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Linear optimization problem subjected to fuzzy relational equations and fuzzy constraints

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Abstract

In this paper, we introduce a new optimization problem with respect to a generalized form of fuzzy relational equations (FRE) in which fuzzy equality replaces ordinary equality in the constraints (FRE-FC). Fuzzy constraints enable us to attain optimal points (called super-optima in this paper) that are better solutions than those resulted from the resolution of the similar problems with ordinary equality constraints. Some structural properties of the FRE-FC problems are studied and a new formulation is presented in which the fuzzy constraints (equations) are precisely modeled. Subsequently, a new PSO-based algorithm is proposed to solve the FRE-FC problems defined by arbitrary continuous t-norms. The proposed algorithm is tested with different test problems generated by ten well-known continuous t-norms used in the literature. Moreover, the generated solutions for these problems, are also compared with some well-known meta-heuristic methods which have been applied to many practical optimization problems. It is shown that the optimal intensity of electromagnetic radiation problem can be formed as a special case of FRE-FC problems in which fuzzy composition is defied by max-product composition.

Keywords: Fuzzy relational equations, fuzzy constraints, continuous t-norms, linear optimization, PSO algorithm.

1 Introduction

The theory of fuzzy relational equations (FRE) as a generalized version of Boolean relation equations was firstly proposed by Sanchez and was applied to problems related to the medical diagnosis [37]. Nowadays, it is well known that many of the issues associated to the body knowledge can be treated as FRE problems [34]. In addition to the medical applications, FRE can also be applied to many other fields such as fuzzy control, discrete dynamic systems, prediction of fuzzy systems, fuzzy decision making, fuzzy pattern recognition, fuzzy clustering, image compression and reconstruction, and so on. Generally, when rules of inference are applied and their corresponding consequences are known, the problem of determining antecedents is simplified and mathematically reduced to solving an FRE [33].

The solvability identification and finding set of solutions are the primary, and the most fundamental, matter concerning the FRE problems. Di Nola et al. proved that the solution set of FRE (if it is nonempty), defined by continuous max-t-norm composition is often a non-convex set. This non-convex set is completely determined by one maximum solution and a finite number of minimal solutions [6]. Such non-convexity property is one of two bottlenecks making a major contribution towards an increase in complexity of FRE-related problems, particularly, in the optimization problems subjected to a system of fuzzy relations. Another bottleneck point is concerned with detecting the minimal solutions for FREs. Chen and Wang [4] presented an algorithm for obtaining the logical representation of all minimal solutions and deduced that a polynomial-time algorithm with the ability to find all minimal solutions of FRE (with max-min composition) may not exist. Also, Markovskii showed that solving max-product FRE is closely related to the covering problem which is a type of NP-hard problem [32]. In fact, the same result holds true for a more general t-norms instead of the minimum and product operators [27]. Over the past decades, the solvability of FRE which

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is defined using different max-t compositions have been investigated by many researchers [14, 15, 36, 40, 41, 44, 50]. Moreover, some other researchers have worked on introducing novel concept and at times improving some of the existing theoretical aspects and applications of fuzzy relational inequalities (FRI) [12, 16, 17, 18, 21, 25, 49]. Li and Yang [25] studied FRI with addition-min composition and presented an algorithm to search for minimal solutions. They applied FRI to data transmission mechanism in a BitTorrent-like Peer-to-Peer file sharing systems. Ghodousian and Khorram [16] focused on the algebraic structure of two fuzzy relational inequalities $A\varphi x \leq b^1$and $D\varphi x \geq b^2$. Their research focuses on the study of a mixed fuzzy system formed by two of the earlier FRIs, where φ is an operator with (closed) convex solutions. Generally, if φ is an operator with closed convex solutions, the set of solutions for $D\varphi x \geq b^2$ is determined by a finite number of maximal solutions as well as the same number for minimal ones. In particular, if φ is a continuous non-decreasing function (specially, a continuous t-norm), all maximal solutions overlap each other [16]. Guo et al. [21] investigated a type of FRI problems and the relationship between minimal solutions and FRI paths. They also introduced some rules for reducing the problems and presented an algorithm for solving optimization problems using FRI constraints.

The optimization problem subject to FRE and FRI is one of the most interesting and on-going research topics amongst similar problems [3, 12, 14, 15, 16, 17, 18, 19, 20, 21, 23, 26, 30, 38, 46, 49]. Fang and Li converted a linear optimization problem subjected to FRE constraints with max-min operation into an integer programming problem and solved it by a branch-and-bound method using jump-tracking technique [9]. Wu et al. worked on improvement of the method employed by Fang and Li; this was done by decreasing the search domain and presented a simplification process by three rules which resulted from a necessary condition [43]. Chang and Shieh presented new theoretical results concerning the linear optimization problem constrained by fuzzy max–min relation equations [3]. They improved an upper bound on the optimal objective value, some rules for simplifying the problem and proposed a rule for reducing the solution tree. Linear optimization problem was further investigated by numerous scholars focusing on max-product operation [23, 31]. Loetamonphong and Fang defined two sub-problems by separating negative from non-negative coefficients in the objective function, and then obtained an optimal solution by combining the optimal solutions of the two sub-problems [31]. The maximum solution of FRE is the optimum for the sub-problem having negative coefficients. Another sub-problem was converted into a binary programming problem and was solved using a branch-and-bound method. Also, in [23] some necessary conditions to test the feasibility and simplification techniques were presented in order to solve FRE with max-product composition. Moreover, generalizations of the linear optimization problem with respect to FRE have been studied; this was done through replacement of max-min and max-product compositions with different fuzzy compositions such as max-average composition [46] or max-t-norm composition [12, 14, 18, 20, 26, 38]. For example, Li and Fang solved the linear optimization problem subjected to a system of sup-t equations by reducing it to a 0-1 integer optimization problem [26]. In [20], a method was presented for solving linear optimization problems with the max-Archimedean t-norm fuzzy relation equation constraint. In [38], the authors solved the same problem whit continuous Archimedean t-norm, and to obtain some optimal variables, they used the covering problem rather than the branch-and-bound methods.

Recently, many interesting forms of generalizations of the linear programming applied to the system of fuzzy relations have been introduced, and developed based on composite operations used in FRE, fuzzy relations used in the definition of the constraints, some developments on the objective function of the problems and other ideas [5, 7, 10, 14, 15, 29, 30, 45]. For example, Wu et al. represented an efficient method to optimize a linear fractional programming problem under FRE with max-Archimedean t-norm composition [45]. Dempe and Ruziyeva generalized the fuzzy linear optimization problem by considering fuzzy coefficients [5]. In addition, Dubey et al. studied linear programming problems involving interval uncertainty modeled using intuitionistic fuzzy set [7]. The linear optimization of bipolar FRE was also the focus of study carried out by some researchers where FRE was defined by max-min composition [10] and max-Łukasiewicz composition [14, 29, 30]. For example, in [29], the authors introduced a linear optimization problem subjected to a system of bipolar FRE defined as $X(A^+, A^-, b) = \{x \in [0,1]^m : x \circ A^+ \vee \tilde{x} \circ A^- = b\}$ where $\tilde{x}_i = 1 - x_i$ for each component of $\tilde{x} = (\tilde{x}_i)_{1\times m}$ and the notations "$\vee$" and "$\circ$" denote max operation and the max-Łukasiewicz composition, respectively. They translated the original problem into a 0-1 integer linear problem which is then solved using well-developed techniques. In a separate, the foregoing bipolar linear optimization problem was solved by an analytical method based on the resolution and some structural properties of the feasible region (using a necessary condition for characterizing an optimal solution and a simplification process for reducing the problem) [30].

The optimization problem subjected to various versions of FRI is widely available in the literature as well [12, 16, 17, 18, 19, 21, 48, 49]. In [16], the authors studied the linear optimization with constraints formed by $X(A, D, b^1, b^2) = \{x \in [0,1]^n : A\varphi x \leq b^1, D\varphi x \geq b^2\}$ where φ represents an operator with convex solutions (e.g., non-decreasing or non-increasing operator). They showed that the feasible region can be expressed as the union of a finite number of convex sets. In particular, if φ is an operator with closed convex solutions such as continuous non-decreasing (non-increasing) operator, the preceding convex sets become closed as well. Therefore, since each t-norm is a non-decreasing function

(resulted directly from the property of the monotonicity in t-norms), continuous t-norms introduce important especial examples of operators with closed convex sets. For this reason, the authors proved that the feasible solutions set defined by a continuous t-norm can be formed as the union of a finite number of compact convex sets. Additionally, because of the identity law of t-norms, it was concluded that the feasible region actually consists of points being between one maximum solution and a finite number of minimal solutions. Yang studied the optimal solution of minimizing a linear objective function subject to a FRI where the constraints defined as $a_{i1} \wedge x_1 + a_{i2} \wedge x_2 + \cdots + a_{in} \wedge x_n \geq b_i$ for $i = 1, ..., m$ and $a \wedge b = \min\{a, b\}$ [48]. He presented an algorithm based on some properties of the minimal solutions of the FRI. Also, in [49], the authors introduced the latticized linear programming problem subject to max-product fuzzy relation inequalities with application in the optimization management model for wireless communication emission base stations. The latticized linear programming problem was defined by minimizing the objective function $z(x) = x_1 \vee x_2 \vee \cdots \vee x_n$ subject to the feasible region $X(A, b) = \{x \in [0, 1]^n \ : \ A \circ x \geq b\}$ where "$\circ$" denotes fuzzy max-product composition. They also presented an algorithm based on the resolution of the feasible region.

In this paper, we investigate FRE-FC problems formulated in the problem below:

$$\begin{array}{l} \min \quad c^T x \\ A\varphi x \approx b \\ x \in [0, 1]^n \end{array} \tag{1}$$

where $A = (a_{ij})_{m \times n}$ is a fuzzy matrix and $b = (b_i)_{m \times 1}$ is a fuzzy vector such that $0 \leq a_{ij} \leq 1$ and $0 \leq b_i \leq 1$ for each $i \in I = \{1, 2, ..., m\}$ and each $j \in J = \{1, 2, ..., n\}$. The constant vector $c = (c_j)_{n \times 1}$ is in R^n and $A\varphi x \approx b$ denotes fuzzy max-φ composition where $\varphi : [0, 1]^2 \to [0, 1]$ is an arbitrary continuous t-norm and "$\approx$" denotes relaxed or fuzzy version of the ordinary equality "$=$". Additionally, if a_i $(i \in I)$ denotes the i'th row of matrix A, then Problem (1) can be expressed as follows:

$$\begin{array}{l} \min \quad c^T x \\ a_i \varphi x \approx b_i \ , \ \ i \in I \\ x \in [0, 1]^n \end{array} \tag{2}$$

where the constraints mean $a_i \varphi x = \max_{j=1}^{n} \{\varphi(a_{ij}, x_j)\} \approx b_i, \forall i \in I$.

Moreover, Problem (1) can be also interpreted as a generalization of the following problem with the ordinary equalities:

$$\begin{array}{l} \min \quad c^T x \\ A\varphi x = b \\ x \in [0, 1]^n \end{array} \tag{3}$$

In the optimization problems, the algorithms attempt to find the best solution (selection) between the feasible solutions (selections satisfying the restrictions). But, in each constrained optimization problem, usually there exist some infeasible points whose objective function values are better than the objective values of optimal feasible solutions. Justifiably, if there exists a point with both a satisfactory objective value and negligible or admissible infeasibility (based on decision maker's view), then it may be a better selection for decision makers instead of the feasible ones with worse objective values. Figure 1 shows a typical case for such these selections where $f(x, y) = x(x - 2)^2 + y(y - 2)^2$ is considered as an objective function with a local minimum at $x^* = (2, 2)$. Figure 1(a) depicts the contours of the function in which the feasible region has been determined with a dotted line. As shown in the figure, point $(2, 2)$ is the best feasible solution, but the objective value is rapidly improved even for infeasible points being near the feasible region. The consideration of such infeasible points as admissible selections for decision makers can be also viewed as perturbing some constraint(s) such that the feasible solutions set becomes greater and includes new points with likely better objective function values. In practice, the applied aspects of the problems sometimes allow some restrictions or limitations to be perturbed while preserving the purposes of the decision makers.

Unlike most optimization algorithms using only the feasible solutions as the search domain, the FRE-FC problem benefits from infeasible points as well as the feasible ones. For this reason, the relaxed or perturbed equalities (denoted by "$\approx$") replace the ordinary ones (denoted by "$=$") in the constraints of Problem (1). Obviously, it is not reasonable to consider all the infeasible points with the same evaluation. For example, an infeasible point (however, with a good objective value) may be so far from the feasible region. Consequently, such a point cannot be reasonably considered as a solution for the problem. Thus, to evaluate the feasibility (or infeasibility) magnitude of the points, the feasibility property is defined as a fuzzy set so that closer points to the feasible set attain larger membership values.

