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Some aspects on computation of scalar valued and fuzzy valued integrals over fuzzy domains

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Abstract

In this note, we consider the problem of computing a scalar-valued, and also a fuzzy-valued integral over fuzzy spatial domains, which is useful, for instance, to calculate fuzzy areas described by a certain property, allowing the density function to be variable, or evaluate magnitudes in the case where the field function is fuzzily known.

Keywords: Integrals over fuzzy domains.

1 Introduction

In the representation of physical information, different types of errors may be present. Many of the values which are processed are very often subject to inexactness and the use of qualitative variables makes it appropriate to consider levels of precision. For instance, a pine forest might not be conformed of pines exclusively on all its extension [14]. Moreover, in the classification of images obtained by satellite, the value which is assigned to each parcel is considered to correspond to a certain type of soil, taking into account the difference between the specific value and a value which is considered as representative of that class.

In the processing of information, some other aspects influence the exactitude of data: the elements which are not relevant are not considered, and this depends on a decision previously taken about what is relevant or not, the dimensions of the objects must satisfy certain minimum requirements, etc. In addition to this, errors are also produced during the process of registration of information itself, or in the mechanism of interpretation of the information obtained (measure or classification errors), and there can also exist accidental omissions in the processing of information [14].

To cite an example, vectorial geographic information systems emphasize the definition of the boundaries of the geographical elements. The changes observed in the boundary are useful to fix the places which have been changing their properties through recent changes.

The monograph [14] describes the main role of the use of information technologies in general, and geographical information systems in particular, to deal with spatial information. The authors indicate the processes which are needed to study the characteristics of geographical areas, by handling and analyzing data, and the difficulty found in the analysis of the error, specially in connection with decision making. Some models of study are proposed in detail. See also [13] for an overview of [14].

For the positioning of different entities on the space, three types of spatial objects are commonly used: dots, lines and polygons. To represent spatial objects which might be subject to uncertainty, we use:

- Fuzzy dots, to indicate objects whose dimensions are very small considering the dimension of the space, where the core is an unitary set.

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- Fuzzy lines (to represent networks), whose core set is a real line.
- Fuzzy polygons, to represent parcels with inexact boundaries. To represent polygons, we may choose a high-dimensional fuzzy set approach, or fix a sequence of fuzzy lines.

Depending on the scaling, we might choose a dot or a polygon to represent a certain object.

The establishment of boundaries between different areas takes into account the different land use purposes, types of socio-economic activities, or some characteristics or appearance. Fuzzy boundaries are useful to differentiate areas and also to analyze the change produced in some region where a certain characteristic is extending to its surroundings. In many cases, there do not exist exact spatial limits for regions to be analyzed, see, for instance, [33]. This way, the identification and analysis of boundaries is an issue of interest in geographic information systems (GIS), since variations of data between areas are not necessarily sharp. This occurs, for instance, in the identification of soil types on the basis of the proportions of different elements. In this type of tasks, exact regions are not appropriate to represent transitional zones. Fuzzy sets are introduced as a tool for modelling the natural variability which appears in the environmental phenomena. See, for instance, [1], for a fuzzy method to estimate forest fire risky areas.

Burrough and McDonnell discuss the application of fuzzy sets to spatial classification and boundary identification problems, see also [4, 5, 6, 7]. In [31], membership degrees are assigned to points of fuzzy boundaries, presenting a technique for modeling broad boundaries.

Some membership functions that are used to deal with certain important issues in the representation of geographical information are [29] sigmoidal or (double) s-shaped functions $m = 1/(1+a(z-c)^2)$ (where a is a parameter that varies the width of the function, z the property being modeled and c the value of this proportion at the function midpoint), J-shaped functions $m = 1/(1+((z-a)/(a-c))^2)$, or linear functions, which, in many cases, are chosen to be symmetric.

The classification problem in GIS or in social sciences is often based on linguistic notions that are inexact. The visualization of fuzzy boundaries favors a better understanding of geographic objects with imprecise boundaries which correspond to different graded categories [5, 6, 17].

The structure in layers of fuzzy regions is useful to model the spatial distribution of regions with inexact boundaries. For instance, studying the classification of soils, it is not appropriate to consider the existence of abrupt changes between regions, and different methods are used to represent the width of transition zones [12].

An application of fuzzy set theory in forest planning models can be found in [3]. In [20], a method is provided to estimate forest area using uncertainty levels. See also [19], where an application of fuzzy representation of geographic boundary to the soil loss model is presented. On the other hand, reference [18] is related with the generation of fuzzy categorical maps from remotely sensed images.

Attending to these applications, the mathematical properties of the integration on fuzzy domains are interesting to deal with the characteristics of inexact regions. In [30], Sugeno introduced the theory of fuzzy measures and fuzzy integrals. Some of its properties were studied in [24, 32], and references [9, 15, 16, 25, 26, 27] show that several integral inequalities also hold for the fuzzy context. Another generalization is given by the concept of uncertain integral [21], and some of their properties are studied in [34]. On the other hand, in [22], it is provided the concept of Gould integrability for interval-valued multifunctions with respect to interval-valued set multifunctions. The Sugeno fuzzy integral for concave functions is considered in [11], and non-linear integrals based on decomposition integrals attending to greediness are given in [28]. Our approach allows to evaluate the integral over a fuzzy set for the case of real-valued as well as fuzzy-valued functions, by reducing the problem to certain calculations in the Euclidean space.

This work is structured as follows. In Section 2, we recall some basic concepts on fuzzy sets. In Section 3, we use fuzzy sets to represent certain objects which are fuzzily known and, in Sections 4 and 5, we provide the definition of the integral of a real-valued, and a fuzzy-valued function on a fuzzy domain, with some examples. In Section 6, we present some examples of integration over two-dimensional fuzzy regions. Finally, in Section 7, we present some conclusions.

2 Fuzzy sets

We call n-dimensional fuzzy set to any function $u : \mathbb{R}^n \longrightarrow [0,1]$. Given a fuzzy set u, the α-cut of u is

$$[u]^\alpha = \{x \in \mathbb{R}^n \ : \ u(x) \geq \alpha\}, \text{ for all } \alpha \in (0,1],$$

and $[u]^0 = \overline{\bigcup_{\alpha\in(0,1]} [u]^\alpha}$ is the support of u.

For fuzzy sets in the space of fuzzy intervals $\mathcal{E}^1$, that is, the space of normal, upper semicontinuous, fuzzy-convex,

$$u(\lambda x_1 + (1-\lambda)x_2) \geq \min\{u(x_1), u(x_2)\}, \text{ for all } x_1,\, x_2 \in \mathbb{R},\, \lambda \in [0,1],$$

and with bounded support [8, 10, 35], the cuts $[u]^\alpha$ are nonempty compact and convex subsets of $\mathbb{R}$, for every $\alpha \in [0,1]$. An analogous space $\mathcal{E}^n$ can be defined for dimension higher than one.

A special type of fuzzy sets is the following $\chi_A : \mathbb{R}^n \longrightarrow [0,1]$, defined as $\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise,} \end{cases} \quad x \in \mathbb{R}^n$, where $A \subset \mathbb{R}^n$, called 'crisp' numbers.

We can define a distance in $\mathcal{E}^n$ by $d_\infty(x,y) = \sup_{\alpha\in[0,1]} d_H([x]^\alpha, [y]^\alpha)$, for x, $y \in \mathcal{E}^n$, where d_H is the Hausdorff distance between nonempty compact convex subsets of $\mathbb{R}^n$. The metric space $(\mathcal{E}^n, d_\infty)$ is complete.

The cuts play an important role in the representation of fuzzy regions. In image processing, for instance, the selection of an specific α-cut is used very often as a defuzzification method, that is, the reduction of a fuzzy set to a crisp set.

3 Objects

We use fuzzy sets to represent imprecise objects of different types. For instance, to represent the point $1 \in \mathbb{R}$, we might use the characteristic function $\chi_{\{1\}}$, which is a crisp (and singleton, in this case) fuzzy interval, or a non-crisp triangular fuzzy interval $(1;1,1)$, given by

$$\varphi(x) := \begin{cases} x, & \text{if } x \in [0,1], \\ -x+2, & \text{if } x \in [1,2], \\ 0, & \text{otherwise,} \end{cases}$$

or, more generally, the non-symmetric triangular fuzzy interval $(1;\delta,\gamma)$, whose expression is

$$\mu(x) := \begin{cases} 1 + \frac{1}{\delta}(x-1), & \text{if } x \in [1-\delta, 1], \\ 1 - \frac{1}{\gamma}(x-1), & \text{if } x \in [1, 1+\gamma], \\ 0, & \text{otherwise.} \end{cases}$$

In the two-dimensional space, a certain object located at the coordinates $(1,1)$ could be represented by the characteristic function of $\chi_{\{(1,1)\}}$, or by a conical set with center of gravity $(1,1)$ at each level,

$$\phi(x,y) := \begin{cases} 1 - \sqrt{(x-1)^2 + (y-1)^2}, & \text{if } (x-1)^2 + (y-1)^2 \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Next, we calculate the expression of a imprecise line L whose core is determined by the exact points of $\mathbb{R}^2$: (x_1, y_1) and (x_2, y_2).

First, we consider the case where $x_1 \neq x_2$ and $y_1 \neq y_2$.

