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Fuzzy implications satisfying the generalized hypothetical syllogism based on the Bandler-Kohout subproduct with semicopulas

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Abstract

Generalized hypothetical syllogism (GHS) plays a very important role in fuzzy inference. Therefore, it is valuable to investigate the GHS based on the Bandler-Kohout subproduct (BK-GHS) in order to measure the availability of fuzzy inference. This paper aims to study the (BK-GHS) property of some well-known fuzzy implications including (S, N)-, QL-, f-, g-, probabilistic and probabilistic S-implications in detail. We use the method of automorphism transformation to investigate (S, N)- and QL-implications and make better use of some essential properties about f-, g-, probabilistic and probabilistic S-implications to make them satisfy (BK-GHS) with semicopulas. With these results, (BK-GHS) can be more effectively applied in practice.

Keywords: Fuzzy implication, generalized hypothetical syllogism, semicopula, Bandler-Kohout subproduct.

1 Introduction

1.1 Motivation

Fuzzy inference has been widely utilized in control systems, expert systems, decision support systems and so on [5, 7, 10, 25, 29]. It owes these applications to which fuzzy inference can obtain some meaningful conclusions from the if-then rules and fuzzy premises. It is well known that fuzzy implications are usually applied to interpret if-then rules in fuzzy inference. This inspires people to develop a variety of fuzzy implications in order to meet the actual needs of fuzzy inference. Some well-known fuzzy implications include R-, (S, N)-, QL-, D-, f-, g-, probabilistic and probabilistic S-implications and so on [3, 8, 9, 12, 16, 18, 19, 26]. Moreover, it is worth noting that Mamdani and Assilian [17] interpret if-then rules by the fuzzy conjunctions (mostly t-norms).

Generalized modus ponens (GMP), generalized modus tollens (GMT) and generalized hypothetical syllogism (GHS) are three basic models of fuzzy inference. They are intuitively represented as, respectively:

Premise 1: IF x is A THEN y is B

Premise 2: x is A'

Conclusion: y is B',

Premise 1: IF x is A THEN y is B

Premise 2: y is not B'

Conclusion: x is not A',

Premise 1: IF x is A THEN y is B
Premise 2: IF y is B' THEN z is C

Conclusion: IF x is A THEN z is C',

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where A and A' are fuzzy sets on the universe X, B and B' are fuzzy sets on the universe Y while C and C' are fuzzy sets on the universe Z.

To obtain the conclusions of GMP, GMT and GHS problems, Zadeh presents the compositional rule of inference (CRI) method [30]. And then the conclusion B' of GMP problem can be calculated as $B'(y) = \sup_{x\in X} T(A'(x), R(x, y))$, where T is a fuzzy conjunction (such as a t-norm). Further, we utilize a fuzzy implication to interpret the premises and conclusion in GHS problem. Based on the CRI method, the conclusion of GHS problem can be rewritten as $I(A(x), C'(z)) = \sup_{y\in Y} T(I(A(x), B'(y)), I(B(y), C(z)))$, where T is a fuzzy conjunction and I is a fuzzy implication. In order to measure the validity of inference scheme solving the GHS problem, the following GHS property is proposed:

$$I(x, y) = \sup_{z\in[0,1]} T(I(x, z), I(z, y)), \quad \forall x, y \in [0, 1].$$

And then many scholars have investigated the GHS equation in depth [4, 14, 22, 21, 23, 28].

Unlike CRI method, the Bandler-Kohout (BK) subproduct is used to compute the conclusion of GMP (or GMT) problem by many researchers [21, 24, 27]. In this case, $B'(y) = \inf_{x\in X} I(A'(x), R(x, y))$, where I is a fuzzy implication. As the BK subproduct method satisfies the same properties as the CRI method does, it becomes an alternative method of GMP problem. Further, let the fuzzy conjunctions be utilized to interpret the premises in GHS problem. This triggers us consider the GHS property based on the BK subproduct as follows:

$$I(x, y) = \inf_{z\in[0,1]} I(B(x, z), B(z, y)), \forall x, y \in [0, 1], \tag{BK-GHS}$$

where I is a fuzzy implication and B is a semicopula.

Miś and Baczyński [21] study the GHS property of R-implications and find that the I_T satisfies (BK-GHS) with t-norm T. Considering the application of fuzzy inference based on BK subproduct method, it is more meaningful to investigate whether other well-known implications satisfy (BK-GHS) with some semicopulas. Therefore, we will study the (BK-GHS) property of some general fuzzy implications such as (S, N)-, QL-, f-, g-, probabilistic and probabilistic S-implications and find some semicopulas such that these fuzzy implications satisfy (BK-GHS) with them.

1.2 Contribution of this paper

As the GHS property can measure the effectiveness of fuzzy inference, it is valuable to research the GHS property based on the BK subproduct for general fuzzy implications. Based on the above, we mainly discuss whether some well-known fuzzy implications, viz., (S, N)-, QL-, f-, g-, probabilistic and probabilistic S-implications, satisfy (BK-GHS) with some semicopulas. Moreover, we find some semicopulas for these implications such that they satisfy (BK-GHS). In short, the contribution of this paper is:

(1) To study (S, N)- and QL-implications satisfying (BK-GHS) with some semicopulas.

(2) To investigate f- and g-implications satisfying (BK-GHS) with some semicopulas.

(3) To seek some semicopulas for probabilistic implications and probabilistic S-implications such that they satisfy (BK-GHS).

This paper is constituted as follows. Section 2 recalls some basic concepts and definitions to facilitate reading. We investigate whether the (S, N)-implications satisfy (BK-GHS) with some semicopulas in Section 3. Section 4 studies the satisfaction of (BK-GHS) property for QL-implications generated by $(S_{\mathrm{LK}})_\varphi$ and a strict negation N such that $N \geq (N_C)_\varphi$. The conclusions that I_f and $I_g \in \mathbb{I}_{\mathbb{G},\infty}$ do not satisfy (BK-GHS) with any semicopula while $I_g \in \mathbb{I}_{\mathbb{G},\aleph}$ satisfies (BK-GHS) with $B = T_{\mathrm{M}}$ or T_{P} are drawn in Section 5. Section 6 shows that a few of probabilistic implications and probabilistic S-implications satisfy (BK-GHS) with some semicopulas.

2 Preliminaries

In order to make this paper as self-contained as possible, this section will review some basic concepts and notations related to the following content.

Definition 2.1. [21] *A function $B : [0, 1]^2 \to [0, 1]$ is called a semicopula if the following conditions are meet:*

(B1) $B(x, 0) = B(0, x) = 0,\ \forall x \in [0, 1]$,

(B2) $B(x, 1) = B(1, x) = x,\ \forall x \in [0, 1]$,

(B3) *B is non-decreasing with respect to both variables.*

Definition 2.2. [2, 6, 20] *A copula (specially, a 2-copula) is a function* $C : [0,1]^2 \to [0,1]$ *which satisfies* (B1)-(B2) *and*

(C3) $C(x_2, y_2) - C(x_2, y_1) - C(x_1, y_2) + C(x_1, y_1) \geq 0$ *for all* $x_1, x_2, y_1, y_2 \in [0,1]$ *such that* $x_1 \leq x_2, y_1 \leq y_2$.

Definition 2.3. [15] *A triangular norm* T *(t-norm for short) is a binary operation on* $[0,1]$*, i.e., a function* $T : [0,1]^2 \to [0,1]$*, which, for all* $x, y, z \in [0,1]$*, satisfies* (B1-B3) *and*

(T4) $T(x, y) = T(y, x)$,

(T5) $T(x, T(y, z)) = T(T(x, y), z)$.

Remark 2.4. *As it is mentioned in* [15]*, the associativity* (T5) *allows us to extend each t-norm* T *in a unique way to an n-ary operation in the usual way by induction, defining for each n-tuple* $(x_1, x_2, \cdots, x_n) \in [0,1]^n$: $\mathop{\mathsf{T}}_{i=1}^{n} x_i = T(\mathop{\mathsf{T}}_{i=1}^{n-1} x_i, x_n) = T(x_1, x_2, \cdots, x_n)$. *In particular, we have* $x_1 = x_2 = \cdots = x_n$*, we shall briefly write* $x_T^{(n)} = T(\underbrace{x, x, \cdots, x}_{n})$. *By convention, we put* $x_T^{(0)} = 1$ *and* $x_T^{(1)} = x$ *for each* $x \in [0,1]$.