Fig. 1 (b) illustrates the notion of the fuzzy set "feasibility" by using grayscale colors varying from white to black in which the brighter parts include the points with larger membership values for "feasibility" (the solid line determines the boundary of the feasible region). Similarly, the quality level or satisfaction amount of the objective value is defined as a fuzzy set. To define the fuzzy set "satisfaction", we initially determine a lower bound z_{best} and an upper bound

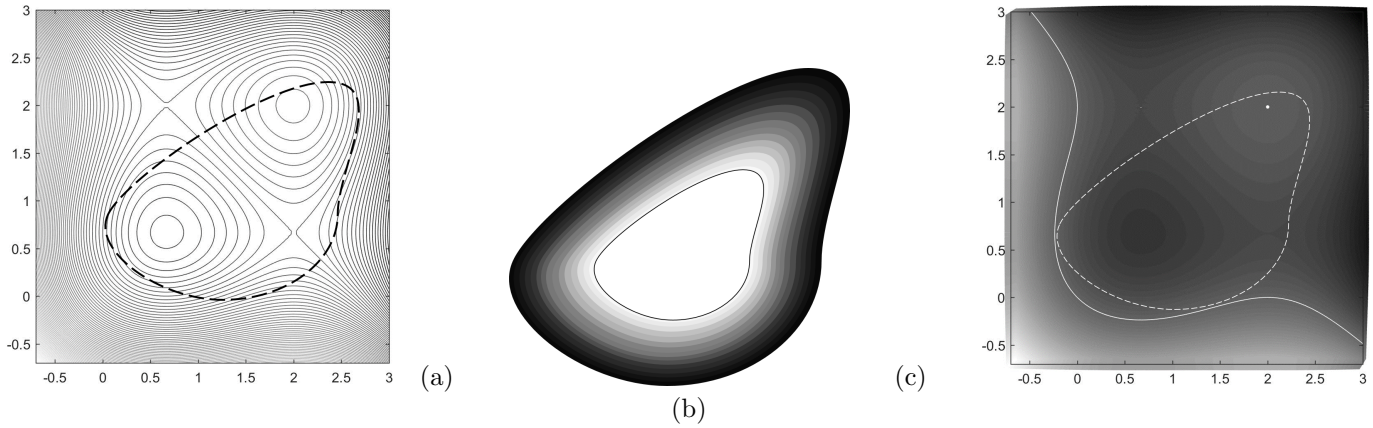


Figure 1: (a) Contours and feasible solutions set (determined by a dotted line). (b) Fuzzy evaluation of feasibility. (c) Fuzzy evaluation of objective values. Dotted line shows the boundary of the feasible region and the white solid line includes all points x such that $f(x) = f(x^*) = 0$.

z_{worst} on the optimal objective value of Problem (3), say z^* (i.e., $z_{best} < z^* < z_{worst}$). Also, we consider value z_{best} (z_{worst}) as the best (worst) case and set its degree of membership (equivalently, its satisfaction amount) equal to one (zero). Actually, the best case z_{best} is an approximate value we wish to be achieved at some near-feasible point (since for each feasible solution x of Problem (3), we have $z_{best} < z^* \leq z = c^T x$). Values z_{best} and z_{worst} may be determined from some previous experiments or by the human experts. Anyway, the satisfaction amount for each objective value $z = c^T x$ (especially, for z^*) is realized by considering the difference between z_{best} and z (see Fig. 1 (c)). Ultimately, total values are obtained by evaluating the feasibility amount of the points as well as the quality level of their objective values and a point with the highest total value is introduced as an optimal solution. Briefly, the target of the FRE-FC problem is to find an infeasible point with the acceptable infeasibility and better objective value than z^* by taking advantage of the flexibility of constraint(s). Following the terminology used in [13, 19], we refer to such these solutions as Super-Optima in this paper.

In [19], the authors studied problem $\min \{ c^T x \ : \ A\varphi x \preceq \mathrm{b} \ , \ \mathrm{x} \in [0,1]^n \}$ with max-min composition. They presented an analytical constructive algorithm to solve the problem and used this fuzzy system to convincingly optimize the quality of education in school education (with minimum cost) while were to be selected by parents. This problem was developed in [13] where the authors presented a modified PSO algorithm for solving a wide range of the fuzzy inequalities $A\varphi x \preceq \mathrm{b}$ defined with arbitrary continuous t-norms. Figure 2 typically shows the structural differences between sets $\{ x \in [0,1]^n \ : \ A\varphi x \leq b \}$, $\{ x \in [0,1]^n \ : \ A\varphi x \preceq \mathrm{b} \}$ and the feasible regions for Problems (1) and (3), where $\hat{x}$ and $\breve{x}_j$ $(j = 1, 2, 3)$ denote maximum and minimal solutions, respectively, and $\mathbf{0}$ is the zero vector.

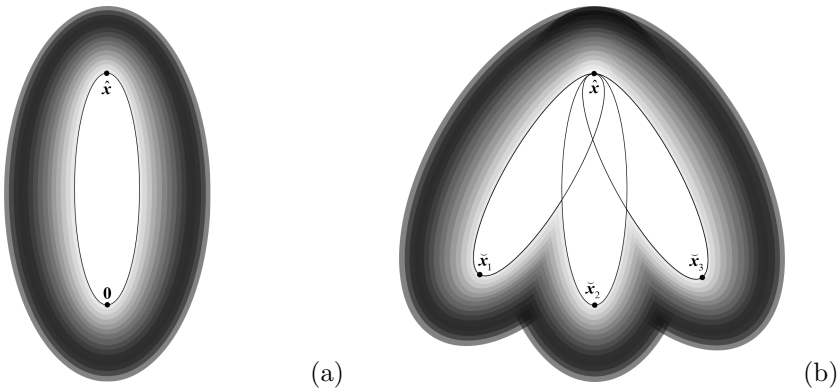


Figure 2: (a) Set $\{ x \in [0,1]^n \ : \ A\varphi x \preceq \mathrm{b} \}$; the solid line depicts the boundary of set $\{ x \in [0,1]^n \ : \ A\varphi x \leq b \}$. (b) Feasible region for Problem (1); the solid lines depict the boundary of the feasible region for Problem (3).

In this paper, we consider Problem (1) in a general case in which φ is an arbitrary continuous t-norm. For this purpose, the paper consists of three main parts. Firstly, some necessary structural details of Problem (3) are derived from its theoretical properties. These results are used throughout the paper. Then, a new formulation of Problem (1)

is introduced which leads to an equivalent problem with a nonlinear objective function (without using any penalty or barrier method) and simple constraints (some lower and upper bounds on the variables). Finally, a new modified PSO algorithm is proposed for tackling the equivalent transformed problem.

The rest of the paper is organized as follows. Section 2 takes a brief look at some basic results on the feasible solutions set of Problem (3). These results provide a proper background to design an efficient modified version of PSO for solving Problem (1). In Section 3, by employing some membership functions, the fuzzy constraints of Problem (1) are precisely defined. These membership functions are used for determining the infeasibility amount of points. Subsequently, Problem (1) is transformed into an equivalent problem in which the major constraints of the problem are placed into the objective function without considering any penalty parameters. A new modified PSO algorithm is described in Section 4. Finally, in Section 5, the experimental results and the application of FRE and FRE-FC problems in the optimal intensity of electromagnetic radiation problem are demonstrated. The proposed algorithm is compared to the original PSO and some other well-known heuristic algorithms.

2 Some basic properties of Problem (3)

This section describes the basic definitions and structural properties concerning Problem (3) that are used throughout the paper. For the sake of simplicity, let $S_\varphi(A,b)$ denotes the feasible solutions set of Problem (3), that is, $S_\varphi(A,b) = \{x \in [0,1]^n \ : \ A \,\varphi\, x = b\}$. So, similar to Problem (2), we can rewrite $S_\varphi(A,b)$ in terms of rows a_i $(i \in I)$ of matrix A as follows:

$$S_\varphi(A,b) = \{x \in [0,1]^n \ : \ \ a_i \varphi x = b_i \ , \ i \in I\}, \tag{4}$$

where the constraints mean $a_i \varphi x = \max_{j=1}^{n} \{\varphi(a_{ij}, x_j)\} = b_i, \forall i \in I$.

Definition 2.1. *For each $i \in I$ and each $j \in J$, let $\Phi_{ij} = \{x_j \in [0,1] \ : \ \varphi\,(a_{ij}, x_j) = b_i\}$. In addition, if $\Phi_{ij} \neq \emptyset$, then we define $\underline{x}\,(a_{ij}, b_i) = \inf\, \Phi_{ij}$ and $\bar{x}\,(a_{ij}, b_i) = \sup\, \Phi_{ij}$.*

Definition 2.2. *Binary operation $\varphi : [0,1]^2 \to [0,1]$ is said to be an operator with closed convex solutions if $\Phi_{ij} \neq \emptyset$ implies $\Phi_{ij} = [\underline{x}\,(a_{ij}, b_i), \bar{x}\,(a_{ij}, b_i)]$; that is, Φ_{ij} is a closed sub-interval of $[0,1]$.*

Remark 2.3. *If $\varphi : [0,1]^2 \to [0,1]$ is a continuous non-decreasing function (specially, a continuous t-norm), then φ is an operator with closed convex solutions. In this case, from the least-upper-bound property of* R*, it is clear that $\underline{x}\,(a_{ij}, b_i)$ and $\bar{x}\,(a_{ij}, b_i)$ exist, if $\Phi_{ij} \neq \emptyset$. Moreover, since φ is non-decreasing , its monotonicity property implies that Φ_{ij} is actually a connected subset of $[0,1]$. Additionally, by the continuity of φ, we must have $\Phi_{ij} = [\underline{x}\,(a_{ij}, b_i), \bar{x}\,(a_{ij}, b_i)]$.*

Corollary 2.4. *Let φ be a continuous t-norm. Then, for each $i \in I$ and $j \in J$, $\Phi_{ij} \neq \emptyset$ iff $a_{ij} \geq b_i$.*

Proof. The identity law and monotonicity of t-norms imply that $\varphi\,(a_{ij}, 0) = 0$ and $\varphi\,(a_{ij}, 1) = a_{ij}$, $\forall i \in I$ and $\forall j \in J$. Let $\Phi_{ij} \neq \emptyset$ and $x' \in \Phi_{ij}$, i.e., $\varphi\,(a_{ij}, x') = b_i$. So, from the identity law and monotonicity of φ, we have $b_i = \varphi\,(a_{ij}, x') \leq \varphi\,(a_{ij}, 1) = a_{ij}$ which means $b_i \leq a_{ij}$. Conversely, if $a_{ij} \geq b_i$, then $b_i \leq a_{ij} = \varphi\,(a_{ij}, 1)$. Thus, $\varphi\,(a_{ij}, 0) \leq b_i$ and $b_i \leq \varphi\,(a_{ij}, 1)$. Due to these facts and the intermediate value theorem for continuous functions, there must exist at least some $x' \in [0,1]$ such that $\varphi\,(a_{ij}, x') = b_i$, i.e., $x' \in \Phi_{ij}$. □

In the case that $a_{ij} \geq b_i$, the values of $\underline{x}\,(a_{ij}, b_i)$ and $\bar{x}\,(a_{ij}, b_i)$ have been listed in Appendix B for ten well-known continuous t-norms (see Tables B.2 and B.3). A more detailed description of the t-norms and their interesting properties are found in [8].

Lemma 2.5 provides a necessary feasibility condition for Problem (3).

Lemma 2.5. *Let φ be an operator with closed convex solutions. If $S_\varphi(A,b) \neq \emptyset$, then for each $i \in I$ there exists some $j \in J$ such that $\Phi_{ij} \neq \emptyset$.*

Proof. See [16, Lemma 2.1]. □

Definition 2.6. *Suppose that φ is an operator with closed convex solutions and $S_\varphi(A,b) \neq \emptyset$. We define $J(i) = \{\, j \in J \ : \ \Phi_{ij} \neq \emptyset \,\}$,$\forall i \in I$, and $I(j) = \{\, i \in I \ : \ \Phi_{ij} \neq \emptyset \,\}$, $\forall j \in J$. Let $e : I \to \bigcup_{i \in I} J(i)$ be a function such that $e(i) \in J(i)$, $\forall i \in I$. Additionally, define $I_j(e) = \{\, i \in I \ : \ e(i) = j \,\}$ and let E denote the set of all the functions e.*

According to Definition 2.6, each $e \in E$ can be equivalently considered as a set of ordered pairs $\{(1, j_1), (2, j_2), ..., (m, j_m)\}$ in witch $e(i) = j_i$, $\forall i \in I$. So, for the sake of simplicity, this set is usually represented only in terms of the second entries $j_1, j_2, \ldots, j_m$ and denoted by the notation $e = [j_1, j_2, ..., j_m]$ as a finite sequence [12].

