x_0)), \mathcal{S}(\mathcal{S}(\mathcal{T}(\tilde{A}(x_0), \mathcal{N}_{\mathcal{S}}(\tilde{A}(x_0))), \mathcal{T}(\tilde{A}(x_0), \mathcal{N}_{\mathcal{S}}(\tilde{B}(y)))))) \\
&= \tilde{B}(y).
\end{aligned}$$

which implies $\tilde{B}(y) = \tilde{B}'(y)$. □

Theorem 4.6. *For intuitionistic fuzzy implications $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$, the QIIP algorithm of IFMP problem given by Theorem 3.9 is recoverable if $\tilde{A}'$ is normal.*

Proof. Suppose $\tilde{A}' = \tilde{A}$ and there exists $x_0 \in U$ such that $\tilde{A}(x_0) = 1_{L^I}$. In the QIIP method of IFMP problem for $\mathcal{I}_1$, we have $\tilde{B}_{1t}(y) \succeq \tilde{B}'_{1t}(y) = \bigvee_{x\in U} (\tilde{A}'_t(x) \vee \tilde{A}'_f(x)) \wedge (\mathcal{I}_{1t}(\tilde{A}'(x), \tilde{A}(x)) \vee \mathcal{I}_{1f}(\tilde{A}'(x), \tilde{A}(x))) \wedge \mathcal{I}_{1t}(\tilde{A}(x), \tilde{B}(y)) = \tilde{B}_{1t}(y)$.

On the other hand, we can obtain that

$$\begin{aligned}\tilde{B}_{1f}(y) &\preceq \tilde{B}'_{1f}(y)\\ &= \bigwedge_{x\in U} \tilde{A}'_t(x) \wedge \mathcal{I}_{1t}(\tilde{A}'(x), \tilde{A}(x)) \wedge \mathcal{I}_{1f}(\tilde{A}(x), \tilde{B}(y))\\ &= \bigwedge_{x\in U} \tilde{A}_t(x) \wedge \mathcal{I}_{1t}(\tilde{A}(x), \tilde{A}(x)) \wedge \mathcal{I}_{1f}(\tilde{A}(x), \tilde{B}(y))\\ &\preceq \bigwedge_{x\in U} 1_{L^I} \wedge \mathcal{I}_{1t}(1_{L^I}, 1_{L^I}) \wedge \mathcal{I}_{1f}(1_{L^I}, \tilde{B}(y))\\ &= \tilde{B}_{1f}(y).\end{aligned}$$

which implies $\tilde{B}'_1(y) = \tilde{B}_1(y)$. The proof of $\mathcal{I}_2$ and $\mathcal{I}_3$ are similar to that of $\mathcal{I}_1$. □

4.2 The continuity of quintuple intuitionistic fuzzy implication for IFMP

Well-known metrics include the Euclidean distance and the Hamming distance. Grzegorzewski has proved that these four distances were topologically equivalent [13]. Therefore, we only consider Grzegorzewski's distance to investigate the continuity of quintuple intuitionistic fuzzy implication for IFMP.

Lemma 4.7. [15] *Let f and g be bounded, real valued functions on U. Then*

$$|\sup_{x\in U} f(x) - \sup_{x\in U} g(x)| \le \sup_{x\in U} |f(x) - g(x)|,$$
$$|\inf_{x\in U} f(x) - \inf_{x\in U} g(x)| \le \inf_{x\in U} |f(x) - g(x)|.$$

Definition 4.8. [14] *The Grzegorzewski's distance measure between two intuitionistic fuzzy sets $\tilde{A}$ and $\tilde{B}$ is defined as follows:*

$$d(\tilde{A}, \tilde{B}) = \max\{|\ pr_1(\tilde{A}) - pr_1(\tilde{B})\ |, |\ pr_2(\tilde{A}) - pr_2(\tilde{B})\ |\}.$$

Definition 4.9. [21] *In the metric space (L^I, d), V $(\subseteq L^I)$ is called a neighborhood of $x_0 \in L^I$ if there exists an open ball $B(x_0, \epsilon) = \{x|\ d(x, x_0) < \epsilon\}$ such that $B(x_0, \epsilon) \subseteq V$.*

Definition 4.10. [21] *We say that a mapping $f : L^I \to L^I$ is continuous if for any neighborhood V of $f(x)$ there exists a neighborhood U of x such that $f(U) \subseteq V$ with respect to the metric d.*

Remark 4.11. *It is obvious from Definition 4.9 and Definition 4.10 that f is continuous if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $d(f(\tilde{A}_1(x), \tilde{A}_2(x))) < \varepsilon$ when $d(\tilde{A}_1(x), \tilde{A}_2(x)) < \delta$ for any intuitionistic fuzzy set on U $\tilde{A}_1$ and $\tilde{A}_2$.*

