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# Diaz-Metcalf type inequality for Sugeno and pseudo-integrals

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**Abstract**

In this paper, we have proved Diaz-Metcalf inequality for fuzzy integrals. More precisely:
If $f, g : [0, 1] \to \mathbb{R}$ are continuous and strictly increasing functions, then the fuzzy integral inequality

$$\int_0^1 f^s d\mu \cdot \int_0^1 g^s d\mu \leq \int_0^1 (f \cdot g)^s \, d\mu,$$

holds, where $s > 1$ and $\mu$ is the Lebesgue measure on $\mathbb{R}$. In addition, we have shown this inequality for pseudo-integrals.

*Keywords:* Diaz-Metcalf type inequality, fuzzy integral, integral inequality, pseudo integral.

## 1 Introduction

Fuzzy measure and fuzzy integral, which were originally introduced by Sugeno in 1974 [21], are important analytical methods of measuring uncertain information [11, 20].
The study of inequalities for the fuzzy integral, which was initiated by Flores-Franulič et al., was discussed in several papers (see [12, 13]). Using the refrences [12, 13], we have the following fuzzy integral inequalities:

For all $a \geq 0$,

$$g(a - p) \geq \int_0^a g d\mu = p,$$

where $\mu$ is a Lebesgue measure on $\mathbb{R}$ and $g : [0, \infty) \to [0, \infty)$ is a continuous and strictly increasing function and $\int_0^a g d\mu = p$. Moreover, if $0 < p < a$, then $g(a - p) = p$. Also, where $g$ is strictly decreasing function, we have

$$g(p) \geq \int_0^a g d\mu = p.$$

Moreover, if $0 < p < a$, then $g(p) = p$.

Diaz-Metcalf inequality [15] is as the following form in classical case:

$$\int_X f^2 d\mu \cdot \int_X g^2 d\mu \leq \frac{(M + m)}{4Mm} \left( \int_X (f \cdot g) \, d\mu \right)^2, \tag{1}$$

where $(X, \mu)$ is a measurable space, $f, g \in L^2(X, \mu)$ and $0 \leq mg \leq f \leq Mg$.

In this paper, by an example, we have shown that the classical version of the Diaz-Metcalf inequality is not valid for fuzzy integrals and by a little change we express and prove the Diaz-Metcalf type inequality for the fuzzy integral as well.

Pseudo-analysis is a generalization of the classical analysis, where instead of the field of real numbers a semiring is taken on a real interval $[a, b] \subseteq [-\infty, +\infty]$ endowed with pseudo-addition $\oplus$ and pseudo-multiplication $\odot$. Many

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authors such as Pap in [16, 17, 18] and Zhang et al. in [23, 24] have generalized some fuzzy integral inequalities. Agahi et al. expressed and proved Berward, Holder, Minkowski and Stolarsky type inequalities for fuzzy, pseudo-integrals and Choquet integrals (see [2, 4, 3]). In the following we state the Chebyshev's inequality for pseudo-integrals from reference [1]:

$$\int_{[0,1]}^{\oplus}(u\odot u)dx\geq\left(\int_{[0,1]}^{\oplus}udx\right)\odot\left(\int_{[0,1]}^{\oplus}vdx\right),$$

where $u,v:[0,1]\to[a,b]$ are measurable functions and $g:[a,b]\to[0,\infty)$ is a generator function for pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$.

Recently, Daraby et al., in [6, 7, 8] studied some related Fritz Carlson type inequalities and related Thunsdorff type and Frank-Pick type inequalities for fuzzy integrals and generalized Stolarsky, Hardy and Feng Qi type inequalities for pseudo-integrals (see [5, 9, 10]).

The article is organized as the following: In Section 2, we give definitions, properties and results of fuzzy measure, fuzzy integrals and pseudo-integrals. In Section 3, firstly, we show by an example that the classical Diaz-Metcalfe inequality is not valid in the fuzzy case, and then we state and prove the fuzzy and pseudo integrals cases. Finally, in Section 4, we discuss the Diaz-Metcalf inequality.

## 2 Preliminary

We denote by $\mathbb{R}$, the set of all real numbers. Let $X$ be a non-empty set and $\Sigma$ be a $\sigma-$algebra of subsets of $X$. Throughout this paper, all considered subsets are supposed to be in $\Sigma$.

**Definition 2.1.** [19] *A set function $\mu:\Sigma\to[0,+\infty]$ is called a fuzzy measure if the following properties are satisfied:*

$(FM1)$ $\mu(\emptyset)=0$,

$(FM2)$ $A\subseteq B\Rightarrow\mu(A)\leq\mu(B)$,

$(FM3)$ $A_1\subseteq A_2\subseteq\ldots\Rightarrow\lim\mu(A_i)=\mu\left(\bigcup_{i=1}^{\infty}A_i\right)$,

$(FM4)$ $A_1\supseteq A_2\supseteq\ldots$ *and* $\mu(A_1)<\infty\Rightarrow\lim\mu(A_i)=\mu\left(\bigcap_{i=1}^{\infty}A_i\right)$.