Definition 2.7. *Suppose that φ is an operator with closed convex solutions, $S_\varphi(A,b) \neq \emptyset$ and $e \in E$. Let $\bar{x}_\varphi = [(\bar{x}_\varphi)_1, ..., (\bar{x}_\varphi)_n]$ and $\underline{x}_\varphi(e) = [\underline{x}_\varphi(e)_1, ..., \underline{x}_\varphi(e)_n]$ be two $n-$dimensional vectors whose components are defined as follows for each $j \in J$:*

$$(\bar{x}_\varphi)_j = \begin{cases} \min\limits_{i \in I(j)} \{\bar{x}\ (a_{ij}, b_i)\}\ , I(j) \neq \emptyset \\ 1 \qquad\qquad\qquad\qquad , I(j) = \emptyset \end{cases} \quad , \quad \underline{x}_\varphi(e)_j = \begin{cases} \max\limits_{i \in I_j(e)} \{\underline{x}\ (a_{ij}, b_i)\}\ , I_j(e) \neq \emptyset \\ 0 \qquad\qquad\qquad\qquad , I_j(e) = \emptyset \end{cases}$$

Theorem 2.8 determines the bounds of the feasible solutions set for Problem (3).

Theorem 2.8. *Suppose that φ is an operator with closed convex solutions and $S_\varphi(A,b) \neq \emptyset$. Then, $S_\varphi(A,b) = \bigcup_{e \in E} [\underline{x}_\varphi(e), \bar{x}_\varphi]$.*

Proof. See [16, Corollary 2.8]. □

Based on Theorem 2.8, the feasible region of Problem (3) is completely determined by the maximum solution $\bar{x}_\varphi$ and a finite number of those minimal solutions $\underline{x}_\varphi(e)$ such that $\underline{x}_\varphi(e) \leq \bar{x}_\varphi$.

The following example illustrates the above-mentioned definitions.

Example 2.9. *Consider the following problem with the minimum t-norm φ:*

$$\begin{bmatrix} 0.83 & 0.14 & 0.32 & 0.5 \\ 0.23 & 0.56 & 0.03 & 0.5 \\ 0.3 & 0.22 & 0.68 & 0.07 \\ 0 & 0.1 & 0 & 0.01 \end{bmatrix} \varphi x = \begin{bmatrix} 0.7 \\ 0.5 \\ 0.3 \\ 0.1 \end{bmatrix},$$

where $\varphi(x,y) = \min\{x,y\}$. By the minimum operator, Definition 2.1, Remark 2.3 and Corollary 2.4, it is easily verified that closed intervals Φ_{ij} are obtained as follows:

$$\Phi_{ij} = \begin{cases} \emptyset & , \ a_{ij} < b_i \\ [b_i, 1] & , \ a_{ij} = b_i \\ \{\, b_i \,\} & , \ a_{ij} > b_i \end{cases}$$

In the following, these intervals are summarized in matrix $\Phi = (\Phi_{ij})_{m_1 \times n}$:

$$\Phi = \begin{bmatrix} \{\, 0.7 \,\} & \emptyset & \emptyset & \emptyset \\ \emptyset & \{\, 0.5 \,\} & \emptyset & [0.5, 1] \\ [0.3, 1] & \emptyset & \{\, 0.3 \,\} & \emptyset \\ \emptyset & [0.1, 1] & \emptyset & \emptyset \end{bmatrix}.$$

From Definition 2.6, we have $I(1) = \{\,1, 3\,\}$, $I(2) = \{\,2, 4\,\}$, $I(3) = \{\,3\,\}$ and $I(4) = \{\,2\,\}$. Also, $J(1) = \{\,1\,\}$, $J(2) = \{\,2, 4\,\}$, $J(3) = \{\,1, 3\,\}$ and $J(4) = \{\,2\,\}$. Hence, from Definition 2.7, vector $\bar{x}_\varphi$ is obtained as $\bar{x}_\varphi = [0.7\,,\, 0.5\,,\, 0.3\,,\, 1]$ and solutions $\underline{x}_\varphi(e)$ $(\forall e \in E)$ are generated as follows:

$$\begin{array}{llll} e_1 = [1,2,1,2]: & \underline{x}_\varphi(e_1) = [0.7, 0.5, 0, 0] & , & e_3 = [1,4,1,2]: \ \underline{x}_\varphi(e_3) = [0.7, 0.1, 0, 0.5], \\ e_2 = [1,2,3,2]: & \underline{x}_\varphi(e_2) = [0.7, 0.5, 0.3, 0] & , & e_4 = [1,4,3,2]: \ \underline{x}_\varphi(e_4) = [0.7, 0.1, 0.3, 0.5]. \end{array}$$

It is to be noted that $\underline{x}_\varphi(e_2)$ and $\underline{x}_\varphi(e_4)$ are not minimal solutions; because, $\underline{x}_\varphi(e_1) \leq \underline{x}_\varphi(e_2)$ and $\underline{x}_\varphi(e_3) \leq \underline{x}_\varphi(e_4)$. Therefore, according to Theorem 2.8, we have $S_\varphi(A,b) = [\underline{x}_\varphi(e_1), \bar{x}_\varphi] \bigcup [\underline{x}_\varphi(e_3), \bar{x}_\varphi]$.

As stated in Theorem 2.8, the feasible region of Problem (3) is formed by the solutions $\bar{x}_\varphi$ and $\underline{x}_\varphi(e)$. The following theorem shows that these solutions also play important roles in characterizing the optimal solution of Problem (3).

Theorem 2.10. *Suppose that φ is a continuous t-norm and $S_\varphi(A,b) \neq \emptyset$. Also, consider problem*

$$\min \{\, c^+ x \ : \ A\varphi x = b \ , \ x \in [0,1]^n \,\}, \tag{5}$$

where $c_j^+ = \max\{\, c_j \,,\, 0 \,\}$, $\forall j \in J$. Then, $\underline{x}_\varphi(e^*)$ is an optimal solution of (5) for some $e^* \in E$. Also, $x^* = (x_1^*, ..., x_n^*)$ as defined below is the optimal solution of Problem (3).

$$x_j^* = \begin{cases} (\bar{x}_\varphi)_j & , \ c_j < 0 \\ \underline{x}_\varphi(e^*)_j & , \ c_j \geq 0 \end{cases} \tag{6}$$

Proof. See [16, Theorem 4.1] where a proof was given for the more general cases. □

Moreover, consider points $x' = (x'_1, ..., x'_n)$ and $x'' = (x''_1, ..., x''_n)$ such that for each $j \in J$:

$$x'_j = \begin{cases} (\bar{x}_\varphi)_j & , \ c_j < 0 \\ 0 & , \ c_j \geq 0 \end{cases} \quad , \quad x''_j = \begin{cases} 0 & , \ c_j \leq 0 \\ (\bar{x}_\varphi)_j & , \ c_j > 0 \end{cases} \tag{7}$$

It is easy to verify that $c^T x' \leq c^T x \leq c^T x'', \forall x \in S_\varphi(A, b)$. That is, $z_{best} = c^T x'$ and $z_{worst} = c^T x''$ define lower and upper bounds on the values $z = c^T x$, respectively. Hereafter, we suppose that φ is a continuous t-norm.

3 Fuzzy equality constraints and equivalent transformed problem

In this section, we consider the constraints as well as the objective function of Problem (1) as fuzzy sets by defining their associated membership functions. Via this approach, all points (whether feasible or not) in Problem (3) are treated as feasible ones in (1) with different degrees of membership in the interval $[0, 1]$. Indeed, a point being in the feasible set of (3) belongs to that of (1) with the degree of membership equal to one, and a farther point from the feasible set of (3) gets a lower degree of membership. To treat i'th fuzzy equation of Problem (2), we employ the following membership functions:

$$\mu\,(a_i \varphi x) = \begin{cases} 1 - \frac{|\, a_i \varphi x - b_i|}{\varepsilon_i} & , \ |\, a_i \varphi x - b_i| \leq \varepsilon_i \\ 0 & , \ otherwise \end{cases} \quad , \quad i \in I \tag{8}$$

where each ε_i $(i \in I)$ is initially determined as the limit of the admissible violation of the i'th equation. From relation (8), i'th membership function is equal to 1 if the i'th constraint of Problem (3) is well satisfied, 0 if the constraint is violated beyond its admissible limit ε_i, and decreasing linearly from 1 to 0.

Definition 3.1. *For each $x \in [0, 1]^n$, the crisp constraints violation vector at point x is an $m \times 1$ vector $CCV(x) = (CCV(x)_i)_{m \times 1}$ whose i'th component is defined as $CCV(x)_i = |\, a_i \varphi x - b_i \,|$. Also, the fuzzy constraints violation vector at point x is an $m \times 1$ vector $FCV(x) = (FCV(x)_i)_{m \times 1}$ whose i'th component is defined as $FCV(x)_i = \max\{\, 0,\ a_i \varphi x - (b_i + \varepsilon_i)\,,\ (b_i - \varepsilon_i) - a_i \varphi x \,\}$.*

From Definition 3.1, $CCV(x)$ and $FCV(x)$ show how much point x violates i'th constraint of Problem (3) (i.e., $a_i \varphi x = b_i$) and that of Problem (1) (i.e., $a_i \varphi x \approx b_i$), respectively.

Similar to (8), for the objective function of Problem (1), we define

$$\mu\,(c^T x) = \begin{cases} 1 & , \ c^T x \leq z_{best} \\ 1 + \frac{c^T x - z_{best}}{z_{best} - z_{worst}} & , \ z_{best} \leq c^T x \leq z_{worst} \\ 0 & , \ c^T x \geq z_{worst} \end{cases} \tag{9}$$

From relation (9), we have $\mu\,(z_{best}) = 1$, $\mu\,(z_{worst}) = 0$ and $0 \leq \mu\,(c^T x) \leq 1$, $\forall x \in S_\varphi(A, b)$. Specially, $0 \leq \mu\,(c^T x^*) \leq 1$ where x^* is the optimal solution of Problem (3). Value $z_{worst} - z_{best}$ determines the length of interval $[z_{best}, z_{worst}]$ in which we want to survey attainability or non-attainability of a better objective function value than $c^T x^*$; that is, we want to find a solution with a better objective function value than $c^T x^*$ in this interval. In other words, this interval is actually an approximate domain in which we guess to be able to find a better objective function value than $c^T x^*$ by partly perturbing some constraints.

By considering relations (8) and (9), we can express both the feasibility and optimality amounts of a point by only one variable as defined in the following definition.

Definition 3.2. *For each $x \in [0, 1]^n$, let $\mu_O(x) = \mu\,(c^T x)$ (i.e., the optimality value of x) and $\mu_F(x) = \min_{i \in I}\{\, \mu\,(a_i \varphi x)\,\}$ (i.e., the feasibility value of x). Then, the total value of x is defined by $\mu_T(x) = \min\{\mu_O(x), \mu_F(x)\}$.*

Definition 3.2 also defines the whole space $S = [0, 1]^n$ as a fuzzy set by associating the membership value $\mu_T(x)$ to each $x \in S$. Based on this fact, our purpose becomes equivalent to finding a point with the greatest total value among all $x \in [0, 1]^n$. In other words, we can express Problem (1) as the following problem

$$\begin{aligned} \max \ & \mu_T(x) \\ & x \in S, \end{aligned} \tag{10}$$

or equivalently

$$\begin{aligned} \max \ & \min\{\, \mu\,(c^T x), \min_{i \in I}\{\, \mu\,(a_i \varphi x)\,\}\,\} \\ & x \in [0, 1]^n, \end{aligned} \tag{11}$$

In Problem (11), any constraint violation is considered in the objective function. Also, the amount of each constraint violation separately reduces the total value of a point.

From relations (8) and (9), it is clear that $\mu_F(x) = 1$ and $\mu_O(x) \leq \mu_O(x^*), \forall x \in S_\varphi(A,b)$. So, Definition 3.2 implies $\mu_T(x) \leq \mu_T(x^*), \forall x \in S_\varphi(A,b)$. On the other hand, the target of Problem (11) is to find solutions whit maximum total values; particularly, solutions with better total values than the optimal solution of Problem (3), i.e., solutions x^{**} such that $\mu_T(x^{**}) > \mu_T(x^*)$. Therefore, we must have $x^{**} \notin S_\varphi(A,b)$. Moreover, since $\mu_F(x^*) = 1$, Definition 3.2 and inequality $\mu_T(x^{**}) > \mu_T(x^*)$ imply $\mu_O(x^{**}) > \mu_O(x^*)$ which means $c^T x^{**} < c^T x^*$; that is, solutions x^{**} must be super-optima. For the sake of expository reference, the definition of a super-optimum is formally presented by the following definition.

Definition 3.3. *Let x^* be the optimal solution of Problem (3). A point x^{**} is said to be a super-optimum if $\mu_T(x^{**}) > \mu_T(x^*)$. Moreover, let X^{**} denote the set of all super-optima, that is, $X^{**} = \{ x^{**} \in [0,1]^n \ : \ \mu_T(x^{**}) > \mu_T(x^*) \}$.*

By Definition 3.1, $S_\varphi(A,b)$ can be equivalently defined as $S_\varphi(A,b) = \{ x \in [0,1]^n \ : \ CCV(x) = \mathbf{0}_{m\times 1} \}$, i.e., all points in space $S = [0,1]^n$ that do not violate any constraint $a_i \varphi x = b_i$, $\forall i \in I$. The following definition (part (a)) provides a similar definition for Problem (1).