Theorem 4.12. *Let $\mathcal{I}$ be an intuitionistic fuzzy implication and $\mathcal{T}$ an intuitionistic fuzzy t-norm $\mathcal{T}$ such that $(\mathcal{T}, \mathcal{I})$ satisfies the residual principle. Then the QIIP algorithm of IFMP problem given by Eq.(3) is continuous at $\tilde{A}$.*

Proof. Suppose $\tilde{A}'_1$ and $\tilde{A}'_2$ be any two inputs. For any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(\mathcal{T}(\tilde{A}'_1(x),\ \mathcal{T}(\mathcal{I}(\tilde{A}'_1(x), \tilde{A}(x)), \mathcal{I}(\tilde{A}(x), \tilde{B}(y))(y))),\ \mathcal{T}(\tilde{A}'_2(x),\ \mathcal{T}(\mathcal{I}_{\mathcal{T}}(\tilde{A}'_2(x), \tilde{A}(x)), \mathcal{I}(\tilde{A}(x), \tilde{B}(y))))) < \varepsilon.$$

if $d(\tilde{A}'_1(x),\ \tilde{A}'_2(x)) < \delta$ for any $x \in U$. Taking $\delta = \varepsilon$, we have $d(\tilde{A}'_1(x),\ \tilde{A}'_2(x)) < \delta$.

From Lemma 4.7 and Definition 4.8, we can obtain that

$$\begin{aligned}d(\tilde{B}'_1(y),\ \tilde{B}'_2(y)) &= \bigvee_{y\in V} \{|\ pr_1(\tilde{B}'_1(y)) - pr_1(\tilde{B}'_2(y))\ |, |\ pr_2(\tilde{B}'_1(y)) - pr_2(\tilde{B}'_2(y))\ |\}\\ &= d(\bigvee_{x\in U} \mathcal{T}(\tilde{A}'_1(x), \mathcal{T}(\mathcal{I}(\tilde{A}'_1(x), \tilde{A}(x)), \mathcal{I}(\tilde{A}(x), \tilde{B}(y)))),\ \bigvee_{x\in U} \mathcal{T}(\tilde{A}'_2(x), \mathcal{T}(\mathcal{I}(\tilde{A}'_2(x), \tilde{A}(x)), \mathcal{I}(\tilde{A}(x), \tilde{B}(y)))))\\ &\le \bigvee_{y\in V} \bigvee_{x\in U} |\ \mathcal{T}(\tilde{A}'_1(x), \mathcal{T}(\mathcal{I}(\tilde{A}'_1(x), \tilde{A}(x)), \mathcal{I}(\tilde{A}(x), \tilde{B}(y)))), \mathcal{T}(\tilde{A}'_2(x), \mathcal{T}(\mathcal{I}(\tilde{A}'_2(x), \tilde{A}(x)), \mathcal{I}(\tilde{A}(x), \tilde{B}(y))))\ |\\ &\le \bigvee_{y\in V} \bigvee_{x\in U} \delta = \varepsilon.\end{aligned}$$

Therefore, there exists $\delta > 0$ such that $d(\tilde{B}'_1, \tilde{B}'_2) < \varepsilon$ if $d(\tilde{A}'_1, \tilde{A}'_2) < \delta$. This means that QIIP algorithm for IFMP problem given by Eq.(3) is continuous at $\tilde{A}$. □

Theorem 4.13. *Let $\mathcal{I}$ be an intuitionistic fuzzy R-implication generated by the intuitionistic fuzzy t-norm $\mathcal{T}$ such that $(\mathcal{T},\mathcal{I}_{\mathcal{T}})$ satisfies the residual principle. Then the QIIP algorithm of IFMP problem given by Eq.(5) is continuous at $\tilde{A}$.*

Proof. It is similar to the proof of Theorem 4.12. □

Theorem 4.14. *The QIIP algorithm for IFMP problem determined by Eq.(6) is continuous at $\tilde{A}$.*

Proof. Suppose $\tilde{A}'_1$ and $\tilde{A}'_2$ be any two inputs. For any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(\mathcal{T}(\tilde{A}'_1(x),\, \mathcal{T}(\mathcal{I}(\tilde{A}'_1(x),\tilde{A}(x)),\mathcal{I}(\tilde{A}(x),\tilde{B}(y))(y))),\, \mathcal{T}(\tilde{A}'_2(x),\, \mathcal{T}(\mathcal{I}_{\mathcal{T}}(\tilde{A}'_2(x),\tilde{A}(x)),\mathcal{I}(\tilde{A}(x),\tilde{B}(y))))) < \varepsilon.$$

if $d(\tilde{A}'_1(x),\, \tilde{A}'_2(x)) < \delta$ for any $x \in U$. Taking $\delta = \varepsilon$, we have $d(\tilde{A}'_1(x),\, \tilde{A}'_2(x)) < \delta$.