*When $\mu$ is a fuzzy measure, the triple $(X,\Sigma,\mu)$ is called a fuzzy measure space.*

If $f$ is a non-negative real-valued function on $X$, we will denote

$$F_\alpha=L_\alpha f=\{x\in X\mid f(x)\geq\alpha\}=\{f\geq\alpha\},$$

the $\alpha$-level of $f$, for $\alpha>0$. $L_0f=\overline{\{x\in X\mid f(x)>0\}}=\mathrm{supp}(f)$ is the support of $f$. We know that:

$$\alpha\leq\beta\Rightarrow\{f\geq\beta\}\subseteq\{f\geq\alpha\}.$$

If $\mu$ is a fuzzy measure on $(X,\Sigma)$, we define the following:

$$\mathfrak{F}^{\sigma}(X)=\{f:X\to[0,\infty)|f\text{ is }\mu-\text{measurable}\}.$$

**Definition 2.2.** [22] *Let $\mu$ be a fuzzy measure on $(X,\Sigma)$. If $f\in\mathfrak{F}^{\mu}(X)$ and $A\in\Sigma$, then the fuzzy integral of $f$ on $A$, with respect to the fuzzy measure $\mu$, is defined as*

$$\oint_A fd\mu=\bigvee_{\alpha\geq0}(\alpha\wedge\mu(A\cap F_\alpha)),$$

*where $\vee$ and $\wedge$ denotes the operations sup and inf on $[0,\infty]$, respectively. In particular, if $A=X$ then*

$$\oint_X fd\mu=\oint fd\mu=\bigvee_{\alpha\geq0}(\alpha\wedge\mu(F_\alpha)).$$

The following properties of the fuzzy integral can be found in [22].

**Proposition 2.3.** [22] *Let $(X, \Sigma, \mu)$ be a fuzzy measure space, with $A, B \in \sum$ and $f, g \in \mathfrak{F}^{\sigma}(X)$. We have*

(1) $\oint_A f d\mu \leq \mu(A)$.

(2) $\oint_A k d\mu = k \wedge \mu(A)$, *for $k$ non-negative constant.*

(3) *If $A \subset B$, then* $\oint_A f d\mu \leq \oint_B f d\mu$.

(4) $\oint_{A\cup B} f d\mu \geq \oint_A f d\mu \vee \oint_B f d\mu$.

(5) *If $\mu(A) < \infty$, then* $\oint_A f d\mu \geq \alpha \Leftrightarrow \mu(A \cap \{f \geq \alpha\}) \geq \alpha$.

(6) $\oint_A f d\mu \leq \alpha \Leftrightarrow \mu(A \cap F_{\alpha^+}) \leq \alpha \Leftarrow \mu(A \cap \{f \geq \alpha\}) \leq \alpha$.

(7) $\oint_A f d\mu > \alpha \Leftrightarrow \mu(A \cap F_{\alpha^+}) > \alpha \Rightarrow \mu(A \cap \{f \geq \alpha\}) \leq \alpha$.

**Definition 2.4.** *Functions $f, g : X \to \mathbb{R}$ are said to be comonotone if for all $x, y \in X$,*

$$(f(x) - f(y))\,(g(x) - g(y)) \geq 0,$$

*and $f$ and $g$ are said to be countermonotone if for all $x, y \in X$,*

$$(f(x) - f(y))\,(g(x) - g(y)) \leq 0.$$

On the other hand, the comonotonicity of functions $f$ and $g$ is equivalent to non-existence of points $x, y \in X$ such that $f(x) < f(y)$ and $g(x) > g(y)$. Similarly, if $f$ and $g$ are conutermonotone then $f(x) < f(y)$ and $g(x) < g(y)$ connot happen.

In the following, we are going to review some well known results of pseudo-operations, pseudo-analysis and pseudo-additive measures and integrals in details, we refer to [14, 22].

Let $[a, b]$ be a closed (in some cases can be considered semi-closed) subinterval of $[-\infty, \infty]$. The full order on $[a, b]$ will be denoted by $\preceq$.

**Definition 2.5.** [22] *The operation $\oplus$ (pseudo-addition) is a function $\oplus : [a, b] \times [a, b] \to [a, b]$ which is commutative, non-decreasing (with respect to $\preceq$ ), associative and with a zero (neutral) element denoted by $\mathbf{0}$, i.e., for each $x \in [a, b], \mathbf{0} \oplus x = x$ holds (usually $\mathbf{0}$ is either $a$ or $b$).*

Let $[a, b]_+ = \{x | x \in [a, b], \mathbf{0} \preceq x\}$.

**Definition 2.6.** [22] *The operation $\odot$ (pseudo-multiplication) is a function $\odot : [a, b] \times [a, b] \to [a, b]$ which is commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x \odot z \preceq y \odot z$ for all $z \in [a, b]_+$, associative and for which there exists a unit element $\mathbf{1} \in [a, b]$, i.e., for each $x \in [a, b], \mathbf{1} \odot x = x$.*

Also $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$, i.e., $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$.