Example 2.5. [15] *The following are four basic t-norms:*

$T_{\mathrm{M}}(x, y) = \min(x, y)$,

$T_{\mathrm{P}}(x, y) = xy$,

$T_{\mathrm{LK}}(x, y) = \max(x + y - 1, 0)$,

$$T_{\mathrm{D}}(x, y) = \begin{cases} 0 & x, y \in [0,1)^2 \\ \min(x, y) & \text{otherwise} \end{cases}.$$

Definition 2.6. [15] *Let* T *be a t-norm.*

i. T *is continuous if it is continuous in both variables,*

ii. T *is idempotent if* $T(x, x) = x$ *for all* $x \in [0,1]$,

iii. T *is strict if it is continuous and strictly monotone,*

iv. T *is nilpotent if it is continuous and each* $x \in (0,1)$ *is a nilpotent element of* T*, i.e., if there exists* $n \in \mathbb{N}$ *such that* $x_T^{(n)} = 0$*, where* $x_T^{(n)} = T(\underbrace{x, x, \cdots, x}_{n})$,

v. T *is positive if* $T(x, y) = 0$ *implies* $x = 0$ *or* $y = 0$.

Definition 2.7. [15] *An element* $x \in (0,1)$ *is called a zero divisor of* T*, if there exist some* $y \in (0,1)$ *such that* $T(x, y) = 0$.

Definition 2.8. [15] *Let* T *be a t-norm and* φ *be a strictly increasing bijection on* $[0,1]$*. The operation* T_φ *defined as* $T_\varphi(x, y) = \varphi^{-1}(T(\varphi(x), \varphi(y)))$ *is a t-norm again. It is called a* φ*-conjugate of* T.

Definition 2.9. [15] *We say* $T_1 \leq T_2$ *if* $T_1(x, y) \leq T_2(x, y)$ *holds for any* $x, y \in [0,1]$.

Definition 2.10. [15] *A triangle conorm (t-conorm for short) is a binary operation* S *on* $[0,1]$*, i.e., a function* $S : [0,1]^2 \to [0,1]$*, which, for all* $x, y, z \in [0,1]$*, satisfies* (B3), (T4), (T5) *and*

(S1) $S(x, 0) = x$.

Example 2.11. [15] *i. The four basic t-conorms are shown as follows:*

$S_{\mathrm{M}}(x, y) = \max(x, y)$,

$S_{\mathrm{P}}(x, y) = x + y - xy$,

$S_{\mathrm{LK}}(x, y) = \min(x + y, 1)$,

$$S_{\mathrm{D}}(x, y) = \begin{cases} 1 & x, y \in (0,1]^2 \\ \max(x, y) & \text{otherwise} \end{cases}.$$

ii. A right-continuous t-conorm $S_{\mathrm{nM}} = \begin{cases} 1 & x + y \geq 1 \\ \max(x, y) & \text{otherwise} \end{cases}$.

Lemma 2.12. [15] *Let* S *be a t-conorm. We have*

i. S *is strict if and only if* $S = (S_{\mathrm{P}})_\varphi$*, i.e., there exists a strictly increasing bijection on* $[0,1]$ φ *such that* $S(x, y) = \varphi^{-1}(\varphi(x) + \varphi(y) - \varphi(x)\varphi(y))$,

ii. S *is nilpotent if and only if* $S = (S_{\mathrm{LK}})_\varphi$*, i.e., there exists a strictly increasing bijection on* $[0,1]$ φ *such that* $S(x, y) = \varphi^{-1}(\min(\varphi(x) + \varphi(y), 1))$.

Definition 2.13. [1, 3] *A function $N : [0,1] \to [0,1]$ is called a fuzzy negation if*
(N1) $N(0) = 1,\ N(1) = 0$,
(N2) N *is decreasing.*

Further, a fuzzy negation N is strict if it satisfies the following properties:
(N3) N is continuous,
(N4) $N(x) > N(y)$ if $x < y$, $\forall x, y \in [0,1]$.

Example 2.14. [1, 3] *i. The greatest negation is given as* $N_\top(x) = \begin{cases} 0 & x = 1 \\ 1 & x < 1 \end{cases}$.

ii. The standard negation is given as $N_C(x) = 1 - x$.

Theorem 2.15. [11] *A continuous t-conorm S satisfies $S(N(x), x) = 1$ for all $x \in [0,1]$ with a strict negation N if and only if there exists a strictly increasing bijection φ on $[0,1]$ such that $S(x,y) = \varphi^{-1}(\min(\varphi(x) + \varphi(y), 1))$ and $N(x) \geq \varphi^{-1}(1 - \varphi(x)), x \in [0,1]$.*

Definition 2.16. [3] *A function $I : [0,1]^2 \to [0,1]$ is a fuzzy implication which satisfies for all $x, y, z \in [0,1]$, the following conditions:*
(I1) *If $x \leq y$ then $I(x,z) \geq I(y,z)$, i.e., $I(\cdot, y)$ is decreasing,*
(I2) *If $y \leq z$ then $I(x,y) \leq I(x,z)$, i.e., $I(x,\cdot)$ is increasing,*
(I3) $I(0,0) = 1$,
(I4) $I(1,1) = 1$,
(I5) $I(1,0) = 0$.

By Definition 2.16, the following properties can be easily obtained:
(LB) Left boundary condition, $I(0,y) = 1, \forall\ y \in [0,1]$,
(RB) Right boundary condition, $I(x,1) = 1, \forall\ x \in [0,1]$.

Definition 2.17. [3] *A fuzzy implication $I : [0,1]^2 \to [0,1]$ is said to satisfy*
(NP) *Left neutrality property, if $I(1,y) = y, \forall\ y \in [0,1]$,*
(IP) *Identity principle, if $I(x,x) = 1, \forall\ x \in [0,1]$.*

Definition 2.18. [3] *The lattice operations $\vee$ and $\wedge$ of fuzzy implications are defined by*
$(I \vee J)(x,y) = \max(I(x,y), J(x,y)),\ x, y \in [0,1]$,
$(I \wedge J)(x,y) = \min(I(x,y), J(x,y)),\ x, y \in [0,1]$.

It is not difficult to find that $I \vee J$ and $I \wedge J$ are still fuzzy implications.

Definition 2.19. [3] *Let I be a fuzzy implication and φ be a strictly increasing bijection on $[0,1]$. The conjugate of fuzzy implication $I_\varphi : [0,1]^2 \to [0,1]$ is defined as $I_\varphi(x,y) = \varphi^{-1}(I(\varphi(x), \varphi(y)))$. It is obvious to see that I_φ is also a fuzzy implication.*

Next, we introduce some well-known fuzzy implications.

Definition 2.20. [3] *A function $I : [0,1]^2 \to [0,1]$ is called an* (S, N)*-implication if there exist a t-conorm S and a fuzzy negation N such that*

$$I_{S,N}(x,y) = S(N(x), y), \qquad x, y \in [0,1].$$

Definition 2.21. [3] *A function $I : [0,1]^2 \to [0,1]$ is called a* QL*-operation if there exist a t-norm T, a t-conorm S and a fuzzy negation N such that*

$$I_{QL}(x,y) = S(N(x), T(x,y)), \qquad x, y \in [0,1].$$

Especially, the QL-operation is called QL-implication if it satisfies (I1).

Definition 2.22. [3] *Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing and continuous mapping with $f(1) = 0$. An f-implication, which is a function $I_f : [0,1]^2 \to [0,1]$ with an f-generator, is defined by*

$$I_f(x,y) = f^{-1}(x \cdot f(y)),$$

with the understanding $0 \cdot \infty = 0$.

The family of all f-implications such that $f(0) < \infty$ and $f(0) = \infty$ are denoted by $\mathbb{I}_{\mathbb{F},\aleph}$ and $\mathbb{I}_{\mathbb{F},\infty}$, respectively.

Definition 2.23. [3] *Let $g : [0,1] \to [0,\infty]$ be a strictly increasing and continuous mapping with $g(0) = 0$. A g-implication, which is a function $I_g : [0,1]^2 \to [0,1]$ with a g-generator, is defined by*

$$I_g(x,y) = g^{(-1)}\left(\frac{g(y)}{x}\right),$$

with the understanding that $\frac{1}{0} = \infty$ and $\infty \cdot 0 = \infty$, where the function $g^{(-1)}(x)$ is the pseudo-inverse of g defined as $g^{(-1)}(x) = \begin{cases} g^{-1}(x) & x \in [0, g(1)] \\ 1 & \text{otherwise} \end{cases}$.

The family of all g-implications such that $g(1) < \infty$ and $g(1) = \infty$ are denoted by $\mathbb{I}_{\mathbb{G},\aleph}$ and $\mathbb{I}_{\mathbb{G},\infty}$, respectively.