We consider a margin of error of $\delta > 0$ and $\gamma > 0$, respectively, on each side. To build the 'imprecise line', we consider the segment line s which joins the points (x_1, y_1) and (x_2, y_2):

$$y = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1),$$

and oblige the cuts of the fuzzy set representing this line L to contain line s longitudinally. To this purpose, we consider the lines passing through (x_1, y_1) and (x_2, y_2), respectively, which are perpendicular to the line s, that is,

$$y = y_1 - \frac{x_2 - x_1}{y_2 - y_1}(x - x_1), \qquad \text{and} \qquad y = y_2 - \frac{x_2 - x_1}{y_2 - y_1}(x - x_2).$$

If we allow a horizontal separation of the vertices of the support from the endpoints of the segment line s of δ and γ, respectively, we get the following points in the plane that determine the support of the line L:

A: $\left(x_1 + \gamma, y_1 - \dfrac{x_2 - x_1}{y_2 - y_1}\gamma\right)$,

B: $\left(x_2 + \gamma, y_2 - \dfrac{x_2 - x_1}{y_2 - y_1}\gamma\right)$,

C: $\left(x_1 - \delta, y_1 + \dfrac{x_2 - x_1}{y_2 - y_1}\delta\right)$,

D: $\left(x_2 - \delta, y_2 + \dfrac{x_2 - x_1}{y_2 - y_1}\delta\right)$.

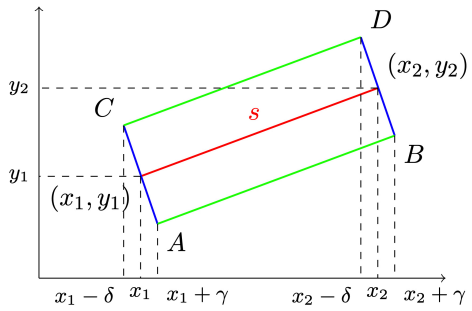


Figure 1: Segment line s in red and perpendicular segment lines in blue.

To obtain the cuts of L, we calculate the expression of the plane which contains the points $\tilde{A}: \left(x_1+\gamma, y_1 - \dfrac{x_2-x_1}{y_2-y_1}\gamma, 0\right)$, $\tilde{B}: \left(x_2+\gamma, y_2 - \dfrac{x_2-x_1}{y_2-y_1}\gamma, 0\right)$, $(x_1, y_1, 1)$ (and, of course, $(x_2, y_2, 1)$), that is,

$$(x, y, z) = (x_1, y_1, 1) + \lambda(x_2 - x_1, y_2 - y_1, 0) + \mu\left(\gamma, -\frac{x_2-x_1}{y_2-y_1}\gamma, -1\right).$$

From this expression, we derive

$$\lambda = \frac{(x-x_1)(x_2-x_1) + (y-y_1)(y_2-y_1)}{(x_2-x_1)^2 + (y_2-y_1)^2}, \tag{1}$$

$$\mu = \frac{1}{\gamma}\left(x - x_1 - \frac{(x-x_1)(x_2-x_1) + (y-y_1)(y_2-y_1)}{(x_2-x_1)^2 + (y_2-y_1)^2}(x_2-x_1)\right),$$

hence the expression is given by

$$z = 1 - \frac{1}{\gamma}\frac{(x-x_1)(y_2-y_1) + (y-y_1)(x_2-x_1)}{(x_2-x_1)^2 + (y_2-y_1)^2}(y_2-y_1).$$

On the other hand, for the expression of the plane which contains the points $\tilde{C}$: $\left(x_1-\delta, y_1 + \dfrac{x_2-x_1}{y_2-y_1}\delta, 0\right)$, $\tilde{D}$: $\left(x_2-\delta, y_2 + \dfrac{x_2-x_1}{y_2-y_1}\delta, 0\right)$, $(x_1, y_1, 1)$ (and, also, $(x_2, y_2, 1)$), we get

$$(x, y, z) = (x_1, y_1, 1) + \lambda(x_2 - x_1, y_2 - y_1, 0) + \mu\left(-\delta, \frac{x_2-x_1}{y_2-y_1}\delta, -1\right).$$

This equation implies that the value of λ coincides with the expression in (1), and

$$\mu = \frac{1}{\delta}\left(x_1 - x + \frac{(x-x_1)(x_2-x_1) + (y-y_1)(y_2-y_1)}{(x_2-x_1)^2 + (y_2-y_1)^2}(x_2-x_1)\right),$$

hence the plane we seek is given by the graph of a two-variable function

$$z = 1 - \frac{1}{\delta}\frac{(x_1-x)(y_2-y_1) + (y-y_1)(x_2-x_1)}{(x_2-x_1)^2 + (y_2-y_1)^2}(y_2-y_1).$$

Hence, the expression of L is given by

$$L(x,y) := \begin{cases} 1 - \frac{1}{\gamma}\dfrac{(x-x_1)(y_2-y_1) + (y-y_1)(x_2-x_1)}{(x_2-x_1)^2 + (y_2-y_1)^2}(y_2-y_1), & \text{in the rectangle } C_1, \\ 1 - \frac{1}{\delta}\frac{(x_1-x)(y_2-y_1) + (y-y_1)(x_2-x_1)}{(x_2-x_1)^2 + (y_2-y_1)^2}(y_2-y_1), & \text{in the rectangle } C_2, \end{cases}$$

where C_1 is the rectangle with edges A, B, (x_1, y_1) and (x_2, y_2), that is, the union $C_1 = C_{11} \cup C_{12} \cup C_{13}$ of the following sets:

$$C_{11} := \{(x,y) \in \mathbb{R}^2 \mid x \in [x_1, x_1+\gamma] \text{ and } y_1 - \frac{x_2-x_1}{y_2-y_1}(x-x_1) \le y \le y_1 + \frac{y_2-y_1}{x_2-x_1}(x-x_1)\},$$

$$C_{12} := \{(x,y) \in \mathbb{R}^2 \mid x \in [x_1+\gamma, x_2] \text{ and } y_1 + \frac{y_2-y_1}{x_2-x_1}(x-x_1) - \frac{(x_2-x_1)^2+(y_2-y_1)^2}{(x_2-x_1)(y_2-y_1)}\gamma \le y \le y_1 + \frac{y_2-y_1}{x_2-x_1}(x-x_1)\},$$

$$C_{13} := \{(x,y) \in \mathbb{R}^2 \mid x \in [x_2, x_2+\gamma] \text{ and } y_1 + \frac{y_2-y_1}{x_2-x_1}(x-x_1) - \frac{(x_2-x_1)^2+(y_2-y_1)^2}{(x_2-x_1)(y_2-y_1)}\gamma \le y \le y_2 - \frac{x_2-x_1}{y_2-y_1}(x-x_2)\},$$

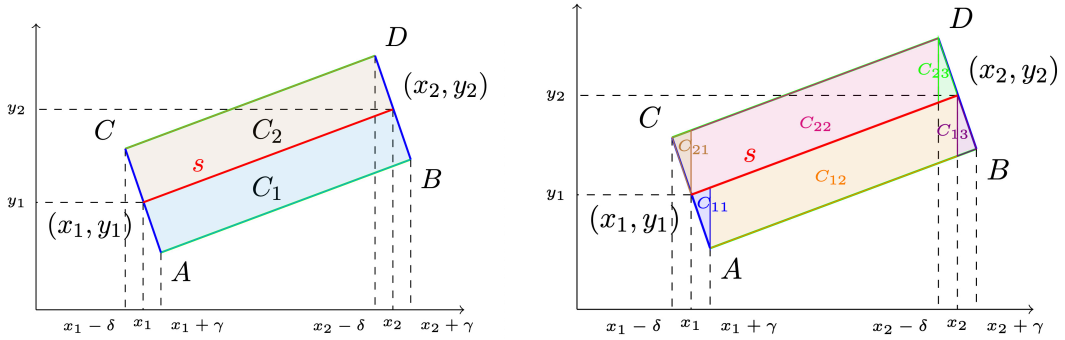
and C_2 is the rectangle with edges C, D, (x_1,y_1) and (x_2,y_2), that is, $C_2 = C_{21} \cup C_{22} \cup C_{23}$, where

$$C_{21} := \{(x,y) \in \mathbb{R}^2 \mid x \in [x_1-\delta, x_1] \text{ and } y_1 - \frac{x_2-x_1}{y_2-y_1}(x-x_1) \le y \le y_1 + \frac{y_2-y_1}{x_2-x_1}(x-x_1) + \delta\frac{(x_2-x_1)^2+(y_2-y_1)^2}{(x_2-x_1)(y_2-y_1)}\},$$

$$C_{22} := \{(x,y) \in \mathbb{R}^2 \mid x \in [x_1, x_2-\delta] \text{ and } y_1 + \frac{y_2-y_1}{x_2-x_1}(x-x_1) \le y \le y_1 + \frac{y_2-y_1}{x_2-x_1}(x-x_1) + \delta\frac{(x_2-x_1)^2+(y_2-y_1)^2}{(x_2-x_1)(y_2-y_1)}\},$$

$$C_{23} := \{(x,y) \in \mathbb{R}^2 \mid x \in [x_2-\delta, x_2] \text{ and } y_1 + \frac{y_2-y_1}{x_2-x_1}(x-x_1) \le y \le y_2 - \frac{x_2-x_1}{y_2-y_1}(x-x_2)\}.$$

Note that the cuts of this fuzzy set are rectangles, thus we call L of orthogonal type. We remark that the core of L is the segment line s joining (x_1,y_1) and (x_2,y_2).



(a) Region C_1 in cyan and C_2 in brown.

