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Original paper

Fuzzy accompanied approximation space under fuzzy relation

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Abstract

If there is a fuzzy relation $\widetilde{R}$ between two spaces U and V, the fuzzy approximations in both spaces based on $\widetilde{R}$ are widely studied, and they basically reflect only the influence from one space to another. In this paper, on each space of U and V, two new fuzzy relations are derived from $\widetilde{R}$, a positive low-value relation and a conservative high-value relation, to reflect the interaction and feedback between the two spaces. So, the fuzzy approximations based on them can reflect the combination of the action and the reaction from one space to another. Therefore, two spaces U and V are closely accompanied, and $(U, V, \widetilde{R})$ is a whole, so it is called a fuzzy accompanied approximation space (FAAS). In an FAAS, the properties of the fuzzy approximation models on each space are studied, the relationships between fuzzy approximation models of two spaces are researched, and examples to show how the approximation operator models in the FAAS to solve practical problems from multiple perspectives are also illustrated. More importantly, when the fuzzy relation $\widetilde{R}$ is a binary relation R or the two spaces are the same, the special cases of FAAS are investigated and some important new models and new results are obtained, which add new ideas and methods to the current research.

Keywords: Positive low-value relation, conservative high-value relation, fuzzy rough set, fuzzy accompanied approximation space.

1 Introduction

In order to deal with the uncertainty, granularity, and incompleteness of knowledge in the information systems, Z. Pawlak firstly proposed the rough set theory based on an equivalence relation on a universe [21]. To meet a wider range of data processing needs, the generalized rough set theories including covering rough set theory [3, 18, 19], binary relation based rough set theory [1, 13, 16, 28], as well as fuzzy rough set theory [10, 17, 23, 29] are gradually established. Moreover, algebraic methods, logic methods, and topological methods have also been introduced [5, 11, 15, 16, 19, 20, 31, 33], and they provide a strong theoretical support for the generalized rough set theory. In the past few decades, the generalized rough set theory has been widely used in all walks of life [2, 4, 6, 7, 12, 14, 27].

Especially worth mentioning is fuzzy rough set theory, which is deeply combined with soft computing, probability, logic and other theories to establish a variety of theoretical cross models, such as fuzzy covering rough set models, fuzzy probability rough set models, fuzzy soft rough set models and so on [8, 22, 25, 30, 32]. So the theoretical structure of fuzzy rough sets is greatly broadened and its application ability is improved from the perspective of synthesis system [24, 26].

For a given fuzzy relation $\widetilde{R}$ between two universes U and V, fuzzy approximation models based on fuzzy relations $\widetilde{R}$ have been widely studied and applied [9, 10]. Unlike the previous literatures, we study the fuzzy approximation models from a different view point in this paper. We take $(U, V, \widetilde{R})$ as a whole and call it fuzzy accompanied approximation space (FAAS for short). Differ from the fuzzy approximations in literature that only consider the one-way effects of one space on another, two new fuzzy relations called positive low-value relation and conservative high-value relation on each space U or V reflecting the interactions between the two spaces are induced by $\widetilde{R}$. So the fuzzy approximations based on them can reflect the feedback after one space acts on another space through $\widetilde{R}$. Particularly, if the fuzzy relation $\widetilde{R}$

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is a general binary relation R, or the two spaces U and V are the same in an FAAS, some important new results can be obtained, which add novel ideas and methods to the current research. In addition, we study the properties of the new approximate models and show their simple applications in the COVID-19 prevention measures and the student competition strategies.

The rest of this paper is organized as follows. In Section 2, we study the matrix knowledge which are the bases of new notions in the following sections. In Section 3, we define the FAAS, establish the new fuzzy relations and new fuzzy approximation models, and investigate their properties, as well as show simple application examples of FAAS. We then investigate the special cases of FAAS in Section 4 and arrive at some conclusions in Section 5.

2 Matrix knowledge

In this paper, a matrix $(a_{ij})_{m\times n}$ is called a fuzzy matrix if $0 \le a_{ij} \le 1$ $(i = 1, 2, \cdots, m, j = 1, 2, \cdots, n)$, and a vector $(a_1, a_2, \cdots, a_n)$ is called a fuzzy vector if $0 \le a_i \le 1$ $(i = 1, 2, \cdots, n)$. In this section, we show some properties of three types products of fuzzy matrices.

Definition 2.1. *Let $\alpha = (a_1, a_2, \cdots, a_n)$, $\beta = (b_1, b_2, \cdots, b_n)$ be two fuzzy vectors. Then $\alpha \cdot \beta$, $\alpha \circ \beta$ and $\alpha * \beta$ are defined as follows:*

$$\alpha \cdot \beta = \bigvee_{i=1}^{n} (a_i \wedge b_i), \quad \alpha \circ \beta = \bigwedge_{i=1}^{n} (a_i \vee b_i), \quad \alpha * \beta = \bigwedge_{i=1}^{n} [(1 - a_i) \vee b_i],$$

where $\vee$ and $\wedge$ denote the max *operation and the* min *operation respectively.*

In fact, the vector products $\alpha \cdot \beta$ and $\alpha * \beta$ were defined in [17] and the vector product $\alpha \circ \beta$ is a new concept. In these three vector products, the two products $\alpha \cdot \beta$ and $\alpha \circ \beta$ have dual forms, and they are the basis of establishing new concepts in this paper, while the product $\alpha * \beta$ will be used for the relevant calculations.

Example 2.2. *For $\alpha = (0.2, 0.3, 0.5)$ and $\beta = (0.3, 0.1, 0.6)$, there are $\alpha\cdot\beta = 0.2\vee 0.1\vee 0.5 = 0.5$, $\alpha\circ\beta = 0.3\wedge 0.3\wedge 0.6 = 0.3$, and $\alpha * \beta = 0.8 \wedge 0.7 \wedge 0.6 = 0.6$. Here we can get $\alpha \cdot \beta > \alpha \circ \beta$, but as the following data shows, that it doesn't always hold.*

For $\alpha' = (0.1, 0.8, 0.9)$ and $\beta' = (0.3, 0.7, 0.8)$, there are $\alpha' \cdot \beta' = 0.3$, $\alpha' \circ \beta' = 0.8$, and $\alpha' * \beta' = 0.7$, so we have $\alpha' \cdot \beta' < \alpha' \circ \beta'$.

This example shows that we do not know exactly which one is bigger between $\alpha \cdot \beta$ and $\alpha \circ \beta$, so both products are necessary in the following study. It follows from Definition 2.1 that $\alpha \cdot \beta$ denotes the positive low-value of α and β, and $\alpha \circ \beta$ denotes the conservative high-value of α and β. In fact, the two products are of great value in practice, and the following example will illustrate some of their practical meanings.

Example 2.3. *Let $\alpha = (0.2, 0.3, 0.5)$ and $\beta = (0.3, 0.1, 0.6)$ denote the weighted average scores of two primary school students A and B in subjects of Chinese, mathematics and foreign languages, respectively.*

Then the lower values of α and β are $0.2 \wedge 0.3 = 0.2$, $0.3 \wedge 0.1 = 0.1$ and $0.5 \wedge 0.6 = 0.5$, and the largest of them is $0.2 \vee 0.1 \vee 0.5 = 0.5$, which represents a positive estimate of the lower level between scores of the two students. While, the higher values of α and β are $0.2 \vee 0.3 = 0.3$, $0.3 \vee 0.1 = 0.3$ and $0.5 \vee 0.6 = 0.6$, and the smallest of them is $0.3 \wedge 0.3 \wedge 0.6 = 0.3$, which denotes a conservative estimate of the higher level between scores of the two students.

If all elements of a vector $\alpha = (a_1, a_2, \cdots, a_n)$ are 0 or 1, then α is called a Boolean vector, which is a special fuzzy vector.

Proposition 2.4. *Let $\alpha = (a_1, a_2, \cdots, a_n)$, $\beta = (b_1, b_2, \cdots, b_n)$ be two Boolean vectors, then*

i) [17] $\alpha \cdot \beta = 1 \Leftrightarrow \exists i, (a_i = 1) \wedge (b_i = 1)$,

ii) $\alpha \circ \beta = 0 \Leftrightarrow \exists i, (a_i = 0) \wedge (b_i = 0)$,

iii) [17] $\alpha * \beta = 1 \Leftrightarrow \forall i, (a_i = 1) \to (b_i = 1)$.

Proof. ii). $\alpha \circ \beta = 0 \Leftrightarrow \bigwedge_{i=1}^{n} (a_i \vee b_i) = 0 \Rightarrow \exists i, a_i \vee b_i = 0 \Leftrightarrow \exists i, (a_i = 0) \wedge (b_i = 0)$. □

For the fuzzy vectors, we have the following property for the products in Definition 2.1.

Proposition 2.5. *Let $\alpha = (a_1, a_2, \cdots, a_n)$, $\beta = (b_1, b_2, \cdots, b_n)$, $\gamma = (c_1, c_2, \cdots, c_n)$ be three fuzzy vectors. Then*

i) $(\alpha \vee \beta) \cdot \gamma = (\alpha \cdot \gamma) \vee (\beta \cdot \gamma)$, $\quad (\alpha \wedge \beta) \cdot \gamma \le (\alpha \cdot \gamma) \wedge (\beta \cdot \gamma)$,

$\alpha \cdot (\beta \vee \gamma) = (\alpha \cdot \beta) \vee (\alpha \cdot \gamma)$, $\quad \alpha \cdot (\beta \wedge \gamma) \le (\alpha \cdot \beta) \wedge (\alpha \cdot \gamma)$,

ii) $(\alpha\vee\beta)\circ\gamma\geq(\alpha\circ\gamma)\vee(\beta\circ\gamma)$, $\quad(\alpha\wedge\beta)\circ\gamma=(\alpha\circ\gamma)\wedge(\beta\circ\gamma)$,
$\alpha\circ(\beta\vee\gamma)\geq(\alpha\circ\beta)\vee(\alpha\circ\gamma)$, $\quad\alpha\circ(\beta\wedge\gamma)=(\alpha\circ\beta)\wedge(\alpha\circ\gamma)$,
iii) $(\alpha\wedge\beta)*\gamma\geq(\alpha*\gamma)\vee(\beta*\gamma)$, $\quad(\alpha\vee\beta)*\gamma=(\alpha*\gamma)\wedge(\beta*\gamma)$,
$\alpha*(\beta\vee\gamma)\geq(\alpha*\beta)\vee(\alpha*\gamma)$, $\quad\alpha*(\beta\wedge\gamma)=(\alpha*\beta)\wedge(\alpha*\gamma)$,
iv) $\alpha\cdot(-\beta)=-(\alpha*\beta)$, $\quad\alpha*(-\beta)=-(\alpha\cdot\beta)$, $\quad\alpha\circ(-\beta)=\beta*\alpha$.

Proof. *i)* $(\alpha\cdot\gamma)\vee(\beta\cdot\gamma)=(\vee_{i=1}^{n}(a_i\wedge c_i))\bigvee(\vee_{i=1}^{n}(b_i\wedge c_i))=\vee_{i=1}^{n}[(a_i\wedge c_i)\vee(b_i\wedge c_i)]=\vee_{i=1}^{n}[(a_i\vee b_i)\wedge c_i]=(\alpha\vee\beta)\cdot\gamma$.
$(\alpha\cdot\gamma)\wedge(\beta\cdot\gamma)=(\vee_{i=1}^{n}(a_i\wedge c_i))\bigwedge(\vee_{i=1}^{n}(b_i\wedge c_i))\geq\vee_{i=1}^{n}[(a_i\wedge c_i)\wedge(b_i\wedge c_i)]=\vee_{i=1}^{n}[(a_i\wedge b_i)\wedge c_i]=(\alpha\wedge\beta)\cdot\gamma$.
Similarly, we can obtain $\alpha\cdot(\beta\vee\gamma)=(\alpha\cdot\beta)\vee(\alpha\cdot\gamma)$ and $\alpha\cdot(\beta\wedge\gamma)\leq(\alpha\cdot\beta)\wedge(\alpha\cdot\gamma)$.
ii) $(\alpha\circ\gamma)\vee(\beta\circ\gamma)=(\wedge_{i=1}^{n}(a_i\vee c_i))\bigvee(\wedge_{i=1}^{n}(b_i\vee c_i))\leq\wedge_{i=1}^{n}[(a_i\vee c_i)\vee(b_i\vee c_i)]=\wedge_{i=1}^{n}[(a_i\vee b_i)\vee c_i]=(\alpha\vee\beta)\circ\gamma$.
$(\alpha\circ\gamma)\wedge(\beta\circ\gamma)=(\wedge_{i=1}^{n}(a_i\vee c_i))\bigwedge(\wedge_{i=1}^{n}(b_i\vee c_i))=\wedge_{i=1}^{n}[(a_i\vee c_i)\wedge(b_i\vee c_i)]=\wedge_{i=1}^{n}[(a_i\wedge b_i)\vee c_i]=(\alpha\wedge\beta)\circ\gamma$.
The other two formulas $\alpha\circ(\beta\vee\gamma)\geq(\alpha\circ\beta)\vee(\alpha\circ\gamma)$ and $\alpha\circ(\beta\wedge\gamma)=(\alpha\circ\beta)\wedge(\alpha\circ\gamma)$ can be proved similarly.
iii) $(\alpha*\gamma)\vee(\beta*\gamma)=\wedge_{i=1}^{n}[(1-a_i)\vee c_i]\bigvee\wedge_{i=1}^{n}[(1-b_i)\vee c_i]\leq\wedge_{i=1}^{n}([(1-a_i)\vee c_i]\vee[(1-b_i)\vee c_i])=\wedge_{i=1}^{n}[(1-a_i)\vee(1-b_i)\vee c_i]=\wedge_{i=1}^{n}[(1-(a_i\wedge b_i))\vee c_i]=(\alpha\wedge\beta)*\gamma$.
$(\alpha*\gamma)\wedge(\beta*\gamma)=\wedge_{i=1}^{n}[(1-a_i)\vee c_i]\bigwedge\wedge_{i=1}^{n}[(1-b_i)\vee c_i]=\wedge_{i=1}^{n}([(1-a_i)\vee c_i]\wedge[(1-b_i)\vee c_i])=\wedge_{i=1}^{n}[(1-a_i)\wedge(1-b_i)\vee c_i]=\wedge_{i=1}^{n}[(1-(a_i\vee b_i))\vee c_i]=(\alpha\vee\beta)*\gamma$.
Two other formulas $\alpha*(\beta\vee\gamma)\geq(\alpha*\beta)\vee(\alpha*\gamma)$ and $\alpha*(\beta\wedge\gamma)=(\alpha*\beta)\wedge(\alpha*\gamma)$ can be obtained using similar methods.
iv) $\alpha\cdot(-\beta)=\vee_{i=1}^{n}[a_i\wedge(1-b_i)]=\vee_{i=1}^{n}[1-((1-a_i)\vee b_i)]=1-\wedge_{i=1}^{n}[(1-a_i)\vee b_i]=-(\alpha*\beta)$,
$\alpha*(-\beta)=\wedge_{i=1}^{n}[(1-a_i)\vee(1-b_i)]=\wedge_{i=1}^{n}[1-(a_i\wedge b_i)]=1-\vee_{i=1}^{n}(a_i\wedge b_i)=-(\alpha\cdot\beta)$,
$\alpha\circ(-\beta)=\wedge_{i=1}^{n}[a_i\vee(1-b_i)]=\beta*\alpha$. □

Next we use vector products to define matrix products which will be used frequently in this article.