Definition 3.4. *(a) Let $\widetilde{S}_\varphi(A,b) = \{x \in [0,1]^n \ : \ b_i - \varepsilon_i \leq a_i \varphi x \leq b_i + \varepsilon_i \, , \forall i \in I \}$. So, we equivalently have $\widetilde{S}_\varphi(A,b) = \{x \in [0,1]^n \ : \ FCV(x) = \mathbf{0}_{m\times 1} \}$; that is, $\widetilde{S}_\varphi(A,b)$ denotes all points in space $S = [0,1]^n$ that do not violate any perturbed constraint $a_i \varphi x \approx b_i$, $\forall i \in I$.*
(b) Let $S'_\varphi(A,b) = \{x \in [0,1]^n \ : \ c^T x \geq z_{worst}\}$.

Based on Definition 3.4(a), $S_\varphi(A,b) \subseteq \widetilde{S}_\varphi(A,b) \subseteq S$. In the following, we summarize some important consequences that are easily resulted from (8), (9) and Definitions 3.2 and 3.4:

$$\begin{array}{ll} 0 \leq \mu_T(x) \leq \mu_T(x^*) & , \ \forall x \in S_\varphi(A,b) \\ 0 \leq \mu_T(x) < 1 & , \ \forall x \in \widetilde{S}_\varphi(A,b) - S_\varphi(A,b) \\ \mu_T(x) = 0 & , \ \forall x \in \left\{ S - \widetilde{S}_\varphi(A,b) \right\} \bigcup S'_\varphi(A,b). \end{array} \tag{12}$$

Based on relation (12), all points of $S - \widetilde{S}_\varphi(A,b)$ and those points of $S'_\varphi(A,b)$ are the most unfavorable points in the sense that their total values are equal to zero. Moreover, we must have $X^{**} \subseteq \widetilde{S}_\varphi(A,b) - S_\varphi(A,b)$. Because, our purpose is to find a solution, say x^{**}, whose total value is strictly greater than the total value of x^* (i.e., $\mu_T(x^{**}) > \mu_T(x^*)$), and the points of $\widetilde{S}_\varphi(A,b) - S_\varphi(A,b)$ are the only candidate solutions that can fulfill this purpose.

Heretofore, we showed that the above-mentioned transformed problem is actually equivalent to Problem (1). In the next section, we will propose an algorithm for solving this new equivalent problem.

4 A modified PSO algorithm for solving the generalized FRE-FC problems

PSO is a stochastic, population-based algorithm motivated based on the simulation of the social behavior (flock of birds) [24]. In PSO, the population is called a swarm and its search points are called particles. These particles move in the search space with a certain velocity. The value and the direction of the velocity vector change according to attractors in the search space. Each particle reacts to two such attractors; one is the best value found by the particle, and the other one is the best value found globally. More precisely, assume an n-dimensional function $f : D \subseteq \mathrm{R}^n \to \mathrm{R}$, and a swarm $S = \{x_1, x_2, ..., x_N\}$ of N particles. Each particle $x_i \in S$ $(i = 1, 2, ..., N)$, its velocity, v_i, and its best previous position, $p_i \in S$, are all $n-$dimensional vectors. Also, assume p^* to be the best previous position among all the particles in the swarm, and t to be the iteration counter. Then, the swarm is updated by applying an operator according to the fitness information obtained from the environment so that the particles can be expected to move towards the better solution. This is accomplished via the equations

$$\begin{array}{l} v_i(t+1) = w(t) v_i(t) + c_1 r_1 (p_i - x_i(t)) + c_2 r_2 (p^* - x_i(t)) \\ x_i(t+1) = x_i(t) + v_i(t+1) \end{array} \ , \tag{13}$$

where $i = 1, 2, ..., N$; c_1 and c_2 are positive constants referred to as cognitive and social parameters, respectively; w is the inertia weight; and r_1 and r_2 are random vectors with components uniformly distributed in $[0,1]$.

In the proposed PSO, we consider two additional directions while updating velocity of each particle. These directions control both the "feasibility" and "satisfaction" values of new points by stressing on some critical points of the feasible region. In other words, the new directions increase the tendency of the algorithm to points having higher total values. Consequently, this modification increases the ability of the algorithm in escaping from local optima and accelerates the convergence of the generated new solutions to a super-optimum.

Relation (13) provides two equations used in the original PSO for generating new positions $x_i(t+1)$ and updating velocity vectors $v_i(t+1)$. Because of the complexity of the objective function of Problem (11), we consider the following additional term for updating the velocity $v_i(t+1)$ as follows

$$c_3\, r_3\, K(t)\, K'(x_i(t))\, (\bar{x}_{\,\varphi} - x_i(t)) + r_4\, K(t)\, (1 - K'(x_i(t)))\ \frac{-c^T}{\| -c^T \|}, \tag{14}$$

where c_3 is a positive constant, r_3 and r_4 are random vectors with components uniformly distributed in $[0,1]$, $-c^T / \| -c^T \|$ is the normalized steepest descent direction and t is the iteration counter. Moreover, $K(t)$ and $K'(x_i(t))$ are defined as follows:

$$K(t) = \max\{\, 0, 1 + \frac{1-t}{0.7\, t_{\max} - 1}\,\} \quad , \quad K'(x_i(t)) = \begin{cases} 1 & , \ \mu_{\,O}(x_i(t)) \geq \mu_{\,F}(x_i(t)) \\ 0 & , \ otherwise, \end{cases} \tag{15}$$

where $t_{\max}$ denotes the maximum number of iterations. As is shown in (14), parameters $K(t)$ and $K'(x_i(t))$ are dynamically changed by each round of iterations. Parameter $K(t)$ is decreasing linearly from 1 to 0 as the iteration counter t is increased from 1 to $0.7\, t_{\max}$, and remains constant, $K(t) = 0$, for $t \geq 0.7\, t_{\max}$. Also, parameter $K'(x_i(t))$ is updated based on the relationship between the optimality and feasibility values of the particle $x_i(t)$. Therefore, adding (14) to (13), we have

$$\begin{aligned} v_i(t+1) = {} & w(t) v_i(t) + c_1 r_1 (p_i - x_i(t)) + c_2 r_2 (p^* - x_i(t)) \\ & + c_3\, r_3\, K(t)\, K'(x_i(t))\, (\bar{x}_{\,\varphi} - x_i(t)) + r_4\, K(t)\, (1 - K'(x_i(t)))\ \tfrac{-c^T}{\| -c^T \|} \quad . \end{aligned} \tag{16}$$

Vector $-c^T / \| -c^T \|$ is a direction along which the values of the objective function of the new particles are improved. On the other hand, direction $\bar{x}_{\,\varphi} - x_i(t)$ increases the value of the feasibility of the new particles by moving them towards $\bar{x}_{\,\varphi}$ that is a feasible point of Problem (3).

From the definition of $K'(x_i(t))$, if the feasibility value of an article $x_i(t)$ is less than or equal to its optimality value (i.e., $\mu_{\,F}(x_i(t)) \leq \mu_{\,O}(x_i(t))$), then $K'(x_i(t)) = 1$. In this case, only the first additional term, i.e., $c_3\, r_3\, K(t)\, K'(x_i(t))\, (\bar{x}_{\,\varphi} - x_i(t))$, is considered in the updating the velocity, and the second one, i.e., $r_4\, K(t)\, (1 - K'(x_i(t)))\ \big(-c^T / \| -c^T \|\big)$, is neglected; since, in this case, $1 - K'(x_i(t)) = 0$. As mentioned before, this manner causes the point $x_i(t)$ to rapidly obtain a higher feasibility value. Otherwise, if the optimality value is less than the feasibility value (i.e., $\mu_{\,O}(x_i(t)) < \mu_{\,F}(x_i(t))$), then $K'(x_i(t)) = 0$ and therefore only the second additional term is computed in order to increase the optimality value of $x_i(t)$. As a result, parameter $K'(x_i(t))$ can enhance search efficient by motivating the algorithm to focus on the points with the higher total values and prevent it from stalling at the others.

In the first iterations, coefficient $K(t)$ is close to 1, and as time passes, $K(t)$ is decreased linearly from 1 to 0. By this manner, in premier iterations when the swarm is most likely to include particles with low total values, the additional terms extremely participate in the updating of the velocity and modify the total values of the particles. Gradually, as the qualities of the new particles are enhanced, the influence of the additional terms fades out. It is to be noted that we have $K(t) = 0$ for the last 30% of iterations. So, when $t \geq 0.7\, t_{\max}$, the additional terms are neglected and therefore relations (16) are converted to relations (13), which in turn increases the exploitation capability of the algorithm in the obtained optimal region.

5 Numerical examples and applications

In this section, we present the experimental results for evaluating the performance of our algorithm. Firstly, in Section 5.1, an application of the FRE (Problem (3)) and FRE-FC (Problem (1)) is provided in optimal intensity of electromagnetic radiation problem. In Section 5.2, the proposed algorithm is applied to solving the transformed equivalent problem (Problem (11)) for the test problems described in Appendix A. The test problems have been randomly generated in different sizes by using 10 well-known continuous t-norms. Table B.1, in Appendix B, presents the continuous t-norms used in our experimental study. Finally, a comparison is made between the proposed PSO, original PSO (OPSO), continuous ant colony optimization (ACO), Harmony Search (HS) and Real Genetic Algorithm (RGA). For this purpose, in Section 5.3, we apply these methods to the ten test problems described in Appendix A. To perform a fair comparison, the same parameters are used for the proposed algorithm, that is, we set $\varepsilon_i = 0.1$ for $i = 1, .., m$.

5.1 An application of FRE and FRE-FC problems

This section considers a type of wireless communication management models. In such wireless communication model, the information is transmitted by the electromagnetic wave, while the electromagnetic wave is emitted from some fixed emission base stations (EBSs). Assume that there are n EBSs, i.e., A_1,A_2,...,A_n, located in different places of a specific area, such as city. The j'th EBS will emit electromagnetic wave with radiation intensity $x_j > 0$, $j = 1, 2, ..., n$. The communication quality level is determined by the intensity of electromagnetic radiation. In order to satisfy the requirement of communication quality level, m testing points, i.e., B_1,B_2,...,B_m are selected to test the intensity of electromagnetic radiation. For example, when the wireless communication is applied to the cell phone network, the testing point is usually the place with higher population density. At the i'th testing point B_i, the intensity of electromagnetic radiation emitted from A_j, denoted by r_{ij}, will belong to $[0, x_j]$ where $i \in I = \{1, ..., m\}$ and $j \in J = \{1, ..., n\}$. Since r_{ij} is related to the distance between B_i and A_j, there exists a real number $a_{ij} \in [0, 1]$ such that $r_{ij} = a_{ij}x_j$. So, the intensity of electromagnetic radiation at B_i is $\max_{j=1}^{n}\{r_{ij}\} = \max_{j=1}^{n}\{a_{ij}x_j\}$. Suppose that requirement of communication quality level at B_i is b_i. Then we get $\max_{j=1}^{n}\{a_{ij}x_j\} = b_i$, $\forall i \in I$. By normalizing the variables and parameters into the unit interval $[0, 1]$, the optimal intensity of electromagnetic radiation model may be reduced into FRE Problem (3) in which φ is defined by the product t-norm and the objective function is the sum of radiation costs, i.e., $\sum_{j\in J} c_j x_j$. As before, if x^* denotes the optimal solution of the obtained FRE problem, then the proposed PSO attempts to find a better solution than x^* by taking advantage of the flexibility of some constraint(s). As is shown in Section 5.2, the associated FRE-FC problem (Problem (1)) can provide better selections for a decision maker, provided that there exists at least a flexible constraint in the problem, i.e., there are some points that violate the constraint(s) and are still permissible to be a solution for the problem based on the decision maker's view.

5.2 Results of the proposed PSO applied to Problem (11)

In this section, the proposed algorithm is applied to the test problems described in Appendix A. Table 1 includes the feasibility values $\mu_F(x^{**})$, optimality values $\mu_O(x^{**})$ and total values $\mu_T(x^{**})$ for the best solutions x^{**} found by the proposed PSO.

In each case, to simplify the comparison between the current optimum x^* (the optimal solution for Problem (3)) and the super-optimum x^{**} found by the proposed PSO algorithm, total values $\mu_T(x^*)$ have been reported. Also, we reported the optimal values of Problem (3) (i.e., $c^T x^*$) associated to each test problem to determine if the proposed PSO can find a super-optimum x^{**} such that $c^T x^{**} < c^T x^*$.