Definition 2.24. [13] *Let C be a copula. A function $I_C : [0,1]^2 \to [0,1]$ given by*

$$I_C(x,y) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{C(x,y)}{x} & \text{if } x > 0 \end{cases}$$

is called a probabilistic implication (based on a copula C).

Definition 2.25. [13] *Let C be a copula. A function $\tilde{I}_C : [0,1]^2 \to [0,1]$ given by*

$$\tilde{I}_C(x,y) = C(x,y) - x + 1,$$

is called a probabilistic S-implication (based on a copula C).

3 (S, N)-implications satisfy (BK-GHS)

This section will discuss whether (S, N)-implications satisfy (BK-GHS) and seek some semicopulas such that they satisfy (BK-GHS). We can easily obtain the following statement.

Lemma 3.1. *Let the fuzzy implication I satisfy* (BK-GHS) *with a semicopula B. Then we have I fulfills* (IP).

Proof. Suppose that I satisfies (BK-GHS) with a semicopula B. Taking $x = y = 1$, we obtain $I(1,1) = 1 = \inf_{z\in[0,1]} I(B(1,z), B(z,1))$. Since (B2), $1 = I(B(1,z), B(z,1)) = I(z,z)$ holds for any $z \in [0,1]$. Hence, I satisfies (IP). □

Theorem 3.2. *I satisfies* (BK-GHS) *with B if and only if I_φ fulfills* (BK-GHS) *with B_φ.*

Proof. ($\Longrightarrow$) Since I satisfies (BK-GHS) with B, we can obtain $\inf_{z\in[0,1]} I(B(\varphi(x),\varphi(z)), B(\varphi(z), \varphi(y))) = I(\varphi(x),\varphi(y))$. Therefore, $\inf_{z\in[0,1]} I_\varphi(B_\varphi(x,z), B_\varphi(z,y)) = \inf_{z\in[0,1]} \varphi^{-1}(I(\varphi(B_\varphi(x,z)), \varphi\ (B_\varphi(z,y)))) = \inf_{z\in[0,1]} \varphi^{-1}(I(\varphi \circ \varphi^{-1}(B(\varphi(x), \varphi(z))), \varphi \circ \varphi^{-1}(B(\varphi(z),\varphi(y))))) = \inf_{z\in[0,1]} \varphi^{-1}(I(B(\varphi\ (x),\varphi(z)), B(\varphi(z),\varphi(y)))) = \varphi^{-1}(\inf_{z\in[0,1]} I(B(\varphi(x),\varphi(z)), B(\varphi(z), \varphi(y)))) = \varphi^{-1}(I(\varphi(x),\varphi(y))) = I_\varphi(x,y)$.

($\Longleftarrow$) Obviously. □

Theorem 3.3. *I satisfies* (BK-GHS) *with $B = T_{\mathrm{M}}$ if and only if I satisfies* (IP).

Proof. ($\Longleftarrow$) Let us consider the following two cases.

i. If $x \le y$. We have $\inf_{z\in[0,1]} I(T_{\mathrm{M}}(x,z), T_{\mathrm{M}}(z,y)) = \inf_{z\in[0,x]} I(z,z) \wedge \inf_{z\in(x,y]} I(x,z) \wedge \inf_{z\in(y,1]} I(x,y)$. We can obtain $I(x,z) \ge I(z,z) = 1$ by (I1) and (IP) if $z \in (x,y]$. Hence, $\inf_{z\in(x,y]} I(x,z) = 1$. We can also get that $I(x,y) \ge I(x,x) = 1$ by (I2) and (IP) if $x \le y$ and $z \in (y,1]$. Therefore, $\inf_{z\in(y,1]} I(x,y) = 1$. Finally, we have $\inf_{z\in[0,1]} I(T_{\mathrm{M}}(x,z), T_{\mathrm{M}}(z,y)) = 1 = I(x,y)$.

ii. If $x > y$. We can obtain $\inf_{z\in[0,1]} I(T_{\mathrm{M}}(x,z), T_{\mathrm{M}}(z,y)) = \inf_{z\in[0,y]} I(z,z) \wedge \inf_{z\in(y,x]} I(z,y) \wedge \inf_{z\in(x,1]} I(x,\ y)$. By (I1), $I(x,y) \le I(z,y) \le I(y,y)$ if $z \in (y,x]$. Therefore, we have $\inf_{z\in(y,x]} I(z,y) = I(x,y)$. Obviously, $\inf_{z\in(x,1]} I(x,y) = I(x,y)$. Finally, we obtain $\inf_{z\in[0,1]} I(T_{\mathrm{M}}(x,z), T_{\mathrm{M}}(z,y)) = I(x,y)$.

($\Longrightarrow$) Let I satisfy (BK-GHS) with $B = T_{\mathrm{M}}$. Thus, I satisfies (IP) by Lemma 3.1. □

Theorem 3.4. *If both I and J satisfy* (BK-GHS) *with $B = T_{\mathrm{M}}$, then $I \wedge J$ and $I \vee J$ also satisfy* (BK-GHS) *with $B = T_{\mathrm{M}}$.*

Proof. Since both I and J satisfy (BK-GHS) with $B = T_{\mathrm{M}}$, $I(x,y) = J(x,y) = 1$ holds if $x \le y$ by Theorem 3.3. Then we have $\max(I(x,y), J(x,y)) = 1$ and $\min(I(x,y), J(x,y)) = 1$ if $x \le y$. Thus, $(I \vee J)(x,y) = 1$ and $(I \wedge J)(x,y) = 1$ if $x \le y$. By Theorem 3.3, $I \wedge J$ and $I \vee J$ satisfy (BK-GHS) with $B = T_{\mathrm{M}}$. □

Next, we consider whether (S, N)-implications generated by continuous t-conorms S and continuous fuzzy negations N satisfy (BK-GHS) with some semicopulas. For convenience, let $B_1 = T_{\mathrm{M}}$, $B_2 = T_{\mathrm{LK}}$ and $B_3 = T_{\mathrm{P}}$. We further denote $B_{ij} = \lambda B_i + (1-\lambda)B_j$ $(i, j = 1, 2, 3, \lambda \in [0,1])$.

Lemma 3.5. *Let $I_{S,N}$ be an* (S, N)*-implication generated by a nilpotent t-conorm and let B be a semicopula. We have*
i. If $N(x) < \varphi^{-1}(1-\varphi(x))$ for any $x \in (0,1)$, then $I_{S,N}$ does not satisfy (BK-GHS) *with any semicopula,*
ii. If $N(x) \ge \varphi^{-1}(1-\varphi(x))$ for any $x \in [0,1]$, then $I_{S,N}$ satisfies (BK-GHS) *with $(B_{ij})_\varphi (i, j = 1, 2, 3)$.*

Proof. $I_{S,N}$ can be rewritten as $I_{S,N}(x,y) = \varphi^{-1}(\min(\varphi(N(x)) + \varphi(y), 1))$.

i. Taking $x = y = 1$, we obtain $\inf\limits_{z\in[0,1]} I_{S,N}(B(x,z), B(z,y)) = \inf\limits_{z\in[0,1]} I_{S,N}(z,z) = \inf\limits_{z\in[0,1]} \varphi^{-1}\left(\min(\varphi(N(z))+\varphi(z), 1)\right)$.