Definition 2.6. [17] *Let $M_{m\times l}=(a_{ij})_{m\times l}$ and $N_{l\times n}=(b_{ij})_{l\times n}$ be two fuzzy matrices. Then three types of matrix products $M\cdot N=(c_{ij})_{m\times n}$, $M\circ N=(d_{ij})_{m\times n}$ and $M*N=(t_{ij})_{m\times n}$ are defined as follows:*
$c_{ij}=(a_{i1},a_{i2},\cdots,a_{il})\cdot(b_{1j},b_{2j},\cdots,b_{lj})$,
$d_{ij}=(a_{i1},a_{i2},\cdots,a_{il})\circ(b_{1j},b_{2j},\cdots,b_{lj})$,
$t_{ij}=(a_{i1},a_{i2},\cdots,a_{il})*(b_{1j},b_{2j},\cdots,b_{lj})$,
for $i=1,2,\cdots,m$, $j=1,2,\cdots,n$.

In particular, if all elements of a_{ij},b_{ij} are 0 or 1, then the matrices M and N are called Boolean matrices and the matrix product $M\cdot N$ is called Boolean product.

3 Fuzzy accompanied approximation spaces

If two spaces U and V are related by a fuzzy relation $\widetilde{R}$, then they actually become a whole and affect each other. Based on $\widetilde{R}$, we will establish two new fuzzy relations on each space of U and V to reflect the mutual influence between the two spaces.

Definition 3.1. *Let $U=\{x_1,x_2,\cdots,x_m\}$, $V=\{y_1,y_2,\cdots,y_n\}$ be two universes and $\widetilde{R}=\{\widetilde{R}(x,y)\in[0,1]:x\in U,y\in V\}$ be a fuzzy relation from U to V, then $(U,V,\widetilde{R})$ is called a fuzzy accompanied approximation space (FAAS for short).*

Correspondingly, the fuzzy relation $\widetilde{R^{-1}}=\{\widetilde{R^{-1}}(y,x)=\widetilde{R}(x,y):y\in V,x\in U\}$ is called the inverse fuzzy relation of $\widetilde{R}$ from V to U.

The matrix $\widetilde{M}=(\widetilde{R}(x_i,y_j))_{m\times n}$ is called matrix representation of the fuzzy relation $\widetilde{R}$ and $\widetilde{M^T}=(\widetilde{R^{-1}}(y_j,x_i))_{n\times m}$ is called matrix representation of the inverse fuzzy relation $\widetilde{R^{-1}}$.

Unlike the fuzzy approximation models based on $\widetilde{R}$ in the literature, which only considers the one-way influence of one space on another space, we define the following two new fuzzy relations on each space in an FAAS, which can reflect the interaction between the two spaces, and then we establish new fuzzy approximation models based on them.

Definition 3.2. *Suppose that $(U,V,\widetilde{R})$ is an FAAS, where $U=\{x_1,x_2,\cdots,x_m\}$, $V=\{y_1,y_2,\cdots,y_n\}$ are two universes. Then the positive low-value relations $\widetilde{\dot{R}_U}$ on U, $\widetilde{\dot{R}_V}$ on V, and the conservative high-value relations $\widetilde{\ddot{R}_U}$ on U, $\widetilde{\ddot{R}_V}$ on V are defined as*
$\widetilde{\dot{R}_U}(x_i,x_j)=\vee_{k=1}^{n}[\widetilde{R}(x_i,y_k)\wedge\widetilde{R}^{-1}(y_k,x_j)]$, $\quad\widetilde{\dot{R}_V}(y_i,y_j)=\vee_{k=1}^{m}[\widetilde{R}^{-1}(y_i,x_k)\wedge\widetilde{R}(x_k,y_j)]$,
and

$\widetilde{\ddot{R}_U}(x_i,x_j)=\wedge_{k=1}^{n}[\widetilde{R}(x_i,y_k)\vee \widetilde{R}^{-1}(y_k,x_j)],\quad \widetilde{\ddot{R}_V}(y_i,y_j)=\wedge_{k=1}^{m}[\widetilde{R}^{-1}(y_i,x_k)\vee \widetilde{R}(x_k,y_j)],$
respectively.

Due to the meanings of matrix products in Definition 2.6, the next proposition is easily obtained.

Proposition 3.3. *Suppose that $(U,V,\widetilde{R})$ is an FAAS, where $U=\{x_1,x_2,\cdots,x_m\}$, $V=\{y_1,y_2,\cdots,y_n\}$ are two universes and $\widetilde{M}=(\widetilde{R}(x_i,y_j))_{m\times n}$ is the matrix representation of the fuzzy relation $\widetilde{R}$. Then the matrix representations of relations $\widetilde{\dot{R}_U},\widetilde{\ddot{R}_U},\widetilde{\dot{R}_V}$ and $\widetilde{\ddot{R}_V}$ are*
$(\widetilde{M}\cdot\widetilde{M^T})_{m\times m},\quad (\widetilde{M}\circ\widetilde{M^T})_{m\times m},\quad (\widetilde{M^T}\cdot\widetilde{M})_{n\times n}$ *and* $(\widetilde{M^T}\circ\widetilde{M})_{n\times n}$ *respectively.*

So, for convenience, each of these new fuzzy relations and its matrix representation are not distinguished, and they can be written as follows.
$\widetilde{\dot{R}_U}=(\widetilde{M}\cdot\widetilde{M^T})_{m\times m},\quad \widetilde{\ddot{R}_U}=(\widetilde{M}\circ\widetilde{M^T})_{m\times m},$
$\widetilde{\dot{R}_V}=(\widetilde{M^T}\cdot\widetilde{M})_{n\times n},\quad \widetilde{\ddot{R}_V}=(\widetilde{M^T}\circ\widetilde{M})_{n\times n}.$

Existing research on fuzzy approximations generally based on the fuzzy relation $\widetilde{R}$, only reflect the influence of one space to another, while the fuzzy relations $\widetilde{\dot{R}_U}$, $\widetilde{\ddot{R}_U}$, $\widetilde{\dot{R}_V}$ and $\widetilde{\ddot{R}_V}$ are combinations of $\widetilde{R}$ and $\widetilde{R^{-1}}$, thus they can reflect the action and reaction of one space to another. So the fuzzy approximations based on these new fuzzy relations can reflect the interactions between two spaces.

Proposition 3.4. *Let $(U,V,\widetilde{R})$ be an FAAS, then all the fuzzy relations $\widetilde{\dot{R}_U}$, $\widetilde{\ddot{R}_U}$, $\widetilde{\dot{R}_V}$ and $\widetilde{\ddot{R}_V}$ are symmetric.*

Proof. We need only to prove that $(\widetilde{M}\circ\widetilde{M^T})_{m\times m}$ is an symmetrical matrix. Suppose that d_{ij} is an element of matrix $\widetilde{M}\circ\widetilde{M^T}$ and c_{ij} is a element of matrix $\widetilde{M}$, then noting that c_{jk} in $\widetilde{M^T}$ is c_{kj} in $\widetilde{M}$, we have

$$d_{ij}=\wedge_{k=1}^{n}[a_{ik}\vee c_{jk}]=\wedge_{k=1}^{n}[c_{jk}\vee c_{ik}]=d_{ji}.$$

Therefor, $(\widetilde{M}\circ\widetilde{M^T})_{m\times m}$ is a symmetrical matrix.

Similarly, we can obtain that all the matrices $(\widetilde{M}\cdot\widetilde{M^T})_{m\times m}$, $(\widetilde{M^T}\cdot\widetilde{M})_{n\times n}$ and $(\widetilde{M^T}\circ\widetilde{M})_{n\times n}$ are symmetrical matrices. So all the fuzzy relations $\widetilde{\dot{R}_U}$, $\widetilde{\ddot{R}_U}$, $\widetilde{\dot{R}_V}$ and $\widetilde{\ddot{R}_V}$ are symmetric. □

It follows from Example 2.2 that, it is not clear which is larger between the positive low-value relation and the conservative high-value relation, so both of them are important to solve some practical problems from multiple aspects.

Under the two new relations, we use the Dubois and Prade's fuzzy rough set models [10] to define two new types of approximation operators on each of the two universes in the FAAS.

Definition 3.5. *Let $U=\{x_1,x_2,\cdots,x_m\}$, $V=\{y_1,y_2,\cdots,y_n\}$ be two universes and $\widetilde{M}$ be the matrix representation of a fuzzy relation $\widetilde{R}$ from U to V. Then for each $\widetilde{X}\in\mathcal{F}(U)$ and each $\widetilde{Y}\in\mathcal{F}(V)$, the fuzzy lower approximations $\widetilde{\dot{P}^-}(\widetilde{X})$ and $\widetilde{\ddot{P}^-}(\widetilde{X})$, the fuzzy upper approximations $\widetilde{\dot{P}^+}(\widetilde{X})$ and $\widetilde{\ddot{P}^+}(\widetilde{X})$, the fuzzy lower approximations $\widetilde{\dot{Q}^-}(\widetilde{Y})$ and $\widetilde{\ddot{Q}^-}(\widetilde{Y})$, the fuzzy upper approximations $\widetilde{\dot{Q}^+}(\widetilde{Y})$ and $\widetilde{\ddot{Q}^+}(\widetilde{Y})$ are defined as follows:*
$\widetilde{\dot{P}^-}(\widetilde{X})(x)=\wedge_{x'\in U}[(1-\widetilde{\dot{R}_U}(x,x')\vee\widetilde{X}(x')],\quad \widetilde{\dot{P}^+}(\widetilde{X})(x)=\vee_{x'\in U}[\widetilde{\dot{R}_U}(x,x')\wedge\widetilde{X}(x')]\ (x\in U),$
$\widetilde{\ddot{P}^-}(\widetilde{X})(x)=\wedge_{x'\in U}[(1-\widetilde{\ddot{R}_U}(x,x')\vee\widetilde{X}(x')],\quad \widetilde{\ddot{P}^+}(\widetilde{X})(x)=\vee_{x'\in U}[\widetilde{\ddot{R}_U}(x,x')\wedge\widetilde{X}(x')]\ (x\in U),$
$\widetilde{\dot{Q}^-}(\widetilde{Y})(y)=\wedge_{y'\in V}[(1-\widetilde{\dot{R}_V}(y,y')\vee\widetilde{Y}(y')],\quad \widetilde{\dot{Q}^+}(\widetilde{Y})(y)=\vee_{y'\in V}[\widetilde{\dot{R}_V}(y,y')\wedge\widetilde{Y}(y')]\ (y\in V),$
$\widetilde{\ddot{Q}^-}(\widetilde{Y})(y)=\wedge_{y'\in V}[(1-\widetilde{\ddot{R}_V}(y,y')\vee\widetilde{Y}(y')],\quad \widetilde{\ddot{Q}^+}(\widetilde{Y})(y)=\vee_{y'\in V}[\widetilde{\ddot{R}_V}(y,y')\wedge\widetilde{Y}(y')]\ (y\in V).$

All matrix representations of these approximations are listed in the next proposition, which follow directly from the Definition 3.2 and Definition 3.5.

Proposition 3.6. *Let $U=\{x_1,x_2,\cdots,x_m\}$, $V=\{y_1,y_2,\cdots,y_n\}$ be two universes and $\widetilde{M}$ be the matrix representation of a fuzzy relation $\widetilde{R}$ from U to V. Then for each $\widetilde{X}\in\mathcal{F}(U)$ and each $\widetilde{Y}\in\mathcal{F}(V)$, the matrix representations of approximations in Definition 3.5 are as follows:*
$\widetilde{\dot{P}^-}(\widetilde{X})=\widetilde{M}\cdot\widetilde{M}^T*\widetilde{X},\quad \widetilde{\dot{P}^+}(\widetilde{X})=\widetilde{M}\cdot\widetilde{M}^T\cdot\widetilde{X},$
$\widetilde{\ddot{P}^-}(\widetilde{X})=\widetilde{M}\circ\widetilde{M}^T*\widetilde{X},\quad \widetilde{\ddot{P}^+}(\widetilde{X})=\widetilde{M}\circ\widetilde{M}^T\cdot\widetilde{X},$
$\widetilde{\dot{Q}^-}(\widetilde{Y})=\widetilde{M}^T\cdot\widetilde{M}*\widetilde{Y},\quad \widetilde{\dot{Q}^+}(\widetilde{Y})=\widetilde{M}^T\cdot\widetilde{M}\cdot\widetilde{Y},$
$\widetilde{\ddot{Q}^-}(\widetilde{Y})=\widetilde{M}^T\circ\widetilde{M}*\widetilde{Y},\quad \widetilde{\ddot{Q}^+}(\widetilde{Y})=\widetilde{M}^T\circ\widetilde{M}\cdot\widetilde{Y}.$

Proof. For each $\widetilde{X} \in F(U)$ and each $x \in U$, according to $\widetilde{M} \cdot \widetilde{M}^T$ and $\widetilde{M} \circ \widetilde{M}^T$ in Definition 3.2, we have

$(\widetilde{M} \cdot \widetilde{M}^T * \widetilde{X})(x) = \wedge_{x' \in U}[\widetilde{(1 - \dot{R}_U(x,x')) \vee \widetilde{X}(x')}] = (\widetilde{\dot{P}^- \widetilde{X}})(x)$,

$(\widetilde{M} \cdot \widetilde{M}^T \cdot \widetilde{X})(x) = \vee_{x' \in U}(\widetilde{\dot{R}_U(x,x')} \wedge \widetilde{X}(x')) = (\widetilde{\dot{P}^+(\widetilde{X})})(x)$.

Hence, $\widetilde{\dot{P}^- \widetilde{X}} = \widetilde{M} \cdot \widetilde{M}^T * \widetilde{X}$ and $\widetilde{\dot{P}^+(\widetilde{X})} = \widetilde{M} \cdot \widetilde{M}^T \cdot \widetilde{X}$ can be followed.

Similarly, for each $\widetilde{Y} \in F(V)$ and each $y \in Y$, we can obtain

$(\widetilde{M}^T \cdot \widetilde{M} * \widetilde{Y})(y) = \wedge_{y' \in U}[\widetilde{(1 - \dot{R}_V(y,y')) \vee \widetilde{Y}(y')}] = (\widetilde{\dot{Q}^- \widetilde{Y}})(y)$,

$(M^T \cdot M \cdot \widetilde{Y})(y) = \vee_{y' \in U}(\widetilde{\dot{R}_V(y,y')} \wedge \widetilde{Y}(y')) = (\widetilde{\dot{Q}^+(\widetilde{Y})})(y)$,

and thus, $\widetilde{\dot{Q}^-(\widetilde{Y})} = \widetilde{M}^T \cdot \widetilde{M} * \widetilde{Y}$ and $\widetilde{\dot{Q}^+(\widetilde{Y})} = \widetilde{M}^T \cdot \widetilde{M} \cdot \widetilde{Y}$ can be followed. Using the same methods we can get

$\widetilde{\ddot{P}^+(\widetilde{X})} = \widetilde{M} \circ \widetilde{M}^T \cdot \widetilde{X}$, $\widetilde{\ddot{P}^-(\widetilde{X})} = \widetilde{M} \circ \widetilde{M}^T * \widetilde{X}$, $\widetilde{\ddot{Q}^+(\widetilde{Y})} = \widetilde{M}^T \circ \widetilde{M} \cdot \widetilde{Y}$, $\widetilde{\ddot{Q}^-(\widetilde{Y})} = \widetilde{M}^T \circ \widetilde{M} * \widetilde{Y}$. □

Proposition 3.7. *Let $\widetilde{(U,V,\widetilde{R})}$ be an FAAS. For any $X, X_1, X_2 \subset U$ and $Y, Y_1, Y_2 \subset U$, there are*

i) $\widetilde{\dot{P}^-}(-\widetilde{X}) = -\widetilde{\dot{P}^+}(\widetilde{X})$, $\widetilde{\dot{Q}^-}(-\widetilde{Y}) = -\widetilde{\dot{Q}^+(\widetilde{Y})}$,

ii) $\widetilde{\dot{P}^-}(\widetilde{U}) = \widetilde{U}$, $\widetilde{\dot{P}^+}(\widetilde{\emptyset}) = \widetilde{\emptyset}$, $\widetilde{\dot{Q}^-}(\widetilde{V}) = \widetilde{V}$, $\widetilde{\dot{Q}^+}(\widetilde{\emptyset}) = \widetilde{\emptyset}$,

iii) $\widetilde{\dot{P}^-(\widetilde{X_1} \cap \widetilde{X_2})} = \widetilde{\dot{P}^-}(\widetilde{X_1}) \cap \widetilde{\dot{P}^-}(\widetilde{X_2})$, $\widetilde{\dot{P}^+}(\widetilde{X_1} \cup \widetilde{X_2}) = \widetilde{\dot{P}^+}(\widetilde{X_1}) \cup \widetilde{\dot{P}^+}(\widetilde{X_2})$,