(b) Regions C_{ij}, $i = 1,2$, $j = 1,2,3$.

Figure 2: Regions.

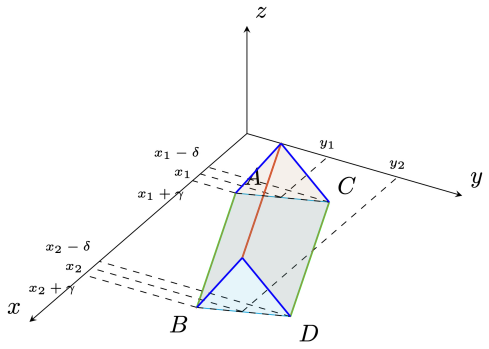


Figure 3: Imprecise line given by an orthogonal fuzzy set L.

Other type of line $\tilde{L}$ can be defined by considering cuts which are not rectangles but romboids. This construction is valid for the points (x_1,y_1) and (x_2,y_2) with $x_1 \neq x_2$ and $y_1 \neq y_2$.

In this case, the line joining $(x_1, y_1 - \gamma)$ and $(x_2, y_2 - \gamma)$, that is,

$$y = y_1 - \gamma + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1),$$

and the line joining $(x_1, y_1 + \delta)$ and $(x_2, y_2 + \delta)$, that is,

$$y = y_1 + \delta + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1),$$

are part of the boundary of the support of $\tilde{L}$, and other lines defining the support have the same direction of the coordinate axis OY, that is, $x = x_1$ and $x = x_2$, respectively.

To obtain the expression of $\tilde{L}$, we calculate the equation of the plane containing the points $(x_1, y_1-\gamma, 0)$, $(x_2, y_2-\gamma, 0)$, $(x_1, y_1, 1)$ (and $(x_2, y_2, 1)$), obtaining

$$(x, y, z) = (x_1, y_1, 1) + \lambda(x_2 - x_1, y_2 - y_1, 0) + \mu\,(0, -\gamma, -1)\,.$$

Hence,

$$\lambda = \frac{x - x_1}{x_2 - x_1}, \qquad \mu = \frac{1}{\gamma}\left(\frac{x - x_1}{x_2 - x_1}(y_2 - y_1) - (y - y_1)\right),$$

and therefore

$$z = 1 - \frac{1}{\gamma}\left(\frac{x - x_1}{x_2 - x_1}(y_2 - y_1) - (y - y_1)\right).$$

Next, we calculate the expression of the plane passing through $(x_1, y_1 + \delta, 0)$, $(x_2, y_2 + \delta, 0)$, $(x_1, y_1, 1)$ (and $(x_2, y_2, 1)$), as follows

$$(x, y, z) = (x_1, y_1, 1) + \lambda(x_2 - x_1, y_2 - y_1, 0) + \mu\,(0, \delta, -1)\,.$$

In this case,

$$\mu = \frac{1}{\delta}\left(y - y_1 - \frac{x - x_1}{x_2 - x_1}(y_2 - y_1)\right),$$

and, thus

$$z = 1 - \frac{1}{\delta}\left(y - y_1 - \frac{x - x_1}{x_2 - x_1}(y_2 - y_1)\right).$$

This expression is valid in the romboid of edges $(x_1, y_1 + \delta)$, $(x_2, y_2 + \delta)$, (x_1, y_1) (and (x_2, y_2)).

The expression of $\tilde{L}$ is, in consequence, given by

$$\tilde{L}(x, y) := \begin{cases} 1 - \frac{1}{\gamma}\left(\frac{x-x_1}{x_2-x_1}(y_2 - y_1) - (y - y_1)\right), \\ \qquad \text{if } x \in [x_1, x_2], y_1 - \delta + \frac{y_2-y_1}{x_2-x_1}(x - x_1) \le y \le y_1 + \delta + \frac{y_2-y_1}{x_2-x_1}(x - x_1), \\ 1 - \frac{1}{\delta}\left(y - y_1 - \frac{x-x_1}{x_2-x_1}(y_2 - y_1)\right), \\ \qquad \text{if } x \in [x_1, x_2], y_1 + \frac{y_2-y_1}{x_2-x_1}(x - x_1) \le y \le y_1 + \delta + \frac{y_2-y_1}{x_2-x_1}(x - x_1). \end{cases}$$

$\tilde{L}$ is an 'oblique' line.

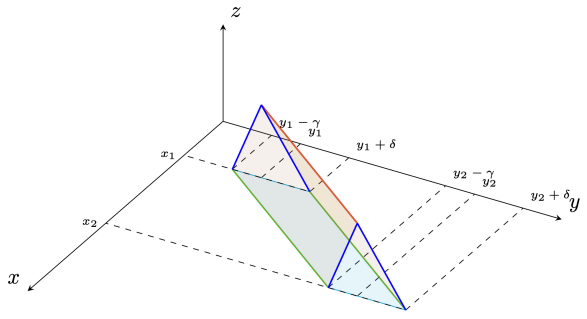


Figure 4: Imprecise line given by an oblique fuzzy set $\tilde{L}$.

Next, we study the cases:

- $x_1 = x_2$ and $y_1 < y_2$;
- $y_1 = y_2$ and $x_1 < x_2$.

In these singular cases, the previous approaches coincide and the cuts are rectangles, since the line joining (x_1, y_1) and (x_1, y_2), respectively (x_1, y_1) and (x_2, y_1), is parallel to one of the coordinate axis.

We start with $x_1 = x_2$ and $y_1 < y_2$. Let us calculate the plane passing through $(x_1-\delta, y_1, 0)$, $(x_1-\delta, y_2, 0)$, $(x_1, y_2, 1)$ (and $(x_1, y_1, 1)$):

$$(x, y, z) = (x_1, y_1, 1) + \lambda(0, y_2 - y_1, 0) + \mu\,(-\delta, 0, -1)\,.$$

Since the values of y are arbitrary, and $\mu = \frac{1}{\delta}(x_1 - x)$, the equation is given by

$$z = 1 - \frac{1}{\delta}(x_1 - x),$$

which depends only on x_1 and $x_1 - \delta$. The set of interest is $x \in [x_1 - \delta, x_1]$ and $y \in [y_1, y_2]$. On the other hand, the plane by $(x_1, y_1, 1)$, $(x_1, y_2, 1)$, $(x_1 + \gamma, y_1, 0)$ (and $(x_1 + \gamma, y_2, 0)$) has equation:

$$(x, y, z) = (x_1, y_1, 1) + \lambda(0, y_2 - y_1, 0) + \mu\,(\gamma, 0, -1)\,.$$

In this case, $\mu = \frac{1}{\gamma}(x - x_1)$, so that the expression is

$$z = 1 - \frac{1}{\gamma}(x - x_1),$$

for the set of interest $x \in [x_1, x_1+\gamma]$ and $y \in [y_1, y_2]$ (rectangle with edges (x_1, y_1), (x_1, y_2), $(x_1+\gamma, y_1)$ (and $(x_1+\gamma, y_2)$).

Similarly, we consider the case where $y_1 = y_2$ and $x_1 < x_2$. The plane through $(x_1, y_1-\delta, 0)$, $(x_2, y_1-\delta, 0)$, $(x_1, y_1, 1)$ (and $(x_2, y_1, 1)$) is given by

$$(x, y, z) = (x_1, y_1, 1) + \lambda(x_2 - x_1, 0, 0) + \mu\,(0, -\delta, -1)\,.$$

Hence

$$z = 1 - \frac{1}{\delta}(y_1 - y),$$

valid for $x \in [x_1, x_2]$ and $y \in [y_1-\delta, y_1]$, the rectangle of edges $(x_1, y_1-\delta)$, (x_2, y_1), (x_1, y_1) and $(x_2, y_1-\delta)$. Considering the points $(x_1, y_1, 1)$, $(x_1, y_1 + \gamma, 0)$, $(x_2, y_1 + \gamma, 0)$ (and $(x_2, y_1, 1)$), it is defined the plane:

$$(x, y, z) = (x_1, y_1, 1) + \lambda(x_2 - x_1, 0, 0) + \mu\,(0, \gamma, -1)\,,$$

hence, $\mu = \frac{1}{\gamma}(y - y_1)$, and

$$z = 1 - \frac{1}{\gamma}(y - y_1),$$

for the set of interest $x \in [x_1, x_2]$ and $y \in [y_1, y_1+\gamma]$ (rectangle with edges (x_1, y_1), (x_2, y_1), $(x_1, y_1+\gamma)$ (and $(x_2, y_1+\gamma)$).

Some interesting properties of lines from the point of view of the representation of information are the lenght, shape, slope, and orientation. The lines obtained can be considered as particular cases of fuzzy polygons.

To represent a flat region, we use a two-dimensional fuzzy set, that is, $u : \mathbb{R}^2 \longrightarrow [0, 1]$. With respect to some properties of this type of regions which will be of interest, we mention the area, perimeter, shape, slope, and orientation.

4 Integral of a scalar function on a fuzzy domain

In the following two sections, we consider the integral of a scalar-valued or a fuzzy-valued function over a fuzzy domain. In this section, we study the case of scalar-valued functions.

In what follows, we also use the notation $\mathcal{F}(X, Y)$ to represent the set of functions from X to Y, and m to denote the Lebesgue measure in $\mathbb{R}^n$.