We shall consider the semiring $([a, b], \oplus, \odot)$ for two important (with completely different behavior) cases. The first case is when pseudo-operations are generated by a monotone and continuous function $g : [a, b] \to [0, \infty)$, i.e., pseudo-operations are given with:

$$x \oplus y = g^{-1}(g(x) + g(x)) \qquad \text{and} \qquad x \odot y = g^{-1}(g(x)g(x)). \tag{2}$$

Then, the pseudo-integral for a function $f : [c, d] \to [a, b]$ reduces to the $g-$integral

$$\int_{[c,d]}^{\oplus} f(x)dx = g^{-1}\left(\int_c^d g(f(x))dx\right). \tag{3}$$

More details on this structure as well as corresponding measures and integrals can be found in [14]. The second class is when $x \oplus y = \max(x, y)$ and $x \odot y = g^{-1}(g(x)g(y))$, the pseudo-integral for a function $f : \mathbb{R} \to [a, b]$ is given by

$$\int_{\mathbb{R}}^{\oplus} f \odot dm = \sup_{x\in\mathbb{R}} (f(x) \odot \psi(x)),$$

where function $\psi$ defines sup-measure $m$. Any sup-measure generated as essential supremum of a continuous density can be obtained as a limit of pseudo-additive measures with respect to generated pseudo-additine. Then for any continuous function $f : [0, \infty] \to [0, \infty]$ the integral $\int^{\oplus} f \odot dm$ can be obtained as a limit of g-integrals.

We denote by $\mu$ the usual Lebesgue measure on $\mathbb{R}$ have

$$m(A) = \text{ess sup}_{\mu}(x | x \in A) = \sup\{a | \mu(x | x \in A, x > a) > 0\}.$$

**Theorem 2.7.** [14] *Let $m$ be a* sup*-measure on $([0,\infty],\mathbb{B}[0,\infty])$, where $\mathbb{B}([0,\infty])$ is the Borel $\sigma$-algebra on $[0,\infty]$, $m(A)=\operatorname{ess\,sup}_{\mu}(\psi(x)|x\in A)$, and $\psi:[0,\infty]\to[0,\infty]$ is a continuous function. Then for any pseudo-addition $\oplus$ with a generator $g$ there exists a family $m_\lambda$ of $\oplus_\lambda$-measure on $([0,\infty],\mathbb{B})$, where $\oplus_\lambda$ is generated by $g^\lambda$ $(\lambda\in(0,\infty))$ such that $\lim\limits_{\lambda\to\infty} m_\lambda = m$.*

**Theorem 2.8.** [14] *Let $([0,\infty],\sup,\odot)$ be a semiring, where $\odot$ is generated by $g$, i.e., we have $x\odot y=g^{-1}(g(x)g(y))$ for every $x,y\in(0,\infty)$. Let $m$ be the same as in Theorem 2.7, Then there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$-measures, where $\oplus_\lambda$ is generated by $g^\lambda,\lambda\in(0,\infty)$ such that for every continuous function $f:[0,\infty]\to[0,\infty]$,*

$$\int^{\sup} f\odot dm=\lim_{\lambda\to\infty}\int^{\oplus_\lambda} f\odot dm_\lambda=\lim_{\lambda\to\infty}(g^\lambda)^{-1}\left(\int g^\lambda(f(x))dx\right). \tag{4}$$

**Theorem 2.9.** [14] *Let $u,v:[0,1]\to[a,b]$ be two measurable functions and let the generator $g:[a,b]\to[0,\infty)$ of the pseudo addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function and $\varphi:[a,b]\to[a,b]$ be a continuous and strictly increasing function such that $\varphi$ commutes with $\odot$. If $u$ and $v$ are comonotone, then the inequality*

$$\varphi^{-1}\left(\int_{[0,1]}^{\oplus}\varphi(u\odot v)dx\right)\geq\left(\varphi^{-1}\left(\int_{[0,1]}^{\oplus}\varphi(u)dx\right)\right)\odot\left(\varphi^{-1}\left(\int_{[0,1]}^{\oplus}\varphi(v)dx\right)\right),$$

*holds. The reverse inequality holds whenever $u$ and $v$ are countermonotone functions.*

# 3 Diaz-Metcalf inequality for fuzzy integrals

At first, by an example we show that the inequality (1) is not valid for Sugeno integrals.

**Example 3.1.** *Let $f(x)=\sqrt{x}$, $g(x)=x$. By choosing $m=\dfrac{1}{2}$ and $M=2$ and a simple calculations, we have:*

$$\begin{aligned}
&\oint_{\frac{1}{4}}^{1} f^2(x)d\mu=\oint_{\frac{1}{4}}^{1} xd\mu=0.5,\\
&\oint_{\frac{1}{4}}^{1} g^2(x)d\mu=\oint_{\frac{1}{4}}^{1} x^2d\mu=0.381966,\\
&\oint_{\frac{1}{4}}^{1} (f\cdot g)^2(x)d\mu=\oint_{0}^{1} x^3d\mu=0.317672,\\
&\frac{(M+m)^2}{4Mm}=\frac{\frac{25}{4}}{4}=\frac{25}{16}.
\end{aligned}$$

*Now, by replacing in* (1)*, we see*

$$(0.5)(0.381966)\leq(0.317672)^2\frac{25}{16}\Rightarrow 0.190983\not\leq 0.157680,$$

*which is a contradiction and so the Inequality* (1) *is not valid.*

**Theorem 3.2.** *Let $f,g:[0,1]\to\mathbb{R}$ be two continuous and comonotone functions. Then the inequality*

$$\oint_0^1 f^s d\mu\cdot\oint_0^1 g^s d\mu\leq\oint_0^1 (f\cdot g)^s\,d\mu, \tag{5}$$