$\widetilde{\dot{Q}^-}(\widetilde{Y_1} \cap \widetilde{Y_2}) = \widetilde{\dot{Q}^-}(\widetilde{Y_1}) \cap \widetilde{\dot{Q}^-}(\widetilde{Y_2})$, $\widetilde{\dot{Q}^+}(\widetilde{Y_1} \cup \widetilde{Y_2}) = \widetilde{\dot{Q}^+}(\widetilde{Y_1}) \cup \widetilde{\dot{Q}^+}(\widetilde{Y_2})$,

iv) $\widetilde{\dot{P}^-}(\widetilde{X_1} \cup \widetilde{X_2}) \supset \widetilde{\dot{P}^-}(\widetilde{X_1}) \cup \widetilde{\dot{P}^-}(\widetilde{X_2})$, $\widetilde{\dot{P}^+}(\widetilde{X_1} \cap \widetilde{X_2}) \supset \widetilde{\dot{P}^+}(\widetilde{X_1}) \cap \widetilde{\dot{P}^+}(\widetilde{X_2})$,

$\widetilde{\dot{Q}^-}(\widetilde{Y_1} \cup \widetilde{Y_2}) \supset \widetilde{\dot{Q}^-}(\widetilde{Y_1}) \cup \widetilde{\dot{Q}^-}(\widetilde{Y_2})$, $\widetilde{\dot{Q}^+}(\widetilde{Y_1} \cap \widetilde{Y_2}) \supset \widetilde{\dot{Q}^+}(\widetilde{Y_1}) \cap \widetilde{\dot{Q}^+}(\widetilde{Y_2})$,

v) $\widetilde{X_1} \subset \widetilde{X_2}$ *implies* $\widetilde{\dot{P}^-}(\widetilde{X_1}) \subset \widetilde{\dot{P}^-}(\widetilde{X_2})$, $\widetilde{\dot{P}^+}(\widetilde{X_1}) \subset \widetilde{\dot{P}^+}(\widetilde{X_2})$, *and*

$\widetilde{Y_1} \subset \widetilde{Y_2}$ *implies* $\widetilde{\dot{Q}^-}(\widetilde{Y_1}) \subset \widetilde{\dot{Q}^-}(\widetilde{Y_2})$, $\widetilde{\dot{Q}^+}(\widetilde{Y_1}) \subset \widetilde{\dot{Q}^+}(\widetilde{Y_2})$.

Proof. *i)* It follows from the Proposition 2.5 that

$\widetilde{\dot{P}^-}(-\widetilde{X}) = \widetilde{M} \cdot \widetilde{M}^T * (-\widetilde{X}) = -(\widetilde{M} \cdot \widetilde{M}^T \cdot \widetilde{X}) = -\widetilde{\dot{P}^+}(\widetilde{X})$,

$\widetilde{\dot{Q}^-}(-\widetilde{Y}) = \widetilde{M}^T \cdot \widetilde{M} * (-\widetilde{Y}) = -(\widetilde{M}^T \cdot \widetilde{M} \cdot \widetilde{Y}) = -\widetilde{\dot{Q}^+}(\widetilde{Y})$.

ii) $\widetilde{\dot{P}^-}(\widetilde{U}) = \widetilde{M} \cdot \widetilde{M}^T * \widetilde{U} = 1_U = \widetilde{U}$, $\widetilde{\dot{P}^+}(\widetilde{\emptyset}) = \widetilde{M} \cdot \widetilde{M}^T \cdot \widetilde{\emptyset} = 0_U = \widetilde{\emptyset}$.

Similarly we have $\widetilde{\dot{Q}^-}(\widetilde{V}) = \widetilde{V}$, and $\widetilde{\dot{Q}^-}(\widetilde{\emptyset}) = \widetilde{\emptyset}$;

iii) It follows from the Proposition 2.5 that

$\widetilde{\dot{P}^-}(\widetilde{X_1} \cap \widetilde{X_2}) = \widetilde{M} \cdot \widetilde{M}^T * (\widetilde{X_1} \wedge \widetilde{X_2}) = (\widetilde{M} \cdot \widetilde{M}^T * \widetilde{X_1}) \wedge (\widetilde{M} \cdot \widetilde{M}^T * \widetilde{X_2}) = \widetilde{\dot{P}^-}(\widetilde{X_1}) \cap \widetilde{\dot{P}^-}(\widetilde{X_2})$.

$\widetilde{\dot{P}^+}(\widetilde{X_1} \cup \widetilde{X_2}) = \widetilde{M} \cdot \widetilde{M}^T \cdot (\widetilde{X_1} \vee \widetilde{X_2}) = (\widetilde{M} \cdot \widetilde{M}^T \cdot \widetilde{X_1}) \vee (\widetilde{M} \cdot \widetilde{M}^T \cdot \widetilde{X_2}) = \widetilde{\dot{P}^+}(\widetilde{X_1}) \cup \widetilde{\dot{P}^+}(\widetilde{X_2})$,

Similarly, we can prove

$\widetilde{\dot{Q}^-}(\widetilde{Y_1} \cap \widetilde{Y_2}) = \widetilde{\dot{Q}^-}(\widetilde{Y_1}) \cap \widetilde{\dot{Q}^-}(\widetilde{Y_2})$ and $\widetilde{\dot{Q}^+}(\widetilde{Y_1} \cup \widetilde{Y_2}) = \widetilde{\dot{Q}^+}(\widetilde{Y_1}) \cup \widetilde{\dot{Q}^+}(\widetilde{Y_2})$.

iv) $\widetilde{\dot{P}^-}(\widetilde{X_1} \cup \widetilde{X_2}) = \widetilde{M} \cdot \widetilde{M}^T * (\widetilde{X_1} \vee \widetilde{X_2}) \supset (\widetilde{M} \cdot \widetilde{M}^T * \widetilde{X_1}) \vee (\widetilde{M} \cdot \widetilde{M}^T * \widetilde{X_2}) = \widetilde{\dot{P}^-}(\widetilde{X_1}) \cup \widetilde{\dot{P}^-}(\widetilde{X_2})$,

$\widetilde{\dot{P}^-}(\widetilde{X_1} \cap \widetilde{X_2}) = \widetilde{M} \cdot \widetilde{M}^T \cdot (\widetilde{X_1} \cap \widetilde{X_2}) \subset (\widetilde{M} \cdot \widetilde{M}^T \cdot \widetilde{X_1}) \wedge (\widetilde{M} \cdot \widetilde{M}^T \cdot \widetilde{X_2}) = \widetilde{\dot{P}^+}(\widetilde{X_1}) \cap \widetilde{\dot{P}^+}(\widetilde{X_2})$.

We can easily get $\widetilde{\dot{Q}^-}(\widetilde{Y_1} \cup \widetilde{Y_2}) \supset \widetilde{\dot{Q}^-}(\widetilde{Y_1}) \cup \widetilde{\dot{Q}^-}(\widetilde{Y_2})$ and $\widetilde{\dot{Q}^+}(\widetilde{Y_1} \cap \widetilde{Y_2}) \subset \widetilde{\dot{Q}^+}(\widetilde{Y_1}) \cap \widetilde{\dot{Q}^+}(\widetilde{Y_2})$ in a similar way.

v) is obvious and thus we have finished the proof. □

Proposition 3.8. *Let $\widetilde{(U,V,\widetilde{R})}$ be an FAAS. For any $X, X_1, X_2 \subset U$ and $Y, Y_1, Y_2 \subset U$, there are*

i) $\widetilde{\ddot{P}^-}(-\widetilde{X}) = -\widetilde{\ddot{P}^+}(\widetilde{X})$, $\widetilde{\ddot{Q}^-}(-\widetilde{Y}) = -\widetilde{\ddot{Q}^+(\widetilde{Y})}$,

ii) $\widetilde{\ddot{P}^-}(\widetilde{U}) = \widetilde{U}$, $\widetilde{\ddot{P}^+}(\widetilde{\emptyset}) = \widetilde{\emptyset}$, $\widetilde{\ddot{Q}^-}(\widetilde{V}) = \widetilde{V}$, $\widetilde{\ddot{Q}^+}(\widetilde{\emptyset}) = \widetilde{\emptyset}$,

iii) $\widetilde{\ddot{P}^-}(\widetilde{X_1} \cap \widetilde{X_2}) = \widetilde{\ddot{P}^-}(\widetilde{X_1}) \cap \widetilde{\ddot{P}^-}(\widetilde{X_2})$, $\widetilde{\ddot{P}^+}(\widetilde{X_1} \cup \widetilde{X_2}) = \widetilde{\ddot{P}^+}(\widetilde{X_1}) \cup \widetilde{\ddot{P}^+}(\widetilde{X_2})$,

$\widetilde{\ddot{Q}^-}(\widetilde{Y_1} \cap \widetilde{Y_2}) = \widetilde{\ddot{Q}^-}(\widetilde{Y_1}) \cap \widetilde{\ddot{Q}^-}(\widetilde{Y_2})$, $\widetilde{\ddot{Q}^+}(\widetilde{Y_1} \cup \widetilde{Y_2}) = \widetilde{\ddot{Q}^+}(\widetilde{Y_1}) \cup \widetilde{\ddot{Q}^+}(\widetilde{Y_2})$,

iv) $\widetilde{\ddot{P}^-}(\widetilde{X_1} \cup \widetilde{X_2}) \supset \widetilde{\ddot{P}^-}(\widetilde{X_1}) \cup \widetilde{\ddot{P}^-}(\widetilde{X_2})$, $\widetilde{\ddot{P}^+}(\widetilde{X_1} \cap \widetilde{X_2}) \supset \widetilde{\ddot{P}^+}(\widetilde{X_1}) \cap \widetilde{\ddot{P}^+}(\widetilde{X_2})$,

$\widetilde{\ddot{Q}^-}(\widetilde{Y_1} \cup \widetilde{Y_2}) \supset \widetilde{\ddot{Q}^-}(\widetilde{Y_1}) \cup \widetilde{\ddot{Q}^-}(\widetilde{Y_2})$, $\widetilde{\ddot{Q}^+}(\widetilde{Y_1} \cap \widetilde{Y_2}) \subset \widetilde{\ddot{Q}^+}(\widetilde{Y_1}) \cap \widetilde{\ddot{Q}^+}(\widetilde{Y_2})$,

v) $\widetilde{X_1} \subset \widetilde{X_2}$ *implies* $\widetilde{\ddot{P}^-}(\widetilde{X_1}) \subset \widetilde{\ddot{P}^-}(\widetilde{X_2})$, $\widetilde{\ddot{P}^+}(\widetilde{X_1}) \subset \widetilde{\ddot{P}^+}(\widetilde{X_2})$, *and*

$\widetilde{Y_1} \subset \widetilde{Y_2}$ *implies* $\widetilde{\ddot{Q}^-}(\widetilde{Y_1}) \subset \widetilde{\ddot{Q}^-}(\widetilde{Y_2})$, $\widetilde{\ddot{Q}^+}(\widetilde{Y_1}) \subset \widetilde{\ddot{Q}^+}(\widetilde{Y_2})$.

Proof. *i*) It follows from the Proposition 2.5 that

$$\widetilde{\ddot{P}^-}(-\widetilde{X}) = \widetilde{M}\circ\widetilde{M}^T * (-\widetilde{X}) = -(\widetilde{M}\circ\widetilde{M}^T\cdot\widetilde{X}) = -\widetilde{\ddot{P}^+}(\widetilde{X}),$$
$$\widetilde{\ddot{Q}^-}(-\widetilde{Y}) = \widetilde{M}^T\circ\widetilde{M} * (-\widetilde{Y}) = -(\widetilde{M}^T\circ\widetilde{M}\cdot\widetilde{Y}) = -\widetilde{\ddot{Q}^+}(\widetilde{Y}).$$

ii) $\widetilde{\ddot{P}^-}(\widetilde{U}) = \widetilde{M}\circ\widetilde{M}^T * \widetilde{U} = 1_U = \widetilde{U}$, $\widetilde{\ddot{P}^+}(\widetilde{\emptyset}) = \widetilde{M}\circ\widetilde{M}^T\cdot\widetilde{\emptyset} = 0_U = \widetilde{\emptyset}$.

Similarly we have $\widetilde{\ddot{Q}^-}(\widetilde{V}) = \widetilde{V}$, and $\widetilde{\ddot{Q}^-}(\widetilde{\emptyset}) = \widetilde{\emptyset}$;

iii) It follows from the Proposition 2.5 that

$$\widetilde{\ddot{P}^-}(\widetilde{X_1}\cap\widetilde{X_2}) = \widetilde{M}\circ\widetilde{M}^T * (\widetilde{X_1}\wedge\widetilde{X_2}) = (\widetilde{M}\circ\widetilde{M}^T * \widetilde{X_1})\wedge(\widetilde{M}\circ\widetilde{M}^T * \widetilde{X_2}) = \widetilde{\ddot{P}^-}(\widetilde{X_1})\cap\widetilde{\ddot{P}^-}(\widetilde{X_2}),$$
$$\widetilde{\ddot{P}^+}(\widetilde{X_1}\cup\widetilde{X_2}) = \widetilde{M}\circ\widetilde{M}^T\cdot(\widetilde{X_1}\vee\widetilde{X_2}) = (\widetilde{M}\circ\widetilde{M}^T\cdot\widetilde{X_1})\vee(\widetilde{M}\circ\widetilde{M}^T\cdot\widetilde{X_2}) = \widetilde{\ddot{P}^+}(\widetilde{X_1})\cup\widetilde{\ddot{P}^+}(\widetilde{X_2}),$$

Similarly, we can prove

$$\widetilde{\ddot{Q}^-}(\widetilde{Y_1}\cap\widetilde{Y_2}) = \widetilde{\ddot{Q}^-}(\widetilde{Y_1})\cap\widetilde{\ddot{Q}^-}(\widetilde{Y_2}) \text{ and } \widetilde{\ddot{Q}^+}(\widetilde{Y_1}\cup\widetilde{Y_2}) = \widetilde{\ddot{Q}^+}(\widetilde{Y_1})\cup\widetilde{\ddot{Q}^+}(\widetilde{Y_2}).$$

iv) $\widetilde{\ddot{P}^-}(\widetilde{X_1}\cup\widetilde{X_2}) = \widetilde{M}\circ\widetilde{M}^T * (\widetilde{X_1}\vee\widetilde{X_2}) \geq (\widetilde{M}\circ\widetilde{M}^T * \widetilde{X_1})\vee(\widetilde{M}\circ\widetilde{M}^T * \widetilde{X_2}) = \widetilde{\ddot{P}^-}(\widetilde{X_1})\cup\widetilde{\ddot{P}^-}(\widetilde{X_2})$,

$$\widetilde{\ddot{P}^-}(\widetilde{X_1}\cap\widetilde{X_2}) = \widetilde{M}\circ\widetilde{M}^T\cdot(\widetilde{X_1}\wedge\widetilde{X_2}) \leq (\widetilde{M}\circ\widetilde{M}^T\cdot\widetilde{X_1})\wedge(\widetilde{M}\circ\widetilde{M}^T\cdot\widetilde{X_2}) = \widetilde{\ddot{P}^+}(\widetilde{X_1})\cap\widetilde{\ddot{P}^+}(\widetilde{X_2}).$$

We can easily get $\widetilde{\ddot{Q}^-}(\widetilde{Y_1}\cup\widetilde{Y_2}) \supset \widetilde{\ddot{Q}^-}(\widetilde{Y_1})\cup\widetilde{\ddot{Q}^-}(\widetilde{Y_2})$ and $\widetilde{\dot{Q}^+}(\widetilde{Y_1}\cap\widetilde{Y_2}) \subset \widetilde{\ddot{Q}^+}(\widetilde{Y_1})\cap\widetilde{\ddot{Q}^+}(\widetilde{Y_2})$ in a similar way.

v) is obvious, and thus the proof is finished. □

Now we give an example to show how to use these fuzzy rough set models in practice to solve problems.