From Lemma 4.7 and Definition 4.8, we can obtain that

$$\begin{aligned}
d(\tilde{B}'_1(y),\, \tilde{B}'_2(y)) &= \bigvee_{y\in V}\{|\, pr_1(\tilde{B}'_1(y)) - pr_1(\tilde{B}'_2(y))\,|, |\, pr_2(\tilde{B}'_1(y)) - pr_2(\tilde{B}'_2(y))\,|\}\\
&= d(\bigvee_{x\in U} \mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(\tilde{A}'_1(x)),\mathcal{S}(\mathcal{S}(\mathcal{T}(\tilde{A}'(x),\mathcal{N}_{\mathcal{S}}(\tilde{A}(x)),\mathcal{T}(\tilde{A}(x),\mathcal{N}_{\mathcal{S}}(\tilde{B}(y)))),\\
&\quad \bigvee_{x\in U} \mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(\tilde{A}'_2(x)),\mathcal{S}(\mathcal{S}(\mathcal{T}(\tilde{A}'(x),\mathcal{N}_{\mathcal{S}}(\tilde{A}(x)),\mathcal{T}(\tilde{A}(x),\mathcal{N}_{\mathcal{S}}(\tilde{B}(y)))))\\
&\le \bigvee_{y\in V}\bigvee_{x\in U} \mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(\tilde{A}'_1(x)),\mathcal{S}(\mathcal{S}(\mathcal{T}(\tilde{A}'(x),\mathcal{N}_{\mathcal{S}}(\tilde{A}(x)),\mathcal{T}(\tilde{A}(x),\mathcal{N}_{\mathcal{S}}(\tilde{B}(y)))),\\
&\quad \mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(\tilde{A}'_2(x)),\mathcal{S}(\mathcal{S}(\mathcal{T}(\tilde{A}'(x),\mathcal{N}_{\mathcal{S}}(\tilde{A}(x)),\mathcal{T}(\tilde{A}(x),\mathcal{N}_{\mathcal{S}}(\tilde{B}(y)))))\\
&\le \bigvee_{y\in V}\bigvee_{x\in U} \delta = \varepsilon.
\end{aligned}$$

Therefore, there exists $\delta > 0$ such that $d(\tilde{B}'_1, \tilde{B}'_2) < \varepsilon$ if $d(\tilde{A}'_1, \tilde{A}'_2) < \delta$. This implies that The QIIP algorithm for IFMP problem determined by Eq.(6) is continuous at $\tilde{A}$. □

Theorem 4.15. *For intuitionistic fuzzy implications $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$, the QIIP algorithm of IFMP problem given by Theorem 3.9 are continuous at $\tilde{A}$.*

Proof. From Lemma 4.7 and Definition 4.8, it is obvious that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned}
d(\tilde{B}'_1(y),\, \tilde{B}'_2(y)) &= \bigvee_{y\in V}\{|\, pr_1(\tilde{B}'_1(y)) - pr_1(\tilde{B}'_2(y))\,|, |\, pr_2(\tilde{B}'_1(y)) - pr_2(\tilde{B}'_2(y))\,|\}\\
&= d(\bigvee_{x\in U} (\tilde{A}'_{1t}(x) \vee \tilde{A}'_{1f}(x)) \wedge (\mathcal{I}_{1t}(\tilde{A}'(x),\tilde{A}(x)) \vee \mathcal{I}_{1f}(\tilde{A}'(x),\tilde{A}(x)) \wedge \mathcal{I}_{1t}(\tilde{A}(x),\tilde{B}(y)),\\
&\quad \bigvee_{x\in U} (\tilde{A}'_{2t}(x) \vee \tilde{A}'_{2f}(x)) \wedge (\mathcal{I}_{2t}(\tilde{A}'(x),\tilde{A}(x)) \vee \mathcal{I}_{2f}(\tilde{A}'(x),\tilde{A}(x)) \wedge \mathcal{I}_{2t}(\tilde{A}(x),\tilde{B}(y)))\\
&\le \bigvee_{y\in V}\bigvee_{x\in U} \delta = \varepsilon.
\end{aligned}$$

Therefore, for any $\varepsilon > 0$, there exists $\delta > 0$, the QIIP algorithm of IFMP problem for $\mathcal{I}_1$ is continuous at $\tilde{A}$. The proof of $\mathcal{I}_2$ and $\mathcal{I}_3$ are similar to that of $\mathcal{I}_1$. □

5 QIIP algorithm for IFMP in medical diagnosis

In this section, we will illustrate the feasibility and effectiveness of the QIIP algorithm for IFMP in a medical diagnosis.