*holds, where $s>1$ and $\mu$ is the Lebesgue measure on $\mathbb{R}$.*

*Proof.* Let $f,g:[0,1]\to\mathbb{R}$ be two strictly increasing functions and let $f^s=F$, $g^s=G$, $(f\cdot g)^s=H$, $\oint_0^1 f^s d\mu=p$, $\oint_0^1 g^s d\mu=q$ and $\oint_0^1 (f\cdot g)^s d\mu=r$. We know that $p,q\in[0,1]$ and $0<r\leq 1$. First we let $p,q\in(0,1)$ and $0<r<1$.

$$\begin{cases} pq\leq p\Rightarrow 1-pq\geq 1-p\Rightarrow F(1-pq)\geq F(1-p)=p,\\ pq\leq q\Rightarrow 1-pq\geq 1-q\Rightarrow G(1-pq)\geq G(1-q)=q.\end{cases}$$

If we suppose that $r < pq$, we then get

$$1-r>1-pq \Rightarrow \left\{ \begin{array}{l} F(1-r)>F(1-pq)\geq F(1-p)\geq p, \\ G(1-r)>G(1-pq)\geq G(1-q)\geq q, \end{array} \right. \Rightarrow r>pq.$$

This relation is in contradiction with

$$\oint_0^1 (f\cdot g)^s d\mu = \oint_0^1 H d\mu = r \Rightarrow H(1-r)=r.$$

If we assume $p=1$ or $q=1$, then the relation $r \geq pq$ is valid.

Suppose $p=1$. Therefore,

$$F(1-p)\geq p \Rightarrow F(0)\geq 1.$$

Based on $F\uparrow$, on interval $[0,1]$ we have $F\geq 1$. Therefore,

$$\oint_0^1 F\cdot G d\mu \geq \oint_0^1 F d\mu \cdot \oint_0^1 G d\mu.$$

Now, suppose that $f,g:[0,1]\to\mathbb{R}$ are two strictly decreasing functions. By assumption in the first case, if $p,q\in(0,1)$ and $0<r<1$, then we have

$$\left\{ \begin{array}{l} pq\leq p \Rightarrow F(pq)\geq F(p)\geq p, \\ pq\leq q \Rightarrow G(pq)\geq G(q)\geq q. \end{array} \right.$$

If we let $r<pq$, we then have

$$\left\{ \begin{array}{l} F(r)>F(pq)\geq F(p)\geq p, \\ G(r)>G(pq)\geq G(q)\geq q, \end{array} \right. \Rightarrow H(r)=F(r)\cdot G(r)>pq,$$

which contradicts

$$(f\cdot g)(r)=H(r)=r.$$

If we assume that $p=1$ or $q=1$, then the inequality $r\geq pq$ is valid.

Assume that $p=1$. Then

$$F(p)\geq p \Rightarrow F(1)\geq 1.$$

Based on $F\downarrow$ on the interval $[0,1]$, $F\geq 1$ and therefore,

$$FG\geq G \Rightarrow \oint_0^1 F\cdot G d\mu \geq \oint_0^1 G d\mu = q = p\cdot q.$$

Thereby

$$\oint_0^1 F\cdot G d\mu \geq \oint_0^1 F d\mu \cdot \oint_0^1 G d\mu.$$

The proof is now complete. □

In the following, we generalize the Theorem 3.2, where the measure is an arbitrary fuzzy measure.

**Theorem 3.3.** *Let $f,g:[0,1]\to\mathbb{R}$ be two continuous and comonotone functions. Then the inequality*

$$\oint_0^1 f^s d\mu \cdot \oint_0^1 g^s d\mu \leq \oint_0^1 (f\cdot g)^s d\mu,$$

*holds, where $s>1$ and $\mu$ is an arbitrary fuzzy measure on $X=[0,1]$.*

*Proof.* We take

$$\left\{ \begin{array}{l} \oint_0^1 f^s d\mu = \oint_0^1 \mu\left([0,1]\cap F_\alpha\right) dm, \\ \\ \oint_0^1 g^s d\mu = \oint_0^1 \mu\left([0,1]\cap G_\alpha\right) dm, \\ \\ \oint_0^1 (f\cdot g)^s d\mu = \oint_0^1 \mu (f\cdot g)^s d\mu = \oint_0^1 \mu\left([0,1]\cap H_\alpha\right) dm, \end{array} \right.$$

where $F_\alpha = \{x \in \mathbb{R} | f^s(x) \geq \alpha\}$, $G_\alpha = \{x \in \mathbb{R} | g^s(x) \geq \alpha\}$, $H_\alpha = \{x \in \mathbb{R} | (f \cdot g)^s(x) \geq \alpha\}$,$\alpha \geq 0$ and $m$ is a Lebesgue measure on $\mathbb{R}$. If we let $A(\alpha) := \mu([0,1] \cap F_\alpha)$, $B(\alpha) := \mu([0,1] \cap B_\alpha)$ and $C(\alpha) := \mu([0,1] \cap H_\alpha)$, Then because $A(\alpha)$, $B(\alpha)$ and $C(\alpha)$ are nondecreasing respect to $\alpha$ and $m$ is a Lebesgue measure, we have

$$\oint_0^1 A(\alpha)dm \cdot \oint_0^1 B(\alpha)dm \leq \oint_0^1 C(\alpha)dm.$$

Since $\dfrac{(M+m)^2}{4Mm} \geq 1$, hence

$$\oint_0^1 A(\alpha)dm \cdot \oint_0^1 B(\alpha)dm \leq \frac{(M+m)^2}{4Mm} \oint_0^1 C(\alpha)dm.$$