Example 3.9. *Let $U = \{x_1, x_2, \cdots, x_9\}$ be a set of students, $V = \{y_1, y_2, \cdots, y_7\}$ be a set of subjects and $\widetilde{R}(x,y)(0 \leq \widetilde{R}(x,y) \leq 1)$ be the student x's ordinary score of subject y. All scores of $\{\widetilde{R}(x_i, y_j) : i = 1, 2, \cdots, 9, j = 1, 2, \cdots, 7\}$ are listed in the following table.*

	y_1	y_2	y_3	y_4	y_5	y_6	y_7
x_1	0.6	0.2	0.3	0.4	0.1	0.3	0.2
x_2	0.7	0.4	0.6	0.3	0.8	0.3	0.1
x_3	0.4	1	0.3	0.3	0.2	0.2	0.4
x_4	0.3	0.2	0.9	0.4	0.4	0.2	0.3
x_5	0.1	0.8	0.3	0.7	0.9	0.3	0.4
x_6	0.2	0.3	0.8	0.4	0.3	0.2	0.2
x_7	0.6	0.3	0.3	0.2	0.4	0.9	0.4
x_8	0.4	0.4	0.3	0.2	0.2	0.9	0.8
x_9	0.4	0.4	0.3	0.3	0.2	0.3	1

Thus

$$\widetilde{R} = \widetilde{M} = \begin{pmatrix} 0.6 & 0.2 & 0.3 & 0.4 & 0.1 & 0.3 & 0.2 \\ 0.7 & 0.4 & 0.6 & 0.3 & 0.8 & 0.3 & 0.1 \\ 0.4 & 1 & 0.3 & 0.3 & 0.2 & 0.2 & 0.4 \\ 0.3 & 0.2 & 0.9 & 0.4 & 0.4 & 0.2 & 0.3 \\ 0.1 & 0.8 & 0.3 & 0.7 & 0.9 & 0.3 & 0.4 \\ 0.2 & 0.3 & 0.8 & 0.4 & 0.3 & 0.2 & 0.2 \\ 0.6 & 0.3 & 0.3 & 0.2 & 0.4 & 0.9 & 0.4 \\ 0.4 & 0.4 & 0.3 & 0.2 & 0.2 & 0.9 & 0.8 \\ 0.4 & 0.4 & 0.3 & 0.3 & 0.2 & 0.3 & 1 \end{pmatrix}.$$

So we can compute

$$\widetilde{\dot{R}_U} = \widetilde{M}\cdot\widetilde{M}^T = \begin{pmatrix} 0.6 & 0.6 & 0.4 & 0.4 & 0.4 & 0.4 & 0.6 & 0.4 & 0.4 \\ 0.6 & 0.7 & 0.4 & 0.6 & 0.6 & 0.6 & 0.6 & 0.4 & 0.4 \\ 0.4 & 0.4 & 1 & 0.3 & 0.8 & 0.3 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.6 & 0.3 & 0.9 & 0.4 & 0.8 & 0.3 & 0.3 & 0.3 \\ 0.4 & 0.6 & 0.8 & 0.4 & 0.9 & 0.4 & 0.6 & 0.4 & 0.4 \\ 0.4 & 0.6 & 0.3 & 0.8 & 0.4 & 0.8 & 0.3 & 0.3 & 0.3 \\ 0.6 & 0.6 & 0.4 & 0.3 & 0.4 & 0.3 & 0.9 & 0.9 & 0.4 \\ 0.4 & 0.4 & 0.4 & 0.3 & 0.4 & 0.3 & 0.9 & 0.9 & 0.8 \\ 0.4 & 0.4 & 0.4 & 0.3 & 0.4 & 0.3 & 0.4 & 0.8 & 0.1 \end{pmatrix},$$

$$\widetilde{\dot{R}_V} = \widetilde{M}^T \cdot \widetilde{M} = \begin{pmatrix} 0.7 & 0.4 & 0.6 & 0.4 & 0.7 & 0.6 & 0.4 \\ 0.4 & 0.8 & 0.4 & 0.7 & 0.8 & 0.4 & 0.4 \\ 0.6 & 0.4 & 0.9 & 0.4 & 0.6 & 0.3 & 0.3 \\ 0.4 & 0.7 & 0.4 & 0.7 & 0.7 & 0.3 & 0.4 \\ 0.7 & 0.8 & 0.6 & 0.7 & 0.9 & 0.4 & 0.4 \\ 0.6 & 0.4 & 0.3 & 0.3 & 0.4 & 0.9 & 0.8 \\ 0.4 & 0.4 & 0.3 & 0.4 & 0.4 & 0.8 & 1 \end{pmatrix},$$

$$\widetilde{\ddot{R}_U} = \widetilde{M} \circ \widetilde{M}^T = \begin{pmatrix} 0.1 & 0.2 & 0.2 & 0.2 & 0.3 & 0.2 & 0.3 & 0.2 & 0.2 \\ 0.2 & 0.1 & 0.3 & 0.3 & 0.3 & 0.2 & 0.3 & 0.3 & 0.3 \\ 0.2 & 0.3 & 0.2 & 0.2 & 0.3 & 0.2 & 0.3 & 0.2 & 0.2 \\ 0.2 & 0.3 & 0.2 & 0.2 & 0.3 & 0.2 & 0.3 & 0.4 & 0.3 \\ 0.3 & 0.3 & 0.3 & 0.3 & 0.1 & 0.2 & 0.3 & 0.3 & 0.3 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.3 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.2 & 0.2 & 0.2 & 0.3 \\ 0.2 & 0.3 & 0.2 & 0.4 & 0.3 & 0.3 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.3 & 0.2 & 0.3 & 0.3 & 0.3 & 0.3 & 0.2 & 0.2 \end{pmatrix},$$

and

$$\widetilde{\ddot{R}_V} = \widetilde{M}^T \circ \widetilde{M} = \begin{pmatrix} 0.1 & 0.3 & 0.3 & 0.4 & 0.3 & 0.2 & 0.2 \\ 0.3 & 0.2 & 0.3 & 0.3 & 0.2 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 \\ 0.4 & 0.3 & 0.3 & 0.2 & 0.2 & 0.3 & 0.3 \\ 0.3 & 0.2 & 0.3 & 0.2 & 0.1 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.3 & 0.3 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.3 & 0.3 & 0.2 & 0.2 & 0.1 \end{pmatrix}.$$

Suppose that all students in U are trained to take part in a competition of comprehensive test of all subjects in V. A fuzzy set $\widetilde{X} = (0.6, 0.5, 0.4, 0.5, 0.4, 0.7, 0.4, 0.3, 0.3)$ represents the average score of each student in all subjects in the preliminary examination and fuzzy set $\widetilde{Y} = (0.6, 0.4, 0.5, 0.4, 0.8, 0.2, 0.3)$ denotes the average scores of all students on each subject in the examination.

We can estimate the grades of students and subjects by the lower and upper approximations of $\widetilde{X}$ and $\widetilde{Y}$ in both conservative and positive ways, and further to design some appropriate measures so as to try their best to get satisfactory results in this competition. Then according to the formulas in Proposition 3.6, we have

$$\widetilde{\dot{P}^-}(\widetilde{X}) = \widetilde{M} \cdot \widetilde{M}^T * \widetilde{X} = \begin{pmatrix} 0.6 & 0.6 & 0.4 & 0.4 & 0.4 & 0.4 & 0.6 & 0.4 & 0.4 \\ 0.6 & 0.7 & 0.4 & 0.6 & 0.6 & 0.6 & 0.6 & 0.4 & 0.4 \\ 0.4 & 0.4 & 1 & 0.3 & 0.8 & 0.3 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.6 & 0.3 & 0.9 & 0.4 & 0.8 & 0.3 & 0.3 & 0.3 \\ 0.4 & 0.6 & 0.8 & 0.4 & 0.9 & 0.4 & 0.6 & 0.4 & 0.4 \\ 0.4 & 0.6 & 0.3 & 0.8 & 0.4 & 0.8 & 0.3 & 0.3 & 0.3 \\ 0.6 & 0.6 & 0.4 & 0.3 & 0.4 & 0.3 & 0.9 & 0.9 & 0.4 \\ 0.4 & 0.4 & 0.4 & 0.3 & 0.4 & 0.3 & 0.9 & 0.9 & 0.8 \\ 0.4 & 0.4 & 0.4 & 0.3 & 0.4 & 0.3 & 0.4 & 0.8 & 0.1 \end{pmatrix} * \begin{pmatrix} 0.6 \\ 0.5 \\ 0.4 \\ 0.5 \\ 0.4 \\ 0.7 \\ 0.4 \\ 0.3 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.4 \\ 0.4 \\ 0.5 \\ 0.4 \\ 0.5 \\ 0.4 \\ 0.3 \\ 0.3 \end{pmatrix},$$

$$\widetilde{\dot{P}^+}(\widetilde{X}) = \widetilde{M} \cdot \widetilde{M}^T \cdot \widetilde{X} = \begin{pmatrix} 0.6 & 0.6 & 0.4 & 0.4 & 0.4 & 0.4 & 0.6 & 0.4 & 0.4 \\ 0.6 & 0.7 & 0.4 & 0.6 & 0.6 & 0.6 & 0.6 & 0.4 & 0.4 \\ 0.4 & 0.4 & 1 & 0.3 & 0.8 & 0.3 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.6 & 0.3 & 0.9 & 0.4 & 0.8 & 0.3 & 0.3 & 0.3 \\ 0.4 & 0.6 & 0.8 & 0.4 & 0.9 & 0.4 & 0.6 & 0.4 & 0.4 \\ 0.4 & 0.6 & 0.3 & 0.8 & 0.4 & 0.8 & 0.3 & 0.3 & 0.3 \\ 0.6 & 0.6 & 0.4 & 0.3 & 0.4 & 0.3 & 0.9 & 0.9 & 0.4 \\ 0.4 & 0.4 & 0.4 & 0.3 & 0.4 & 0.3 & 0.9 & 0.9 & 0.8 \\ 0.4 & 0.4 & 0.4 & 0.3 & 0.4 & 0.3 & 0.4 & 0.8 & 0.1 \end{pmatrix} \cdot \begin{pmatrix} 0.6 \\ 0.5 \\ 0.4 \\ 0.5 \\ 0.4 \\ 0.7 \\ 0.4 \\ 0.3 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.6 \\ 0.4 \\ 0.7 \\ 0.5 \\ 0.7 \\ 0.6 \\ 0.4 \\ 0.4 \end{pmatrix},$$

$$\widetilde{\dot{Q}^-}(\widetilde{Y}) = \widetilde{M}^T \cdot \widetilde{M} * \widetilde{Y} = \begin{pmatrix} 0.7 & 0.4 & 0.6 & 0.4 & 0.7 & 0.6 & 0.4 \\ 0.4 & 0.8 & 0.4 & 0.7 & 0.8 & 0.4 & 0.4 \\ 0.6 & 0.4 & 0.9 & 0.4 & 0.6 & 0.3 & 0.3 \\ 0.4 & 0.7 & 0.4 & 0.7 & 0.7 & 0.3 & 0.4 \\ 0.7 & 0.8 & 0.6 & 0.7 & 0.9 & 0.4 & 0.4 \\ 0.6 & 0.4 & 0.3 & 0.3 & 0.4 & 0.9 & 0.8 \\ 0.4 & 0.4 & 0.3 & 0.4 & 0.4 & 0.8 & 1 \end{pmatrix} * \begin{pmatrix} 0.6 \\ 0.4 \\ 0.5 \\ 0.4 \\ 0.8 \\ 0.2 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.4 \\ 0.5 \\ 0.4 \\ 0.4 \\ 0.2 \\ 0.2 \end{pmatrix},$$

$$\widetilde{\dot{Q}^+}(\widetilde{Y}) = \widetilde{M}^T \cdot \widetilde{M} \cdot \widetilde{Y} = \begin{pmatrix} 0.7 & 0.4 & 0.6 & 0.4 & 0.7 & 0.6 & 0.4 \\ 0.4 & 0.8 & 0.4 & 0.7 & 0.8 & 0.4 & 0.4 \\ 0.6 & 0.4 & 0.9 & 0.4 & 0.6 & 0.3 & 0.3 \\ 0.4 & 0.7 & 0.4 & 0.7 & 0.7 & 0.3 & 0.4 \\ 0.7 & 0.8 & 0.6 & 0.7 & 0.9 & 0.4 & 0.4 \\ 0.6 & 0.4 & 0.3 & 0.3 & 0.4 & 0.9 & 0.8 \\ 0.4 & 0.4 & 0.3 & 0.4 & 0.4 & 0.8 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0.6 \\ 0.4 \\ 0.5 \\ 0.4 \\ 0.8 \\ 0.2 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 0.7 \\ 0.8 \\ 0.6 \\ 0.7 \\ 0.8 \\ 0.6 \\ 0.4 \end{pmatrix},$$

$$\widetilde{\ddot{P}^-}(\widetilde{X}) = \widetilde{M} \circ \widetilde{M}^T * \widetilde{X} = \begin{pmatrix} 0.1 & 0.2 & 0.2 & 0.2 & 0.3 & 0.2 & 0.3 & 0.2 & 0.2 \\ 0.2 & 0.1 & 0.3 & 0.3 & 0.3 & 0.2 & 0.3 & 0.3 & 0.3 \\ 0.2 & 0.3 & 0.2 & 0.2 & 0.3 & 0.2 & 0.3 & 0.2 & 0.2 \\ 0.2 & 0.3 & 0.2 & 0.2 & 0.3 & 0.2 & 0.3 & 0.4 & 0.3 \\ 0.3 & 0.3 & 0.3 & 0.3 & 0.1 & 0.2 & 0.3 & 0.3 & 0.3 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.3 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.2 & 0.2 & 0.2 & 0.3 \\ 0.2 & 0.3 & 0.2 & 0.4 & 0.3 & 0.3 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.3 & 0.2 & 0.3 & 0.3 & 0.3 & 0.3 & 0.2 & 0.2 \end{pmatrix} * \begin{pmatrix} 0.6 \\ 0.5 \\ 0.4 \\ 0.5 \\ 0.4 \\ 0.7 \\ 0.4 \\ 0.3 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 0.7 \\ 0.7 \\ 0.7 \\ 0.6 \\ 0.7 \\ 0.7 \\ 0.7 \\ 0.6 \\ 0.7 \end{pmatrix},$$

$$\widetilde{\ddot{P}^+}(\widetilde{X}) = \widetilde{M} \circ \widetilde{M}^T \cdot \widetilde{X} = \begin{pmatrix} 0.1 & 0.2 & 0.2 & 0.2 & 0.3 & 0.2 & 0.3 & 0.2 & 0.2 \\ 0.2 & 0.1 & 0.3 & 0.3 & 0.3 & 0.2 & 0.3 & 0.3 & 0.3 \\ 0.2 & 0.3 & 0.2 & 0.2 & 0.3 & 0.2 & 0.3 & 0.2 & 0.2 \\ 0.2 & 0.3 & 0.2 & 0.2 & 0.3 & 0.2 & 0.3 & 0.4 & 0.3 \\ 0.3 & 0.3 & 0.3 & 0.3 & 0.1 & 0.2 & 0.3 & 0.3 & 0.3 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.3 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.2 & 0.2 & 0.2 & 0.3 \\ 0.2 & 0.3 & 0.2 & 0.4 & 0.3 & 0.3 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.3 & 0.2 & 0.3 & 0.3 & 0.3 & 0.3 & 0.2 & 0.2 \end{pmatrix} \cdot \begin{pmatrix} 0.6 \\ 0.5 \\ 0.4 \\ 0.5 \\ 0.4 \\ 0.7 \\ 0.4 \\ 0.3 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.3 \\ 0.3 \\ 0.3 \\ 0.3 \\ 0.3 \\ 0.3 \\ 0.3 \\ 0.3 \end{pmatrix},$$