First, we consider a function $f : [a, b] \longrightarrow \mathbb{R}$, and a fuzzy interval $A \in \mathcal{E}^1$, whose cuts $[A]^\alpha$ are nonempty compact and convex real sets, for every $\alpha \in [0, 1]$. The problem of interest is to calculate the integral of f on the fuzzy domain

A, which represents a region with a fuzzy boundary, varying from the core $[A]^1$ and its support $[A]^0$, which must be included in the domain of f, $[a,b]$.

In the simplest case, where $A=\chi_{[c,d]}$, with $[c,d]\subseteq[a,b]$, we suppose that f is measurable and integrable on $[c,d]$ in the sense of Lebesgue, then we define $\int_A f$ as a function of the variable $\alpha\in[0,1]$, that is, $\int_A f\in\mathcal{F}([0,1],\mathbb{R})$, in the following way:

$$\left(\int_{\chi_{[c,d]}} f\right)(\alpha)=\int_{\left[\chi_{[c,d]}\right]^\alpha} f(x)\,dx=\int_{[c,d]} f(x)\,dx,$$

where the integral is understood in the sense of Lebesgue.

Proposition 4.1. *If $A=\chi_{[c,d]}$, and f is integrable on $[c,d]$ in the sense of Lebesgue, then $\int_A f$ is a constant function on $[0,1]$ (and equal to $\int_{[c,d]} f(x)\,dx$).*

Similarly, we can define the integral of a real function $f:[a,b]\longrightarrow\mathbb{R}$ on an arbitrary fuzzy set $A\in\mathcal{E}^1$. We present directly a more general concept for the integral of a real n-variable function on an n-dimensional fuzzy set $A\in\mathcal{E}^n$.

Definition 4.2. *If $A\in\mathcal{E}^n$, and $f:C\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}$ is measurable on $[A]^0\subseteq C$ (and hence on each $[A]^\alpha$, for every $\alpha\in[0,1]$), then we can define*

$$\left(\int_A f\right)(\alpha)=\int_{[A]^\alpha} f(x)\,dx,$$

where the integral on the right-hand side is calculated in the sense of Lebesgue. We say that f is integrable on A if the n-dimensional integral $\int_{[A]^0} f(x)\,dx$ is finite. In that case, we define the mapping

$$\begin{array}{rccl}\int_A f: & [0,1] & \longrightarrow & \mathbb{R}\\ & \alpha & \longrightarrow & \left(\int_A f\right)(\alpha)=\int_{[A]^\alpha} f(x)\,dx.\end{array}$$

In the crisp case, χ_B, where $B\subseteq\mathbb{R}^n$, we have

$$\left(\int_{\chi_B} f\right)(\alpha)=\int_B f(x)\,dx,\ \forall\alpha,$$

obtaining a constant function.

Besides, if $A\in\mathcal{E}^n$ is such that its core $[A]^1$ is unitary, then it has measure zero in $\mathbb{R}^n$, and

$$\left(\int_A f\right)(1)=\int_{[A]^1} f(x)\,dx=0.$$

Proposition 4.3. *Let $A\in\mathcal{E}^1$ and f be continuous on the compact set $[A]^0$ (or f be integrable on A and f bounded on the compact set $[A]^0$). If the fuzzy interval $A\in\mathcal{E}^1$ is continuous, that is, the functions A_L and A_R given by $[A]^\alpha=[A_L(\alpha),A_R(\alpha)]$, for $\alpha\in[0,1]$, are continuous with respect to the membership grade α, then $\int_A f$ is (uniformly) continuous on $[0,1]$.*

Proof. Suppose that $\beta\in(0,1]$ is fixed and $\alpha<\beta$, then by the properties of the fuzzy intervals, $[A]^\beta\subseteq[A]^\alpha$, and

$$\left|\left(\int_A f\right)(\alpha)-\left(\int_A f\right)(\beta)\right|=\left|\int_{[A]^\alpha} f(x)\,dx-\int_{[A]^\beta} f(x)\,dx\right|=\left|\int_{[A]^\alpha\setminus[A]^\beta} f(x)\,dx\right|\leq m([A]^\alpha\setminus[A]^\beta)k,$$

where k is a bound for f on the compact set $[A]^0$.

Since $m([A]^\alpha\setminus[A]^\beta)=m([A]^\alpha\cap([A]^0\setminus[A]^\beta))\overset{\alpha\uparrow\beta}{\to}m([A]^\beta\cap([A]^0\setminus[A]^\beta))=m(\emptyset)=0$, we obtain the left continuity of the integral mapping. On the other hand, if $\alpha>\beta$, then $[A]^\alpha\subseteq[A]^\beta$, so that

$$\left|\left(\int_A f\right)(\alpha)-\left(\int_A f\right)(\beta)\right|=\left|\int_{[A]^\beta} f(x)\,dx-\int_{[A]^\alpha} f(x)\,dx\right|=\left|\int_{[A]^\beta\setminus[A]^\alpha} f(x)\,dx\right|\leq m([A]^\beta\setminus[A]^\alpha)k,$$

and $m([A]^\beta\setminus[A]^\alpha)=m([A]^\beta\cap([A]^0\setminus[A]^\alpha))\overset{\alpha\downarrow\beta}{\to}m([A]^\beta\cap([A]^0\setminus[A]^\beta))=m(\emptyset)=0$ again. □

Proposition 4.4. *Let $A\in\mathcal{E}^1$ and f be continuous on the compact set $[A]^0$ (or f be integrable on A and f bounded on the compact set $[A]^0$). Then $\int_A f$ is left-continuous on $(0,1]$ and right-continuous at $\alpha=0$.*

Proposition 4.5. *Let $A \in \mathcal{E}^n$ and f be integrable on A. If f is a nonnegative function on A, $f \geq 0$ on $[A]^0$, then $\int_A f$ is a (nonnegative) decreasing function on $[0,1]$.*

Proof. The sign of the function is clearly deduced. On the other hand, for $\alpha \leq \beta$, $[A]^\beta \subseteq [A]^\alpha$, and

$$\left(\int_A f\right)(\beta) = \int_{[A]^\beta} f(x)\,dx \leq \int_{[A]^\alpha} f(x)\,dx = \left(\int_A f\right)(\alpha).$$

□

Similarly, we obtain the following result.

Proposition 4.6. *Let $A \in \mathcal{E}^n$ and f be integrable on A. If f is a nonpositive function on A, $f \leq 0$ on $[A]^0$, then $\int_A f$ is a (nonpositive) increasing function on $[0,1]$.*

Proposition 4.7. *Let $A \in \mathcal{E}^n$ and f be integrable on A. If $f = f^+ - f^-$, where f^+ and f^- are nonnegative on A, then $\left(\int_A f\right)(\alpha) = \int_{[A]^\alpha} f^+(x)\,dx - \int_{[A]^\alpha} f^-(x)\,dx$.*

4.1 Particular cases

Consider a triangular fuzzy interval $A = (r; \delta, \gamma) \in \mathcal{E}^1$, with $r \in \mathbb{R}$, and δ, $\gamma > 0$, that is,

$$A(x) := \begin{cases} \frac{1}{\delta}(x - r + \delta), & \text{if } x \in [r-\delta, r], \\ 1 - \frac{1}{\gamma}(x - r), & \text{if } x \in [r, r+\gamma], \\ 0, & \text{otherwise,} \end{cases}$$

whose cuts are $[A]^\alpha = [r - \delta(1-\alpha), r + \gamma(1-\alpha)]$, for all $\alpha \in [0,1]$.

To integrate a function $f : [a,b] \longrightarrow \mathbb{R}$, with $[A]^0 = [r-\delta, r+\gamma] \subseteq [a,b]$, over the fuzzy set A, we have to calculate

$$\left(\int_A f\right)(\alpha) = \int_{[A]^\alpha} f(x)\,dx = \int_{[r-\delta(1-\alpha), r+\gamma(1-\alpha)]} f(x)\,dx.$$

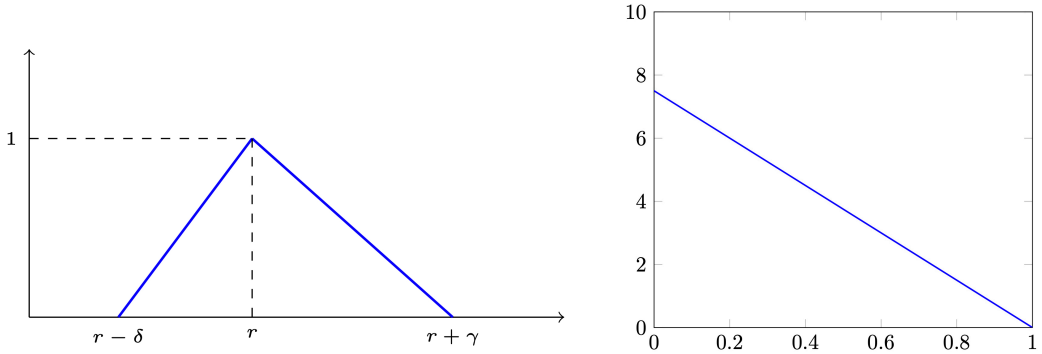
If F is a primitive of f (it exists, for instance, if f is continuous), then, for every $\alpha \in [0,1]$,

$$\left(\int_A f\right)(\alpha) = \int_{[A]^\alpha} f(x)\,dx = \int_{[r-\delta(1-\alpha), r+\gamma(1-\alpha)]} f(x)\,dx = F(r+\gamma(1-\alpha)) - F(r-\delta(1-\alpha)).$$

Note that, if f is constant (equal to k), then

$$\left(\int_A f\right)(\alpha) = k(\gamma+\delta)(1-\alpha),\ \forall \alpha \in [0,1],$$

and the function is linear on α, attaining the values 0 at $\alpha = 1$ and $k(\gamma+\delta)$ at $\alpha = 0$.