Additionally, for each test problem, Table 2 presents the optimal solutions x^* of Problem (3), the best solutions x^{**} found by the current algorithm and the crisp constraints violation vectors at points x^{**}.

Table 1
A comparison between the quality of the best solutions x^{**} found by the proposed PSO and that of the optimal solutions x^* of Problem (3) for the test Problems A.1 - A.10.

Test problems	$c^T x^{**}$	$\mu_F(x^{**})$	$\mu_O(x^{**})$	$\mu_T(x^{**})$	$c^T x^*$	$\mu_T(x^*)$
A.1	- 13.6287	0.965	0.9833	0.965	- 9.2764	0.7494
A.2	- 4.1903	0.6579	0.6573	0.6573	- 2.24	0.5574
A.3	4.5836	0.6105	0.7915	0.6105	12.8909	0.4649
A.4	- 12.5029	1	1	1	- 12.5029	1
A.5	- 13.1852	0.8020	0.8013	0.8013	- 9.5647	0.6715
A.6	7.2159	0.5958	0.5971	0.5958	14.2525	0.2604
A.7	- 4.132	0.7336	0.7322	0.7322	- 3.0025	0.6552
A.8	- 1.3343	0.7368	0.7336	0.7336	11.8478	0.3468
A.9	- 8.6549	0.8900	0.9269	0.8900	- 6.5085	0.8189
A.10	- 5.8309	0.6726	0.6727	0.6726	- 4.4724	0.6160

As shown in Table 1, values $\mu_T(x^{**})$ are strictly greater than $\mu_T(x^*)$ in all the cases. Thus, $x^{**} \in X^{**}$ that means the proposed PSO could find super-optima for their associated test problems (Definition 3.3). Also, objective function

values $c^T x^{**}$ are strictly less than $c^T x^*$ for all the test problems.

Therefore, the proposed PSO produces better solutions, that is, solutions with less objective values compared to the optimal solutions of Problem (3). The only exception is test Problem A.4, where $x^{**}= x^*$.

For test Problem A.4, since $\mu_T(x^*) = 1$ (and therefore $z_{best} = z^*$), the modified PSO has introduced x^* as the best solution. In this case, in order to find a super-optimum, we have to consider another upper bound z_{best} such that $z_{best} < z^*$ (e.g., $z_{best} - \varepsilon$ for some arbitrarily small $\varepsilon > 0$).

Table 2 includes solutions x^*, x^{**} and the crisp constraints violation vectors $CCV\,(x^{**})$ for the test problems. The results, in Table 2, show that the proposed PSO produces optimal solutions with admissible infeasibilities; more precisely, for each $i \in I$, we have $CCV(x^{**})_i \leq \varepsilon_i = 0.1$.

Table 2
The best solutions x^{**} and their crisp constraints violation vectors $CCV1\,(x^{**})$ (related to inequalities $A\varphi x \leq b^1$) and $CCV2\,(x^{**})$ (related to inequalities $D\psi x \geq b^2$).

A.1	$x^* = [0.8741\ ,\ 0.6345\ ,\ 0.6041\ ,\ 0.6998\ ,\ 0.4198\ ,\ 0.0729\ ,\ 0.8971]$ $x^{**} = [0.8743\ ,\ 0.6525\ ,\ 0.6541\ ,\ 0.7028\ ,\ 0.0409\ ,\ 0.0200\ ,\ 1]$ $CCV\,(x^{**}) = [0.0295\ ,\ 0.0176\ ,\ 0.0002\ ,\ 0.0035\ ,\ 0.003\ ,\ 0.0030\ ,\ 0.0196]$
A.2	$x^* = [0.5116\ ,\ 0.9487\ ,\ 0.7180\ ,\ 0.9461\ ,\ 0.8378\ ,\ 0.9567\ ,\ 0.7347]$ $x^{**} = [0.0008\ ,\ 0.8687\ ,\ 0.7648\ ,\ 0.9830\ ,\ 0.7847\ ,\ 0.9912\ ,\ 0.7815]$ $CCV\,(x^{**}) = [0.0342\ ,\ 0.0328\ ,\ 0.0341\ ,\ 0.0335\ ,\ 0.0339\ ,\ 0.0328\ ,\ 0.0174]$
A.3	$x^* = [0.9161\ ,\ 0\ ,\ 0.9842\ ,\ 0.9298\ ,\ 0.8949\ ,\ 0.8918\ ,\ 0]$ $x^{**} = [0.9551\ ,\ 0.0057\ ,\ 0.9845\ ,\ 0.9058\ ,\ 0.0039\ ,\ 0.0640\ ,\ 0.1207]$ $CCV\,(x^{**}) = [0.0233\ ,\ 0.0240\ ,\ 0.0390\ ,\ 0.0390\ ,\ 0.0003]$
A.4	$x^* = [0.8330\ ,\ 0.2982\ ,\ 0.8432\ ,\ 0.0226\ ,\ 0.0302\ ,\ 0.4661\ ,\ 0.0299]$ $x^{**} = [0.8330\ ,\ 0.2982\ ,\ 0.8432\ ,\ 0.0226\ ,\ 0.0302\ ,\ 0.4661\ ,\ 0.0299]$ $CCV\,(x^{**}) = [0\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ 0]$
A.5	$x^* = [0.8215\ ,\ 0.6510\ ,\ 0.8606\ ,\ 0.4342\ ,\ 0.3055\ ,\ 0.2816\ ,\ 0.5306\ ,\ 0.4018]$ $x^{**} = [0.8200\ ,\ 0.7022\ ,\ 0.8427\ ,\ 0.4526\ ,\ 0.3306\ ,\ 0.0005\ ,\ 0.5461\ ,\ 0.4187]$ $CCV\,(x^{**}) = [0.008\ ,\ 0.0168\ ,\ 0.0097\ ,\ 0.0198\ ,\ 0.0155\ ,\ 0.0181\ ,\ 0.0015]$
A.6	$x^* = [0.9139\ ,\ 0.4836\ ,\ 0.9722\ ,\ 0.4671\ ,\ 0\ ,\ 0.3148\ ,\ 0.7626\ ,\ 0.0662]$ $x^{**} = [0.8994\ ,\ 0.4828\ ,\ 0.0205\ ,\ 0\ ,\ 0.7697\ ,\ 0.3762\ ,\ 0.752\ ,\ 0.0799]$ $CCV\,(x^{**}) = [0.0075\ ,\ 0.0137\ ,\ 0.0404\ ,\ 0.0124\ ,\ 0.0004\ ,\ 0.0403\ ,\ 0.0135]$
A.7	$x^* = [0.2482\ ,\ 0\ ,\ 0.2996\ ,\ 0.7599\ ,\ 0.4587\ ,\ 0.6288\ ,\ 0.3443\ ,\ 0.6849]$ $x^{**} = [0.2233\ ,\ 0.0003\ ,\ 0.3057\ ,\ 0.7740\ ,\ 0.3703\ ,\ 0.6608\ ,\ 0.3721\ ,\ 0.6937]$ $CCV\,(x^{**}) = [0.0061\ ,\ 0.0128\ ,\ 0.0263\ ,\ 0.0266\ ,\ 0.0265\ ,\ 0.0067\ ,\ 0.0249]$
A.8	$x^* = [0.8369\ ,\ 0.9166\ ,\ 0.6153\ ,\ 0.9492\ ,\ 0.9470\ ,\ 0\ ,\ 0.6391\ ,\ 0.8394]$ $x^{**} = [0\ ,\ 0.9235\ ,\ 4.5847\mathrm{e}-06\ ,\ 0.9436\ ,\ 0.9450\ ,\ 0\ ,\ 0.6312\ ,\ 0.8463]$ $CCV\,(x^{**}) = [0.0253\ ,\ 0.0148\ ,\ 0.0121\ ,\ 0.02601\ ,\ 0.0040\ ,\ 0.0163\ ,\ 0.0263]$
A.9	$x^* = [0.9805\ ,\ 0.9647\ ,\ 0.9703\ ,\ 0.0985\ ,\ 0.7406\ ,\ 0.6549\ ,\ 0.8512\ ,\ 0]$ $x^{**} = [0.9891\ ,\ 0.9913\ ,\ 0\ ,\ 0.0120\ ,\ 0.7521\ ,\ 0.6495\ ,\ 0.8378\ ,\ 0]$ $CCV\,(x^{**}) = [0.0083\ ,\ 0.0110\ ,\ 0.0077\ ,\ 0.0108\ ,\ 0.0051\ ,\ 0.0039\ ,\ 0.0110]$
A.10	$x^* = [0\ ,\ 0.3008\ ,\ 0.482\ ,\ 0.3891\ ,\ 0.8452\ ,\ 0.7735\ ,\ 0.237\ ,\ 0.7055]$ $x^{**} = [0\ ,\ 0.3590\ ,\ 0.4496\ ,\ 0.3984\ ,\ 0.8774\ ,\ 0.7743\ ,\ 0.2532\ ,\ 0.6738]$ $CCV\,(x^{**}) = [0.0317\ ,\ 0.0322\ ,\ 0.0324\ ,\ 0.0327\ ,\ 0.0093\ ,\ 0.0008\ ,\ 0.0162]$

5.3 Proposed PSO versus OPSO, ACO, HS and RGA algorithms

In this section, we apply the proposed PSO, original PSO (OPSO), continuous ant colony optimization (CACO), harmony search (HS) and real genetic algorithm (RGA) to Problem (11) defined by the test problems described in Appendix A. For this purpose, 30 experiments are performed for each test problem. The maximum number of iterations is equal to 100 and the population size is set to 10. In order to ensure a fair comparison, we carefully used the parameters suggested by the authors for each algorithm. The parameters of OPSO algorithm that are used in each case are as follows. Swarm size is set to 10, $c_1 = c_2 = 2$ and inertia factor w is decreasing linearly from 1 to 0 by a damping factor of 0.99 [24]. Also, for the proposed PSO, we set $c_1 = c_2 = c_3 = 2$. In RGA, arithmetic crossover, Gaussian mutation and roulette wheel selection are used as described in [22]. The crossover and mutation probabilities were set to 0.3 and 0.1, respectively. For CACO, we set $m = 10$ (number of ants used in an iteration), $\xi = 0.85$ (the speed of convergence), $q = 10^{-4}$ (locality of the search process) and $k = 50$ (archive size) [39], and for HS, $HMS = 10$ (size of harmony memory), $HMCR = 0.825$ (harmony memory consideration rate) and $PAR = 0.35$ (Pitch Adjustment Rate) [11].

In Table 3, the results have been averaged over 30 runs and the average best-so-far (Avg), median best-so-far (Mdn) in the last iterations and the standard deviations (Sd) are reported. The results, in Table 3, show the ability of the proposed PSO to detect the optimal solutions of Problem (11). According to this table, the current method produces significantly better solutions (solutions with larger total values) than those obtained by the other algorithms for all the test problems. This is due to the attractive role of the two additional directions considered in the proposed PSO. These directions dynamically control the movements of the particles and motivate the algorithm to uniformly find solutions with better qualities. Indeed, these directions can be interpreted as some global information about the search space, which enable the algorithm to escape from the solutions with very small total values. In contrast, the other algorithms do not benefit this general controlling effect over the particles. As a result, solutions found by them may have very small total values. Also, the proposed PSO tends to find the global optima faster than the other algorithms and hence has a higher convergence rate. Figures 3-5 depict the good convergence rate of the proposed PSO for some test problems.

Table 3
A comparison between the results found by the proposed PSO, OPSO, CACO, HS and RGA algorithms for the test problems of Appendix A. The results have been averaged over 30 runs. Maximum number of iterations=100.

Test problems		proposed PSO	OPSO	CACO	HS	RGA
A.1	Avg	0.8832	0.047	0.1555	0.0476	0.0305
	Mdn	0.8837	0	0	0	0
	Sd	0.0027	0.0664	0.0757	0.0751	0.0585
A.2	Avg	0.6475	0.0734	0.2632	0.1808	0.0336
	Mdn	0.6495	0	0	0.0229	0
	Sd	0.0031	0.0550	0.0866	0.0798	0.0660
A.3	Avg	0.6053	0.1739	0.4897	0.4290	0.0790
	Mdn	0.6105	0	0.5982	0.6001	0
	Sd	0.0037	0.0898	0.0785	0.0777	0.1792
A.4	Avg	0.9891	0.0105	0.1015	0.0039	0
	Mdn	1	0	0	0	0
	Sd	0.0024	0.6370	0.5966	0.3395	0.5059
A.5	Avg	0.7508	0.0114	0.1330	0.1695	0.0686
	Mdn	0.7532	0	0	0.0504	0
	Sd	0.0032	0.0641	0.0780	0.0714	0.0555
A.6	Avg	0.4718	0	0.1664	0.1197	0.0209
	Mdn	0.4850	0	0	0	0
	Sd	0.0053	0.1979	0.2188	0.1707	0.1510
A.7	Avg	0.6963	0.0177	0.2648	0.2771	0.0518
	Mdn	0.7048	0	0.0059	0.1271	0
	Sd	0.0033	0.0900	0.1270	0.0952	0.0907
A.8	Avg	0.5046	0	0.0711	0.01190	0

	Mdn	0.6613	0	0	0	0
	Sd	0.1397	0.1153	0.1252	0.0928	0.0746
A.9	Avg	0.7820	0.1955	0.5172	0.6724	0.0610
	Mdn	0.7946	0	0.5658	0.6873	0
	Sd	0.0026	0.1282	0.0496	0.0069	0.0849
A.10	Avg	0.6421	0.0509	0.1974	0.1690	0.0712
	Mdn	0.6476	0	0	0.0924	0
	Sd	0.0044	0.0463	0.0667	0.0645	0.0504

These results confirm that the proposed PSO can desirably avoid trapping in the unfavorable points

$$x \in \left\{S - \widetilde{S}_{\varphi}(A,b)\right\} \bigcup S'_{\varphi}(A,b),$$

having zero total values (i.e., $\mu_{T}(x) = 0$). Table 3 and Figures 3-5 imply that the proposed algorithm is very efficient both in locating the optimal region and in local searching. Therefore, the algorithm could be a robust global search algorithm for FRE-FC problems even though the feasible domain is generally non-convex.