$$\widetilde{\ddot{Q}^-}(\widetilde{Y}) = \widetilde{M}^T \circ \widetilde{M} * \widetilde{Y} = \begin{pmatrix} 0.1 & 0.3 & 0.3 & 0.4 & 0.3 & 0.2 & 0.2 \\ 0.3 & 0.2 & 0.3 & 0.3 & 0.2 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 \\ 0.4 & 0.3 & 0.3 & 0.2 & 0.2 & 0.3 & 0.3 \\ 0.3 & 0.2 & 0.3 & 0.2 & 0.1 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.3 & 0.3 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.3 & 0.3 & 0.2 & 0.2 & 0.1 \end{pmatrix} * \begin{pmatrix} 0.6 \\ 0.4 \\ 0.5 \\ 0.4 \\ 0.8 \\ 0.2 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.7 \\ 0.7 \\ 0.6 \\ 0.7 \\ 0.7 \\ 0.7 \end{pmatrix},$$

$$\widetilde{\ddot{Q}^+}(\widetilde{Y}) = \widetilde{M}^T \circ \widetilde{M} \cdot \widetilde{Y} = \begin{pmatrix} 0.1 & 0.3 & 0.3 & 0.4 & 0.3 & 0.2 & 0.2 \\ 0.3 & 0.2 & 0.3 & 0.3 & 0.2 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 \\ 0.4 & 0.3 & 0.3 & 0.2 & 0.2 & 0.3 & 0.3 \\ 0.3 & 0.2 & 0.3 & 0.2 & 0.1 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.3 & 0.3 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.3 & 0.3 & 0.2 & 0.2 & 0.1 \end{pmatrix} \cdot \begin{pmatrix} 0.6 \\ 0.4 \\ 0.5 \\ 0.4 \\ 0.8 \\ 0.2 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.3 \\ 0.3 \\ 0.4 \\ 0.3 \\ 0.3 \\ 0.3 \end{pmatrix}.$$

Thus all approximations are obtained as follows.

$\widetilde{\dot{P}^-}(\widetilde{X}) = (0.4, 0.4, 0.4, 0.5, 0.4, 0.5, 0.4, 0.3, 0.3)$, $\widetilde{\dot{P}^+}(\widetilde{X}) = (0.6, 0.6, 0.4, 0.7, 0.5, 0.7, 0.6, 0.4, 0.4)$,
$\widetilde{\ddot{P}^-}(\widetilde{X}) = (0.7, 0.7, 0.7, 0.6, 0.7, 0.7, 0.7, 0.6, 0.7)$, $\widetilde{\ddot{P}^+}(\widetilde{X}) = (0.3, 0.3, 0.3, 0.3, 0.3, 0.3, 0.3, 0.3, 0.3)$,
$\widetilde{\dot{Q}^-}(\widetilde{Y}) = (0.4, 0.4, 0.5, 0.4, 0.4, 0.2, 0.2)$, $\widetilde{\dot{Q}^+}(\widetilde{Y}) = (0.7, 0.8, 0.6, 0.7, 0.8, 0.6, 0.4)$,

$\widetilde{\ddot{Q}^-}(\widetilde{Y}) = (0.6, 0.7, 0.7, 0.6, 0.7, 0.7, 0.7)$, $\widetilde{\ddot{Q}^+}(\widetilde{Y}) = (0.4, 0.3, 0.3, 0.4, 0.3, 0.3, 0.3)$.

Then we can compare $\widetilde{X}$, $\widetilde{\dot{P}^-}(\widetilde{X})$, $\widetilde{\dot{P}^+}(\widetilde{X})$, $\widetilde{\ddot{P}^-}(\widetilde{X})$, $\widetilde{\ddot{P}^+}(\widetilde{X})$ *and compare* $\widetilde{Y}$, $\widetilde{\dot{Q}^-}(\widetilde{Y})$, $\widetilde{\dot{Q}^+}(\widetilde{Y})$, $\widetilde{\ddot{Q}^-}(\widetilde{Y})$, $\widetilde{\ddot{Q}^+}(\widetilde{Y})$.

$$\begin{pmatrix} \widetilde{\dot{P}^-}(\widetilde{X}) & 0.4 & 0.4 & 0.4 & 0.5 & 0.4 & 0.5 & 0.4 & 0.3 & 0.3 \\ & & & \vert & \vert & \vert & & \vert & \vert & \vert \\ \widetilde{X} & 0.6 & 0.5 & 0.4 & 0.5 & 0.4 & 0.7 & 0.4 & 0.3 & 0.3 \\ & \vert & & \vert & & & \vert & & & \\ \widetilde{\dot{P}^+}(\widetilde{X}) & 0.6 & 0.6 & 0.4 & 0.7 & 0.5 & 0.7 & 0.6 & 0.4 & 0.4 \end{pmatrix},$$

$$\begin{pmatrix} \widetilde{\ddot{P}^-}(\widetilde{X}) & 0.7 & 0.7 & 0.7 & 0.6 & 0.7 & 0.7 & 0.7 & 0.6 & 0.7 \\ & & & & & & \vert & & & \\ \widetilde{X} & 0.6 & 0.5 & 0.4 & 0.5 & 0.4 & 0.7 & 0.4 & 0.3 & 0.3 \\ & & & & & & & & \vert & \vert \\ \widetilde{\ddot{P}^+}(\widetilde{X}) & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 \end{pmatrix},$$

$$\begin{pmatrix} \widetilde{\dot{Q}^-}(\widetilde{Y}) & 0.4 & 0.4 & 0.5 & 0.4 & 0.4 & 0.2 & 0.2 \\ & & \vert & \vert & \vert & & \vert & \\ \widetilde{Y} & 0.6 & 0.4 & 0.5 & 0.4 & 0.8 & 0.2 & 0.3 \\ & & & & & \vert & & \\ \widetilde{\dot{Q}^+}(\widetilde{Y}) & 0.7 & 0.8 & 0.6 & 0.7 & 0.8 & 0.6 & 0.4 \end{pmatrix},$$

$$\begin{pmatrix} \widetilde{\ddot{Q}^-}(\widetilde{Y}) & 0.6 & 0.7 & 0.7 & 0.6 & 0.7 & 0.7 & 0.7 \\ & \vert & & & & & & \\ \widetilde{Y} & 0.6 & 0.4 & 0.5 & 0.4 & 0.8 & 0.2 & 0.3 \\ & & & & \vert & & & \vert \\ \widetilde{\ddot{Q}^+}(\widetilde{Y}) & 0.4 & 0.3 & 0.3 & 0.4 & 0.3 & 0.3 & 0.3 \end{pmatrix}.$$

By comparing these values in these tables, some conclusions are obtained which can be used as reference for decision-making of further training programs.

(1) Since, at x_2*, no one of these lower and upper approximations* $\widetilde{\dot{P}^-}(\widetilde{X})$, $\widetilde{\dot{P}^+}(\widetilde{X})$, $\widetilde{\ddot{P}^-}(\widetilde{X})$, $\widetilde{\ddot{P}^+}(\widetilde{X})$ *is equal to* $\widetilde{X}$ *and the largest of them is 0.7 which is much larger than the value 0.5 of* $\widetilde{X}$*, so the student* x_2 *is promising more progress. While, at each of* x_1, x_4, x_5, x_7*, there is one of these lower and upper approximations is equal to* $\widetilde{X}$*, so the four students* x_1, x_4, x_5, x_7 *are expected to get some improvements and reach their higher marks with an effective training. Whereas, there are very little likelihood of improvements for the students* x_3, x_6, x_8, x_9*, because there are two of these lower and upper approximations equal to* $\widetilde{X}$ *at each of* x_3, x_6, x_8, x_9*, so the four students are expected to keep their original achievements.*

(2) In order to improve the 9 students' scores, the subjects $y_1, y_2, y_3, y_5, y_6, y_7$ *can be mainly considered, because, at each of them, there is only one of their lower and upper approximations equal to* $\widetilde{Y}$*. Especially at* y_2 *and* y_6*, some of approximations have the highest score 0.8, so the students have greater probability of achieving higher scores in the subjects* y_2 *and* y_6*. However, since there are two of these approximations are equal to* $\widetilde{Y}$ *at subject* y_4*, so all the students must try to keep their original achievements in this subject.*

4 The special cases

In an FAAS, if the fuzzy relation $\widetilde{R}$ is a general binary relation R, or the spaces U and V are the same, then what are the results? We then discuss these two special cases in this section.

4.1 Fuzzy relation $\widetilde{R}$ being a binary relation R in an FAAS $(U, V, \widetilde{R})$

If the fuzzy relation $\widetilde{R}$ is a binary relation R, then the next proposition directly follows from Proposition 3.3.

Proposition 4.1. *Let* (U, V, R) *be an FAAS, where* $U = \{x_1, x_2, \cdots, x_m\}$, $V = \{y_1, y_2, \cdots, y_n\}$ *are two universes, and* R *is a binary relation from* U *to* V *which can be regarded as a special fuzzy relation with all fuzzy values are* 0 *or* 1. *If* $M = (R(x_i, y_j))_{m\times n}$ *is the Boolean matrix representation of the binary relation* R*, then the Boolean matrix representations of the positive low-value relation* $\dot{R}_U$ *and the conservative high-value relation* $\ddot{R}_U$ *on* U *are* $M \cdot M^T$ *and* $M \circ M^T$ *respectively, and the Boolean matrix representations of the positive low-value relation* $\dot{R}_V$ *and the conservative high-value relation* $\ddot{R}_V$ *on* V *are* $M^T \cdot M$ *and* $M^T \circ M$ *respectively.*

Example 4.2. Let (U, V, R) be an FAAS, where $X = \{x_1, x_2, \cdots, x_{11}\}$, $Y = \{y_1, y_2, \cdots, y_7\}$, and R is a binary relation denoted by the Boolean matrix M as follows.

$$M = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

Then, using Proposition 2.4, we can quickly compute the following matrices which represent $\dot{R}_U,\ \ddot{R}_U,\ \dot{R}_V$ and $\ddot{R}_V$.

$$M \cdot M^T = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \end{pmatrix},\ M \circ M^T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$M^T \cdot M = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},\ M^T \circ M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In this section, the subset X of U is regarded as a special fuzzy subset, so the symbols $X(x)$ is frequently used, and its meaning is defined as below:

$X(x) = 1 \Leftrightarrow x \in X, \quad X(x) = 0 \Leftrightarrow x \notin X.$

If the fuzzy relation $\widetilde{R}$ is a general binary relation R, then all the models in Definition 3.6 can be translated into the classical models based on the binary relation.

Proposition 4.3. *Let* (U, V, R) *be an FAAS, where* R *is a binary relation from* U *to* V*. For each* $x \in U,\ y \in V$*, set*

$\dot{R}_U(x) = \{z \in U : \dot{R}_U(x, z) = 1\}, \quad \ddot{R}_U(x) = \{z \in U : \ddot{R}_U(x, z) = 1\},$

$\dot{R}_V(y) = \{z \in V : \dot{R}_V(y, z) = 1\}, \quad \ddot{R}_V(y) = \{z \in V : \ddot{R}_V(y, z) = 1\}.$

Then for $X \subseteq U, \quad Y \subseteq V$*, there are:*

$\widetilde{\dot{P}^-}(X) = \{x \in U : \dot{R}_U(x) \subseteq X\}, \quad \widetilde{\dot{P}^+}(X) = \{x \in U : \dot{R}_U(x) \cap X \neq \emptyset\},$

$\widetilde{\ddot{P}^-}(X) = \{x \in U : \ddot{R}_U(x) \subseteq X\}, \quad \widetilde{\ddot{P}^+}(X) = \{x \in U : \ddot{R}_U(x) \cap X \neq \emptyset\},$

$\widetilde{\dot{Q}^-}(Y) = \{y \in V : \dot{R}_V(y) \subseteq Y\}, \quad \widetilde{\dot{Q}^+}(Y) = \{y \in V : \dot{R}_V(y) \cap Y \neq \emptyset\},$

$\widetilde{\ddot{Q}^-}(Y) = \{y \in V : \ddot{R}_V(y) \subseteq Y\}, \quad \widetilde{\ddot{Q}^+}(Y) = \{y \in V : \ddot{R}_V(y) \cap Y \neq \emptyset\}.$

Proof. We need only to prove $\widetilde{\dot{P}^-}(X) = \{x \in U : \dot{R}_U(x) \subseteq X\}$ and $\widetilde{\dot{P}^+}(X) = \{x \in U : \dot{R}_U(x) \cap X \neq \emptyset\}$.

$$x \in \widetilde{\dot{P}^-}(X) \Leftrightarrow \widetilde{\dot{P}^-}(X)(x) = 1 \Leftrightarrow \wedge_{x' \in U}[(1 - \dot{R}_U(x, x') \vee X(x')] = 1 \Leftrightarrow \forall x' \in U,\ (\dot{R}_U(x, x') = 1) \rightarrow (X(x') = 1)$$
$$\Leftrightarrow \forall x' \in U,\ (x' \in \dot{R}_U(x)) \rightarrow (x' \in X) \Leftrightarrow \dot{R}_U(x) \subseteq X.$$

So, $\widetilde{\dot{P}^-}(X) = \{x \in U : \dot{R}_U(x) \subseteq X\}$.

$$x \in \widetilde{\dot{P}^+}(X) \Leftrightarrow \vee_{x' \in U}[\dot{R}_U(x, x') \wedge X(x') = 1] \Leftrightarrow \exists x' \in U, (\dot{R}_U(x, x') = 1) \wedge (X(x') = 1)$$
$$\Leftrightarrow \dot{R}_U(x) \cap X \neq \emptyset.$$

So, $\widetilde{\dot{P}^+}(X) = \{x \in U : \dot{R}_U(x) \cap X \neq \emptyset\}$.

The other formulas are similarly verifiable. □

Example 4.4. *During the COVID-19 outbreak, there is a group U of special service workers (couriers, food deliverers, etc.) and their usual workplaces V (express drop-off, mailroom, etc.). No two members of the group U are in direct contact with each other. If two persons have been to the same workplace of V, then they are considered to have indirect contact. Similarly, in these workplaces of V, if the same person of U has visited two workplaces, then the two workplaces are also considered to have indirect contact.*

Let R be a binary relation from U to V, where $R(x, y) = 1$ denotes that the person $x \in U$ has visited the workplace $y \in V$, and $R(x, y) = 0$ denotes that the person $x \in U$ has not visited the workplace $y \in V$.

Let (U, V, R) be an FAAS the same as that in Example 4.2. Due to the matrix representations of $\dot{R}_U$, $\ddot{R}_U$, $\dot{R}_V$ and $\ddot{R}_V$ in Proposition 4.3, we can get the following subsets.