(a) Triangular fuzzy interval.

(b) Integral of the constant function $f(x) = k = 3$ on the triangular fuzzy interval $A = (1; 0.5, 2)$.

Figure 5: Triangular fuzzy interval and example of integral.

For the case where the triangular number is symmetric, that is, $\delta = \gamma$, then

$$\left(\int_A f\right)(\alpha) = F(r+\gamma(1-\alpha)) - F(r-\gamma(1-\alpha)),$$

and, if f is constant equal to k, then $\left(\int_A f\right)(\alpha) = 2k\gamma(1-\alpha)$, $\forall \alpha \in [0,1]$, which means that the error at $\alpha = 0$ is multiplied by k.

For trapezoidal fuzzy intervals $(r, s; \delta, \gamma)$, where $r, s \in \mathbb{R}$, $r < s$, and δ, $\gamma > 0$ (see Figure 6(a) for an example of a symmetric trapezoidal fuzzy interval), that is,

$$B(x) := \begin{cases} 1, & x \in [r,s], \\ \frac{1}{\delta}(x-r+\delta), & \text{if } x \in [r-\delta, r], \\ \frac{1}{\gamma}(s+\gamma-x), & \text{if } x \in [s, s+\gamma], \\ 0, & \text{otherwise}, \end{cases}$$

whose cuts are

$$[B]^\alpha = [r-\delta(1-\alpha), s+\gamma(1-\alpha)],$$

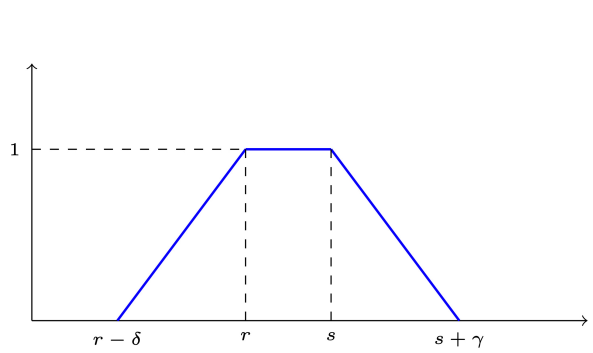
for all $\alpha \in [0,1]$, the integrals are calculated as:

$$\left(\int_B f\right)(\alpha) = \int_{[B]^\alpha} f(x)\,dx = \int_{[r-\delta(1-\alpha), s+\gamma(1-\alpha)]} f(x)\,dx = F(s+\gamma(1-\alpha)) - F(r-\delta(1-\alpha)),$$

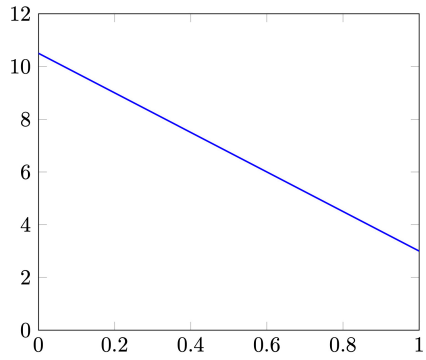
if F is a primitive of f. For f constant (equal to k),

$$\left(\int_B f\right)(\alpha) = k(s-r+(\gamma+\delta)(1-\alpha)),\ \forall \alpha \in [0,1],$$

which is a linear function on α, attaining the values $k(s-r)$ at $\alpha = 1$ and $k(s-r+\gamma+\delta)$ at $\alpha = 0$.



(a) A symmetric trapezoidal fuzzy interval ($\delta = \gamma$).



(b) Integral of the constant function $f(x) = k = 3$ on the trapezoidal fuzzy interval $B = (1, 2; 0.5, 2)$.

Figure 6: Symmetric trapezoidal fuzzy interval and example of integral.

Obviously, if the trapezoidal fuzzy interval B is symmetric, then $\gamma = \delta$, and

$$\left(\int_B f\right)(\alpha) = F(s+\gamma(1-\alpha)) - F(r-\gamma(1-\alpha)),\ \forall \alpha \in [0,1],$$

and, for $f \equiv k$, we have $\left(\int_B f\right)(\alpha) = k(s-r+2\gamma(1-\alpha))$, $\forall \alpha \in [0,1]$. This situation coincides with the case of triangular fuzzy intervals if $s = r$.

In the case of functions $f : C \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, if $A \in \mathcal{E}^n$ is such that $[A]^0 \subseteq C$, and f is constant (and equal to k), then

$$\left(\int_A f\right)(\alpha) = \int_{[A]^\alpha} f(x)\,dx = k\int_{[A]^\alpha} 1\,dx = k\,m([A]^\alpha),\ \forall \alpha \in [0,1],$$

and, therefore, the integral function is:

- identically zero for $k = 0$,
- decreasing in α, if $k > 0$, and
- increasing in α, if $k < 0$.

5 Integral of a fuzzy-valued function on a fuzzy domain

We recall the concept of integral of a fuzzy function $f : [a,b] \longrightarrow \mathcal{E}^n$ based on the notion of Aumann integrals of set-valued functions [2]. The above-mentioned generalization, called Aumann integral for fuzzy-set-valued functions, was given by Puri and Ralescu [23].

It is said that a function $f : [a,b] \longrightarrow \mathcal{E}^n$ is strongly measurable if, for all $\alpha \in (0,1]$, the set-valued mapping $f_\alpha : [a,b] \longrightarrow \mathcal{P}_K(\mathbb{R}^n)$ defined by $f_\alpha(t) = [f(t)]^\alpha$ is Lebesgue-measurable, taking the space $\mathcal{P}_K(\mathbb{R}^n)$ of nonempty, compact, convex subsets of $\mathbb{R}^n$, considering the Hausdorff distance d_H. The integral of f over $[a,b]$, which is denoted by $\int_{[a,b]} f(t)\,dt$ or $\int_a^b f(t)\,dt$, is defined level wise by

$$\left[\int_{[a,b]} f(t)\,dt\right]^\alpha = \int_{[a,b]} f_\alpha(t)\,dt = \left\{\int_{[a,b]} g(t)\,dt \;\mid\; g : [a,b] \longrightarrow \mathbb{R}^n \text{ is a measurable selection for } f_\alpha\right\},$$

for each $\alpha \in (0,1]$.

Definition 5.1. *The function f is integrable on $[a,b]$ if $\int_{[a,b]} f(t)\,dt \in \mathcal{E}^n$.*

A continuous function $f : [a,b] \longrightarrow \mathcal{E}^n$ is integrable on $[a,b]$.

In the previous definition, if $f : [a,b] \longrightarrow \mathbb{R}^n$, then we can consider the function $\hat{f} : t \in [a,b] \longrightarrow \chi_{\{f(t)\}} \in \mathcal{E}^n$, which has a unique selection f and $[\int_{[a,b]} \hat{f}(s)\,ds]^\alpha = \{\int_a^b f(s)\,ds\}$, so that $\int_{[a,b]} \hat{f}(s)\,ds = \chi_{\{\int_a^b f(s)\,ds\}}$ is crisp.

Using the notion of integral of a fuzzy function on a crisp set and the considerations in the previous section, we consider the integration of a fuzzy-valued function over a fuzzy domain.

First, we consider, $f : [a,b] \subseteq \mathbb{R} \longrightarrow \mathcal{E}^1$, or, more generally, $f : [a,b] \subseteq \mathbb{R} \longrightarrow \mathcal{E}^n$, $n \in \mathbb{N}$, and study the problem of integrating that function on the fuzzy domain $A \in \mathcal{E}^1$, with $[A]^0 \subseteq [a,b]$.

We consider that a function $f : [a,b] \longrightarrow \mathcal{E}^n$ is integrable on $A \in \mathcal{E}^1$ if it is integrable on $[A]^\alpha$, for every $\alpha \in [0,1]$. For this, it suffices to consider that f is integrable on $[A]^0$, that is, f is strongly measurable on $[A]^0$ and such that $\int_{[A]^0} f \in \mathcal{E}^n$.

Definition 5.2. *Let $A \in \mathcal{E}^1$ be such that $[A]^0 \subseteq [a,b]$. We say that $f : [a,b] \longrightarrow \mathcal{E}^n$ is strongly measurable on A if, for all $\beta \in (0,1]$, the set-valued mapping $f_\beta : [a,b] \longrightarrow \mathcal{P}_K(\mathbb{R}^n)$ defined by $f_\beta(t) = [f(t)]^\beta$ is Lebesgue-measurable on $[A]^0$, taking the space $\mathcal{P}_K(\mathbb{R}^n)$ of nonempty, compact, convex subsets of $\mathbb{R}^n$, and considering the Hausdorff distance d_H. The integral of f over A, denoted by $\int_A f$, is a fuzzy-valued function of the real variable $\alpha \in [0,1]$, defined in the following way: $\int_A f \in \mathcal{F}([0,1], \mathcal{E}^n)$ is given, for each $\alpha \in [0,1]$, by $\left(\int_A f\right)(\alpha) = \int_{[A]^\alpha} f(t)\,dt \in \mathcal{E}^n$, this fuzzy set is defined level wise by*

$$\left[\left(\int_A f\right)(\alpha)\right]^\beta = \left[\int_{[A]^\alpha} f(t)\,dt\right]^\beta = \int_{[A]^\alpha} f_\beta(t)\,dt = \left\{\int_{[A]^\alpha} g(t)\,dt \;\mid\; g : [A]^\alpha \longrightarrow \mathbb{R}^n \text{ is a measurable selection for } f_\beta\right\},$$

for each $\beta \in (0,1]$.