Table 4 contains the average error of 30 runs (over the last iterations) and the mean square error from all executions of all the methods over all the test problems. And the error is given by $(1/m)\ \sum_{i=1}^{m} CCV(x^{**})_i$, that is, the average of all the constraint violations related to equations $A\varphi x = b$.

From this table, we can note which method has the lower error on average. Additionally, the results of Tables 3 and 4 demonstrate that the proposed PSO produces high quality solutions with low average error (good accuracy) and low standard deviation (good stability) for all the test problems.

Table 4
Errors averaged over the last iterations of 30 runs.

Test problems	Proposed PSO	OPSO	CACO	HS	RGA
A.1	0.006392	0.15364	0.13589	0.15222	0.17119
A.2	0.024808	0.11329	0.11296	0.10901	0.16348
A.3	0.027765	0.12658	0.057322	0.074142	0.22909
A.4	0.0008	1.1038	1.0364	1.224	1.1446
A.5	0.014981	0.20491	0.1484	0.10194	0.1427
A.6	0.035064	0.39781	0.28405	0.18826	0.30377
A.7	0.01985	0.27418	0.15255	0.099988	0.18446
A.8	0.085196	0.36365	0.20483	0.21911	0.2235
A.9	0.013674	0.15652	0.035121	0.017891	0.17844
A.10	0.023457	0.14342	0.10762	0.0853	0.11302
MSE	0.0011271	0.17236	0.12894	0.16502	0.16599

Table 5 presents the total values $\mu_{T}(x^{**})$ for the best solutions x^{**} obtained by the algorithms. As this table illustrates the modified PSO provides better results (larger values $\mu_{T}(x^{**})$) than the other algorithms for all the test problems.

Furthermore, in Table 5, shaded boxes have been used to show those cases in which super-optima were not reached. As shown in this table, RGA and OPSO have the worst results; actually, they could find super-optima (i.e., solutions x^{**} such that $\mu_{T}(x^{**}) > \mu_{T}(x^{*})$) only for 10% and 20% of the test problems, respectively.

The performance of HS is mediocre in the sense that it could reach the required accuracy for 50% of the test problems. CACO was successful for 90% of the problems and the proposed PSO could find super-optima for all the test problems.

Table 5
A comparison between the total values $\mu_T(x^{**})$ for the best solutions x^{**} found by the proposed PSO, OPSO, CACO, HS and RGA algorithms for the test problems of Appendix A.

Test problems	Proposed PSO	OPSO	CACO	HS	RGA
A.1	0.965	0.705	0.8606	0.4548	0.6135
A.2	0.6573	0.574	0.6481	0.6490	0.5114
A.3	0.6105	0.6105	0.6105	0.6102	0.4542
A.4	1	0.3158	0.5074	0.1172	0
A.5	0.8013	0.3406	0.7459	0.6175	0.5799
A.6	0.5958	0	0.5750	0.5946	0.2852
A.7	0.7322	0.5322	0.7152	0.7053	0.4225
A.8	0.7336	0	0.5706	0.3034	0
A.9	0.8901	0.744	0.8246	0.8644	0.6880
A.10	0.6726	0.6150	0.6523	0.5933	0.4887

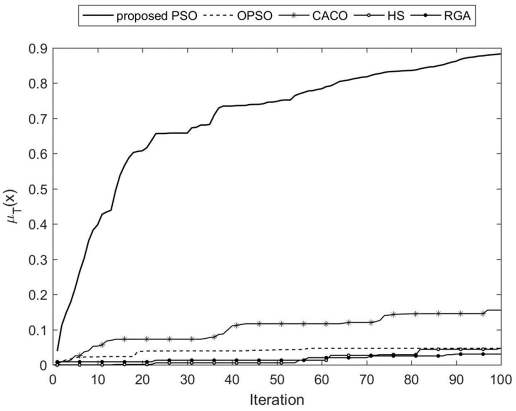


Figure 3: The performance of the proposed PSO, original PSO, ACO and DE algorithms on test Problem A.1 defined with max-min composition.

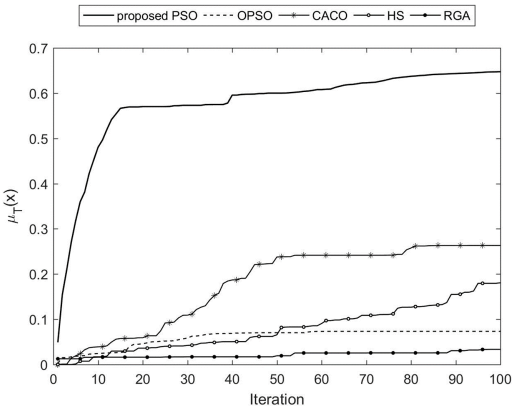


Figure 4: The performance of the proposed PSO, original PSO, ACO and DE algorithms on test Problem A.1 defined max-product composition.

6 Conclusion

In this paper, we investigated the most general version of fuzzy relational equations with fuzzy constraints (FRE-FC) defined by arbitrary continuous t-norms. The linear optimization problem with respect to non-convex regions formed as FRE-FC was studied and a new formulation for the optimization problem was introduced by using some theoretical

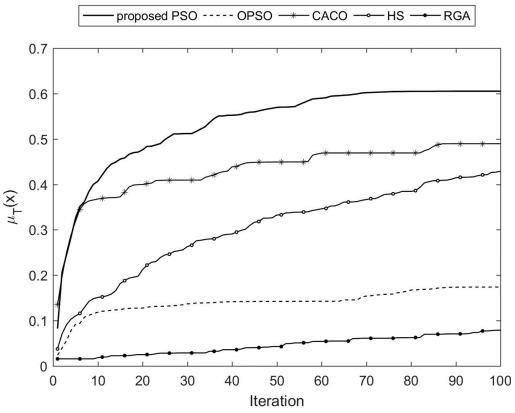


Figure 5: The performance of the proposed PSO, original PSO, ACO and DE algorithms on test Problem A.1 defined max-Lukasiewicz composition.

aspects of the fuzzy relational equalities. Based on the new formulation, a new modified PSO algorithm was presented without using the optimal solution of the associated FRE problem, minimal solutions of the feasible region and penalty coefficients in the objective function. This was done, by introducing two additional directions in the velocity of each particle. These new directions were resulted from some structural properties of the feasible region. Some experiments were performed by the proposed method in linear optimization problems subjected to the FRE-FC regions defined by ten well-known continuous t-norms. We conclude that the proposed PSO algorithm can find a super-optimum satisfying all the pre-determined requirements for all the cases. Finally, a comparison was made between the proposed PSO, original PSO, continuous ant colony optimization, harmony search and real genetic algorithm by applying them to ten test problems. The results showed that the proposed PSO can find better solutions with better convergence rate by neglecting some unfavorable points in the search space and pushing the articles toward new solutions with the higher feasibility and optimality values. Also, from the results of the standard deviation, MSE and times of the execution of the proposed method, we observe that our algorithm has a good stability and can find optimal solutions with high qualities in less than a second. As future works, we aim at testing our algorithm in other type of FRE-FC problems where the objective function is nonlinear.

Appendix A

Test Problem A.1

$$\varphi = T_{Min}$$

$$c^T = [-8.6119\ ,\ 6.481\ ,\ -9.5333\ ,\ 0.4055\ ,\ -0.4286\ ,\ -6.477\ ,\ 0.2970]$$

$$A = \begin{bmatrix} 0.5287 & 0.3577 & 0.3043 & 0.3136 & 0.8729 & 0.6115 & 0.9266 \\ 0.0258 & 0.0265 & 0.0471 & 0.0553 & 0.0466 & 0.8188 & 0.0429 \\ 0.9686 & 0.6331 & 0.8074 & 0.7938 & 0.7898 & 0.1482 & 0.1597 \\ 0.5649 & 0.3047 & 0.6076 & 0.5584 & 0.1176 & 0.4624 & 0.1005 \\ 0.1679 & 0.4168 & 0.3502 & 0.0724 & 0.7064 & 0.0471 & 0.0009 \\ 0.0791 & 0.0835 & 0.3114 & 0.8678 & 0.8205 & 0.9455 & 0.5260 \\ 0.2197 & 0.7448 & 0.8523 & 0.4762 & 0.7927 & 0.8516 & 0.2105 \end{bmatrix}$$

$$b^T = [0.8971\ ,\ 0.0729\ ,\ 0.8741\ ,\ 0.6041\ ,\ 0.4198\ ,\ 0.6998\ ,\ 0.6345]$$

Test Problem A.2

$$\varphi = T_P$$

$$c^T = [1.569\ ,\ 2.2491\ ,\ -8.7847\ ,\ -1.7848\ ,\ 6.8038\ ,\ -2.0295\ ,\ -1.2788]$$

$$A = \begin{bmatrix} 0.1152 & 0.0971 & 0.1128 & 0.4121 & 0.5572 & 0.1889 & 0.7300 \\ 0.1897 & 0.7039 & 0.1420 & 0.8863 & 0.2691 & 0.3896 & 0.1156 \\ 0.1213 & 0.4266 & 0.3743 & 0.1028 & 0.3889 & 0.0623 & 0.3546 \\ 0.2712 & 0.0874 & 0.5163 & 0.1171 & 0.6308 & 0.3963 & 0.3614 \\ 0.0795 & 0.6246 & 0.3682 & 0.3114 & 0.0593 & 0.9843 & 0.7980 \\ 0.5686 & 0.2501 & 0.7006 & 0.2931 & 0.0455 & 0.3268 & 0.0325 \\ 0.3501 & 0.1862 & 0.0234 & 0.0205 & 0.1480 & 0.0324 & 0.0217 \end{bmatrix}$$

$$b^T = [0.5363\ ,\ 0.8385\ ,\ 0.4047\ ,\ 0.5285\ ,\ 0.9417\ ,\ 0.5030\ ,\ 0.1791]$$