$\dot{R}_U(x_1) = \{x_1, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11}\}$, $\dot{R}_U(x_2) = \{x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{10}, x_{11}\}$,
$\dot{R}_U(x_3) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}\}$, $\dot{R}_U(x_4) = \{x_1, x_2, x_3, x_4, x_7, x_8, x_9, x_{10}, x_{11}\}$,
$\dot{R}_U(x_5) = \{x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}\} = \dot{R}_U(x_6)$,
$\dot{R}_U(x_7) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11}\}$, $\dot{R}_U(x_8) = \{x_2, x_3, x_4, x_5, x_6, x_8, x_{10}, x_{11}\}$,
$\dot{R}_U(x_9) = \{x_1, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11}\}$, $\dot{R}_U(x_{10}) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$,
$\dot{R}_U(x_{11}) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{11}\}$,
$\ddot{R}_U(x_1) = \{x_2\}$, $\ddot{R}_U(x_2) = \{x_1, x_7, x_9\}$, $\ddot{R}_U(x_3) = \{x_{10}\}$, $\ddot{R}_U(x_4) = \{x_5\}$, $\ddot{R}_U(x_5) = \{x_4\}$,
$\ddot{R}_U(x_6) = \emptyset$, $\ddot{R}_U(x_7) = \{x_2, x_8\}$, $\ddot{R}_U(x_8) = \{x_7\}$, $\ddot{R}_U(x_9) = \{x_2\}$, $\ddot{R}_U(x_{10}) = \{x_3, x_{11}\}$,
$\ddot{R}_U(x_{11}) = \{x_{10}\}$,

$\dot{R}_V(y_1) = \dot{R}_V(y_2) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7\}$,
$\dot{R}_V(y_3) = \{y_1, y_2, y_3, y_5, y_6, y_7\}$, $\dot{R}_V(y_4) = \{y_1, y_2, y_4, y_5, y_6, y_7\}$,
$\dot{R}_V(y_5) = \dot{R}_V(y_6) = \dot{R}_V(y_7) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7\}$,
$\ddot{R}_V(y_1) = \{y_2\}$, $\ddot{R}_V(y_2) = \{y_1\}$, $\ddot{R}_V(y_3) = \{y_4\}$,
$\ddot{R}_V(y_4) = \{y_3\}$, $\ddot{R}_V(y_5) = \ddot{R}_V(y_6) = \ddot{R}_V(y_7) = \emptyset$.

According to the meanings of these relations $\dot{R}_U$, $\ddot{R}_U$, $\dot{R}_V$ and $\ddot{R}_V$, let's explain what the above subsets mean. If $x_i \in \dot{R}_U(x_j)$, then two persons x_i and x_j are in indirect contact, and if $y_i \in \dot{R}_V(y_j)$, then the two workplaces y_i and y_j are in indirect contact. That is to say, $\dot{R}_U(x_j)$ denotes the group of some persons in U, every one of which has indirect contact with x_j, and $\dot{R}_V(y_j)$ denotes the class of some workplaces in V, every one of which has indirect contact with y_j.

Moreover, if $x_i \in \ddot{R}_U(x_j)$, then every workplace is visited by persons x_i or x_j, so x_i and x_j may have been in indirect contact, and if $y_i \in \ddot{R}_V(y_j)$, then every person visited workplace y_i or y_j, so y_i and y_j may have been in indirect contact. That is to say, $\ddot{R}_U(x_j)$ denotes the group of some persons of U, each of which is likely to have indirect contact with x_j, and $\ddot{R}_V(y_j)$ denotes the class of some workplaces of V, each of which is likely to have indirect contact with y_j.

According to nucleic acid test results, three persons x_2, x_8, x_9 are positive, and six workplaces $y_1, y_2, y_3, y_5, y_6, y_7$ are positive. Based on the analysis of approximate sets of $X = \{x_2, x_8, x_9\}$ and $Y = \{y_1, y_2, y_3, y_5, y_6, y_7\}$, we will give epidemic prevention levels and make epidemic prevention auxiliary decisions.

For X and Y, we use Proposition 4.3 to compute all the lower and upper approximations in the FAAS (U, V, R) as follows.

$\widetilde{\dot{P}^-}(X) = \{x_8, x_9\}$, $\widetilde{\dot{P}^+}(X) = U$, $\widetilde{\ddot{P}^-}(X) = \{x_1, x_6, x_7, x_9\}$, $\widetilde{\ddot{P}^+}(X) = \{x_1, x_2, x_7, x_9\}$,

$\widetilde{\dot{Q}^-}(Y)=Y$, $\widetilde{\dot{Q}^+}(Y)=V$, $\widetilde{\ddot{Q}^-}(Y)=\{y_1,y_2,y_4,y_5,y_6,y_7\}$, $\widetilde{\ddot{Q}^+}(Y)=\{y_1,y_2,y_3\}$.
So, we have the following decisions for reference on epidemic prevention measures.

1) *Based on how many times every person appear in these approximations* $\widetilde{\dot{P}^-}(X)$, $\widetilde{\dot{P}^+}(X)$, $\widetilde{\ddot{P}^-}(X)$, $\widetilde{\ddot{P}^+}(X)$ *and* X, *we can get the following order in which these persons need to be observed from critical to non-critical:*

$$\{x_9\}\rightsquigarrow\{x_8\}\rightsquigarrow\{x_2\}\rightsquigarrow\{x_1,x_7\}\rightsquigarrow\{x_6\}\rightsquigarrow\{x_3,x_4,x_5,x_{10},x_{11}\}.$$

2) *Based on how many times every workplace appear in these approximations* $\widetilde{\dot{Q}^-}(Y)$, $\widetilde{\dot{Q}^+}(Y)$, $\widetilde{\ddot{Q}^-}(Y)$, $\widetilde{\ddot{Q}^+}(Y)$ *and* Y, *we can obtain the following order in which these workplaces need to be observed from critical to non-critical:*

$$\{y_1,y_2,y_3\}\rightsquigarrow\{y_5,y_6,y_7\}\rightsquigarrow\{y_4\}.$$

4.2 $U=V$ in an FAAS $(U,V,\widetilde{R})$

If $U=V$ in FAAS, the fuzzy approximations of the following definition based on the fuzzy relation $\widetilde{R}$ have been widely studied and applied.

Definition 4.5. [10] *If* $U=V$ *in an FAAS* $(U,V,\widetilde{R})$, *then the fuzzy approximations of each* $\widetilde{X}\in F(U)$ *under* $\widetilde{R}$ *and* $\widetilde{R^{-1}}$ *are defined as follows:*
$\widetilde{P^-}(\widetilde{X})(x)=\wedge_{x'\in U}[(1-\widetilde{R}(x,x')\vee\widetilde{X}(x')]$, $\widetilde{P^+}(\widetilde{X})(x)=\vee_{x'\in U}[\widetilde{R}(x,x')\wedge\widetilde{X}(x')]$ $(x\in U)$;
$\widetilde{Q^-}(\widetilde{X})(x)=\wedge_{x'\in U}[(1-\widetilde{R^{-1}}(x,x')\vee\widetilde{X}(x')]$, $\widetilde{Q^+}(\widetilde{X})(x)=\vee_{x'\in U}[\widetilde{R^{-1}}(x,x')\wedge\widetilde{X}(x')]$ $(x\in U)$.

Now, we can define the new fuzzy approximation models.

Definition 4.6. *If* $U=V$ *in an FAAS* $(U,V,\widetilde{R})$, *the fuzzy approximations of each* $\widetilde{X}\in F(U)$ *are defined as follows:*

$$\widetilde{\dot{P}^-}(\widetilde{X})(x)=\wedge_{x'\in U}[(1-\widetilde{\dot{R}_U}(x,x')\vee\widetilde{X}(x')],\quad \widetilde{\dot{P}^+}(\widetilde{X})(x)=\vee_{x'\in U}[\widetilde{\dot{R}_U}(x,x')\wedge\widetilde{X}(x')]\ (x\in U),$$

$$\widetilde{\dot{Q}^-}(\widetilde{X})(x)=\wedge_{x'\in U}[(1-\widetilde{\dot{R}_V}(x,x')\vee\widetilde{X}(x')],\quad \widetilde{\dot{Q}^+}(\widetilde{X})(x)=\vee_{x'\in U}[\widetilde{\dot{R}_V}(x,x')\wedge\widetilde{X}(x')]\ (x\in U);$$

$$\widetilde{\ddot{P}^-}(\widetilde{X})(x)=\wedge_{x'\in U}[(1-\widetilde{\ddot{R}_U}(x,x')\vee\widetilde{X}(x')],\quad \widetilde{\ddot{P}^+}(\widetilde{X})(x)=\vee_{x'\in U}[\widetilde{\ddot{R}_U}(x,x')\wedge\widetilde{X}(x')]\ (x\in U),$$

$$\widetilde{\ddot{Q}^-}(\widetilde{X})(x)=\wedge_{x'\in U}[(1-\widetilde{\ddot{R}_V}(x,x')\vee\widetilde{X}(x')],\quad \widetilde{\ddot{Q}^+}(\widetilde{X})(x)=\vee_{x'\in U}[\widetilde{\ddot{R}_V}(x,x')\wedge\widetilde{X}(x')]\ (x\in U).$$

Noting that, even $U=V$ in an FAAS, the $\widetilde{\dot{R}_U}$ and $\widetilde{\dot{R}_V}$ are generally different according to their definitions, as well as $\widetilde{\ddot{R}_U}$ and $\widetilde{\ddot{R}_V}$.

Similar to Proposition 3.6, we have the following proposition.

Proposition 4.7. *Let* $U=\{x_1,x_2,\cdots,x_n\}$ *be a universe and* $\widetilde{M}$ *be the matrix representation of a fuzzy relation* $\widetilde{R}$ *on* U. *Then for each* $\widetilde{X}\in\mathcal{F}(U)$, *there are:*
$\widetilde{P^-}(\widetilde{X})=\widetilde{M}*\widetilde{X}$, $\widetilde{P^+}(\widetilde{X})=\widetilde{M}\cdot\widetilde{X}$,
$\widetilde{Q^-}(\widetilde{X})=\widetilde{M}^T*\widetilde{X}$, $\widetilde{Q^+}(\widetilde{X})=\widetilde{M}^T\cdot\widetilde{X}$,
$\widetilde{\dot{P}^-}(\widetilde{X})=\widetilde{M}\cdot\widetilde{M}^T*\widetilde{X}$, $\widetilde{\dot{P}^+}(\widetilde{X})=\widetilde{M}\cdot\widetilde{M}^T\cdot\widetilde{X}$,
$\widetilde{\ddot{P}^-}(\widetilde{X})=\widetilde{M}\circ\widetilde{M}^T*\widetilde{X}$, $\widetilde{\ddot{P}^+}(\widetilde{X})=\widetilde{M}\circ\widetilde{M}^T\cdot\widetilde{X}$,
$\widetilde{\dot{Q}^-}(\widetilde{X})=\widetilde{M}^T\cdot\widetilde{M}*\widetilde{X}$, $\widetilde{\dot{Q}^+}(\widetilde{X})=\widetilde{M}^T\cdot\widetilde{M}\cdot\widetilde{X}$,
$\widetilde{\ddot{Q}^-}(\widetilde{X})=\widetilde{M}^T\circ\widetilde{M}*\widetilde{X}$, $\widetilde{\ddot{Q}^+}(\widetilde{X})=\widetilde{M}^T\circ\widetilde{M}\cdot\widetilde{X}$.

Proof. For each $\widetilde{X}\in F(U)$ and each $x\in U$, according to $\widetilde{M}=(\widetilde{R}(x_i,x_j))_{m\times m}$, we have
$(\widetilde{M}*\widetilde{X})(x)=\wedge_{x'\in U}[(1-\widetilde{R}(x,x'))\vee\widetilde{X}(x')]=(\widetilde{P^-}(\widetilde{X}))(x)$,
$(\widetilde{M}\cdot\widetilde{X})(x)=\vee_{x'\in U}(\widetilde{R}(x,x')\wedge\widetilde{X}(x'))=(\widetilde{P^+}(\widetilde{X}))(x)$.
Hence, $\widetilde{P^-}(\widetilde{X})=\widetilde{M}*\widetilde{X}$ and $\widetilde{P^+}(\widetilde{X})=\widetilde{M}\cdot\widetilde{X}$ can be followed.

Similarly, according to $\widetilde{M}^T=(\widetilde{R}^{-1}(x_i,x_j))_{m\times m}$, we can obtain
$(\widetilde{M}^T*\widetilde{X})(x)=\wedge_{x'\in U}[(1-\widetilde{R}^{-1}(x,x'))\vee\widetilde{X}(x')]=(\widetilde{Q^-}\widetilde{X})(y)$,
$(\widetilde{M}^T\cdot\widetilde{X})(x)=\vee_{x'\in U}(\widetilde{R}^{-1}(x,x')\wedge\widetilde{X}(x'))=(\widetilde{Q^+}\widetilde{X})(x)$,
and thus, $\widetilde{Q^-}(\widetilde{X})=\widetilde{M}^T*\widetilde{X}$ and $\widetilde{Q^+}(\widetilde{X})=\widetilde{M}^T\cdot\widetilde{X}$ can be followed.

Other formulas can be derived directly from Proposition 3.6. □

Example 4.8. *Let $U = \{x_1, x_2, \cdots, x_9\}$ denote a group of student candidates preparing to take part in a competition. $\widetilde{R}$ is a fuzzy relation on U, where $\widetilde{R}(x_i, x_j)(0 \leq \widetilde{R}(x_i, x_j) \leq 1)$ denotes the score of the student x_i evaluating the student x_j.*

$\widetilde{R}$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
x_1	*0.6*	*0.2*	*0.3*	*0.4*	*0.1*	*0.3*	*0.2*	*0.3*	*0.4*
x_2	*0.7*	*0.4*	*0.6*	*0.3*	*0.8*	*0.3*	*0.1*	*0.6*	*0.3*
x_3	*0.4*	*1*	*0.3*	*0.3*	*0.2*	*0.2*	*0.4*	*0.3*	*0.3*
x_4	*0.3*	*0.2*	*0.9*	*0.4*	*0.4*	*0.2*	*0.3*	*0.9*	*0.4*
x_5	*0.1*	*0.8*	*0.3*	*0.7*	*0.9*	*0.3*	*0.4*	*0.3*	*0.7*
x_6	*0.2*	*0.3*	*0.8*	*0.4*	*0.3*	*0.2*	*0.2*	*0.8*	*0.4*
x_7	*0.6*	*0.3*	*0.3*	*0.2*	*0.4*	*0.9*	*0.4*	*0.3*	*0.2*
x_8	*0.4*	*0.4*	*0.3*	*0.2*	*0.2*	*0.9*	*0.8*	*0.3*	*0.2*
x_9	*0.4*	*0.4*	*0.3*	*0.3*	*0.2*	*0.3*	*1*	*0.3*	*0.3*

A fuzzy set $\widetilde{X} = (0.6, 0.5, 0.4, 0.5, 0.4, 0.7, 0.4, 0.3, 0.3)$ is the result of a preliminary exam. We need to make a comprehensive evaluation on the students from the positive and conservative aspects, and select 3 students to participate in the final competition. So all the lower and upper approximations of $\widetilde{X}$ are calculated as the following steps.