5.1 Numerical examples

Example 5.1. *Construct an intuitionistic fuzzy rule-based system to automatically predict the risk of suffering from a Cardio-Vascular Disease (CVD) in Zhejiang China using the data consisting of 827 clinical cases obtained from health centers of Zhoushan from 2016 to 2020.*

In this example, we will consider a two-input single-output model, i.e. "if $\tilde{A}$ and $\tilde{S}$ then $\tilde{B}$", where $\tilde{A}$ symbolled as patient's age, $\tilde{S}$ symbolled as systolic blood pressure and $\tilde{B}$ symbolled as the risk of suffering from Cardio-Vascular Disease. The risk prediction of Cardio-Vascular Disease mainly includes the following three basic steps:

Step 1. Input-output variable fuzzification.

i. Cluster analysis on raw data. The example space is depicted in Figure 1. The results obtained by clustering the data using the k-means++, k-means, fuzzy k-means algorithms in Ref [2, 19, 9] are displayed in Figures 2-4. Figure 5 depicts the total time for different clustering methods. According to the running speed, the k-means, fuzzy k-means, and k-means++ algorithms take 2.831s, 1.321s, and 0.433s, respectively. The k-means++ algorithm consistently outperforms the algorithm of k-means and fuzzy k-means in running time and the classification results of the k-means++ algorithm are more obvious, that is, the accuracy is higher, and there is almost no crossover. Consequently, we use the k-means++ algorithm to cluster the input and output variables in this example. The centers of various types are shown in Tabes 1-3. Risks are divided according to the three cluster centers described above.

- Low Risk patients, whose risk ranges in $[0, 2.3555]$;
- Middle Risk patients, whose risk ranges in $[2.3555, 4.6845]$;
- High Risk patients, whose risk ranges in $[4.6845, 7.2874]$.

ii. Establish the intuitionistic fuzzy membership functions of input variables and output variables. The corresponding images are described in Figure 6.

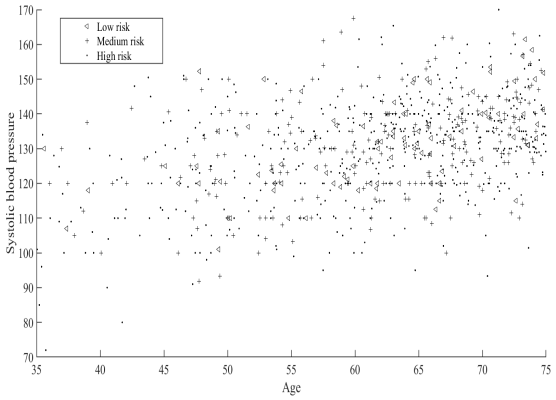


Figure 1: The example space.

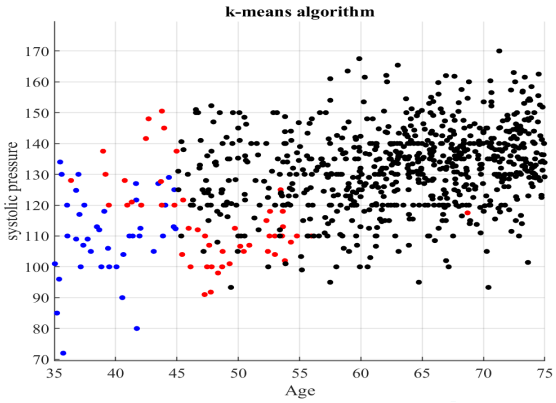


Figure 2: The example of k-means.

Table 1: The cluster center of the original data.

Cluster center	Age	Systolic blood pressure	Risk	Normalized risk
I	48.8333	114.6176	3.0766	0.1154
II	64.2003	131.4158	5.3368	0.2409
III	66.4361	140.9671	5.6850	0.2603

iii. Bring the original data into the corresponding intuitionistic fuzzy membership function, the corresponding intuitionistic fuzzy set image is described in Figure 7.

iv. Establish fuzzy rules. According to the established standard of the fuzzy system, the fuzzy rule table is established and characterized in Table 4. A_{Low}, A_{Middle} and A_{High} represent low age, medium age and high age, respectively,

while S_{Low}, S_{Middle} and S_{High} represent low systolic blood pressure, medium systolic blood pressure and high systolic blood pressure respectively.

Step 2. Compute $\tilde{B}'$ by QIIP algorithm of IFMP based on multiple fuzzy rules.

i. Composite input intuitionistic fuzzy sets. Fuzzify the values of all the elements to the corresponding intuitionistic

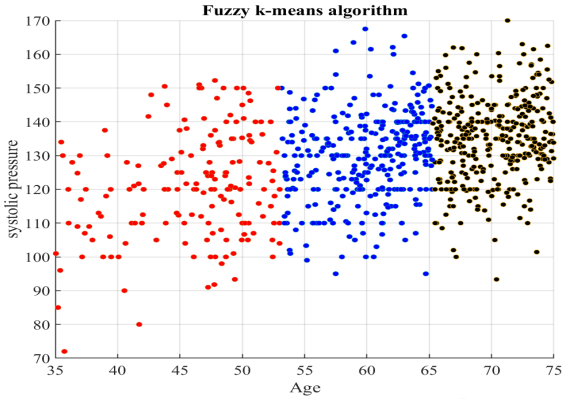


Figure 3: The example of fuzzy k-means.