Test Problem A.3

$$\varphi = T_L$$

$$c^T = [0.7358\ ,\ 5.2422\ ,\ 3.0487\ ,\ -0.7754\ ,\ 2.7865\ ,\ 8.3467\ ,\ 8.3467]$$

$$A = \begin{bmatrix} 0.8522 & 0.2376 & 0.3586 & 0.7260 & 0.8891 & 0.2771 & 0.1316 \\ 0.4673 & 0.8176 & 0.1173 & 0.5350 & 0.1426 & 0.0020 & 0.2892 \\ 0.9707 & 0.4058 & 0.7248 & 0.1826 & 0.6193 & 0.8108 & 0.9630 \\ 0.8412 & 0.4663 & 0.7011 & 0.1124 & 0.6848 & 0.9434 & 0.4656 \\ 0.0785 & 0.9515 & 0.9997 & 0.0028 & 0.4982 & 0.6384 & 0.3852 \end{bmatrix}$$

$$b^T = [0.7840\ ,\ 0.4648\ ,\ 0.8868\ ,\ 0.8352\ ,\ 0.9839]$$

Test Problem A.4

$$\varphi = T_F^s\ \ (s = 2)$$

$$c^T = [-3.4986\ ,\ -0.74759\ ,\ -6.6782\ ,\ -4.0893\ ,\ -3.7398\ ,\ -7.4822\ ,\ -1.4248]$$

$$A = \begin{bmatrix} 0.7281 & 0.7902 & 0.9953 & 3.7319 & 3.2348 & 0.4420 & 0.2382 \\ 0.6692 & 1.2252 & 0.5667 & 0.3056 & 3.0033 & 0.0970 & 0.3301 \\ 0.0061 & 0.0441 & 0.0069 & 0.0758 & 0.5072 & 0.0054 & 0.5111 \\ 0.2067 & 0.3460 & 0.0664 & 2.9037 & 1.7853 & 0.5744 & 0.9343 \\ 0.0030 & 0.0199 & 0.0082 & 0.6132 & 0.1881 & 0.0010 & 0.3507 \\ 0.0901 & 0.9337 & 0.2309 & 1.6215 & 3.1454 & 0.4806 & 0.9565 \end{bmatrix}$$

$$b^T = [0.8390\ ,\ 0.5471\ ,\ 0.0128\ ,\ 0.2467\ ,\ 0.0120\ ,\ 0.2738]$$

Test Problem A.5

$$\varphi = T_Y^p\ \ (p = 2)$$

$$c^T = [1.0774\ ,\ -9.0426\ ,\ 6.7309\ ,\ -7.7235\ ,\ -6.5123\ ,\ 8.8219\ ,\ -9.684\ ,\ -5.8716]$$

$$A = \begin{bmatrix} 0.0435 & 0.0555 & 0.1488 & 0.2714 & 0.2047 & 0.6267 & 0.3194 & 0.2752 \\ 0.1495 & 0.4336 & 0.0635 & 0.6648 & 0.6100 & 0.2811 & 0.3213 & 0.9451 \\ 0.2675 & 0.1008 & 0.7718 & 0.0747 & 0.1574 & 0.7820 & 0.6325 & 0.8005 \\ 0.1245 & 0.1986 & 0.0332 & 0.0330 & 0.4692 & 0.0962 & 0.0637 & 0.0066 \\ 0.3358 & 0.2144 & 0.3903 & 0.4959 & 0.5513 & 0.1610 & 0.9640 & 0.0663 \\ 0.1220 & 0.3677 & 0.0197 & 0.8984 & 0.3886 & 0.7078 & 0.3703 & 0.3268 \\ 0.9448 & 0.8626 & 0.1840 & 0.8934 & 0.2138 & 0.6546 & 0.9129 & 0.9503 \end{bmatrix}$$

$$b^T = [0.1904\ ,\ 0.3993\ ,\ 0.7326\ ,\ 0.1259\ ,\ 0.5292\ ,\ 0.4251\ ,\ 0.8132]$$

Test Problem A.6

$$\varphi = T_H^\alpha\ \ (\alpha = 2)$$

$$c^T = [-0.0207\ ,\ 7.0163\ ,\ 9.9106\ ,\ 4.9204\ ,\ 6.4095\ ,\ -3.3868\ ,\ 0.1762\ ,\ -1.854]$$

$$A = \begin{bmatrix} 0.2049 & 0.0773 & 0.2904 & 0.1781 & 0.2710 & 0.0364 & 0.6074 & 0.0767 \\ 0.2003 & 0.0072 & 0.0841 & 0.8016 & 0.4733 & 0.2531 & 0.2409 & 1.5668 \\ 0.0432 & 0.2902 & 0.0834 & 0.3294 & 0.3147 & 0.6609 & 0.1286 & 1.1558 \\ 0.6979 & 0.1751 & 0.3239 & 0.9019 & 0.0791 & 0.7741 & 0.6155 & 0.2101 \\ 0.0182 & 0.5373 & 0.0800 & 0.2726 & 0.1790 & 0.0183 & 0.1294 & 0.6273 \\ 0.1778 & 0.0600 & 0.2494 & 0.0040 & 0.2977 & 0.5861 & 0.2469 & 0.6972 \\ 0.0514 & 0.1384 & 0.0167 & 0.0765 & 0.0814 & 0.2829 & 0.0922 & 0.9899 \end{bmatrix}$$

$$b^T = [0.4237\ ,\ 0.3386\ ,\ 0.1688\ ,\ 0.6216\ ,\ 0.2097\ ,\ 0.2375\ ,\ 0.0649]$$

Test Problem A.7

$$\varphi = T_D^\lambda\ \ (\lambda = 2)$$

$$c^T = [5.3948\ ,\ 6.2066\ ,\ -1.5944\ ,\ -3.5802\ ,\ 8.1026\ ,\ -4.8531\ ,\ -1.7767\ ,\ -1.7472]$$

$$A = \begin{bmatrix} 0.5702 & 0.4757 & 0.8816 & 0.1956 & 0.2975 & 0.2049 & 0.0977 & 0.0816 \\ 0.8757 & 0.8618 & 0.3322 & 0.8994 & 0.0104 & 0.0250 & 0.9379 & 0.6662 \\ 0.1859 & 0.3957 & 0.5540 & 0.2316 & 0.0413 & 0.2350 & 0.7120 & 0.2155 \\ 0.4771 & 0.3078 & 0.0616 & 0.5245 & 0.5947 & 0.7889 & 0.8649 & 0.0086 \\ 0.8202 & 0.7877 & 0.3791 & 0.0691 & 0.3408 & 0.1502 & 0.1618 & 0.0610 \\ 0.7950 & 0.1145 & 0.9573 & 0.5570 & 0.7174 & 0.3553 & 0.3520 & 0.7773 \\ 0.8853 & 0.7159 & 0.1801 & 0.1777 & 0.1001 & 0.1286 & 0.1916 & 0.1229 \end{bmatrix}$$

$$b^T = [0.2992\ ,\ 0.7490\ ,\ 0.3393\ ,\ 0.6067\ ,\ 0.3062\ ,\ 0.6485\ ,\ 0.2480]$$

Test Problem A.8

$$\varphi = T_{SS}^{p} \ \ (p = 2)$$

$$c^T = [9.3908\ ,\ -3.6503\ ,\ 8.5187\ ,\ 8.1411\ ,\ -6.5879\ ,\ 4.2487\ ,\ 2.2398\ ,\ -0.9847]$$

$$A = \begin{bmatrix} 0.3937 & 0.2372 & 0.4513 & 0.3860 & 0.2277 & 0.6380 & 0.7531 & 0.4817 \\ 0.5974 & 0.3687 & 0.4158 & 0.0403 & 0.3316 & 0.9390 & 0.8440 & 0.3536 \\ 0.6171 & 0.5829 & 0.3588 & 0.3178 & 0.0344 & 0.9683 & 0.0226 & 0.7211 \\ 0.5633 & 0.0769 & 0.0979 & 0.3276 & 0.3442 & 0.2711 & 0.1937 & 0.2618 \\ 0.1654 & 0.0704 & 0.5820 & 0.0502 & 0.5766 & 0.4708 & 0.7445 & 0.7098 \\ 0.1976 & 0.2769 & 0.7885 & 0.2353 & 0.3052 & 0.8182 & 0.3438 & 0.3721 \\ 0.2897 & 0.4605 & 0.5192 & 0.3646 & 0.1443 & 0.9708 & 0.5984 & 0.0733 \end{bmatrix}$$

$$b^T = [0.2235\ ,\ 0.3475\ ,\ 0.4739\ ,\ 0.1328\ ,\ 0.4788\ ,\ 0.0163\ ,\ 0.2283]$$

Test Problem A.9

$$\varphi = T_{SW}^{\lambda} \ \ (\lambda = 2)$$

$$c^T = [-6.0952\ ,\ -3.2494\ ,\ 2.0945\ ,\ -0.1672\ ,\ 1.1542\ ,\ -1.4943\ ,\ 0.8345\ ,\ 6.86]$$

$$A = \begin{bmatrix} 0.9507 & 0.1824 & 0.6716 & 0.3393 & 0.4980 & 0.0681 & 0.3028 & 0.8800 \\ 0.4767 & 0.3615 & 0.2162 & 0.6696 & 0.9345 & 0.3157 & 0.7011 & 0.7735 \\ 0.2093 & 0.0504 & 0.2526 & 0.5754 & 0.0571 & 0.4407 & 0.3550 & 0.2951 \\ 0.0847 & 0.1121 & 0.0232 & 0.2573 & 0.0123 & 0.1752 & 0.1558 & 0.1141 \\ 0.5341 & 0.3756 & 0.6194 & 0.7273 & 0.3675 & 0.9322 & 0.5919 & 0.6010 \\ 0.4135 & 0.7710 & 0.7931 & 0.7238 & 0.1424 & 1.1018 & 0.1037 & 0.7139 \\ 0.0202 & 0.0266 & 0.0358 & 0.9970 & 0.2024 & 0.1484 & 0.1082 & 0.0085 \end{bmatrix}$$

$$b^T = [0.9318\ ,\ 0.6864\ ,\ 0.2702\ ,\ 0.0977\ ,\ 0.6027\ ,\ 0.7675\ ,\ 0.0973]$$

Test Problem A.10

$$\varphi = T_{DP}^{\gamma} \ \ (\gamma = 0.5)$$

$$c^T = [6.4807\ ,\ -9.0013\ ,\ 5.1807\ ,\ -7.8281\ ,\ -6.8893\ ,\ -1.4281\ ,\ -4.2431\ ,\ 9.5203]$$

$$A = \begin{bmatrix} 0.2136 & 0.9211 & 0.7602 & 0.9743 & 0.5270 & 0.1832 & 0.7746 & 0.7225 \\ 0.3926 & 0.8714 & 0.8990 & 0.8519 & 0.9890 & 0.9241 & 0.9913 & 0.9607 \\ 0.0904 & 0.4955 & 0.7030 & 0.4822 & 0.0557 & 0.0067 & 0.4828 & 0.1609 \\ 0.5129 & 0.2813 & 0.1706 & 0.1760 & 0.0094 & 0.0642 & 0.2088 & 0.0025 \\ 0.0965 & 0.4940 & 0.4028 & 0.5738 & 0.2913 & 0.0587 & 0.4544 & 0.0102 \\ 0.1734 & 0.8883 & 0.8364 & 0.7743 & 0.4088 & 0.8070 & 0.8091 & 0.9511 \\ 0.2147 & 0.3333 & 0.2378 & 0.2563 & 0.1611 & 0.0557 & 0.5638 & 0.0464 \end{bmatrix}$$

$$b^T = [0.7055\ ,\ 0.8452\ ,\ 0.4820\ ,\ 0.1692\ ,\ 0.3891\ ,\ 0.7735\ ,\ 0.2370]$$

Appendix B

Table B.1
Summary of the test continuous t-norms used in the definition of max-φ composition in problem (11).

t-norm	Function	Parameter
Minimum	$T_{Min}(x,y) = \min\{x,y\}$	–
Product	$T_P(x,y) = xy$	–
Lukasiewicz	$T_L(x,y) = \max\{x+y-1,0\}$	–
Frank	$T_F^s(x,y) = \begin{cases} T_{Min}(x,y) & ,\ if\ \ s=0 \\ T_P(x,y) & ,\ if\ \ s=1 \\ \log_s[1+((s^x-1)(s^y-1)/(s-1))]\ , & if\ s>0, s\neq 1 \end{cases}$	$s \geq 0$
Yager	$T_Y^p(x,y) = \max\{1-[(1-x)^p+(1-y)^p]^{1/p}, 0\}$	$p > 0$
Hamacher	$T_H^\alpha(x,y) = \begin{cases} 0 & ,\ if\ \alpha = x = y = 0 \\ \frac{xy}{\alpha+(1-\alpha)(x+y-xy)} & ,\ otherwise \end{cases}$	$\alpha \geq 0$

Dombi	$T_D^{\lambda}(x,y) = \begin{cases} 0 & , \ if\ x=0\ or\ y=0 \\ \frac{1}{1+[((1-x)/x)^{\lambda}+((1-y)/y)^{\lambda}]^{1/\lambda}} & , \ otherwise \end{cases}$	$\lambda > 0$
Schweizer - Sklar	$T_{SS}^{p}(x,y) = \begin{cases} (x^p+y^p-1)^{1/p} & , \ if\ -\infty < p < 0 \\ \varphi_P(x,y) & , \ if\ p=0 \\ (\max\{0, x^p+y^p-1\})^{1/p} & , \ if\ 0 < p < \infty \end{cases}$	$-\infty < p < \infty$
Sugeno - Weber	$T_{SW}^{\lambda}(x,y) = \max\{(x+y-1+\lambda xy)/(1+\lambda)\, , 0\}$	$\lambda > -1$
Dubois - Prade	$T_{DP}^{\gamma}(x,y) = \begin{cases} T_{Min}(x,y) & , \ if\ \gamma=0 \\ T_P(x,y) & , \ if\ \gamma=1 \\ (xy)/\max\{x,y,\gamma\} & , \ otherwise \end{cases}$	$\gamma \in [0,1]$

Table B.2
Values of $\underline{x}\ (a_{ij}, b_i)$ for $i \in I$ and $j \in J$, when $a_{ij} \geq b_i$.