Since $N(x) < \varphi^{-1}(1-\varphi(x))$, we have $\inf\limits_{z\in[0,1]} \varphi^{-1}(\min(\varphi(N(z))+\varphi(z), 1)) < 1 = I(1,1)$. Therefore, $I_{S,N}$ does not satisfy (BK-GHS) with any semicopula by Lemma 3.1.

ii-1. If $N(x) = \varphi^{-1}(1-\varphi(x))$. We only prove that $I_{S,N}$ satisfies (BK-GHS) with $(B_{32})_\varphi = (\lambda T_{\mathrm{P}} + (1-\lambda)T_{\mathrm{LK}})_\varphi$. Other cases can similarly be proved. Let us consider the following two options.

ii-1-1. If $x \le y$. Obviously, $B(x,z) \le B(y,z) = B(z,y)$ holds. Therefore, $\inf\limits_{z\in[0,1]} I_{S,N}(B(x,z),\, B(z,y)) = 1$.

ii-1-2. If $x > y$. This case means that $1-\varphi(x) < 1-\varphi(y)$. Let $\varphi^{-1}(1-\varphi(x)) = z_1$ and $\varphi^{-1}(1-\varphi(y)) = z_2$. $\inf\limits_{z\in[0,1]} I_{S,N}(B(x,z), B(z,y)) = I_{S,N}(0,0) \wedge \inf\limits_{z\in(0,z_1]} I_{S,N}((\lambda T_{\mathrm{P}})_\varphi(x,z), (\lambda T_{\mathrm{P}})_\varphi\ (z,y)) \wedge \inf\limits_{z\in(z_1,z_2]} I_{S,N}((\lambda T_{\mathrm{P}}(x,z) + (1-\lambda)T_{\mathrm{LK}}(x,z))_\varphi, (\lambda T_{\mathrm{P}})_\varphi(z,y)) \wedge \inf\limits_{z\in(z_2,1]} I_{S,N}((\lambda T_{\mathrm{P}}(x,z) + (1-\lambda)T_{\mathrm{LK}}(x,z))_\varphi, (\lambda T_{\mathrm{P}}(z,y) + (1-\lambda)T_{\mathrm{LK}}(z,y))_\varphi) = 1 \wedge \inf\limits_{z\in(0,z_1]} \varphi^{-1}(1-\varphi(\varphi^{-1}(\lambda\varphi(x)\ \varphi(z))) + \varphi(\varphi^{-1}(\lambda\varphi(z)\varphi(y)))) \wedge \inf\limits_{z\in(z_1,z_2]} \varphi^{-1}(1-\varphi(\varphi^{-1}(\lambda\varphi(x)\varphi(z) + (1-\lambda)(\varphi(x) + \varphi(z)-1))) + \varphi(\varphi^{-1}(\lambda\varphi(z)\ \varphi(y)))) \wedge \inf\limits_{z\in(z_2,1]} \varphi^{-1}(1-\varphi(\varphi^{-1}(\lambda\varphi(x)\varphi(z)+(1-\lambda)(\varphi(x)+\varphi(z)-1))) + \varphi(\varphi^{-1}(\lambda\varphi(z)\varphi(y)+(1-\lambda)(\varphi(z)+\varphi(y)-1)))) = 1 \wedge \inf\limits_{z\in(0,z_1]} \varphi^{-1}(1-\lambda\varphi(x)\ \varphi(z)+\lambda\varphi(z)\varphi(y)) \wedge \inf\limits_{z\in(z_1,z_2]} \varphi^{-1}(1-(\lambda\varphi(x)\varphi(z)+(1-\lambda)(\varphi(x)+\varphi(z)-1))+\lambda\varphi(z)\varphi(y)) \wedge \inf\limits_{z\in(z_2,1]} \varphi^{-1}(1-(\lambda\varphi(x)\varphi(z)+(1-\lambda)(\varphi(x)+\varphi(z)-1))+(\lambda\varphi(z)\varphi(y)+(1-\lambda)(\varphi(z)+\varphi(y)-1))) = 1 \wedge \inf\limits_{z\in(0,z_1]} \varphi^{-1}(1+\lambda(\varphi(y)-\varphi(x))\varphi(z)) \wedge \inf\limits_{z\in(z_1,z_2]} \varphi^{-1}(1+(\lambda(\varphi(y)-\varphi(x))-(1-\lambda))\varphi(z)-(1-\lambda)(\varphi(x)-1)) \wedge \inf\limits_{z\in(z_2,1]} \varphi^{-1}(1+\lambda(\varphi(y)-\varphi(x))\varphi(z) + (1-\lambda)(\varphi(y)-\varphi(x)))$.

Since $x > y$ and φ is a strictly increasing bijection on [0,1], we have $\varphi(y) - \varphi(x) < 0$. Therefore, we can obtain $\lambda(\varphi(y)-\varphi(x))\varphi(z)$ and $(\lambda(\varphi(y)-\varphi(x))-(1-\lambda))\varphi(z)$ are decreasing on $z \in [0,1]$ for any fixed $\lambda \in [0,1]$ and $x, y \in [0,1]$.

Hence, we can easily have $\inf\limits_{z\in(0,z_1]} \varphi^{-1}(1+\lambda(\varphi(y)-\varphi(x))\varphi(z)) = \varphi^{-1}(1+\lambda(\varphi(y)-\varphi(x))\varphi(z_1)) = \varphi^{-1}(1+\lambda(1-\varphi(x))(\varphi(y)-\varphi(x))) \ge \varphi^{-1}(1-\varphi(x)+\varphi(y))$, $\inf\limits_{z\in(z_1,z_2]} \varphi^{-1}(1+(\lambda(\varphi(y)-\varphi(x))-(1-\lambda))\varphi(z)-(1-\lambda)(\varphi(x)-1)) = \varphi^{-1}(1+(\lambda(\varphi(y)-\varphi(x))-(1-\lambda))\varphi(z_2)-(1-\lambda)(\varphi(x)-1)) = \varphi^{-1}(1+(1-(1-\lambda)\varphi(y))(\varphi(y)-\varphi(x))) \ge \varphi^{-1}(1-\varphi(x)+\varphi(y))$ and $\inf\limits_{z\in(z_2,1]} \varphi^{-1}(1+\lambda(\varphi(y)-\varphi(x))\varphi(z)+(1-\lambda)(\varphi(y)-\varphi(x))) = \varphi^{-1}(1-\varphi(x)+\varphi(y))$.

Consequently, $\inf\limits_{z\in[0,1]} I_{S,N}(B(x,z), B(z,y)) = 1 \wedge \varphi^{-1}(1+\lambda(1-\varphi(x))(\varphi(y)-\varphi(x))) \wedge \varphi^{-1}(1+(1-(1-\lambda)\varphi(y))(\varphi(y)-\varphi(x))) \wedge \varphi^{-1}(1-\varphi(x)+\varphi(y)) = \varphi^{-1}(1-\varphi(x)+\varphi(y))$.

ii-2. Since $N(x) \ge \varphi^{-1}(1-\varphi(x))$, we have $\varphi^{-1}(\min(\varphi(N(B(x,z)) + \varphi(B(z,y)), 1)) \ge \varphi^{-1}(\min(1-\varphi(B(x,z)) + \varphi(B(z,y)), 1))$. Therefore, $\inf\limits_{z\in[0,1]} \varphi^{-1}(\min(\varphi(N(B(x,z)) + \varphi(B(z,y)), 1)) \ge \inf\limits_{z\in[0,1]} \varphi^{-1}(\min(1-\varphi(B(x,z)) + \varphi(B(z,y)), 1))$. Thus, $\inf\limits_{z\in[0,1]} \varphi^{-1}(\min(\varphi(N(B(x,z))+\varphi(B(z,y)), 1)) = \inf\limits_{z\in[0,1]} \varphi^{-1}(\min(1-\varphi(B(x,z))+\varphi(B(z,y)), 1)) = I(x,y)$. □

Theorem 3.6. *Let $I_{S,N}$ be a continuous* (S, N)*-implication and let B be a semicopula. Then $I_{S,N}$ fulfills* (IP) *if and only if it satisfies* (BK-GHS) *with $(B_{ij})_\varphi$ $(i, j = 1, 2, 3)$.*

Proof. ($\Longrightarrow$) According to Theorem 2.4.17 in [3], we have the fact that a continuous (S, N)-implication $I_{S,N}$ satisfies (IP) if and only if S is a nilpotent t-conorm and $N(x) \ge N_S(x) = \varphi^{-1}(1-\varphi(x))$ for all $x \in [0,1]$. On the other

hand, $I_{S,N}$ which is generated by a nilpotent t-conorm and $N(x) \geq N_S(x) = \varphi^{-1}(1-\varphi(x))$ satisfies (BK-GHS) with $(B_{ij})_\varphi (i,j=1,2,3)$ for all $x \in [0,1]$ by Lemma 3.5. Thus, the continuous (S, N)-implication $I_{S,N}$ satisfies (BK-GHS) with $(B_{ij})_\varphi$ $(i,j=1,2,3)$ if it fulfills (IP).

($\Longleftarrow$) Obviously, $I_{S,N}$ satisfies (IP) by Lemma 3.1. □

Above all, we obtain the fact that continuous (S, N)-implications satisfy (BK-GHS) if and only if S is a nilpotent t-conorm. Now, we move on discussing the satisfaction of (BK-GHS) for (S, N)-implications generated by some continuous t-conorms and noncontinuous fuzzy negations.

Theorem 3.7. *Let $I_{S,N}$ be an* (S, N)*-implication generated by S_{M} and let B be a semicopula. Then $I_{S,N}$ satisfies* (BK-GHS) *with B if and only if* $I_{S,N}(x,y) = I_{\mathrm{WB}}(x,y) = \begin{cases} 1 & x<1 \\ y & x=1 \end{cases}$.