The proof is now complete. □

**Example 3.4.** *Let $f, g : [0,1] \to \mathbb{R}$ where $f(x) = \dfrac{1}{2}\sqrt{x}$, $g(x) = \dfrac{1}{4}x$, $h(x) = f(x) \cdot g(x)$, $s = 2$ and $\mu$ be a Lebesgue measure on $\mathbb{R}$. by a simple calculations, we get*

$$\oint_0^1 f^2 d\mu = \sup_{\alpha \in [0, \frac{1}{4}]} [\alpha \wedge (1 - 4\alpha)] = \frac{1}{5} = 0.2,$$
$$\oint_0^1 g^2 d\mu = \sup_{\alpha \in [0, \frac{1}{16}]} \left[\alpha \wedge (1 - 4\sqrt{\alpha})\right] = \frac{18 - \sqrt{320}}{2} = 0.05,$$
$$\oint_0^1 (f \cdot g)^2 d\mu = \oint_0^1 h^2 d\mu = \sup_{\alpha \in [0, \frac{1}{64}]} \left[\alpha \wedge (1 - 4\sqrt[3]{\alpha})\right] = 0.015.$$

*Thus, we get*

$$0.015 = \oint_0^1 h^2 d\mu \geq \oint_0^1 f^2 d\mu \cdot \oint_0^1 g^d \mu = 0.2 \cdot 0.05.$$

**Example 3.5.** *Let $X = [0,1]$ and let the fuzzy measure $\mu$ be defined as $\mu(A) = m^2(A)$, where $m$ is the Lebesgue measure. And let*

$$f(x) = \begin{cases} \sqrt{x}, & x \in [0, \frac{1}{4}] \\ \dfrac{\sqrt{2}}{2}, & x \in (\frac{1}{4}, \frac{1}{2}) \\ \sqrt{x}, & x \in [\frac{1}{2}, 1] \end{cases} \Rightarrow f^2(x) = \begin{cases} x, & x \in [0, \frac{1}{4}] \\ \dfrac{1}{2}, & x \in (\frac{1}{4}, \frac{1}{2}) \\ x, & x \in [\frac{1}{2}, 1] \end{cases}$$

*and*

$$g(x) = \begin{cases} x, & x \in [0, \frac{1}{2}) \\ \sqrt{x}, & x \in [\frac{1}{2}, 1] \end{cases} \Rightarrow g^2(x) = \begin{cases} x^2, & x \in [0, \frac{1}{2}) \\ x, & x \in [\frac{1}{2}, 1] \end{cases}$$

*Then*

$$\oint_0^1 f^2 d\mu = \bigvee \left(\alpha \wedge m^2([0,1] \cap F_\alpha)\right) = \frac{1}{2},$$
$$\oint_0^1 g^2 d\mu = \bigvee \left(\alpha \wedge m^2([0,1] \cap G_\alpha)\right) = \frac{1}{4},$$
$$\frac{1}{4} = \oint_0^1 (f \cdot g)^2 d\mu > \left(\oint_0^1 f^2 d\mu\right)\left(\oint_0^1 g^2 d\mu\right) = \frac{1}{2} \cdot \frac{1}{4}.$$

**Remark 3.6.** *If $f$, and $g$ be countermonotone functions, then in the fuzzy case, it may not change in an unequal direction. This is illustrated by an example in the following.*

**Example 3.7.** *Let $f, g : [0,1] \to \mathbb{R}$, $f(x) = x + 1$, $g(x) = \dfrac{1}{x+1}$ and $\mu$ be the Lebesgue measure on $\mathbb{R}$ and $s = 3$. By a simple calculation, we have*

$$f^3(x) = (x+1)^3 = F,$$
$$g^3(x) = \left(\frac{1}{x+1}\right)^3 = G,$$
$$(f \cdot g)^3(x) = 1 = H.$$

*Then*

$$\begin{aligned}
&\oint_0^1 F d\mu = \sup\left[\alpha \wedge \left(-\sqrt{3}[\alpha]\right)\right) = 1,\\
&\oint_0^1 G d\mu = \sup\left[\alpha \wedge \left(\frac{1}{\sqrt{3}[\alpha]} - 1\right)\right] = 0.38,\\
&\oint_0^1 H d\mu = \oint_0^1 1 d\mu = 1.
\end{aligned}$$

*Therefore,*

$$1 = \oint_0^1 H d\mu \geq \oint_0^1 F d\mu \cdot \oint_0^1 G d\mu = 1 \times 0.38.$$

# 4 Diaz-Matcalf inequality for pseudo-integrals

In this section, we state and prove Diaz-Metcalf inequality for the pseudo-integral.