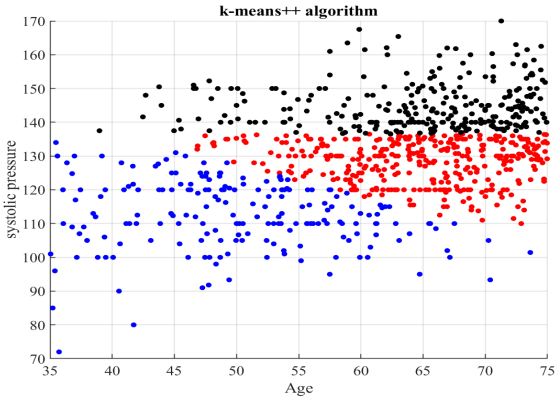


Figure 4: The example of k-means++.

Function Name	Calls	Total Time	Self Time*	Total Time Plot (dark band = self time)
kmeans	1	2.831 s	1.751 s	

Function Name	Calls	Total Time	Self Time*	Total Time Plot (dark band = self time)
fuzzyKmeans	1	1.321 s	0.268 s	

Function Name	Calls	Total Time	Self Time*	Total Time Plot (dark band = self time)
kmeansjj	1	0.433 s	0.090 s	

Figure 5: The total time spent by different clustering methods

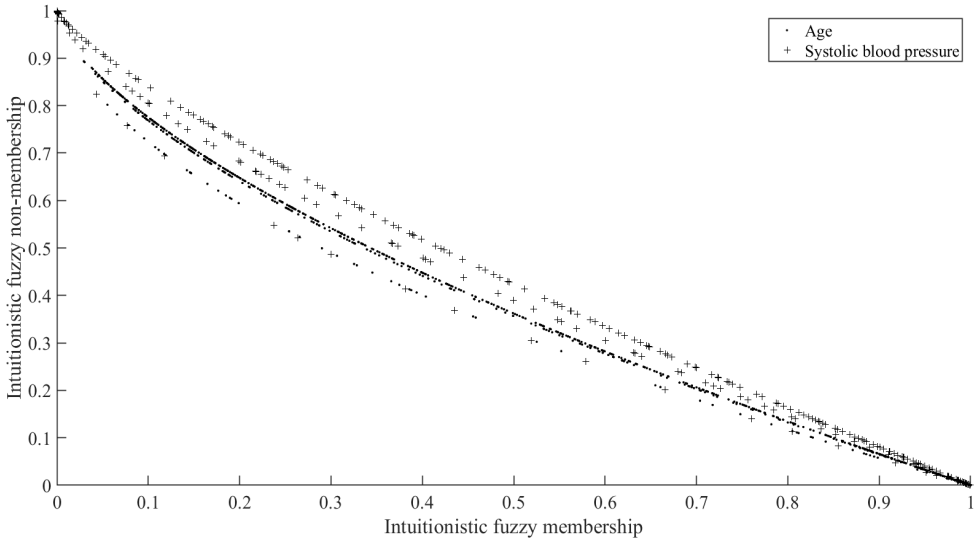


Figure 6: Intuitionistic fuzzy set image.

Table 2: Standard deviation of input and output data.

Standard deviation	Age	Systolic blood pressure	Risk
I	4.2335	7.1814	0.7938
II	4.3694	4.7698	0.4654
III	3.1375	7.6257	1.8517

Table 3: The mean value of clustering data.

Mean value	Age	Systolic blood pressure	Risk
I	43.6618	105.7078	2.3555
II	57.8674	123.9621	4.6845
III	69.6802	141.4831	7.2874