Proof. ($\Longrightarrow$) Let $I_{S,N}$ satisfy (BK-GHS) with B. By Lemma 3.1, $I_{S,N}$ fulfills (IP). This implies that $x=1$ or $N(x)=1$ holds for any $x \in [0,1)$. Therefore, $N = N_\top$. This means that $I_{S,N}(x,y) = I_{\mathrm{WB}}(x,y) = \begin{cases} 1 & x<1 \\ y & x=1 \end{cases}$.

($\Longleftarrow$) Let us consider the following two options.

i. If $x<1$. This case means that $\inf\limits_{z\in[0,1]} I_{\mathrm{WB}}(B(x,z),B(z,y)) = 1$.

ii. If $x=1$. This case implies that $\inf\limits_{z\in[0,1]} I_{\mathrm{WB}}(B(x,z),B(z,y)) = \inf\limits_{z\in[0,1]} I_{\mathrm{WB}}(B(1,z),B(z,y)) = \inf\limits_{z\in[0,1)} I_{\mathrm{WB}}(z,B(z,y))$ $\wedge I_{\mathrm{WB}}(1,y) = 1 \wedge y = y$ □

Theorem 3.8. *Let $I_{S,N}$ be an* (S, N)*-implication generated by a strict t-conorm S and let B be a semicopula. $I_{S,N}$ satisfies* (BK-GHS) *with B if and only if* $I_{S,N}(x,y) = I_{\mathrm{WB}}(x,y) = \begin{cases} 1 & x<1 \\ y & x=1 \end{cases}$.

Proof. Since $I_{S,N}$ is generated by a strict t-conorm S, it is rewritten as $I_{S,N}(x,y) = (S_{\mathrm{P}})_\varphi(N(x),\, y) = \varphi^{-1}(\varphi(N(x)) + \varphi(y) - \varphi(N(x))\varphi(y))$.

($\Longrightarrow$) Let $I_{S,N}$ satisfy (BK-GHS). By Lemma 3.1, $I_{S,N}$ fulfills (IP). Then we obtain $\varphi^{-1}(\varphi(N\ (x)) + \varphi(x) - \varphi(N(x))\varphi(x)) = 1$. This implies that $x=1$ or $N(x)=1$ holds for any $x \in [0,1)$. Therefore, $N = N_\top$. That is, $I_{S,N}(x,y) = I_{\mathrm{WB}}(x,y) = \begin{cases} 1 & x<1 \\ y & x=1 \end{cases}$.

($\Longleftarrow$) Obviously. □

In the rest of the section, we consider the (S, N)-implications generated by a right-continuous t-conorm S_{nM} and a t-conorm S_{D}, respectively.

Lemma 3.9. *Let $I_{S,N}$ be an* (S, N)*-implication generated by a right-continuous t-conorm S_{nM} and a continuous fuzzy negation N and let B be a semicopula.*

i. If $N(x) < 1-x$ for any $x \in (0,1)$, then $I_{S,N}$ does not satisfy (BK-GHS) *with any semicopula,*

ii. If $N(x) \geq 1-x$ for any $x \in [0,1]$, then $I_{S,N}$ satisfies (BK-GHS) *with $B = T_{\mathrm{M}}$.*

Proof. In this case, $I_{S,N}$ can be rewritten as $I_{S,N}(x,y) = \begin{cases} 1 & N(x)+y \geq 1 \\ \max(N(x),y) & N(x)+y<1 \end{cases}$.

i. Assume that $I_{S,N}$ satisfies (BK-GHS) with B. We can obtain $\inf\limits_{z\in[0,1]} I_{S,N}(B(1,z),B(z,1)) = \inf\limits_{z\in[0,1]} I_{S,N}(z,z) = I(1,1) = 1$. Since $N(x) < 1-x$, we have $I_{S,N}(z,z) = \max(N(z),z) < 1$. Thus, $\inf\limits_{z\in[0,1]} I_{S,N}(z,z) < 1$. This is a contradiction. This means that $I_{S,N}$ does not satisfy (BK-GHS) with any semicopula.

ii. Since $I_{S,N}$ can be rewritten as $I_{S,N}(x,y) = \begin{cases} 1 & N(x)+y \geq 1 \\ \max(N(x),y) & N(x)+y<1 \end{cases}$, we have $I_{S,N}(x,x) = 1$. This means that $I_{S,N}$ satisfies (IP). The conclusion can be immediately obtained by Theorem 3.3. □

Theorem 3.10. *Let $I_{S,N}$ be an* (S, N)*-implication generated by a right-continuous t-conorm S_{nM} and a continuous fuzzy negation N and let B be a semicopula. Then $I_{S,N}$ fulfills* (IP) *if and only if it satisfies* (BK-GHS) *with $B = T_{\mathrm{M}}$.*

Proof. ($\Longrightarrow$) Since $I_{S,N}$ satisfies (IP), $S(N(x),x)=1$ holds for all $x\in[0,1]$ by Lemma 2.4.16. According to Proposition 2.3.11 in [3], we have the fact that $I_{S,N}$ satisfies $S(N(x),x)=1$ if and only if $N(x)\geq N_{S_{\mathrm{nM}}}(x)=1-x$ for all $x\in[0,1]$. On the other hand, $I_{S,N}$ which is generated by a right-continuous t-conorm S_{nM} and $N(x)\geq 1-x$ satisfies (BK-GHS) with $B=T_{\mathrm{M}}$ for any $x\in[0,1]$ by Lemma 3.9. Therefore, $I_{S,N}$ satisfies (BK-GHS) with $B=T_{\mathrm{M}}$ if it fulfills (IP).

($\Longleftarrow$) Obviously, $I_{S,N}$ satisfies (IP) by Lemma 3.1. □

Lemma 3.11. *Let $I_{S,N}$ be an* (S, N)*-implication generated by S_{D} and let B be a semicopula without zero divisor. Then $I_{S,N}$ satisfies* (BK-GHS) *with B.*

Proof. Let us consider the following four options.

i. If $x=1, y>0$. This case implies that $\inf\limits_{z\in[0,1]} I_{S,N}(B(1,z),B(z,y)) = \inf\limits_{z\in[0,1]} I_{S,N}(z,B(z,y)) = I_{S,N}(0,0)\wedge \inf\limits_{z\in(0,1)} I_{S,N}(z,B(z,y))\wedge I_{S,N}(1,B(1,y)) = 1\wedge 1\wedge y = y$.

ii. If $x=1, y=0$. This case means that $\inf\limits_{z\in[0,1]} I_{S,N}(B(1,z),B(z,0)) = \inf\limits_{z\in[0,1]} I_{S,N}(z,0) = \inf\limits_{z\in[0,1]} N(z) = 0 = I_{S,N}(1,0)$.

iii. If $x<1, y>0$. In this case, we have $\inf\limits_{z\in[0,1]} I_{S,N}(B(x,z),B(z,y)) = I_{S,N}(0,0)\wedge \inf\limits_{z\in(0,1]} I_{S,N}(B(x,z),B(z,y)) = 1\wedge 1 = 1$.

iv. If $x<1, y=0$. In this case, we obtain $\inf\limits_{z\in[0,1]} I_{S,N}(B(x,z),B(z,0)) = \inf\limits_{z\in[0,1]} I_{S,N}(B(x,z),0)) = \inf\limits_{z\in[0,1]} N(B(x,z))$. Since N is decreasing, $N(B(x,z))\geq N(B(x,1)) = N(x)$ holds for any $z\in[0,1]$. This means that $\inf\limits_{z\in[0,1]} N(B(x,z)) = N(B(x,1)) = N(x) = I_{S,N}(x,y)$. □

4 QL-implications satisfy (BK-GHS)

In this section, we will search some semicopulas such that QL-implications satisfy (BK-GHS) with them. Let us firstly consider the QL-implications generated by some noncontinuous fuzzy negations.