**Theorem 4.1.** *Let $f, h : [0,1] \to [a,b]$ be two measurable comonotone functions and $g : [a,b] \to [0,\infty)$ be a continuous and monotone function. Then the inequality*

$$\int_{[0,1]}^{\oplus} (f \odot h)^s dx \geq \int_{[0,1]}^{\oplus} f^s dx \odot \int_{[0,1]}^{\oplus} h^s dx, \tag{6}$$

*holds, where $s > 1$.*

*Proof.*

$$\begin{aligned}
\int_{[0,1]}^{\oplus} (f \odot h)^s dx &= \int_{[0,1]}^{\oplus} (f^s \odot h^s)\, dx = \int_{[0,1]}^{\oplus} (f^s \odot h^s)\, dx\\
&= g^{-1}\left[\int_0^1 g\,(f^s \odot h^s)\, dx\right] = g^{-1}\int_0^1 g(f^s) \times g(h^s) dx.
\end{aligned}$$

Using the Chebyshev Inequality, we get

$$\begin{aligned}
\int_{[0,1]}^{\oplus} (f \odot h)^s\, dx &\geq g^{-1}\left[\int_0^1 g(f^s)dx \times \int_0^1 g(h^s)dx\right]\\
&= g^{-1}\left[g\left(g^{-1}\left(\int_0^1 g(f^s)dx\right)\right) \times g\left(g^{-1}\left(\int_0^1 g(h^s)dx\right)\right)\right]\\
&= g^{-1}\left(g\left(\int_{[0,1]}^{\oplus} f^s dx\right) \times g\left(\int_{[0,1]}^{\oplus} h^s dx\right)\right)\\
&= \int_{[0,1]}^{\oplus} f^s dx \odot \int_{[0,1]}^{\oplus} h^s dx.
\end{aligned}$$

The proof is now complete. □

**Example 4.2.** *Let $f, h : [0,1] \to [0,1]$ are defined as $f(x) = \sqrt{x}$ and $h(x) = \sqrt{\frac{1}{2}x}$ respectively. Let $g : [0,1] \to' 0, \infty)$ where $g(x) = x^2$ and let $s = 2$. A straightforward calculation shows that*

$$\frac{\sqrt{20}}{2} = \int_{[0,1]}^{\oplus} \left(f^2 \odot h^2\right) dx \geq \int_{[0,1]}^{\oplus} f^2 dx \odot \int_{[0,1]}^{\oplus} h^2 dx = \frac{1}{6}.$$

Now, we generalize the Diaz-Metcalf type inequality by the semiring $([0,1], \max, \odot)$, where $\odot$ is generated.

**Theorem 4.3.** *Let $f, h : [0,1] \to [a,b]$ be two measurable comonotone functions and $([0,1], \sup, \odot)$ be a semiring and $m$ be the same as Theorems 2.7 and 2.8. If $g$ is the continuous and bounded function, then the following inequality is holds*

$$\int_{[0,1]}^{\sup} (f \odot h)^s dx \geq \left( \int_{[0,1]}^{\sup} f^s \odot dm \right) \odot \left( \int_{[0,1]}^{\sup} h^s \odot dm \right),$$

*where $f$ and $h$ are comonotone functions and $s > 1$.*

*Proof.* We have

$$\begin{aligned}
\int_{[0,1]}^{\sup} (f \odot h)^s \odot dm &= \lim_{\lambda \to \infty} (f \odot h)^s \odot dm_\lambda \\
&= \lim_{\lambda \to \infty} \left(g^\lambda\right)^{-1} \left( \int_0^1 g^\lambda \left(f^s \odot h^s\right) dx \right) \\
&= \lim_{\lambda \to \infty} \left(g^\lambda\right)^{-1} \left( \int_0^1 g^\lambda (f^s \odot h^s)(x) dx \right) \\
&\geq \lim_{\lambda \to \infty} \left( \left(g^\lambda\right)^{-1} \left( \int_0^1 g^\lambda (f^s(x)) dx \right) \odot \left(g^\lambda\right)^{-1} \left( \int_0^1 g^\lambda (h^s(x)) dx \right) \right) \\
&= \left( \lim_{\lambda \to \infty} \left(g^\lambda\right)^{-1} \left( \int_0^1 g^\lambda (f^s(x) \right) dx \right) \odot \left( \lim_{\lambda \to \infty} \left(g^\lambda\right)^{-1} \left( \int_0^1 g^\lambda (h^s(x)) dx \right) \right) \\
&= \left( \int_{[0,1]}^{\sup} f^s \odot dm \right) \odot \left( \int_{[0,1]}^{\sup} h^s \odot dm \right).
\end{aligned}$$

The proof is now complete. □

**Example 4.4.** *Let $g^\lambda(x) = e^{\lambda x}$ and $\psi(x)$ be the form Theorem 2.8. Then*

$$x \oplus y = \lim_{\lambda \to \infty} \frac{1}{\lambda} \ln \left( e^{\lambda x} + e^{\lambda y} \right) = \max(x, y),$$

*and*

$$x \odot y = \lim_{\lambda \to \infty} \frac{1}{\lambda} \ln \left( e^{\lambda x} \cdot e^{\lambda y} \right) = x + y.$$

*Therefore, we get*

$$\sup \left( f^s(x) + h^s(x)\psi(x) \right) \geq \sup \left( f^s(x) + \psi(x) \right) + \sup \left( h^s(x) + \psi(x) \right).$$

# 5 Further discussion

**Theorem 5.1.** *Let $f, h : [0,1] \to [a,b]$ be two measurable functions. Let a generator $g : [a,b] \to [0,\infty)$ of the pseudo addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. Let $\varphi : [a,b] \to [a,b]$ be a continuous and strictly increasing function that commutes with $\odot$. If $f$ and $h$ are comonotone, then the inequality*

$$\varphi^{-1} \left( \int_{[0,1]}^{\oplus} \varphi \left( f \odot h \right)^s dm \right) \geq \left( \varphi^{-1} \left( \int_{[0,1]}^{\oplus} \varphi(f^s) dx \right) \right) \odot \left( \varphi^{-1} \left( \int_{[0,1]}^{\oplus} \varphi(h^s) dx \right) \right),$$