Since

$$\widetilde{M} = \begin{pmatrix} 0.6 & 0.2 & 0.3 & 0.4 & 0.1 & 0.3 & 0.2 & 0.3 & 0.4 \\ 0.7 & 0.4 & 0.6 & 0.3 & 0.8 & 0.3 & 0.1 & 0.6 & 0.3 \\ 0.4 & 1 & 0.3 & 0.3 & 0.2 & 0.2 & 0.4 & 0.3 & 0.3 \\ 0.3 & 0.2 & 0.9 & 0.4 & 0.4 & 0.2 & 0.3 & 0.9 & 0.4 \\ 0.1 & 0.8 & 0.3 & 0.7 & 0.9 & 0.3 & 0.4 & 0.3 & 0.7 \\ 0.2 & 0.3 & 0.8 & 0.4 & 0.3 & 0.2 & 0.2 & 0.8 & 0.4 \\ 0.6 & 0.3 & 0.3 & 0.2 & 0.4 & 0.9 & 0.4 & 0.3 & 0.2 \\ 0.4 & 0.4 & 0.3 & 0.2 & 0.2 & 0.9 & 0.8 & 0.3 & 0.2 \\ 0.4 & 0.4 & 0.3 & 0.3 & 0.2 & 0.3 & 1 & 0.3 & 0.3 \end{pmatrix},$$

$$\widetilde{M} \cdot \widetilde{M}^T = \begin{pmatrix} 0.6 & 0.6 & 0.4 & 0.4 & 0.4 & 0.4 & 0.6 & 0.4 & 0.4 \\ 0.6 & 0.7 & 0.4 & 0.6 & 0.8 & 0.6 & 0.6 & 0.4 & 0.4 \\ 0.4 & 0.4 & 1 & 0.3 & 0.8 & 0.3 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.6 & 0.3 & 0.9 & 0.4 & 0.8 & 0.4 & 0.3 & 0.3 \\ 0.4 & 0.8 & 0.8 & 0.4 & 0.9 & 0.4 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.6 & 0.3 & 0.8 & 0.4 & 0.8 & 0.3 & 0.3 & 0.3 \\ 0.6 & 0.6 & 0.4 & 0.4 & 0.4 & 0.3 & 0.9 & 0.9 & 0.4 \\ 0.4 & 0.4 & 0.4 & 0.3 & 0.4 & 0.3 & 0.9 & 0.9 & 0.8 \\ 0.4 & 0.4 & 0.4 & 0.3 & 0.4 & 0.3 & 0.4 & 0.8 & 0.1 \end{pmatrix},$$

$$\widetilde{M}^T \cdot \widetilde{M} = \begin{pmatrix} 0.7 & 0.4 & 0.6 & 0.4 & 0.7 & 0.6 & 0.4 & 0.6 & 0.4 \\ 0.4 & 1 & 0.4 & 0.7 & 0.8 & 0.4 & 0.4 & 0.4 & 0.7 \\ 0.6 & 0.4 & 0.9 & 0.4 & 0.6 & 0.3 & 0.3 & 0.9 & 0.4 \\ 0.4 & 0.7 & 0.4 & 0.7 & 0.7 & 0.3 & 0.4 & 0.4 & 0.7 \\ 0.7 & 0.8 & 0.6 & 0.7 & 0.9 & 0.3 & 0.4 & 0.6 & 0.7 \\ 0.6 & 0.4 & 0.3 & 0.3 & 0.4 & 0.9 & 0.8 & 0.3 & 0.3 \\ 0.4 & 0.4 & 0.3 & 0.3 & 0.4 & 0.8 & 0.8 & 0.3 & 0.4 \\ 0.6 & 0.4 & 0.9 & 0.4 & 0.6 & 0.3 & 0.3 & 0.9 & 0.4 \\ 0.4 & 0.7 & 0.4 & 0.7 & 0.7 & 0.3 & 0.4 & 0.4 & 0.7 \end{pmatrix},$$

$$\widetilde{M} \circ \widetilde{M}^T = \begin{pmatrix} 0.1 & 0.2 & 0.2 & 0.2 & 0.3 & 0.2 & 0.3 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.3 & 0.3 & 0.3 & 0.2 & 0.3 & 0.3 & 0.3 \\ 0.2 & 0.3 & 0.2 & 0.2 & 0.3 & 0.2 & 0.3 & 0.2 & 0.2 \\ 0.2 & 0.3 & 0.2 & 0.2 & 0.3 & 0.3 & 0.3 & 0.4 & 0.3 \\ 0.3 & 0.3 & 0.3 & 0.3 & 0.1 & 0.2 & 0.3 & 0.3 & 0.3 \\ 0.2 & 0.2 & 0.3 & 0.3 & 0.2 & 0.2 & 0.3 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.2 & 0.2 & 0.3 \\ 0.2 & 0.3 & 0.2 & 0.4 & 0.3 & 0.3 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.3 & 0.2 & 0.3 & 0.3 & 0.3 & 0.3 & 0.2 & 0.2 \end{pmatrix},$$

$$\widetilde{M}^T \circ \widetilde{M} = \begin{pmatrix} 0.1 & 0.3 & 0.3 & 0.4 & 0.3 & 0.2 & 0.2 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.3 & 0.3 & 0.2 & 0.2 & 0.2 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 \\ 0.4 & 0.3 & 0.3 & 0.2 & 0.2 & 0.3 & 0.3 & 0.3 & 0.2 \\ 0.3 & 0.2 & 0.3 & 0.2 & 0.2 & 0.2 & 0.2 & 0.3 & 0.3 \\ 0.2 & 0.2 & 0.3 & 0.3 & 0.2 & 0.2 & 0.2 & 0.3 & 0.3 \\ 0.2 & 0.2 & 0.3 & 0.3 & 0.2 & 0.2 & 0.1 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 \\ 0.4 & 0.3 & 0.3 & 0.2 & 0.3 & 0.3 & 0.3 & 0.3 & 0.2 \end{pmatrix},$$

then

$$\widetilde{P^-}(\widetilde{X}) = \widetilde{M} * \widetilde{X} = \begin{pmatrix} 0.6 & 0.2 & 0.3 & 0.4 & 0.1 & 0.3 & 0.2 & 0.3 & 0.4 \\ 0.7 & 0.4 & 0.6 & 0.3 & 0.8 & 0.3 & 0.1 & 0.6 & 0.3 \\ 0.4 & 1 & 0.3 & 0.3 & 0.2 & 0.2 & 0.4 & 0.3 & 0.3 \\ 0.3 & 0.2 & 0.9 & 0.4 & 0.4 & 0.2 & 0.3 & 0.9 & 0.4 \\ 0.1 & 0.8 & 0.3 & 0.7 & 0.9 & 0.3 & 0.4 & 0.3 & 0.7 \\ 0.2 & 0.3 & 0.8 & 0.4 & 0.3 & 0.2 & 0.2 & 0.8 & 0.4 \\ 0.6 & 0.3 & 0.3 & 0.2 & 0.4 & 0.9 & 0.4 & 0.3 & 0.2 \\ 0.4 & 0.4 & 0.3 & 0.2 & 0.2 & 0.9 & 0.8 & 0.3 & 0.2 \\ 0.4 & 0.4 & 0.3 & 0.3 & 0.2 & 0.3 & 1 & 0.3 & 0.3 \end{pmatrix} * \begin{pmatrix} 0.6 \\ 0.5 \\ 0.4 \\ 0.5 \\ 0.4 \\ 0.7 \\ 0.4 \\ 0.3 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.4 \\ 0.4 \\ 0.3 \\ 0.3 \\ 0.3 \\ 0.4 \\ 0.4 \\ 0.4 \end{pmatrix},$$

$$\widetilde{P^+}(\widetilde{X}) = \widetilde{M} \cdot \widetilde{X} = \begin{pmatrix} 0.6 & 0.2 & 0.3 & 0.4 & 0.1 & 0.3 & 0.2 & 0.3 & 0.4 \\ 0.7 & 0.4 & 0.6 & 0.3 & 0.8 & 0.3 & 0.1 & 0.6 & 0.3 \\ 0.4 & 1 & 0.3 & 0.3 & 0.2 & 0.2 & 0.4 & 0.3 & 0.3 \\ 0.3 & 0.2 & 0.9 & 0.4 & 0.4 & 0.2 & 0.3 & 0.9 & 0.4 \\ 0.1 & 0.8 & 0.3 & 0.7 & 0.9 & 0.3 & 0.4 & 0.3 & 0.7 \\ 0.2 & 0.3 & 0.8 & 0.4 & 0.3 & 0.2 & 0.2 & 0.8 & 0.4 \\ 0.6 & 0.3 & 0.3 & 0.2 & 0.4 & 0.9 & 0.4 & 0.3 & 0.2 \\ 0.4 & 0.4 & 0.3 & 0.2 & 0.2 & 0.9 & 0.8 & 0.3 & 0.2 \\ 0.4 & 0.4 & 0.3 & 0.3 & 0.2 & 0.3 & 1 & 0.3 & 0.3 \end{pmatrix} \cdot \begin{pmatrix} 0.6 \\ 0.5 \\ 0.4 \\ 0.5 \\ 0.4 \\ 0.7 \\ 0.4 \\ 0.3 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.6 \\ 0.5 \\ 0.4 \\ 0.5 \\ 0.4 \\ 0.6 \\ 0.4 \\ 0.4 \end{pmatrix},$$

$$\widetilde{Q^-}(\widetilde{X}) = \widetilde{M}^T * \widetilde{X} = \begin{pmatrix} 0.6 & 0.7 & 0.4 & 0.3 & 0.1 & 0.2 & 0.6 & 0.4 & 0.4 \\ 0.2 & 0.4 & 1 & 0.2 & 0.8 & 0.3 & 0.3 & 0.4 & 0.4 \\ 0.3 & 0.6 & 0.3 & 0.9 & 0.3 & 0.8 & 0.3 & 0.3 & 0.3 \\ 0.4 & 0.3 & 0.3 & 0.4 & 0.7 & 0.4 & 0.2 & 0.2 & 0.3 \\ 0.1 & 0.8 & 0.2 & 0.4 & 0.9 & 0.3 & 0.4 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.2 & 0.2 & 0.3 & 0.2 & 0.9 & 0.9 & 0.3 \\ 0.2 & 0.1 & 0.4 & 0.3 & 0.4 & 0.2 & 0.4 & 0.8 & 1 \\ 0.3 & 0.6 & 0.3 & 0.9 & 0.3 & 0.8 & 0.3 & 0.3 & 0.3 \\ 0.4 & 0.3 & 0.3 & 0.4 & 0.7 & 0.4 & 0.2 & 0.2 & 0.3 \end{pmatrix} * \begin{pmatrix} 0.6 \\ 0.5 \\ 0.4 \\ 0.5 \\ 0.4 \\ 0.7 \\ 0.4 \\ 0.3 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.4 \\ 0.3 \\ 0.4 \\ 0.4 \\ 0.3 \\ 0.3 \\ 0.3 \\ 0.4 \end{pmatrix},$$

$$\widetilde{Q^+}(\widetilde{X}) = \widetilde{M}^T \cdot \widetilde{X} = \begin{pmatrix} 0.6 & 0.7 & 0.4 & 0.3 & 0.1 & 0.2 & 0.6 & 0.4 & 0.4 \\ 0.2 & 0.4 & 1 & 0.2 & 0.8 & 0.3 & 0.3 & 0.4 & 0.4 \\ 0.3 & 0.6 & 0.3 & 0.9 & 0.3 & 0.8 & 0.3 & 0.3 & 0.3 \\ 0.4 & 0.3 & 0.3 & 0.4 & 0.7 & 0.4 & 0.2 & 0.2 & 0.3 \\ 0.1 & 0.8 & 0.2 & 0.4 & 0.9 & 0.3 & 0.4 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.2 & 0.2 & 0.3 & 0.2 & 0.9 & 0.9 & 0.3 \\ 0.2 & 0.1 & 0.4 & 0.3 & 0.4 & 0.2 & 0.4 & 0.8 & 1 \\ 0.3 & 0.6 & 0.3 & 0.9 & 0.3 & 0.8 & 0.3 & 0.3 & 0.3 \\ 0.4 & 0.3 & 0.3 & 0.4 & 0.7 & 0.4 & 0.2 & 0.2 & 0.3 \end{pmatrix} \cdot \begin{pmatrix} 0.6 \\ 0.5 \\ 0.4 \\ 0.5 \\ 0.4 \\ 0.7 \\ 0.4 \\ 0.3 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.4 \\ 0.5 \\ 0.4 \\ 0.5 \\ 0.4 \\ 0.4 \\ 0.5 \\ 0.4 \end{pmatrix},$$

$$\widetilde{\dot{P}^-}(\widetilde{X}) = \widetilde{M} \cdot \widetilde{M}^T * \widetilde{X} = \begin{pmatrix} 0.6 & 0.6 & 0.4 & 0.4 & 0.4 & 0.4 & 0.6 & 0.4 & 0.4 \\ 0.6 & 0.7 & 0.4 & 0.6 & 0.8 & 0.6 & 0.6 & 0.4 & 0.4 \\ 0.4 & 0.4 & 1 & 0.3 & 0.8 & 0.3 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.6 & 0.3 & 0.9 & 0.4 & 0.8 & 0.4 & 0.3 & 0.3 \\ 0.4 & 0.8 & 0.8 & 0.4 & 0.9 & 0.4 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.6 & 0.3 & 0.8 & 0.4 & 0.8 & 0.3 & 0.3 & 0.3 \\ 0.6 & 0.6 & 0.4 & 0.4 & 0.4 & 0.3 & 0.9 & 0.9 & 0.4 \\ 0.4 & 0.4 & 0.4 & 0.3 & 0.4 & 0.3 & 0.9 & 0.9 & 0.8 \\ 0.4 & 0.4 & 0.4 & 0.3 & 0.4 & 0.3 & 0.4 & 0.8 & 0.1 \end{pmatrix} * \begin{pmatrix} 0.6 \\ 0.5 \\ 0.4 \\ 0.5 \\ 0.4 \\ 0.7 \\ 0.4 \\ 0.3 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.4 \\ 0.4 \\ 0.3 \\ 0.4 \\ 0.3 \\ 0.3 \\ 0.3 \\ 0.3 \end{pmatrix},$$

$$\widetilde{\dot{P}^{+}}(\widetilde{X}) = \widetilde{M}\cdot\widetilde{M}^{T}\cdot\widetilde{X} = \begin{pmatrix} 0.6 & 0.6 & 0.4 & 0.4 & 0.4 & 0.4 & 0.6 & 0.4 & 0.4 \\ 0.6 & 0.7 & 0.4 & 0.6 & 0.8 & 0.6 & 0.6 & 0.4 & 0.4 \\ 0.4 & 0.4 & 1 & 0.3 & 0.8 & 0.3 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.6 & 0.3 & 0.9 & 0.4 & 0.8 & 0.4 & 0.3 & 0.3 \\ 0.4 & 0.8 & 0.8 & 0.4 & 0.9 & 0.4 & 0.4 & 0.4 & 0.4 \\ 0.4 & 0.6 & 0.3 & 0.8 & 0.4 & 0.8 & 0.3 & 0.3 & 0.3 \\ 0.6 & 0.6 & 0.4 & 0.4 & 0.4 & 0.3 & 0.9 & 0.9 & 0.4 \\ 0.4 & 0.4 & 0.4 & 0.3 & 0.4 & 0.3 & 0.9 & 0.9 & 0.8 \\ 0.4 & 0.4 & 0.4 & 0.3 & 0.4 & 0.3 & 0.4 & 0.8 & 0.1 \end{pmatrix} \cdot \begin{pmatrix} 0.6 \\ 0.5 \\ 0.4 \\ 0.5 \\ 0.4 \\ 0.7 \\ 0.4 \\ 0.3 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.6 \\ 0.4 \\ 0.7 \\ 0.5 \\ 0.7 \\ 0.6 \\ 0.4 \\ 0.4 \end{pmatrix},$$

$$\widetilde{\dot{Q}^{-}}(\widetilde{X}) = \widetilde{M}^{T}\cdot\widetilde{M} * \widetilde{X} = \begin{pmatrix} 0.7 & 0.4 & 0.6 & 0.4 & 0.7 & 0.6 & 0.4 & 0.6 & 0.4 \\ 0.4 & 1 & 0.4 & 0.7 & 0.8 & 0.4 & 0.4 & 0.4 & 0.7 \\ 0.6 & 0.4 & 0.9 & 0.4 & 0.6 & 0.3 & 0.3 & 0.9 & 0.4 \\ 0.4 & 0.7 & 0.4 & 0.7 & 0.7 & 0.3 & 0.4 & 0.4 & 0.7 \\ 0.7 & 0.8 & 0.6 & 0.7 & 0.9 & 0.3 & 0.4 & 0.6 & 0.7 \\ 0.6 & 0.4 & 0.3 & 0.3 & 0.4 & 0.9 & 0.8 & 0.3 & 0.3 \\ 0.4 & 0.4 & 0.3 & 0.3 & 0.4 & 0.8 & 0.8 & 0.3 & 0.4 \\ 0.6 & 0.4 & 0.9 & 0.4 & 0.6 & 0.3 & 0.3 & 0.9 & 0.4 \\ 0.4 & 0.7 & 0.4 & 0.7 & 0.7 & 0.3 & 0.4 & 0.4 & 0.7 \end{pmatrix} * \begin{pmatrix} 0.6 \\ 0.5 \\ 0.4 \\ 0.5 \\ 0.4 \\ 0.7 \\ 0.4 \\ 0.3 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.3 \\ 0.3 \\ 0.3 \\ 0.3 \\ 0.3 \\ 0.3 \\ 0.3 \\ 0.3 \end{pmatrix},$$