Lemma 4.1. *Let I_{QL} be a* QL*-implication generated by the triple $(T,S,N_\top)$ and let B be a semicopula. Then I_{QL} satisfies* (BK-GHS) *with any semicopula.*

Proof. Let B be a semicopula. According to Remark 2.6.6 (iv) in [3], we know that QL-implication obtained from the triple $(T,S,N_\top)$ is the Weber implication I_{WB}. This means that I_{QL} satisfies (BK-GHS) with any semicopula by Theorem 3.7. □

Let I_{QL} be a fuzzy implication generated by a t-conorm S, a t-norm T and a strict negation N. Then $S(N(x),x)=1$ holds for all $x\in[0,1]$ by [3]. In the rest of this section, we only discuss the case where t-conorm S is continuous. Therefore, we have the following results, respectively, i.e.,

$$S(x,y)=(S_{\mathrm{LK}})_\varphi=\varphi^{-1}(\min(\varphi(x)+\varphi(y),1)),$$

$$N(x)\geq\varphi^{-1}(1-\varphi(x)), x\in[0,1].$$

Theorem 4.2. *Let $I_{QL}(x,y)=(S_{\mathrm{LK}})_\varphi(N(x),T_{\mathrm{M}}(x,y))$. If $N(x)\geq\varphi^{-1}(1-\varphi(x))$ for any $x\in[0,1]$, then I_{QL} satisfies* (BK-GHS) *with $(B_{ij})_\varphi$ $(i,j=1,2,3)$.*

Proof. In this case, I_{QL} can be rewritten as $I_{QL}=(I_{\mathrm{LK}})_\varphi$. This is concluded by Theorem 3.2 and Lemma 3.5. □

Next, we consider I_{QL} generated by a nilpotent t-conorm $(S_{\mathrm{LK}})_\varphi$, a nilpotent t-norm $(T_{\mathrm{LK}})_\psi$ and a strict fuzzy negation N, where φ and ψ are two strictly increasing bijections on [0,1], respectively. For each $y\in[0,1]$, we define $F_y(x)=\varphi(N(x))+\varphi((T_{\mathrm{LK}})_\psi(x,y)))$. According to Corollary 4.27 in [26], I_{QL} is a fuzzy implication if and only if $F_1(x)\geq 1$, i.e., $N(x)\geq\varphi^{-1}(1-\varphi(x))$ for all $x\in[0,1]$ and F_y is decreasing in [0,1] for all $y\in[0,1)$. Based on the above argument, we have the following result.

Theorem 4.3. *Let $I_{QL}(x,y)=(S_{\mathrm{LK}})_\varphi(N(x),(T_{\mathrm{LK}})_\psi(x,y))$. Then I_{QL} does not satisfy* (BK-GHS) *with any semicopula.*

Proof. Let B be a semicoupla. Assume that I_{QL} satisfies (BK-GHS) with B. Taking $x = y = 1$, we have $\inf_{z\in[0,1]} I_{QL}(B(1, z), B(z, 1)) = I(1, 1) = 1$. Since ψ is a strictly increasing bijection on [0,1], there exists $z_0 \in (0, 1)$ such that $2\psi(z_0) = 1$, This implies that $\inf_{z\in[0,1]} I(z, z) = 1 \wedge \inf_{z\in(0,z_0]} N(z) \wedge \inf_{z\in(z_0,1]} \varphi^{-1}(\min(\varphi(N(z)) + \varphi \circ \psi^{-1}(2\psi(z) - 1)), 1)$. Since N is strict, we have $N(x) < 1$ holds for any $x \in (0, 1]$. Therefore, $\inf_{z\in(0,z_0]} N(z) < 1$ holds. This means that $1 \wedge \inf_{z\in(0,z_0]} N(z) \wedge \inf_{z\in(z_0,1]} \varphi^{-1}(\min(\varphi(N(z)) + \varphi \circ \psi^{-1}(2\psi(z) - 1)), 1) < 1$. This is a contradiction. □

Then we consider I_{QL} generated by a nilpotent t-conorm $(S_{\mathrm{LK}})_\varphi$, a strict t-norm $(T_{\mathrm{P}})_\psi$ and a strict fuzzy negation N, where φ and ψ are two strictly increasing bijections on [0,1]. According to Theorem 4.8 in [26], I_{QL} is a fuzzy implication if $f(x) = \psi^{-1}(\frac{\psi\circ\varphi^{-1}(1-\varphi(N(x)))}{\psi(x)})$ is increasing on $(0, 1]$. Therefore, we have the following result.

Theorem 4.4. *Let $I_{QL}(x, y) = (S_{\mathrm{LK}})_\varphi(N(x), (T_{\mathrm{P}})_\psi(x, y))$. If $N(x) < \varphi^{-1}(1 - \varphi \circ \psi^{-1}(\psi^2(x)))$, I_{QL} does not satisfy* (BK-GHS) *with any semicopula.*

Proof. I_{QL} can be rewritten as $I_{QL}(x, y) = \varphi^{-1}(\min(\varphi(N(x)) + \varphi \circ \psi^{-1}(\psi(x)\psi(y))), 1)$. Let B be a semicopula. Assume that I_{QL} satisfies (BK-GHS) with B. Taking $x = y = 1$, we have $\inf_{z\in[0,1]} I(B(1, z), B(z, 1)) = I(1, 1) = 1$. On the other hand, $\inf_{z\in[0,1]} I(B(1, z), B(z, 1)) = \inf_{z\in[0,1]} I(z, z) = \inf_{z\in[0,1]} \varphi^{-1}(\min(\varphi(N(z)) + \varphi \circ \psi^{-1}(\psi(z)\psi(z))), 1)$ holds. Since $N(x) < \varphi^{-1}(1-\varphi\circ\psi^{-1}(\psi^2(x)))$, we obtain $\varphi(N(z))+\varphi\circ\psi^{-1}(\psi(z)\psi(z)) < 1$. Thus, $\inf_{z\in[0,1]} \varphi^{-1}(\min(\varphi(N(z))+\varphi\circ\psi^{-1}(\psi(z)\psi(z))), 1) < 1$. This is a contradiction. □

Remark 4.5. *For the case of $N(x) \geq \varphi^{-1}(1 - \varphi \circ \psi^{-1}(\psi^2(x)))$, it is still difficult to verify whether I_{QL} satisfies* (BK-GHS) *with B. Hence, we only consider the following special case.*

Lemma 4.6. *Let $\varphi(x) = x^{n_1}, \psi(y) = y^{n_2}, n_1, n_2 \in \mathbb{N}$. If there exists $r(\in \mathbb{N}) \geq 2$ such that $N(x) = \varphi^{-1}(1 - \varphi \circ \psi^{-1}(\psi^r(x)))$, then I_{QL} satisfies* (BK-GHS) *with $B = T_{\mathrm{M}}$.*

Proof. In this case, I_{QL} can be rewritten as $I_{QL}(x, y) = \varphi^{-1}(\min(1 - x^{rn_1} + (xy)^{n_1}, 1))$. Let us discuss the following three options.

i. If $x \leq y$, $x^r \leq x^2 \leq xy$. In this case, $x^{rn_1} \wedge z^{rn_1} \leq (x \wedge z)^{n_1} \cdot (z \wedge y)^{n_1}$ holds for any $z \in [0, 1]$. Therefore, $\inf_{z\in[0,1]} I_{QL}(B(x, z), B(z, y)) = 1$.

ii. If $x > y, x^r \leq xy \leq x^2$. This case implies that $\inf_{z\in[0,1]} I_{QL}(B(x, z), B(z, y)) = \inf_{z\in[0,y]} I_{QL}(z, z) \wedge \inf_{z\in(y,x]} I_{QL}(z, y) \wedge \inf_{z\in(x,1]} I_{QL}(x, y) = 1 \wedge 1 \wedge 1 = 1$.

iii. If $x > y, x^2 > x^r > xy$. In this case, we have $\inf_{z\in[0,1]} I_{QL}(B(x, z), B(z, y)) = \inf_{z\in[0,y]} I_{QL}(z, z) \wedge \inf_{z\in(y,x]} I_{QL}(z, y) \wedge \inf_{z\in(x,1]} I_{QL}(x, y)$. For any fixed $r \geq 2$ and $y \in [0, 1]$, we define a function f on [0,1] as follows: $f(z) = 1 - z^{rn_1} + (zy)^{n_1}$. It is not difficult to see that $z_0 = (r^{-1} \cdot y^{n_1})^{((r-1)n_1)^{-1}}$ is the maximum value point on [0,1] by Extreme Value Theorem. Since $x^{rn_1} > (xy)^{n_1}$, we obtain $x > (y^{n_1})^{(r-1)n_1{}^{-1}}$. Thus, $z_0 < r^{-(r-1)n_1^{-1}} \cdot x < x$. Moreover, we have $(r^{-1}y^{n_1})^{((r-1)n_1)^{-1}} \leq (0.5 \cdot y^{n_1})^{((r-1)n_1)^{-1}} \leq y^{n_1} \leq y$. Therefore, $z_0 \leq y < x$ holds. Since $f(y) = 1 - y^{rn_1} + y^{2n_1} \geq 1$ and $f(x) = 1-x^{rn_1}+x^{n_1}y^{n_1} < 1$, $z = x$ is the minimum value point on $(y, x]$. Hence, $\inf_{z\in[0,y]} I_{QL}(z, z)\wedge \inf_{z\in(y,x]} I_{QL}(z, y)\wedge \inf_{z\in(x,1]} I_{QL}(x, y) = 1 \wedge \varphi^{-1}(1 - \varphi(x^r) + \varphi(xy)) \wedge \varphi^{-1}(1 - \varphi(x^r) + \varphi(xy)) = I_{QL}(x, y)$. □

5 f- and g-implications satisfy (BK-GHS)

In this section, we continue to study whether f- and g-implications satisfy (BK-GHS) with some semicopulas.