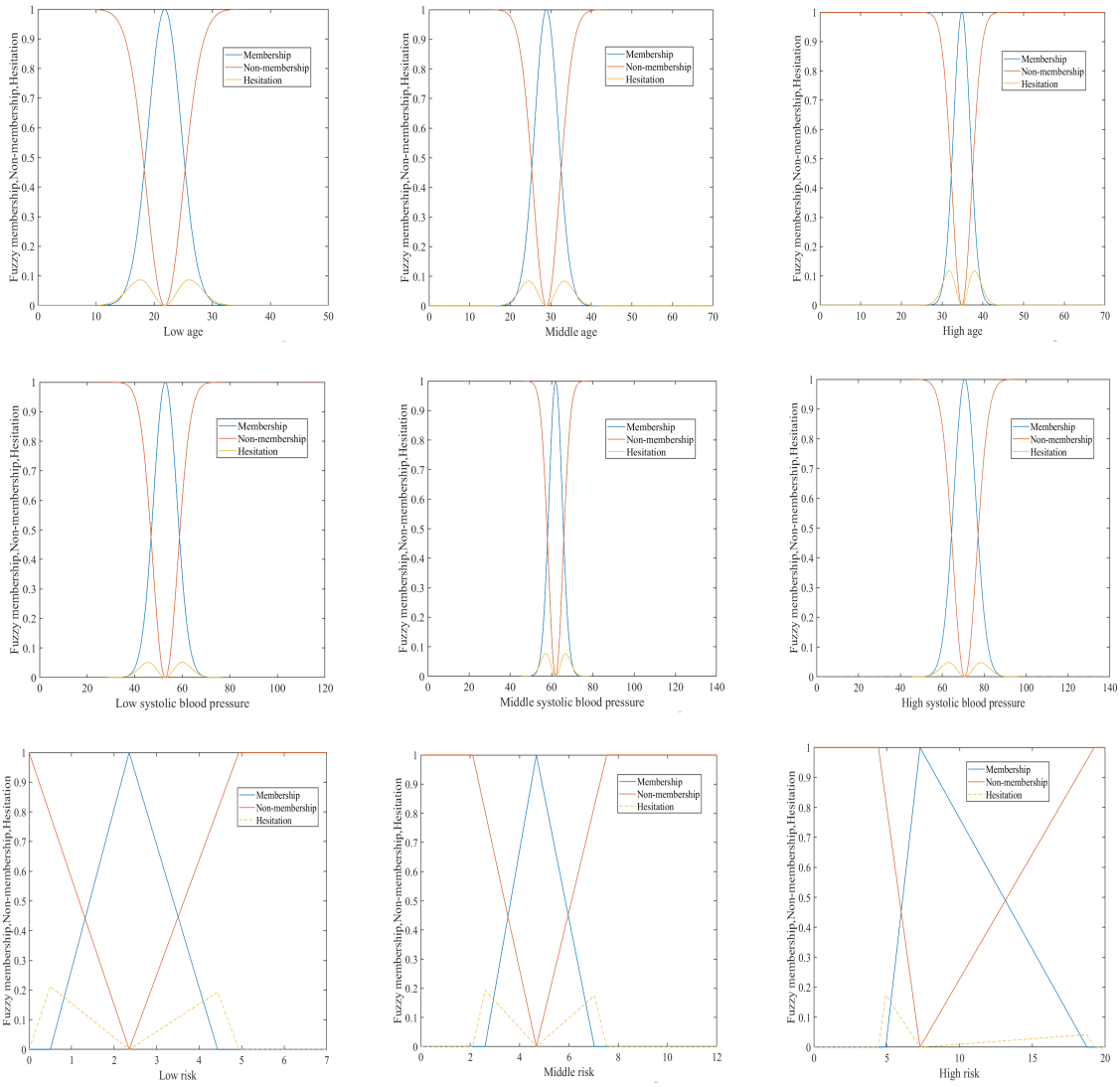


Figure 7: The corresponding images of input and output variables.

fuzzy sets of using the following rules:

$$\mathcal{O}(\tilde{A}_{1i}(x_j), \tilde{S}_{2i}(x_j)) = \tilde{D}_k(x), \quad i = 1, 2, 3, \;\; j = 1, 2, ..., 827, \;\; k = 1, 2, ..., 9.$$

$\mathcal{O}$ represents arithmetic operations and we select the meet operator $\wedge$.

ii. Transform fuzzy rules into fuzzy relations. For each rule, the associated fuzzy relations are built: $R_k(\tilde{D}_k(x_m), \tilde{B}_k(y_m)) = \mathcal{I}(\tilde{D}_k(x_m), \tilde{B}_k(y_m))$ and $R_k(\tilde{D}'_k(x_m), \tilde{D}_k(x_m)) = \mathcal{I}(\tilde{D}'_k(x_m), \tilde{D}_k(x_m))$, where $k = 1, 2, ..., 9$, $m = 1, 2, ..., n$. $\mathcal{I}$ is the intuitionistic fuzzy implication on L^I.

iii. Fuzzify the given fact. Using the single-point fuzzification method, i.e.

$$\tilde{D}'_k(x_m) = \begin{cases} \tilde{D}_k(x_m), & x'_m = x_m, \\ 0, & x'_m \neq x_m. \end{cases}$$

iv. Calculate $\tilde{B}'_k(y)$ by Quintuple intuitionistic fuzzy implications reasoning algorithms. Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm on L^I, for each rule, we calculate:

$$\tilde{B}'_k(y_m) = \bigvee_{x \in U} \mathcal{T}(\tilde{D}'_k(x_m), \mathcal{T}(\mathcal{I}(\tilde{D}'_k(x_m), \tilde{D}_k(x_m)), \mathcal{I}(\tilde{D}_k(x_m), \tilde{B}_k(y_m))).$$

v. Total output of the fuzzy system. Take the larger result obtained in **iv**, i.e.