Definition 5.3. *We say that $f : [a,b] \longrightarrow \mathcal{E}^n$ is integrable on $A \in \mathcal{E}^1$ if $\left(\int_A f\right)(\alpha) \in \mathcal{E}^n$, for every $\alpha \in [0,1]$, that is, if the function $\int_A f$ is well-defined.*

It is obvious that a continuous function $f : [a,b] \longrightarrow \mathcal{E}^n$ is integrable on $A \in \mathcal{E}^1$, if $[A]^0 \subseteq [a,b]$.

Next, we show an example to illustrate the applications of the notion presented.

Example 5.4. *Consider the function* $f : [0,2] \longrightarrow \mathcal{E}^1$ *given by* $f(t) = t(1;1,1)$, *for* $t \in [0,2]$, *whose cuts are given by*

$$[f(t)]^\beta = t[\beta, 2-\beta] = [t\beta, t(2-\beta)].$$

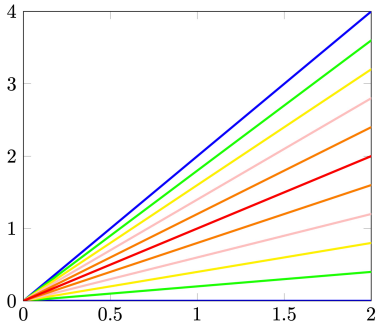


Figure 7: Representation of the cuts of $f(t)$, for several equidistant values of $\beta \in [0,1]$ (from $\beta = 0$ in blue to $\beta = 1$ in red).

Consider the triangular fuzzy interval $A = \left(1; \frac{1}{2}, \frac{1}{2}\right) \in \mathcal{E}^1$, *with cuts* $[A]^\alpha = \left[\frac{1}{2} + \frac{1}{2}\alpha, \frac{3}{2} - \frac{1}{2}\alpha\right]$, *for all* $\alpha \in [0,1]$, *and satisfying* $[A]^0 = \left[\frac{1}{2}, \frac{3}{2}\right] \subseteq [0,2]$. *We calculate the integral of the function* f *over the fuzzy interval* A, $\int_A f \in \mathcal{F}([0,1], \mathcal{E}^1)$. *For each* $\alpha \in [0,1]$, *we take* $\left(\int_A f\right)(\alpha) = \int_{[A]^\alpha} f(t)\, dt \in \mathcal{E}^1$, *defined level wise by*

$$\left[\left(\int_A f\right)(\alpha)\right]^\beta = \left[\int_{[A]^\alpha} f(t)\, dt\right]^\beta = \int_{[A]^\alpha} f_\beta(t)\, dt = \left\{\int_{[A]^\alpha} g(t)\, dt \;\middle|\; g : [A]^\alpha \longrightarrow \mathbb{R} \text{ is a measurable selection for } f_\beta\right\},$$

for each $\beta \in (0,1]$, *and check that this expression defines a fuzzy interval for each* $\alpha \in [0,1]$ *fixed. Indeed, for* $\alpha \in (0,1]$, *we get*

$$\begin{aligned}\left[\left(\int_A f\right)(\alpha)\right]^\beta &= \int_{\frac{1}{2}+\frac{1}{2}\alpha}^{\frac{3}{2}-\frac{1}{2}\alpha} f_\beta(t)\, dt = \int_{\frac{1}{2}+\frac{1}{2}\alpha}^{\frac{3}{2}-\frac{1}{2}\alpha} [t\beta, t(2-\beta)]\, dt \\ &= [\beta(1-\alpha), (2-\beta)(1-\alpha)] = [1-\alpha-(1-\alpha)(1-\beta), 1-\alpha+(1-\alpha)(1-\beta)]\,.\end{aligned}$$

This proves that, for each $\alpha \in (0,1]$, $\left(\int_A f\right)(\alpha) = (1-\alpha; 1-\alpha, 1-\alpha)$. *For* $\alpha = 1$, *we have* $[A]^1 = \{1\}$, *and, therefore,* $\left(\int_A f\right)(1) = \chi_{\{0\}}$. *In Figure 8, we represent the value of the integral for several* $\alpha \in (0,1]$.

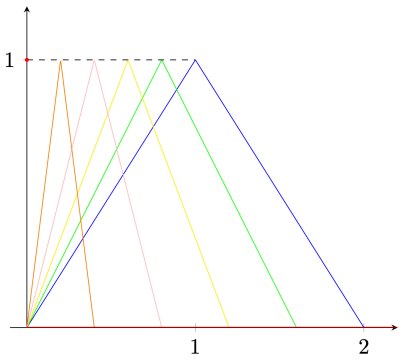


Figure 8: Representation of $\left(\int_A f\right)(\alpha)$, for several equidistant values of $\alpha \in [0,1]$ (from $\alpha = 0$ in blue to $\alpha = 1$ in red).

6 Applications to integrals over 2-dimensional fuzzy regions

As we have explained, the concepts previously defined are useful to deal with the calculus of the integral of a certain magnitude over a boundary which is not established exactly. In this section, we consider domains of dimension higher than one.

Suppose that the function $\varrho : C \subseteq \mathbb{R}^2 \longrightarrow \mathcal{E}^1$ represents the density of a certain property in relation with a portion of territory which has an imprecise boundary, represented by a fuzzy region $A \in \mathcal{E}^2$ with $[A]^0 \subseteq C$. To calculate the area corresponding to that property, we consider the following integral

$$\left(\int_A \varrho\right)(\alpha) = \int_{[A]^\alpha} \varrho(x)\, dx,\ \forall \alpha \in [0,1].$$

If the portion of territory is circular, and we represent it, for instance, using the family of circles of center $(p_1, p_2) \in \mathbb{R}^2$ with radius $1 - \epsilon\alpha$, where $\epsilon \in (0,1)$ is fixed, then each one of the cuts of A is given by the equation

$$(x-p_1)^2 + (y-p_2)^2 \leq (1-\epsilon\alpha)^2.$$

Then

$$\left(\int_A \varrho\right)(\alpha) = \int_{(x-p_1)^2+(y-p_2)^2 \leq (1-\epsilon\alpha)^2} \varrho\, dm,\ \forall \alpha \in [0,1],$$

where m represents the Lebesgue measure in $\mathbb{R}^2$.

If function ϱ satisfies the appropriate conditions, we can use Fubini's Theorem to calculate the previous quantity by the following expression:

$$\left(\int_A \varrho\right)(\alpha) = \int_{p_1-(1-\epsilon\alpha)}^{p_1+(1-\epsilon\alpha)} \int_{p_2-\sqrt{(1-\epsilon\alpha)^2-(x-p_1)^2}}^{p_2+\sqrt{(1-\epsilon\alpha)^2-(x-p_1)^2}} \varrho(x,y)\, dy\, dx,\ \forall \alpha \in [0,1].$$

Computationally, we might choose a partition of the interval $[0,1]$ to evaluate the corresponding integrals for a sequence of values of α.

Next, we consider a similar problem but representing the region with fuzzy boundary as a pyramidal fuzzy region P with support given by the rectangle with edges $(r-\delta, s-\tilde{\delta})$, $(r-\delta, s+\tilde{\gamma})$, $(r+\gamma, s-\tilde{\delta})$, $(r+\gamma, s+\tilde{\gamma})$, and with core the unitary set $\{(r,s)\}$. Here $(r,\ s) \in \mathbb{R}^2$, and $\delta, \gamma, \tilde{\delta}, \tilde{\gamma} > 0$ are fixed.

To obtain the expression of the membership function relative to P, we have to calculate different expressions on the following four regions:

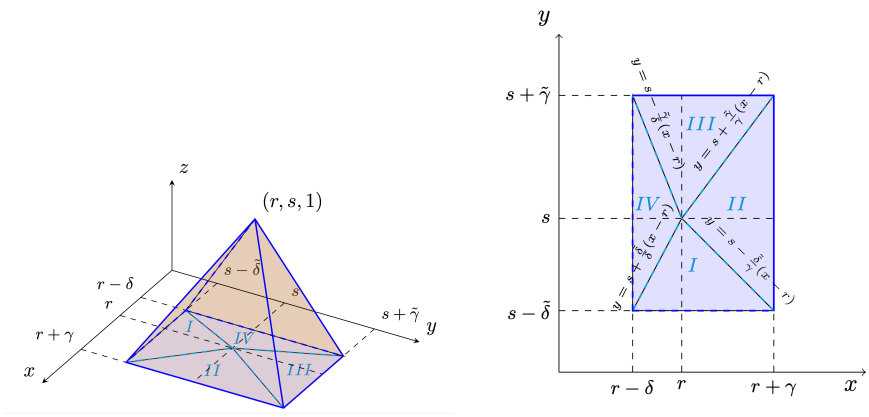
$$\text{Region I} : \left\{(x,y) \in \mathbb{R}^2 \ \middle|\ s - \tilde{\delta} \leq y \leq \min\left\{s + \frac{\tilde{\delta}}{\delta}(x-r), s - \frac{\tilde{\delta}}{\gamma}(x-r)\right\}\right\},$$

$$\text{Region II} : \left\{(x,y) \in \mathbb{R}^2 \ \middle|\ r \leq x \leq r+\gamma,\ s - \frac{\tilde{\delta}}{\gamma}(x-r) \leq y \leq s + \frac{\tilde{\gamma}}{\gamma}(x-r)\right\},$$

$$\text{Region III} : \left\{(x,y) \in \mathbb{R}^2 \ \middle|\ \max\left\{s - \frac{\tilde{\gamma}}{\delta}(x-r), s + \frac{\tilde{\gamma}}{\gamma}(x-r)\right\} \leq y \leq s + \tilde{\gamma}\right\},$$

and

$$\text{Region IV} : \left\{(x,y) \in \mathbb{R}^2 \ \middle|\ r-\delta \leq x \leq r,\ s + \frac{\tilde{\delta}}{\delta}(x-r) \leq y \leq s - \frac{\tilde{\gamma}}{\delta}(x-r)\right\}.$$



(a) Pyramidal fuzzy region P.