*holds, where $s > 1$.*

*Proof.* Since $\varphi$ commutes with $\odot$, then we have

$$\int_{[0,1]}^{\oplus} \varphi \left( f^s \odot h^s \right) dx = \int_{[0,1]}^{\oplus} \left( \varphi(f^s) \odot \varphi(h^s) \right) dx. \tag{7}$$

If $f$ and $h$ are comonotone functions and $\varphi$ is a continuous and strictly function, then $\varphi(f^s)$ and $\varphi(h^s)$ are also comonotone. From (7) and using Chebyshev's Theorem, we have

$$\int_{[0,1]}^{\oplus} \varphi \left( f^s \odot h^s \right) dx \geq \left( \int_{[0,1]}^{\oplus} \varphi(f^s) dx \right) \odot \left( \int_{[0,1]}^{\oplus} \varphi(h^s) dx \right).$$

It follows that

$$\begin{aligned}\varphi^{-1}\int_{[0,1]}^{\oplus}\varphi\left(f^s\odot h^s\right)dx &\geq \varphi^{-1}\left[\left(\int_{[0,1]}^{\oplus}\varphi(f^s)dx\right)\odot\left(\int_{[0,1]}^{\oplus}\varphi(h^s)dx\right)\right]\\ &= \left[\left(\varphi^{-1}\left(\int_{[0,1]}^{\oplus}\varphi(f^s)dx\right)\right)\odot\left(\varphi^{-1}\left(\int_{[0,1]}^{\oplus}\varphi(h^s)dx\right)\right)\right],\end{aligned}$$

where $\varphi$ commutes with $\odot$, hence (7) is valid.

Similarly, if $f$ and $h$ are countermonotone functions and $\varphi$ is a continuous and strictly increasing function, then $\varphi(f^s)$ and $\varphi(h^s)$ are also countermonotone.

From (7) and using Chebyshev's Theorem, we have

$$\varphi^{-1}\left(\int_{[0,1]}^{\oplus}\varphi\left(f^s\odot h^s\right)dx\right)\leq\left(\varphi^{-1}\left(\int_{[0,1]}^{\oplus}\varphi(f^s)dx\right)\right)\odot\left(\varphi^{-1}\left(\int_{[0,1]}^{\oplus}\varphi(h^s)dx\right)\right),$$

where $\varphi$ commutes with $\odot$. Thereby, the theorem is proved. □

Now, we consider the second class, when $x\oplus y=\max(x,y)$ and $x\odot y=g^{-1}\left(g(x)g(y)\right)$.

**Theorem 5.2.** *Let $f,h:[0,1]\to[a,b]$ be two continuous functions and $\odot$ be represented by an increasing multiplicative generator $g$ and $\varphi:[a,b]\to[a,b]$ be a continuous and strictly increasing function such that $\varphi$ commutes with $\odot$ be the same as in Theorems 2.7 and 2.8. If $f$ and $h$ are comonotone, then the inequality*

$$\varphi^{-1}\left(\int_{[0,1]}^{\sup}\varphi\left(f^s\odot h^s\right)\odot dm\right)\geq\left(\varphi^{-1}\left(\int_{[0,1]}^{\sup}\varphi(f^s)\odot dm\right)\right)\odot\left(\varphi^{-1}\left(\int_{[0,1]}^{\sup}\varphi(f^s)\odot dm\right)\right),\tag{8}$$

*holds, where $s>1$.*

*Proof.* Theorem 2.8, implies that

$$\begin{aligned}\int_{[0,1]}^{\sup}\varphi\left(f^s\odot h^s\right)\odot dm &= \lim_{\lambda\to\infty}\int_{[0,1]}^{\oplus_\lambda}\varphi\left(f^s\odot h^s\right)\odot dm_\lambda\\ &= \lim_{\lambda\to\infty}\left(g^\lambda\right)^{-1}\left(\int_0^1 g^\lambda\left(\left(f^s\odot h^s\right)\right)(x)dx\right).\end{aligned}$$

If $f$ and $h$ are comonotone functions and $\varphi$ is a continuous and strictly functions, then using Theorem 2.9 and the Theorem 2.8, we have

$$\begin{aligned}&\varphi^{-1}\left(\int_{[0,1]}^{\sup}\varphi\left(f^s\odot h^s\right)\odot dm\right)\\ =\;&\varphi^{-1}\left(\lim_{\lambda\to\infty}\left(g^\lambda\right)^{-1}\left(\int_0^1 g^\lambda\left(f^s\odot h^s\right)(x)\right)dx\right)\\ =\;&\lim_{\lambda\to\infty}\left(\varphi^{-1}\left(\left(g^\lambda\right)^{-1}\left(\int_0^1 g^\lambda\left(\varphi\left(f^s\odot h^s\right)(x)\right)dx\right)\right)\right)\\ \geq\;&\lim_{\lambda\to\infty}\left[\left(\left(g^\lambda\right)^{-1}\left(\varphi^{-1}\left(\int_0^1 g^\lambda\left(\varphi(f^s(x))\right)\right)\right)\right)\odot\left(\left(g^\lambda\right)^{-1}\left(\varphi^{-1}\left(\int_0^1 g^\lambda\left(\varphi(h^s(x))\right)\right)\right)\right)\right]\\ =\;&\lim_{\lambda\to\infty}\left(\left(g^\lambda\right)^{-1}\left(\varphi^{-1}\left(\int_0^1 g^\lambda\left(\varphi\left(f^s(x)\right)\right)\right)dx\right)\right)\odot\lim_{\lambda\to\infty}\left(\left(g^\lambda\right)^{-1}\left(\varphi^{-1}\left(\int_0^1 g^\lambda\left(\varphi\left(h^s(x)\right)\right)\right)dx\right)\right)\\ =\;&\lim_{\lambda\to\infty}\left(g^\lambda\right)^{-1}\left(\int_0^1 g^\lambda\left(\left(f^s\odot h^s\right)\right)(x)dx\right),\end{aligned}$$

where $\varphi$ commutes with $\odot$. Hence (8) is valid.