$$\tilde{B}'(y) = \bigvee_{k}^{n} \tilde{B}'_k(y_m).$$

Step 3. Defuzzification. Considering that the intuitionistic fuzzy inference obtains a fuzzy quantity, it is necessary to convert the fuzzy quantity into a clear quantity. We will use the method of taking the minimum value of the maximum degree of membership as an example. Results using the different intuitionistic fuzzy t-norms [30] are shown in Table 5.

Taking the Gödel t-norm as an example, the analysis results are as follows:

(1) When the age is 43.77 years old and the systolic blood pressure is 120.00 mmHg, the output risk index is 1.8123, then the risk of suffering from a Cardio-Vascular Disease is low;

Table 4: Fuzzy rules of Cardio-Vascular Disease.

Systolic blood pressure	Age		
	A_{Low}	A_{Middle}	A_{High}
S_{Low}	Low	Low	Low
S_{Middle}	Low	Middle	Middle
S_{High}	Middle	Middle	High

Table 5: The inference results using the different intuitionistic fuzzy t-norms.

Intuitionistic fuzzy t-norm	$A_{Low} = 43.77$	$A_{Middle} = 50.64$	$A_{High} = 73.18$
	$S_{Middle} = 120.00$	$S_{High} = 146.30$	$S_{High} = 156.67$
Gödel	1.8123	2.7820	5.4989
Łukasiewicz	1.9466	2.7997	5.5044
Product	1.8123	2.7820	5.4989
Nilpotent minimum	1.8123	2.7820	5.4989
Example 2.2	1.8123	2.7820	5.4989

(2) When the age is 50.64 years old and the systolic blood pressure is 146.30 mmHg, the output risk index is 2.7820, then the risk of suffering from a Cardio-Vascular Diseaseis is middle;
(3) When the age is 73.18 years old and the systolic blood pressure is 156.67 mmHg, the output risk index is 5.4989, then the risk of suffering from a Cardio-Vascular Diseaseis is high.

Although we can obtain different QIIP algorithm solutions by selecting different intuitionistic fuzzy t-norm, the medical diagnosis is the same in this example. With the increase in age and systolic blood pressure, the risk of suffering from a Cardio-Vascular is gradually increasing. Further, to demonstrate the effectiveness of different intuitionistic fuzzy t-norms, we analyze variance using Gödel intuitionistic fuzzy t-norm, and Łukasiewicz intuitionistic fuzzy t-norm as examples. The results are listed in Table 6.

Table 6: Variance.

Intuitionistic fuzzy t-norm	$A_{Low} = 43.77$	$A_{High} = 50.64$	$A_{High} = 73.18$
	$S_{Low} = 120.00$	$S_{Middle} = 146.30$	$S_{High} = 156.67$
Gödel	0.005436	0.039755	0.028834
Łukasiewicz	0.002018	0.059693	0.029683

Łukasiewicz intuitionistic fuzzy t-norm provides the best results for different degrees of risk. On the whole, the systematic error of the obtained results are small, indicating that the QIIP algorithms have certain reliability and practicability in medical diagnosis.

5.2 Further comparison and discussion

To justify the relevance of the theory presented, we compare the QIIP method with other methods in this subsection.

i. Comparison with the resolution method of Ref [18]
The implementations of the Ref [18] are described as follows:

Step 1. Calculate QIIP solution $\tilde{B}'(y)$ of IFMP problem. The results are shown in Table 7.

Table 7: The QIIP solution $\tilde{B}'(y)$ of IFMP problem.

t-norm	$A_{Low} = 43.77$	$A_{High} = 50.64$	$A_{High} = 73.18$
	$S_{Low} = 120.00$	$S_{Middle} = 146.30$	$S_{High} = 156.67$
Gödel	$(0.4637, 0.4637)$	$(0.0462, 0.9757)$	$(0.0144, 0.9757)$
Łukasiewicz	$(0.1509, 0.4637)$	$(0.0000, 0.9133)$	$(0.0000, 0.9757)$
Produce	$(0.4637, 0.4637)$	$(0.0462, 0.9757)$	$(0.0144, 0.9757)$
Nilpotent minimum	$(0.4637, 0.4637)$	$(0.0462, 0.9757)$	$(0.0144, 0.9757)$

Step 2. According to the method of Ref [7], the intuitionistic fuzzy sets of the symptoms characteristic of the diagnosis and the symptoms characteristic of the patient in Table 7 are transformed into interval fuzzy sets by using the mapping g.