t-norm	$\underline{x}\ (a_{ij}, b_i)$	t-norm	$\underline{x}\ (a_{ij}, b_i)$
Minimum	b_i	Hamacher	$[(\alpha+(1-\alpha)\, a_{ij})\, b_i]/[a_{ij}-(1-\alpha)\, (1-a_{ij})\, b_i]$
Product	b_i/a_{ij}	Dombi	$\frac{1}{1+[(1-b_i)/b_i)^{\lambda}-((1-a_{ij})/a_{ij})^{\lambda}]^{1/\lambda}}$
Lukasiewicz	$1+b_i-a_{ij}$	Schweizer - Sklar	$(1+(b_i)^p-{a_{ij}}^p)^{1/p}$
Frank	$\log_s[1+((s^{b_i}-1)\,(s-1))/(s^{a_{ij}}-1)]$	Sugeno - Weber	$[(1+\lambda)b_i+(1-a_{ij})]/(1+\lambda a_{ij})$
Yager	$1-[(1-b_i)^p-(1-a_{ij})^p]^{1/p}$	Dubois - Prade	$\begin{cases} \max\{\, b_i, \gamma\} & , \ if\ \ a_{ij}=b_i \\ b_i & , \ if\ \ a_{ij} > b_i\ ,\ a_{ij} \geq \gamma \\ (\gamma\, b_i)/a_{ij} & , \ if\ \ b_i < a_{ij} < \gamma \end{cases}$

Table B.3
Values of $\bar{x}\ (a_{ij}, b_i)$ for $i \in I$ and $j \in J$, when $a_{ij} \geq b_i$.

t-norm	$\bar{x}\ (a_{ij}, b_i)$	t-norm	$\bar{x}\ (a_{ij}, b_i)$
Minimum	$\begin{cases} 1 & , \ if\ a_{ij}=b_i \\ b_i & , \ if\ a_{ij} > b_i \end{cases}$	Hamacher	$[(\alpha+(1-\alpha)\, a_{ij})\, b_i]/[a_{ij}-(1-\alpha)\, (1-a_{ij})\, b_i]$
Product	b_i/a_{ij}	Dombi	$\frac{1}{1+[(1-b_i)/b_i)^{\lambda}-((1-a_{ij})/a_{ij})^{\lambda}]^{1/\lambda}}$
Lukasiewicz	$1+b_i-a_{ij}$	Schweizer - Sklar	$(1+(b_i)^p-{a_{ij}}^p)^{1/p}$
Frank	$\log_s[1+((s^{b_i}-1)\,(s-1))/(s^{a_{ij}}-1)]$	Sugeno - Weber	$[(1+\lambda)b_i+(1-a_{ij})]/(1+\lambda a_{ij})$
Yager	$1-[(1-b_i)^p-(1-a_{ij})^p]^{1/p}$	Dubois - Prade	$\begin{cases} 1 & , \ if\ \ a_{ij}=b_i \\ b_i & , \ if\ \ a_{ij} > b_i\ ,\ a_{ij} \geq \gamma \\ (\gamma\, b_i)/a_{ij} & , \ if\ \ b_i < a_{ij} < \gamma \end{cases}$

References

[1] O. A. Abdul-Rahman, M. Munetomo, *An adaptive parameter binary-real coded genetic algorithm for constraint optimization problems: Performance analysis and estimation of optimal control parameters*, Information Sciences, **233** (2013), 54-86.

[2] M. S. Bazaraa, H. D. Sherali, C. M. Shetty, *Nonlinear programming: Theory and algorithms*, John Wiley and Sons, New York, NY, 2006.

[3] C. W. Chang, B. S. Shieh, *Linear optimization problem constrained by fuzzy max-min relation equations*, Information Sciences, **234** (2013), 71-79.

[4] L. Chen, P. P. Wang, *Fuzzy relation Equations (i): The general and specialized solving algorithms*, Soft Computing, **6**(5) (2002), 428-435.

[5] M. Cornejo, D. Lobo, J. Medina, *On the solvability of bipolar max-product fuzzy relation equations with the standard negation*, Fuzzy Sets and Systems, **410** (2021), 1-18.

[6] S. Dempe, A. Ruziyeva, *On the calculation of a membership function for the solution of a fuzzy linear optimization problem*, Fuzzy Sets and Systems, **188** (2012), 58-67.

[7] A. Di Nola, S. Sessa, W. Pedrycz, E. Sanchez, *Fuzzy relational equations and their applications in knowledge engineering*, Dordrecht: Kluwer Academic Press, 1989.

[8] D. Dubey, S. Chandra, A. Mehra, *Fuzzy linear programming under interval uncertainty based on IFS representation*, Fuzzy Sets and Systems, **188** (2012), 68-87.

[9] D. Dubois, H. Prade, *Fundamentals of fuzzy sets*, Kluwer, Boston, 2000.

[10] S. C. Fang, G. Li, *Solving fuzzy relational equations with a linear objective function*, Fuzzy Sets and Systems, **103** (1999), 107-113.

[11] S. Freson, B. De Baets, H. De Meyer, *Linear optimization with bipolar max-min constraints*, Information Sciences, **234** (2013), 3-15.

[12] Z. W. Geem, J. H. Kim, G. V. Loganathan, *A new heuristic optimization algorithm: Harmony search*, Simulation, **76**(2) (2001), 60-68.

[13] A. Ghodousian, *An algorithm for solving linear optimization problems subjected to the intersection of two fuzzy relational inequalities defined by Frank family of t-norms*, International Journal in Foundations of Computer Science and Technology, **8**(3) (2018), 1-20.

[14] A. Ghodousian, *Optimization of linear problems subjected to the intersection of two fuzzy relational inequalities defined by Dubois-Prade family of t-norms*, Information Sciences, **503** (2019), 291-306.

[15] A. Ghodousian, A. Babalhavaeji, *An efficient genetic algorithm for solving nonlinear optimization problems defined with fuzzy relational equations and max-Łukasiewicz composition*, Applied Soft Computing, **69** (2018), 475-492.

[16] A. Ghodousian, E. Khorram, *Fuzzy linear optimization in the presence of the fuzzy relation inequality constraints with max-min composition*, Information Sciences, **178** (2008), 501-519.

[17] A. Ghodousian, E. Khorram, *Linear optimization with an arbitrary fuzzy relational inequality*, Fuzzy Sets and Systems, **206** (2012), 89-102.

[18] A. Ghodousian, M. Naeeimib, A. Babalhavaeji, *Nonlinear optimization problem subjected to fuzzy relational equations defined by Dubois-Prade family of t-norms*, Computers and Industrial Engineering, **119** (2018), 167-180.

[19] A. Ghodousian, M. Raeisian Parvari, *A modified PSO algorithm for linear optimization problem subject to the generalized fuzzy relational inequalities with fuzzy constraints (FRI-FC)*, Information Sciences, **418-419** (2017), 317-345.

[20] F. F. Guo, L. P. Pang, D. Meng, Z. Q. Xia, *An algorithm for solving optimization problems with fuzzy relational inequality constraints*, Information Sciences, **252** (2013), 20-31.

[21] F. F. Guo, Z. Q. Xia, *An algorithm for solving optimization problems with one linear objective function and finitely many constraints of fuzzy relation inequalities*, Fuzzy Optimization and Decision Making, **5** (2006), 33-47.

[22] S. M. Guu, Y. K. Wu, *Minimizing a linear objective function with fuzzy relation equation constraints*, Fuzzy Optimization and Decision Making, **12** (2002), 1568-4539.

[23] S. M. Guu, Y. K. Wu, *Minimizing a linear objective function under a max-t-norm fuzzy relational equation constraint*, Fuzzy Sets and Systems, **161** (2010), 285-297.

[24] R. L. Haupt, E. Haupt, *Practical genetic algorithms*, Second ed., John Wiley and Sons, 2004.

[25] J. Kennedy, R. C. Eberhart, *Particle swarm optimization*, in: Proceedings of IEEE International Conference on Neural Networks, **4** (1995), 1942-1948.

[26] F. Kouchakinejad, M. Mashinchi, R. Mesiar, *Solution-set invariant matrices and vectors in fuzzy relation inequalities based on max-aggregation function composition*, Iranian Journal of Fuzzy Systems, **13**(7) (2016), 91-100.

[27] P. K. Li, S. C. Fang, *On the resolution and optimization of a system of fuzzy relational equations with sup-t composition*, Fuzzy Optimization and Decision Making, **7** (2008), 169-214.

[28] P. K. Li, Y. Liu, *Linear optimization with bipolar fuzzy relational equation constraints using Łukasiewicz triangular norm*, Soft Computing, **18** (2014), 1399-1404.

[29] J. X. Li, S. J. Yang, *Fuzzy relation inequalities about the data transmission mechanism in bittorrent-like peer-to-peer file sharing systems*, in: Proceedings of the 9^{th} International Conference on Fuzzy Systems and Knowledge discovery, (2012), 452-456.

[30] J. L. Lin, *On the relation between fuzzy max-Archimedean t-norm relational equations and the covering problem*, Fuzzy Sets and Systems, **160** (2009), 2328-2344.

[31] C. H. Lin, *A rough penalty genetic algorithm for constrained optimization*, Information Sciences, **241** (2013), 119-137.

[32] C. C. Liu, Y. Y. Lur, Y. K. Wu, *Linear optimization of bipolar fuzzy relational equations with max-Łukasiewicz composition*, Information Sciences, **360** (2016), 149-162.

[33] J. Loetamonphong, S. C. Fang, *Optimization of fuzzy relation equations with max-product composition*, Fuzzy Sets and Systems, **118** (2001), 509-517.

[34] A. V. Markovskii, *On the relation between equations with max-product composition and the covering problem*, Fuzzy Sets and Systems, **153** (2005), 261-273.

[35] H. D. Mazraeh, A. Molai, *Resolution of nonlinear optimization problems subject to bipolar max-min fuzzy relation equation constraints using genetic algorithm*, Iranian Journal of Fuzzy Systems, (2018), 109-131.

[36] M. Mizumoto, H. J. Zimmermann, *Comparison of fuzzy reasoning method*, Fuzzy Sets and Systems, **8** (1982), 253-283.

[37] W. Pedrycz, *Granular computing: Analysis and design of intelligent systems*, CRC Press, Boca Raton, 2013.

[38] C. Pornsing, K. Saichareon, T. Karot, *A modified particle swarm optimization for engineering constrained optimization problems*, International Journal of Computer Science and Electronics Engineering, **3**(1) (2015), 2320-4028.

[39] X. B. Qu, X. P. Wang, M. H. Lei, *Conditions under which the solution sets of fuzzy relational equations over complete Brouwerian lattices form lattices*, Fuzzy Sets and Systems, **234** (2014), 34-45.

[40] E. Sanchez, *Solution in composite fuzzy relation equations: Application to medical diagnosis in Brouwerian logic, in: M.M. Gupta. G.N. Saridis, B.R. Games (Eds.)*, Fuzzy Automata and Decision Processes, North-Holland, New York, (1977), 221-234.

[41] B. S. Shieh, *Minimizing a linear objective function under a fuzzy max-t-norm relation equation constraint*, Information Sciences, **181** (2011), 832-841.

[42] K. Socha, M. Dorigo, *Ant colony optimization for continuous domain*, European Journal of Operational Research, **185** (2008), 1155-1173.

[43] F. Sun, *Conditions for the existence of the least solution and minimal solutions to fuzzy relation equations over complete Brouwerian lattices*, Information Sciences, **205** (2012), 86-92.

[44] F. Sun, X. P. Wang, X. B. Qu, *Minimal join decompositions and their applications to fuzzy relation equations over complete Brouwerian lattices*, Information Sciences, **224** (2013), 143-151.

[45] C. L. Sun, J. C. Zeng, J. S. Pan, *An improved vector particle swarm optimization for constrained optimization problems*, Information Sciences, **181** (2011), 1153-1163.

[46] S. Wang, H. Li, *Resolution of fuzzy relational inequalities with Boolean semi-tensor product composition*, Mathematics, **9** (2021), 937.

[47] Y. K. Wu, *Optimization of fuzzy relational equations with max-av composition*, Information Sciences, **177** (2007), 4216-4229.

[48] Y. K. Wu, S. M. Guu, *Minimizing a linear function under a fuzzy max-min relational equation constraints*, Fuzzy Sets and Systems, **150** (2005), 147-162.

[49] Y. K. Wu, S. M. Guu, *An efficient procedure for solving a fuzzy relation equation with max-Archimedean t-norm composition*, IEEE Transactions on Fuzzy Systems, **16** (2008), 73-84.

[50] Y. K. Wu, S. M. Guu, J. Y. Liu, *Reducing the search space of a linear fractional programming problem under fuzzy relational equations with max-Archimedean t-norm composition*, Fuzzy Sets and Systems, **159** (2008), 3347-3359.

[51] S. J. Yang, *An algorithm for minimizing a linear objective function subject to the fuzzy relation inequalities with addition-min composition*, Fuzzy Sets and Systems, **255** (2014), 41-51.

[52] X. P. Yang, X. G. Zhou, B. Y. Cao, *Latticized linear programming subject to max-product fuzzy relation inequalities with application in wireless communication*, Information Sciences, **358-359** (2016), 44-55.

[53] C. T. Yeh, *On the minimal solutions of max-min fuzzy relation equations*, Fuzzy Sets and Systems, **159** (2008), 23-39.