Theorem 5.1. *Let $I_g \in \mathbb{I}_{\mathbb{G},\infty}$. Then I_g does not satisfy* (BK-GHS) *with any semicopula.*

Proof. Let B be a semicopula. Suppose that I_g satisfies (BK-GHS) with B. Then we have $\inf_{z\in[0,1]} I_g(B(1, z), B(z, 1)) = \inf_{z\in[0,1]} I(z, z) = \inf_{z\in[0,1]} g^{-1}(\frac{g(z)}{z})$. Since g is strictly increasing and continuous, there exists a $z_1 \in (0, 1)$ such that

$\frac{g(z_1)}{z_1} < \infty$. Therefore, $\inf_{z\in[0,1]} g^{-1}(\frac{g(z)}{z}) < 1$. This implies that $\inf_{z\in[0,1]} I_g(B(1,z),\ B(z,1)) < 1 = I_g(1,1)$. This is a contradiction. Thus, there is no semicopula such that I_g satisfies (BK-GHS) with it. □

Lemma 5.2. *Let $g(x) = cx^\lambda$ for any $\lambda \in (0,1]$ and $c \in (0,\infty)$. Then I_g satisfies* (BK-GHS) *with $B = T_{\mathrm{M}}$ or T_{P}.*

Proof. In this case, $I_g = g^{-1}(\min(\frac{g(y)}{x}, g(1))) = g^{-1}(\min(\frac{cy^\lambda}{x}, g(1))) = \min(x^{-\lambda^{-1}}y, 1)$. We only prove that I_g satisfies (BK-GHS) with $B = T_{\mathrm{P}}$. The other case can be similarly proved. Let us discuss the following two options.

i. If $x^{-\lambda^{-1}}y \geq 1$. This means that $\inf_{z\in[0,1]} I_g(T_{\mathrm{P}}(x,z), T_{\mathrm{P}}(z,y)) = 1$

ii. If $x^{-\lambda^{-1}}y < 1$. Assume that $(xz_1)^{-\lambda^{-1}}(z_1y) = 1$. We can obtain $z_1 = (x^{-\lambda^{-1}}y)^{\frac{\lambda}{1-\lambda}}$. Since $\frac{\lambda}{1-\lambda} \in (0,\infty)$, we have $z_1 \in [0,1)$. This implies that $\inf_{z\in[0,1]} I_g(T_{\mathrm{P}}(x,z), T_{\mathrm{P}}(z,y)) = 1 \wedge \inf_{z\in(z_1,1]} (xz)^{-\lambda^{-1}}(zy) = x^{-\lambda^{-1}}y$. □

Theorem 5.3. *Let $I_g \in \mathbb{I}_{\mathbb{G},\aleph}$, $x \leq \frac{g(x)}{g(1)}$ for any $x \in [0,1]$ and let B be a semicopula. Then I_g satisfies* (BK-GHS) *with $B = T_{\mathrm{M}}$ or T_{P}.*

Proof. In this case, $I_g(x,y) = g^{-1}(\min(\frac{g(y)}{x}, g(1))) = \begin{cases} 1 & \frac{g(y)}{x} \geq g(1) \\ g^{-1}(\frac{g(y)}{x}) & \frac{g(y)}{x} < g(1) \end{cases}$. We only prove that I_g satisfies (BK-GHS) with $B = T_{\mathrm{M}}$. The other case can be similarly proved. Let us consider the following three options.

i. If $x \leq y$. Since g is strictly increasing and continuous, we have $\frac{g(y)}{x} \geq \frac{g(x)}{x} \geq g(1)$. Therefore, $\inf_{z\in[0,1]} I_g(B(x,z), B(z, y)) = \inf_{z\in[0,x]} I_g(z,z) \wedge \inf_{z\in(x,y]} I_g(x,z) \wedge \inf_{z\in(y,1]} I_g(x,y)$. This implies that $\inf_{z\in[0,x]} I_g(z,z) \wedge \inf_{z\in(x,y]} I_g(x,z) \wedge \inf_{z\in(y,1]} I_g(x,y) = 1 \wedge 1 \wedge 1 = 1$.

ii. If $x > y$ and $g(1) \leq \frac{g(y)}{x}$. This case means that $\inf_{z\in[0,1]} I_g(T_{\mathrm{M}}(x,z), T_{\mathrm{M}}(z,y)) = \inf_{z\in[0,y]} I_g(z,\ z) \wedge \inf_{z\in(y,x]} I_g(z,y) \wedge \inf_{z\in(x,1]} I_g(x,y)$. Since g is strictly increasing and continuous, $\frac{g(y)}{z} \geq \frac{g(y)}{x} \geq g(1)$ holds for any $z \in (y,x]$. Thus, $\inf_{z\in[0,y]} I_g(z,z) \wedge \inf_{z\in(y,x]} I_g(z,\ y) \wedge \inf_{z\in(x,1]} I_g(x,y) = 1 \wedge 1 \wedge 1 = 1$.

iii. If $x > y$ and $g(1) > \frac{g(y)}{x}$. We further consider two cases.

iii-1. If $g(1) = \frac{g(x)}{x}$. This case means that $\inf_{z\in[0,1]} I_g(T_{\mathrm{M}}(x,z), T_{\mathrm{M}}(z,y)) = \inf_{z\in[0,y]} I_g(z,z) \wedge \inf_{z\in(y,x]} I_g(z,y) \wedge \inf_{z\in(x,1]} I_g(x,y)$. Since g is strictly increasing and continuous, $\frac{g(y)}{x} \leq \frac{g(y)}{z} < \frac{g(y)}{y} = g(1)$ holds for any $z \in (y,x]$. Therefore, $\inf_{z\in[0,y]} I_g(z,z) \wedge \inf_{z\in(y,x]} I_g(z,y) \wedge \inf_{z\in(x,1]} I_g(x,y) = 1 \wedge \inf_{z\in(y,x]} g^{-1}(\frac{g(y)}{x}) \wedge \inf_{z\in(x,1]} g^{-1}(\frac{g(y)}{x}) = g^{-1}(\frac{g(y)}{x})$.

iii-2. If $g(1) < \frac{g(x)}{x}$. This implies that $\inf_{z\in[0,1]} I_g(T_{\mathrm{M}}(x,z), T_{\mathrm{M}}(z,y)) = \inf_{z\in[0,y]} I_g(z,z) \wedge \inf_{z\in(y,x]} I_g(z,y) \wedge \inf_{z\in(x,1]} I_g(x,y)$. For any fixed $0 < y < 1$, we define a function F as $F(z) = g^{-1}(\frac{g(y)}{z})$. Obviously, F is decreasing and continuous on [0,1]. It is not difficult to see that $F(y) = g^{-1}(\frac{g(y)}{y}) > 1$ and $F(x) = g^{-1}(\frac{g(y)}{x}) < 1$ hold. By Intermediate Value Theorem, there exists a $z_0 \in (y,x)$ such that $g^{-1}(\frac{g(y)}{g(z_0)}) = 1$. Therefore, $\inf_{z\in[0,y]} I_g(z,z) \wedge \inf_{z\in(y,z_0]} I_g(z,y) \wedge \inf_{z\in(z_0,x]} I_g(z,y) \wedge \inf_{z\in(x,1]} I_g(x,y) = 1 \wedge 1 \wedge \inf_{z\in(z_0,x]} g^{-1}(\frac{g(y)}{x}) \wedge \inf_{z\in(x,1]} g^{-1}(\frac{g(y)}{x}) = g^{-1}(\frac{g(y)}{x})$. □

Theorem 5.4. *Let $I_g \in \mathbb{I}_{\mathbb{G},\aleph}$ and $x \leq \frac{g(x)}{g(1)}$ for any $x \in [0,1]$. Then I_g fulfills* (IP) *if and only if it satisfies* (BK-GHS) *with $B = T_{\mathrm{M}}$ or T_{P}.*