Step 3. Calculate QIP solution $\tilde{B}''(y)$ of the IVFMP problem. The results are shown in Table 8.
Comparing the data in Table 7 and Table 8, it is easy to know that solution $\tilde{B}'(y)$ of the IFMP problem is equal to solution $\tilde{B}''(y)$ of the IVFMP problem. The results show that the solutions based on the two fuzzy sets are in one-to-one correspondence. The difference between is that interval-valued fuzzy sets can effectively reduce the loss of ambiguous information, while intuitionistic fuzzy sets considers the information of true membership degree and false membership degree at the same time, which makes it has a stronger expressive ability when dealing with uncertain information more

in line with human intuition.

ii. Comparison with the solution method of Ref [17]

The implementations of the Ref [17] are described as follows:

Step 1. Building intuitionistic fuzzy sets of the symptoms characteristic of the diagnosis and the symptoms characteristic of the patient. The results are shown in Table 9.

Step 2. According to the method provided by the Ref [17], calculate the distance between the symptoms and the diagnosis of three patients in Table 5.

The distance measurement method in Ref [17] not only satisfies the axiomatic definition of distance measurement but also considers the hesitation function of intuitionistic fuzzy sets, overcoming the counter-intuitive situation. However, in Example 5.1, we established a fuzzy inference of "double input and single output" , substituting the patient symptom data into Ref [17] for calculation, and obtained the largest non-negative eigenvalues of the positive definite Hermitian matrix $\prod^T \prod$ and the distance metric are all 0. The final result is also 0. It means that the method of Ref [17] can easily lead to unreasonable results when the number of symptoms $n \leq 2$.

iii. Comparison with the standard fuzzy sets

The implementations of the standard fuzzy sets are similar to Example 5.1 and the inference results obtained are described in Table 10.

It can be seen that the diagnostic result is the same as the intuitionistic fuzzy set. Comparing the data in Table 6, the variance of the standard fuzzy sets is larger than that of the intuitionistic fuzzy set that the QIIP method has less error in dealing with medical diagnosis problems.

Table 8: The QIP solution $\tilde{B}''(y)$ of the IVFMP problem.

t-norm	$A_{Low} = 43.77$	$A_{High} = 50.64$	$A_{High} = 73.18$
	$S_{Low} = 120.00$	$S_{Middle} = 146.30$	$S_{High} = 156.67$
Gödel	$[0.4637, 0.5363]$	$[0.0462, 0.0867]$	$[0.0144, 0.0243]$
Łukasiewicz	$[0.1509, 0.5363]$	$[0.0000, 0.0867]$	$[0.0000, 0.0243]$
Produce	$[0.4637, 0.5363]$	$[0.0462, 0.0867]$	$[0.0144, 0.0243]$
Nilpotent minimum	$[0.4637, 0.5363]$	$[0.0462, 0.0867]$	$[0.0144, 0.0243]$

Table 9: The intuitionistic fuzzy sets of input and output variables.

Input and output variables	$A_{Low} = 43.77$	$A_{High} = 50.64$	$A_{High} = 73.18$
	$S_{Low} = 120.00$	$S_{Middle} = 146.30$	$S_{High} = 156.67$
$\tilde{A}_{1i}(x_j)$	$(0.9992, 0.0006)$	$(0.0462, 0.0867)$	$(0.2297, 0.6565)$
$\tilde{S}_{1i}(x_j)$	$(0.4637, 0.4637)$	$(0.6489, 0.0867)$	$(0.0144, 0.9757)$
$\tilde{B}_i(y_j)$	$(0.8084, 0.1509)$	$(0.8779, 0.1007)$	$(0.6761, 0.3104)$

Table 10: The inference results obtained from standard fuzzy set.

Index	$A_{Low} = 43.77$	$A_{High} = 50.64$	$A_{High} = 73.18$
	$S_{Low} = 120.00$	$S_{Middle} = 146.30$	$S_{High} = 156.67$
Inference results	1.181500	2.506400	4.733700
Variances	0.002485	0.086509	0.264611

6 Conclusions

This paper mainly shows the QIIP algorithms for IFMP and IFMT problems based on Atanassov's intuitionistic fuzzy implications. We first represent the expression of the QIIP algorithms for intuitionistic fuzzy R-implication, S-implication, and several intuitionistic fuzzy implications from Ref.[5]. Based on such expressions, the recovery property and continuity of the QIIP algorithms for IFMP and IFMT problems are investigated. Finally, an example for the risk prediction of Cardio-Vascular Disease to illustrate its application is presented and the feasibility, rationality, and superiority of the proposed method are demonstrated by comparing with it other methods. In the future, we wish to investigate the ability of Atanassov's intuitionistic fuzzy inference system based on the QIIP algorithms.

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