(b) Regions I, II, III and IV.

Figure 9: Pyramidal fuzzy region and regions I–IV.

The expression of P on the different regions can be obtained easily. For instance, in Region I, considering the plane through $(r, s, 1)$, $(r-\delta, s-\tilde{\delta}, 0)$, and $(r+\gamma, s-\tilde{\delta}, 0)$, we get

$$P(x,y) = 1 + \frac{y-s}{\tilde{\delta}}, \text{ for } (x,y) \text{ in Region I.}$$

By a straightforward calculus, obtaining the expressions of the lines through the point $(r, s, 1)$ and, respectively, $(r, s-\tilde{\delta}, 0)$, $(r+\gamma, s, 0)$, $(r, s+\tilde{\gamma}, 0)$, $(r-\delta, s, 0)$, we find that the restrictions for (x, y) to belong to the α-cut of P are

$$s - \tilde{\delta}(1-\alpha) \leq y, \text{ in Region I,}$$

$$x \leq r + \gamma(1-\alpha), \text{ in Region II,}$$

$$y \leq s + \tilde{\gamma}(1-\alpha), \text{ in Region III,}$$

and

$$r - \delta(1-\alpha) \leq x, \text{ in Region IV.}$$

This proves that the cuts of the pyramidal fuzzy set P are given by the rectangles

$$[P]^\alpha = \left\{(x,y) \in \mathbb{R}^2 \mid r-\delta(1-\alpha) \leq x \leq r+\gamma(1-\alpha),\ s-\tilde{\delta}(1-\alpha) \leq y \leq s+\tilde{\gamma}(1-\alpha)\right\}.$$

Note that the previous expression is reduced to the point $\{(r,s)\}$ for $\alpha = 1$, and to $[r-\delta, r+\gamma] \times \left[s-\tilde{\delta}, s+\tilde{\gamma}\right]$ for $\alpha = 0$.

If we want to calculate the area of the region P subject to a density function $\varrho : C \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$, where $[P]^0 \subseteq C$, we must compute

$$\left(\int_P \varrho\right)(\alpha) = \int_{[P]^\alpha} \varrho(x)\,dx = \int_{r-\delta(1-\alpha)}^{r+\gamma(1-\alpha)} \int_{s-\tilde{\delta}(1-\alpha)}^{s+\tilde{\gamma}(1-\alpha)} \varrho(x,y)\,dy\,dx, \quad \forall \alpha \in [0,1].$$

For ϱ a constant function equal to k, we have, for every $\alpha \in [0,1]$,

$$\left(\int_P \varrho\right)(\alpha) = k(\tilde{\gamma}+\tilde{\delta})(\gamma+\delta)(1-\alpha)^2.$$

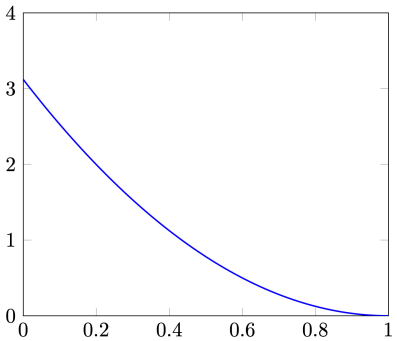


Figure 10: Area of the region P subject to a constant density function $\varrho(x) = k = 3$, where P is determined by $r = s = 1$, $\delta = 0.3$, $\gamma = 0.5$, $\tilde{\delta} = 0.5$, and $\tilde{\gamma} = 0.8$.

Finally, consider the boundary of the region to be a truncated pyramid Q, whose support is the rectangle with edges $(r-\delta, s-\tilde{\delta})$, $(r+\gamma, s-\tilde{\delta})$, $(r+\gamma, s+\tilde{\gamma})$, $(r-\delta, s+\tilde{\gamma})$, and with core the rectangle with edges $(r-\mu, s-\tilde{\mu})$, $(r+\nu, s-\tilde{\mu})$, $(r+\nu, s+\tilde{\nu})$, $(r-\mu, s+\tilde{\nu})$, where $\delta > \mu > 0$, $\tilde{\delta} > \tilde{\mu} > 0$, $\gamma > \nu > 0$, and $\tilde{\gamma} > \tilde{\nu} > 0$.

Considering the same Regions I–IV in the previous example, and obtaining the expressions of the lines passing through the points $(r, s-\tilde{\delta}, 0)$ and $(r, s-\tilde{\mu}, 1)$, $(r+\gamma, s, 0)$ and $(r+\nu, s, 1)$, $(r, s+\tilde{\gamma}, 0)$ and $(r, s+\tilde{\nu}, 1)$, $(r-\delta, s, 0)$ and $(r-\mu, s, 1)$, respectively, we find that the cuts of Q are given by the rectangles

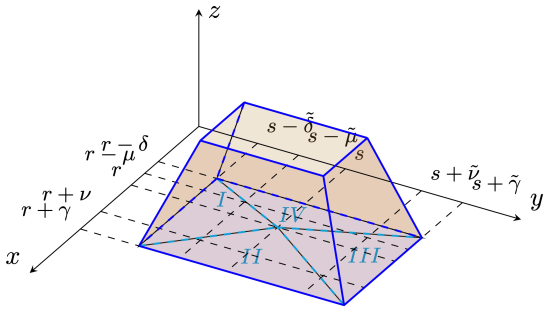
$$[Q]^\alpha = \big\{(x,y) \in \mathbb{R}^2 \mid r-\mu-(\delta-\mu)(1-\alpha) \leq x \leq r+\nu+(\gamma-\nu)(1-\alpha),$$
$$s-\tilde{\mu}-(\tilde{\delta}-\tilde{\mu})(1-\alpha) \leq y \leq s+\tilde{\nu}+(\tilde{\gamma}-\tilde{\nu})(1-\alpha)\big\}.$$

Thus, the area of region Q subject to the density function $\varrho : C \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$, with $[Q]^0 \subseteq C$, is calculated as

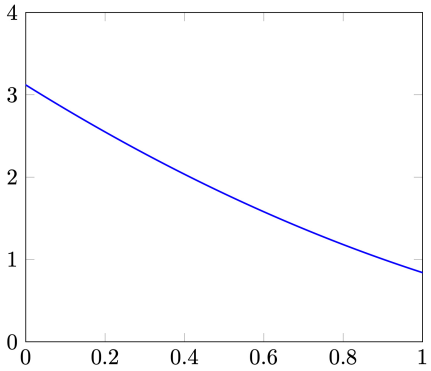
$$\left(\int_Q \varrho\right)(\alpha) = \int_{[Q]^\alpha} \varrho(x)\,dx = \int_{r-\mu-(\delta-\mu)(1-\alpha)}^{r+\nu+(\gamma-\nu)(1-\alpha)} \int_{s-\tilde{\mu}-(\tilde{\delta}-\tilde{\mu})(1-\alpha)}^{s+\tilde{\nu}+(\tilde{\gamma}-\tilde{\nu})(1-\alpha)} \varrho(x,y)\,dy\,dx, \quad \forall \alpha \in [0,1].$$

For $\varrho \equiv k$, we obtain, for every $\alpha \in [0,1]$,

$$\left(\int_Q \varrho\right)(\alpha) = k\left(\nu+\mu+(\gamma-\nu+\delta-\mu)(1-\alpha)\right)\left(\tilde{\nu}+\tilde{\mu}+(\tilde{\gamma}-\tilde{\nu}+\tilde{\delta}-\tilde{\mu})(1-\alpha)\right).$$



(a) Truncated pyramidal fuzzy region Q.



(b) Area of the region Q subject to a constant density function $\varrho(x) = k = 3$, where Q is determined by $r = s = 1$, $\delta = 0.3 > \mu = 0.1$, $\gamma = 0.5 > \nu = 0.3$, $\tilde{\delta} = 0.5 > \tilde{\mu} = 0.2$, $\tilde{\gamma} = 0.8 > \tilde{\nu} = 0.5$.

Figure 11: Truncated pyramidal fuzzy region and example of area.

If the bounded set $[A]^0$ is divided into grid cells and the density function is given by a membership value between 0 and 1 which depends on each cell, the corresponding integral on each cut $[A]^\alpha$, $\alpha \in [0,1]$, is the integral of a piece wise constant function, and it is calculated as the finite sum

$$\int_{[A]^\alpha} \varrho(x)\,dx = \sum_{i,j} \varrho_{i,j} m([A]^\alpha \cap C_{ij}),$$

where ϱ_{ij} is the value of the density on the cell C_{ij}, and m is the Lebesgue measure.