Proof. Follows from Theorem 3.2.8 in [3], Theorem 5.3 and Lemma 3.1. □

Theorem 5.5. *Let $I_f \in \mathbb{I}_{\mathbb{F},\aleph}$. Then I_f does not satisfy* (BK-GHS) *with any semicopula.*

Proof. Let B be a semicopula. Assume that I_f satisfy (BK-GHS) with B. Taking $x = y = 1$, we have $\inf_{z\in[0,1]} I_f(z,z) = \inf_{z\in[0,1]} f^{-1}(z \cdot f(z))$. Since f is decreasing, $\inf_{z\in[0,1]} f^{-1}(z \cdot f(z)) = \sup_{z\in[0,1]} z \cdot f(z)$ holds. On the other hand, the continuity of f implies that there exists a $z_1 \in (0,1)$ such that $z_1 \cdot f(z_1) > 0$. Thus, we obtain $\inf_{z\in[0,1]} I_f(z,z) < 1 = I(1,1)$. This is a contradiction. This means that there is no I_f satisfying (BK-GHS) with a semicopula. □

Proposition 4.4.1 in [3] shows $\mathbb{I}_{\mathbb{F},\infty} = \mathbb{I}_{\mathbb{G},\infty}$. The following result can be easily obtained.

Lemma 5.6. *For $I_f \in \mathbb{I}_{\mathbb{F},\infty}$, I_f does not satisfy* (BK-GHS) *with any semicopula.*

6 Probabilistic and probabilistic S-implications satisfy (BK-GHS)

This section aims to study whether probabilistic and probabilistic S-implications satisfy (BK-GHS) with some semicopulas.

Theorem 6.1. *Let I_C be a probabilistic implication. Then I_C satisfies* (BK-GHS) *with $B = \lambda T_{\mathrm{P}} + (1-\lambda)T_{\mathrm{M}}$ if and only if C is idempotent, where $\lambda \in [0,1]$.*

Proof. ($\Longrightarrow$) Let I_C satisfy (BK-GHS). By Lemma 3.1, I_C fulfills (IP). Thus, we have $I_C(x,x) = \begin{cases} 1 & x = 0 \\ \frac{C(x,x)}{x} & x > 0 \end{cases} = 1$ for any $x \in [0,1]$. This implies that $C(x,x) = x$ holds for any $x \in [0,1]$, i.e., C is idempotent.

($\Longleftarrow$) According to Example 9.5 (ii) in [15], C is idempotent if and only if $C = T_{\mathrm{M}}$. Therefore, $I_C(x,y) = \begin{cases} 1 & x = 0 \\ \frac{\min(x,y)}{x} & x > 0 \end{cases} = \begin{cases} 1 & x \le y \\ \frac{y}{x} & x > y \end{cases}$. Let us consider the following two options.

i. If $x \le y$. This case means that $\lambda xz+(1-\lambda)\min(x,z) \le \lambda zy+(1-\lambda)\min(z,y)$. Hence, $\inf\limits_{z\in[0,1]} I_C(B(x,z),B(z,y)) = 1 = I_C(x,y)$.

ii. If $x > y$. In this case, we have $\lambda xz + (1-\lambda)\min(x,z) \ge \lambda zy + (1-\lambda)\min(z,y)$. Obviously, we have $z = 0$ if $\lambda xz + (1-\lambda)\min(x,z) = \lambda zy + (1-\lambda)\min(z,y)$ for any $x, y \in [0,1]$. Therefore, $\inf\limits_{z\in[0,1]} I_C(B(x,z),B(z,y)) = I_C(0,0) \wedge \inf\limits_{z\in(0,y]} I_C(\lambda xz+(1-\lambda)z, \lambda zy+(1-\lambda)z) \wedge \inf\limits_{z\in(y,x]} I_C(\lambda xz+(1-\lambda)z, \lambda zy+(1-\lambda)y) \wedge \inf\limits_{z\in(x,1]} I_C(\lambda xz+(1-\lambda)x, \lambda zy+(1-\lambda)y) = 1 \wedge \inf\limits_{z\in(0,y]} \frac{\lambda zy+(1-\lambda)z}{\lambda xz+(1-\lambda)z} \wedge \inf\limits_{z\in(y,x]} \frac{\lambda zy+(1-\lambda)y}{\lambda xz+(1-\lambda)z} \wedge \inf\limits_{z\in(x,1]} \frac{\lambda zy+(1-\lambda)y}{\lambda xz+(1-\lambda)x}$.

Since $\frac{\lambda zy+(1-\lambda)z}{\lambda xz+(1-\lambda)z} = \frac{z(\lambda y+(1-\lambda))}{z(\lambda x+(1-\lambda))} = \frac{\lambda y+(1-\lambda)}{\lambda x+(1-\lambda)}$, $\frac{\lambda zy+(1-\lambda)y}{\lambda xz+(1-\lambda)z} - \frac{y}{x} > 0$ and $\frac{\lambda zy+(1-\lambda)y}{\lambda xz+(1-\lambda)x} = (\lambda + \frac{1-\lambda}{z}) \cdot \frac{y}{\lambda x+(1-\lambda)}$ hold, we have $\inf\limits_{z\in[0,1]} I_C(B(x,z),B(z,y)) = 1 \wedge \frac{\lambda y+(1-\lambda)}{\lambda x+(1-\lambda)} \wedge \frac{y}{x} \wedge \frac{y}{x} = \frac{y}{x}$. □

Theorem 6.2. *Let $\tilde{I}_C$ be a probabilistic S-implication. Then $\tilde{I}_C$ satisfies* (BK-GHS) *with $B = B_{ij}$ $(i,j = 1,2,3)$ if and only if C is idempotent.*

Proof. ($\Longrightarrow$) Let $\tilde{I}_C$ satisfy (BK-GHS). By Lemma 3.1, $\tilde{I}_C$ fulfills (IP). Therefore, We have $\tilde{I}_C(x,x) = C(x,x)-x+1 = 1$. This implies that $C(x,x) = x$ holds for any $x \in [0,1]$, viz., C is idempotent.

($\Longleftarrow$) Let C be idempotent. According to Example 9.5 (ii) in [15], C is idempotent for each $x \in [0,1]$ if and only if $C = T_{\mathrm{M}}$. Therefore, $\tilde{I}_C(x,y) = \min(x,y) - x + 1 = \min(1-x+y,1)$. This means that $\tilde{I}_C$ satisfies (BK-GHS) with B if and only if $(\tilde{I}_C)_\varphi$ satisfies (BK-GHS) with B_φ by Theorem 3.2. On the other hand, $I(x,y) = \varphi^{-1}(\min(1-\varphi(x)+\varphi(y),1))$ satisfies (BK-GHS) with $(B_{ij})_\varphi$ $(i,j = 1,2,3)$ by Lemma 3.5 ii. Therefore, $\tilde{I}_C$ satisfies (BK-GHS) with B_{ij}. □

7 Conclusions

In order to make better use of fuzzy implications and semicopulas in approximate reasoning and decision-making, we have mainly discussed the satisfaction of (BK-GHS) for some well-known fuzzy implications with some semicopulas and obtained the following results:

(1) Revealing that the continuous (S, N)-implication satisfies (BK-GHS) if and only if it fulfills (IP). Further, some semicopulas are found for some noncontinuous (S, N)-implications such that they satisfy (BK-GHS).

(2) Investigating whether QL-implications generated by $(S_{\mathrm{LK}})_\varphi$ and a strict negation N such that $N \ge (N_C)_\varphi$ satisfy (BK-GHS) with some semicopulas.

(3) Presenting that I_f and $I_g \in \mathbb{I}_{\mathbb{G},\infty}$ do not satisfy (BK-GHS) with any semicopula while $I_g \in \mathbb{I}_{\mathbb{G},\aleph}$ satisfies (BK-GHS).

(4) Showing that probabilistic and probabilistic S-implications satisfy (BK-GHS) if and only if C is idempotent.

The above results help to measure the effectiveness of approximate reasoning. However, we need to study the following questions in the future.

(1) Do there exist some semicopulas for noncontinuous (S, N)- and QL-implications such that they satisfy (BK-GHS)?

(2) What happen to the other fuzzy implications such as T-power based implications and (U, N)-implications?

(3) Do there exist some fuzzy implicatons satisfying $\inf\limits_{z\in[0,1]} I_1(B(x,z),B(z,y)) = I_2(x,y)$ with some semicopulas?

Declarations

The authors declare that they have no conflict of interest. All authors contributed equally.

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