□

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## References

[1] H. Agahi, R. Mesiar, Y. Ouyang, *Chebyshev type inequalities for pseudo-integrals*, Nonlinear Analysis: Theory, Methods and Applications, **72**(6) (2010), 2737-2743.

[2] H. Agahi, R. Mesiar, Y. Ouyang, E. Pap, M. Strboja, *Berwald type inequality for Sugeno integral*, Applied Mathematics and Computation, **217** (2010), 4100-4108.

[3] H. Agahi, R. Mesiar, Y. Ouyang, E. Pap, M. Strboja, *On Stolarsky inequality for Sugeno and Choquet integrals*, Information Sciences, **266** (2014), 134-139.

[4] H. Agahi, Y. Ouyang, R. Mesiar, E. Pap, M. Strboja, *Hölder and Minkowski type inequalities for pseudo-integral*, Applied Mathematics and Computation, **217** (2011), 8630-8639.

[5] B. Daraby, *Generalization of the Stolarsky type inequality for pseudo-integrals*, Fuzzy Sets and Systems, **194** (2012), 90-96.

[6] B. Daraby, L. Arabi, *Related Fritz Carlson type inequalities for Sugeno integrals*, Soft Computing, **17**(10) (2013), 1745-1750.

[7] B. Daraby, H. G. Asll, I. Sadeqi, *General related inequalities to Carlson-type inequality for the Sugeno integral*, Applied Mathematics and Computation, **305** (2017), 323-329.

[8] B. Daraby, R. Mesiar, F. Rostampour, A. Rahimi, *Related Thunsdorff type and Frank-Pick type inequalities for Sugeno integral*, Applied Mathematics and Computation, **414** (2022), 126683.

[9] B. Daraby, F. Rostampour, A. Rahimi, *Hardy's type inequality for pseudo integrals*, Acta Universitatis Apulensis, **42** (2015), 53-65.

[10] B. Daraby, A. Shafiloo, A. Rahimi, *Generalizations of the Feng Qi type inequality for pseudo-integral*, Gazi University Journal of Science, **28**(4) (2015), 695-702.

[11] S. Ezghari, A. Zahi, K. Zenkouar, *A new nearest neighbor classification method based on fuzzy set theory and aggregation operators*, Expert Systems with Applications, **80** (2017), 58-74.

[12] A. Flores-Franulič, H. Román-Flores, *A Chebyshev type inequality for fuzzy integrals*, Applied Mathematics and Computation, **190**(2) (2007), 1178-1184.

[13] A. Flores-Franulič, H. Román-Flores, Y. Chalco-Cano, *Markov type inequalities for fuzzy integrals*, Applied Mathematics and Computation, **207**(1) (2009), 242-247.

[14] R. Mesiar, E. Pap, *Idempotent integral as limit of g-integrals*, Fuzzy Sets and Systems, **102**(3) (1999), 385-392.

[15] M. S. Moslehian, R. Nakamoto, Y. Seo, *A Diaz-Metcalf type inequality for positive linear maps and its applications*, Electronic Journal of Linear Algebra, **22** (2011), 179-190.

[16] E. Pap, *g-Calculus*, University u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat, **23**(1) (1993), 145-156.

[17] E. Pap, *Null-additive set functions*, Kluwer Academic Publishers, Mathematics and its Applications, **337**, Dordrecht/Boston/London, 1995.

[18] E. Pap, *Pseudo-additive measures and their applications*, In Handbook of Measure Theory, North-Holland, (2002), 1403-1468.

[19] D. Ralescu, G. Adams, *The fuzzy integral*, Journal of Mathematical Analysis and Applications, **75**(2) (1980), 562-570.

[20] J. A. Rodger, G. A. George, *Triple bottom line accounting for optimizing natural gas sustainability: A statistical linear programming fuzzy ILOWA optimized sustainment model approach to reducing supply chain global cybersecurity vulnerability through information and communications technology*, Journal of Cleaner Production, **142** (2017), 1931-1949.

[21] M. Sugeno, *Theory of fuzzy integrals and its applications*, Doct. Thesis, Tokyo Institute of Technology, 1974.

[22] Z. Wang, G. J. Klir, *Fuzzy measure theory*, Springer Science and Business Media, 1992.

[23] D. Zhang, E. Pap, *Jensen's inequalities for pseudo-integrals*, Iranian Journal of Fuzzy Systems, **18**(3) (2021), 99-109.

[24] D. Zhang, E. Pap, R. Mesiar, *Jensen's inequality for Choquet integral revisited and a note on Jensen's inequality for generalized Choquet integral*, Fuzzy Sets and Systems, **430** (2022), 79-87.