The type of calculations described in this work can be useful to estimate, for instance, the fuzzy area described by a forest, when the density of trees is variable, but also for estimating the integral of a fuzzy interval valued function over some domain, when the field function is fuzzily known.

If $\varrho : C \subseteq \mathbb{R}^n \longrightarrow \mathcal{E}^1$, and $A \in \mathcal{E}^n$ is such that $[A]^0 \subseteq C$, the integral on each cut of A has to be computed, considering also the different β-cuts of ϱ, since $\left(\int_A \varrho\right)(\alpha) \in \mathcal{E}^1$, for every α. Therefore, in practice, it is necessary to select an admissible certainty level for the density function, as well as the domain.

In some cases, we might be interested on calculating the mean value of the density function $\varrho : C \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ on the set $A \in \mathcal{E}^n$. If all the cuts of A have positive measure (this happens, for instance, with the fuzzy set Q), then we calculate the mean value of ϱ on A as the function of the variable $\alpha \in [0,1]$ defined by

$$mv(\varrho, A)(\alpha) = \frac{1}{m([A]^\alpha)} \int_{[A]^\alpha} \varrho\,dm, \ \forall \alpha \in [0,1],$$

where m denotes the Lebesgue measure.

For a general domain $A \in \mathcal{E}^n$, we have to exclude, for the validity of the previous expression, the values of $\alpha \in [0,1]$ such that $m([A]^\alpha) = 0$. At the unitary cuts, we define $mv(\varrho, A)(\alpha) = \varrho(x)$, where $[A]^\alpha = \{x\}$.

The calculus of the integral of a function on a fuzzy set can be made at a finite sequence of elements $\alpha_i \in [0,1]$ to provide an idea of the behavior of the function on the whole interval. In the case of the integral of a real function, we may choose a certain value $\alpha \in (0,1)$ as admissible for certainty and, if we are integrating a fuzzy function on a fuzzy set, we choose a certain level α to fix the certainty on the boundary and a level β which fixes the certainty on the function. In this type of calculus, we can consider membership values greater than 0.5 as a representation of members of the region given by the fuzzy set A.

7 Conclusions

We have presented some concepts and examples in relation with the computation of the integral for scalar-valued functions over a fuzzy spatial domain. This type of concepts are applicable for the estimation of the area of a region satisfying a certain property, what can be done by considering a non-constant density function and evaluating the global contribution to the property. We have also checked that a similar notion can be defined for fuzzy-valued functions. This fact has applications to the evaluation of magnitudes for field functions that are fuzzily known.

In particular, after some preliminary notions in Section 2, we have presented, in Section 3, the notion of imprecise line, of orthogonal and oblique type. In Section 4, we have defined the integral of a scalar-valued function on a fuzzy domain, showing some examples of integration over triangular fuzzy intervals and trapezoidal fuzzy intervals. Then, in Section 5, we have presented the notion of the integral of a fuzzy-valued function on a fuzzy domain, and illustrated its calculation with an example. Finally, in Section 6, we have presented some applications to the integration over domains with higher dimension. In particular, we have considered circular-type 2-dimensional fuzzy regions, pyramidal fuzzy regions, and truncated pyramidal fuzzy regions, and we have shown some applications to the calculation of areas.

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References

[1] Z. Akyürek, T. A. Yanar, *A fuzzy-based tool for spatial reasoning: A case study on estimating forest fire risky areas*, Proceedings of International Symposium on Spatio-temporal Modeling, Spatial Reasoning, Analysis, Data Mining and Data Fusion, ISPRS Archives - Volume XXXVI-2/W25, (2005), 1-6. `https://www.isprs.org/proceedings/xxxvi/2-W25/default.aspx`.

[2] J. Aumann, *Integrals of set-valued functions*, Journal of Mathematical Analysis and Applications, **12** (1965), 1-12.

[3] M. Boyland, J. Nelson, F. L. Bunnell, R. G. D'Eon, *An application of fuzzy set theory for seral-class constraints in forest planning models*, Forest Ecology and Management, **223** (2006), 395-402.

[4] P. A. Burrough, *Principles of geographical information systems for land resource assessment*, Clarendon Press, Oxford, 1986.

[5] P. A. Burrough, *Fuzzy mathematical methods for soil survey and land evaluation*, Journal of Soil Science, **40** (1989), 477-492.

[6] P. A. Burrough, A. U. Frank, *Geographic objects with indeterminate boundaries*, Taylor and Francis, London, 1996.

[7] P. A. Burrough, R. A. McDonnell, *Principles of geographical information systems*, Oxford University Press, Oxford, 1998.

[8] P. Diamond, P. E. Kloeden, *Metric spaces of fuzzy sets: Theory and applications*, World Scientific, Singapore, 1994.

[9] D. Dubois, H. Fargier, A. Rico, *Double Sugeno integrals and the commutation problem*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, **27** (2019), 1-38.

[10] D. Dubois, H. Prade, *Fuzzy sets and systems: Theory and application*, Academic Press, New York, 1980.

[11] M. Eshaghi Gordji, S. Abbaszadeh, C. Park, *The Sugeno fuzzy integral of concave functions*, Iranian Journal of Fuzzy Systems, **16**(2) (2019), 197-204.

[12] M. H. Greve, M. B. Sørensen, *Soil boundary fuzziness: Classification and representation in GIS*, In Proceedings of the World Congress of Computers in Agriculture and Natural Resources, (2002), 325-331.

[13] G. Guerra, A. Fernando, *Reseña "SIG: Sistemas de información geográfica" de Javier Gutiérrez Puebla y Michael Gould*, Geoenseñanza, **14** (2009), 151-154. Available at: `https://www.redalyc.org/articulo.oa?id=36021235009`.

[14] J. Gutiérrez Puebla, M. Gould, *Sig: Sistemas de información Geográfica*, Síntesis, Madrid, 1994.

[15] R. Halas, R. Mesiar, J. Pocs, *A new characterization of the discrete Sugeno integral*, Information Fusion, **29** (2016), 84-86.

[16] R. Halas, R. Mesiar, J. Pocs, V. Torra, *A note on some algebraic properties of discrete Sugeno integrals*, Fuzzy Sets and Systems, **355** (2019), 110-120.

[17] B. Jiang, *Visualisation of fuzzy boundaries of geographical objects*, Cartography, **27** (1998), 41-46.

[18] Z. Jingxiong, R. P. Kirby, *A comparison of alternative criteria for defining fuzzy boundaries on fuzzy categorical maps*, Geo-Spatial Information Science, **3**(2) (2000), 26-34.

[19] G. S. Lee, K. H. Lee, *Application of fuzzy representation of geographic boundary to the soil loss model*, Hydrology and Earth System Sciences Discuss, **3** (2006), 115-133.

[20] S. Leyk, N. Zimmermann, *Improving land change detection based on uncertain survey maps using fuzzy sets*, Landscape Ecology, **22** (2007), 257-272.

[21] B. Liu, *Fuzzy process, hybrid process and uncertain process*, Journal of Uncertain Systems, **2**(1) (2008), 3-16.

[22] E. Pap, A. Iosif, A. Gavriluţ, *Integrability of an interval-valued multifunction with respect to an interval-valued set multifunction*, Iranian Journal of Fuzzy Systems, **15**(3) (2018), 47-63.

[23] M. L. Puri, D. A. Ralescu, *Fuzzy random variables*, Journal of Mathematical Analysis and Applications, **114** (1986), 409-422.

[24] D. Ralescu, G. Adams, *The fuzzy integral*, Journal of Mathematical Analysis and Applications, **75** (1980), 562-570.

[25] H. Román-Flores, Y. Chalco-Cano, *Sugeno integral and geometric inequalities*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, **15** (2007), 1-11.

[26] H. Román-Flores, A. Flores-Franulič, Y. Chalco-Cano, *A Jensen type inequality for fuzzy integrals*, Information Sciences, **177** (2007), 3192-3201.

[27] H. Román-Flores, A. Flores-Franulič, Y. Chalco-Cano, *A Hardy-type inequality for fuzzy integrals*, Applied Mathematics and Computation, **204** (2008), 178-183.

[28] A. Šeliga, *Greedy decomposition integrals*, Iranian Journal of Fuzzy Systems, **17**(5) (2020), 23-28.

[29] M. J. Smith, M. F. Goodchild, P. A. Longley, *Geospatial analysis, a comprehensive guide to principles techniques and software tools*, The Winchelsea Press, 6th Edition, (2018), `https://www.spatialanalysisonline.com/extractv6.pdf`.

[30] M. Sugeno, *Theory of fuzzy integrals and its applications*, Ph.D. Thesis, Tokyo Institute of Technology, 1974.

[31] J. Verstraete, B. Van der Cruyssen, R. De Caluwe, *Assigning membership degrees to points of fuzzy boundaries*, In: Proceedings IEEE PeachFuzz 2000: 19th International Conference of the North American Fuzzy Information Processing Society, NAFIPS - Atlanta, Georgia, (2000), 444-447.

[32] Z. Wang, G. Klir, *Fuzzy measure theory*, Plenum, New York, 1992.

[33] T. Yang, B. Hillier, *The fuzzy boundary: The spatial definition of urban areas*, In: A. S. Kubat, O. Ertekin, Y. I. Guney, E. Eyuboglu (eds.), 6th International Space Syntax Symposium, Istanbul Technical University, Turkey, (2007), 16 pages.

[34] C. You, N. Xiang, *Some properties of uncertain integral*, Iranian Journal of Fuzzy Systems, **15**(2) (2018), 133-142.

[35] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338-353.