$$\widetilde{\dot{Q}^{+}}(\widetilde{X}) = \widetilde{M}^{T}\cdot\widetilde{M}\cdot\widetilde{X} = \begin{pmatrix} 0.7 & 0.4 & 0.6 & 0.4 & 0.7 & 0.6 & 0.4 & 0.6 & 0.4 \\ 0.4 & 1 & 0.4 & 0.7 & 0.8 & 0.4 & 0.4 & 0.4 & 0.7 \\ 0.6 & 0.4 & 0.9 & 0.4 & 0.6 & 0.3 & 0.3 & 0.9 & 0.4 \\ 0.4 & 0.7 & 0.4 & 0.7 & 0.7 & 0.3 & 0.4 & 0.4 & 0.7 \\ 0.7 & 0.8 & 0.6 & 0.7 & 0.9 & 0.3 & 0.4 & 0.6 & 0.7 \\ 0.6 & 0.4 & 0.3 & 0.3 & 0.4 & 0.9 & 0.8 & 0.3 & 0.3 \\ 0.4 & 0.4 & 0.3 & 0.3 & 0.4 & 0.8 & 0.8 & 0.3 & 0.4 \\ 0.6 & 0.4 & 0.9 & 0.4 & 0.6 & 0.3 & 0.3 & 0.9 & 0.4 \\ 0.4 & 0.7 & 0.4 & 0.7 & 0.7 & 0.3 & 0.4 & 0.4 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 0.6 \\ 0.5 \\ 0.4 \\ 0.5 \\ 0.4 \\ 0.7 \\ 0.4 \\ 0.3 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.5 \\ 0.6 \\ 0.5 \\ 0.6 \\ 0.6 \\ 0.4 \\ 0.6 \\ 0.5 \end{pmatrix},$$

$$\widetilde{\ddot{P}^{-}}(\widetilde{X}) = \widetilde{M}\circ\widetilde{M}^{T} * \widetilde{X} = \begin{pmatrix} 0.1 & 0.2 & 0.2 & 0.2 & 0.3 & 0.2 & 0.3 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.3 & 0.3 & 0.3 & 0.2 & 0.3 & 0.3 & 0.3 \\ 0.2 & 0.3 & 0.2 & 0.2 & 0.3 & 0.2 & 0.3 & 0.2 & 0.2 \\ 0.2 & 0.3 & 0.2 & 0.2 & 0.3 & 0.3 & 0.3 & 0.4 & 0.3 \\ 0.3 & 0.3 & 0.3 & 0.3 & 0.1 & 0.2 & 0.3 & 0.3 & 0.3 \\ 0.2 & 0.2 & 0.3 & 0.3 & 0.2 & 0.2 & 0.3 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.2 & 0.2 & 0.3 \\ 0.2 & 0.3 & 0.2 & 0.4 & 0.3 & 0.3 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.3 & 0.2 & 0.3 & 0.3 & 0.3 & 0.3 & 0.2 & 0.2 \end{pmatrix} * \begin{pmatrix} 0.6 \\ 0.5 \\ 0.4 \\ 0.5 \\ 0.4 \\ 0.7 \\ 0.4 \\ 0.3 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 0.7 \\ 0.7 \\ 0.7 \\ 0.6 \\ 0.7 \\ 0.7 \\ 0.7 \\ 0.6 \\ 0.7 \end{pmatrix},$$

$$\widetilde{\ddot{P}^{+}}(\widetilde{X}) = \widetilde{M}\circ\widetilde{M}^{T}\cdot\widetilde{X} = \begin{pmatrix} 0.1 & 0.2 & 0.2 & 0.2 & 0.3 & 0.2 & 0.3 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.3 & 0.3 & 0.3 & 0.2 & 0.3 & 0.3 & 0.3 \\ 0.2 & 0.3 & 0.2 & 0.2 & 0.3 & 0.2 & 0.3 & 0.2 & 0.2 \\ 0.2 & 0.3 & 0.2 & 0.2 & 0.3 & 0.3 & 0.3 & 0.4 & 0.3 \\ 0.3 & 0.3 & 0.3 & 0.3 & 0.1 & 0.2 & 0.3 & 0.3 & 0.3 \\ 0.2 & 0.2 & 0.3 & 0.3 & 0.2 & 0.2 & 0.3 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.2 & 0.2 & 0.3 \\ 0.2 & 0.3 & 0.2 & 0.4 & 0.3 & 0.3 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.3 & 0.2 & 0.3 & 0.3 & 0.3 & 0.3 & 0.2 & 0.2 \end{pmatrix} \cdot \begin{pmatrix} 0.6 \\ 0.5 \\ 0.4 \\ 0.5 \\ 0.4 \\ 0.7 \\ 0.4 \\ 0.3 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.3 \\ 0.3 \\ 0.3 \\ 0.3 \\ 0.3 \\ 0.3 \\ 0.3 \\ 0.3 \end{pmatrix},$$

$$\widetilde{\ddot{Q}^{-}}(\widetilde{X}) = \widetilde{M}^{T}\circ\widetilde{M} * \widetilde{X} = \begin{pmatrix} 0.1 & 0.3 & 0.3 & 0.4 & 0.3 & 0.2 & 0.2 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.3 & 0.3 & 0.2 & 0.2 & 0.2 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 \\ 0.4 & 0.3 & 0.3 & 0.2 & 0.2 & 0.3 & 0.3 & 0.3 & 0.2 \\ 0.3 & 0.2 & 0.3 & 0.2 & 0.2 & 0.2 & 0.2 & 0.3 & 0.3 \\ 0.2 & 0.2 & 0.3 & 0.3 & 0.2 & 0.2 & 0.2 & 0.3 & 0.3 \\ 0.2 & 0.2 & 0.3 & 0.3 & 0.2 & 0.2 & 0.1 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 \\ 0.4 & 0.3 & 0.3 & 0.2 & 0.3 & 0.3 & 0.3 & 0.3 & 0.2 \end{pmatrix} * \begin{pmatrix} 0.6 \\ 0.5 \\ 0.4 \\ 0.5 \\ 0.4 \\ 0.7 \\ 0.4 \\ 0.3 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.7 \\ 0.7 \\ 0.6 \\ 0.7 \\ 0.7 \\ 0.7 \\ 0.7 \\ 0.7 \end{pmatrix},$$

$$\widetilde{\ddot{Q}^{+}}(\widetilde{X}) = \widetilde{M}^{T} \circ \widetilde{M} \cdot \widetilde{X} = \begin{pmatrix} 0.1 & 0.3 & 0.3 & 0.4 & 0.3 & 0.2 & 0.2 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.3 & 0.3 & 0.2 & 0.2 & 0.2 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 \\ 0.4 & 0.3 & 0.3 & 0.2 & 0.2 & 0.3 & 0.3 & 0.3 & 0.2 \\ 0.3 & 0.2 & 0.3 & 0.2 & 0.2 & 0.2 & 0.2 & 0.3 & 0.3 \\ 0.2 & 0.2 & 0.3 & 0.3 & 0.2 & 0.2 & 0.2 & 0.3 & 0.3 \\ 0.2 & 0.2 & 0.3 & 0.3 & 0.2 & 0.2 & 0.1 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 \\ 0.4 & 0.3 & 0.3 & 0.2 & 0.3 & 0.3 & 0.3 & 0.3 & 0.2 \end{pmatrix} \cdot \begin{pmatrix} 0.6 \\ 0.5 \\ 0.4 \\ 0.5 \\ 0.4 \\ 0.7 \\ 0.4 \\ 0.3 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.3 \\ 0.3 \\ 0.3 \\ 0.3 \\ 0.3 \\ 0.3 \\ 0.3 \\ 0.4 \end{pmatrix}.$$

Now we can compare them as

$$\begin{pmatrix} \widetilde{\ddot{P}^{-}}(\widetilde{X}) & 0.7 & 0.7 & 0.7 & 0.6 & 0.7 & 0.7 & 0.7 & 0.6 & 0.7 \\ \widetilde{\dot{P}^{-}}(\widetilde{X}) & 0.4 & 0.4 & 0.4 & 0.3 & 0.4 & 0.3 & 0.3 & 0.3 & 0.3 \\ \widetilde{P^{-}}(\widetilde{X}) & 0.6 & 0.4 & 0.4 & 0.3 & 0.3 & 0.3 & 0.4 & 0.4 & 0.4 \\ \widetilde{X} & 0.6 & 0.5 & 0.4 & 0.5 & 0.4 & 0.7 & 0.4 & 0.3 & 0.3 \\ \widetilde{P^{+}}(\widetilde{X}) & 0.6 & 0.6 & 0.5 & 0.4 & 0.5 & 0.4 & 0.6 & 0.4 & 0.4 \\ \widetilde{\dot{P}^{+}}(\widetilde{X}) & 0.6 & 0.6 & 0.4 & 0.7 & 0.5 & 0.7 & 0.6 & 0.4 & 0.4 \\ \widetilde{\ddot{P}^{+}}(\widetilde{X}) & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 \end{pmatrix},$$

$$\begin{pmatrix} \widetilde{\ddot{Q}^{-}}(\widetilde{X}) & 0.6 & 0.7 & 0.7 & 0.6 & 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ \widetilde{\dot{Q}^{-}}(\widetilde{X}) & 0.4 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 \\ \widetilde{Q^{-}}(\widetilde{X}) & 0.4 & 0.4 & 0.3 & 0.4 & 0.4 & 0.3 & 0.3 & 0.3 & 0.4 \\ \widetilde{X} & 0.6 & 0.5 & 0.4 & 0.5 & 0.4 & 0.7 & 0.4 & 0.3 & 0.3 \\ \widetilde{Q^{+}}(\widetilde{X}) & 0.6 & 0.4 & 0.5 & 0.4 & 0.5 & 0.4 & 0.4 & 0.5 & 0.4 \\ \widetilde{\dot{Q}^{+}}(\widetilde{X}) & 0.6 & 0.5 & 0.6 & 0.5 & 0.6 & 0.6 & 0.4 & 0.6 & 0.5 \\ \widetilde{\ddot{Q}^{+}}(\widetilde{X}) & 0.4 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.4 \end{pmatrix}.$$

By calculating the average of the above 12 approximations, we obtained the following fuzzy set:

$$\overline{\widetilde{X}} = (\underline{0.52},\ \underline{0.47},\ 0.42,\ 0.4,\ 0.45,\ \underline{0.46},\ 0.44,\ 0.4,\ 0.42).$$

Then we compare the scores to select the higher three and underline them. Hence we can make the decision that the students x_1, x_2 and x_6 will be selected for the final round.

Next, we discuss the following properties under some special conditions.

Proposition 4.9. *Let $(U, V, \widetilde{R})$ be an FAAS, where $U = V$. If $\widetilde{R}$ is a symmetric fuzzy relation on U, then*

i) $\widetilde{R} = \widetilde{R^{T}}$,

ii) $\widetilde{\dot{R}_U} = \widetilde{\dot{R}_V}$,

iii) $\widetilde{\ddot{R}_U} = \widetilde{\ddot{R}_V}$.

Proof. *i*) If $\widetilde{R}$ is a symmetric fuzzy relation on U, then for each $x, y \in U$, $\widetilde{R}(x,y) = \widetilde{R}(y,x)$, so $\widetilde{R} = \widetilde{R^T}$ is obvious.

ii) It follows from *i*) that $\widetilde{M} = \widetilde{M^T}$, then $\widetilde{M} \cdot \widetilde{M^T} = \widetilde{M^T} \cdot \widetilde{M}$ is followed. So, $\widetilde{\dot{R}_U} = \widetilde{\dot{R}_V}$ can be obtained.

iii) According to $\widetilde{M} = \widetilde{M^T}$, there is $\widetilde{M} \circ \widetilde{M^T} = \widetilde{M^T} \circ \widetilde{M}$, so we can obtain $\widetilde{\ddot{R}_U} = \widetilde{\ddot{R}_V}$. □

Proposition 4.10. *Let* $(U, V, \widetilde{R})$ *be an FAAS, where* $U = V$*. If* $\widetilde{R}$ *is a symmetric fuzzy relation on* U*, then for each* $\widetilde{X} \in F(U)$*, the following conclusions can be derived:*

i) $\widetilde{P^-}(\widetilde{X}) = \widetilde{Q^-}(\widetilde{X})$, $\widetilde{P^+}(\widetilde{X}) = \widetilde{Q^+}(\widetilde{X})$,

ii) $\widetilde{\dot{P}^-}(\widetilde{X}) = \widetilde{\dot{Q}^-}(\widetilde{X})$, $\widetilde{\dot{P}^+}(\widetilde{X}) = \widetilde{\dot{Q}^+}(\widetilde{X})$,

iii) $\widetilde{\ddot{P}^-}(\widetilde{X}) = \widetilde{\ddot{Q}^-}(\widetilde{X})$, $\widetilde{\ddot{P}^+}(\widetilde{X}) = \widetilde{\ddot{Q}^+}(\widetilde{X})$.

Proof. *i*) According to Proposition 4.7, and noting $\widetilde{M} = \widetilde{M^T}$, there are

$$\widetilde{P^-}(\widetilde{X}) = \widetilde{M} * \widetilde{X} = \widetilde{M^T} * \widetilde{X} = \widetilde{Q^-}(\widetilde{X}),$$
$$\widetilde{P^+}(\widetilde{X}) = \widetilde{M} \cdot \widetilde{X} = \widetilde{M^T} \cdot \widetilde{X} = \widetilde{Q^+}(\widetilde{X}).$$

ii) Similar to *i*), there are

$$\widetilde{\dot{P}^-}(\widetilde{X}) = \widetilde{M} \cdot \widetilde{M^T} * \widetilde{X} = \widetilde{M^T} \cdot \widetilde{M} * \widetilde{X} = \widetilde{\dot{Q}^-}(\widetilde{X}),$$
$$\widetilde{\dot{P}^+}(\widetilde{X}) = \widetilde{M} \cdot \widetilde{M^T} \cdot \widetilde{X} = \widetilde{M^T} \cdot \widetilde{M} \cdot \widetilde{X} = \widetilde{\dot{Q}^+}(\widetilde{X}).$$

iii) Similarly, the next two formulas can be obtained:

$$\widetilde{\ddot{P}^-}(\widetilde{X}) = \widetilde{M} \circ \widetilde{M^T} * \widetilde{X} = \widetilde{M^T} \circ \widetilde{M} * \widetilde{X} = \widetilde{\ddot{Q}^-}(\widetilde{X}),$$
$$\widetilde{\ddot{P}^+}(\widetilde{X}) = \widetilde{M} \circ \widetilde{M^T} \cdot \widetilde{X} = \widetilde{M^T} \circ \widetilde{M} \cdot \widetilde{X} = \widetilde{\ddot{Q}^+}(\widetilde{X}).$$

□

4.3 $U = V$ and $\widetilde{R}$ being a binary relation R in an FAAS $(U, V, \widetilde{R})$

If an FAAS $(U, V, \widetilde{R})$ is a special case that has both properties in above two subsections, i.e., $U = V$ and $\widetilde{R}$ being a binary relation R on U, then the FAAS is a general binary relation based approximation space (U, R). Combining Definition 4.6 and Proposition 4.3, the following property can be easily obtained.

Proposition 4.11. *Let* (U, V, R) *be an FAAS, where* $U = V$ *and* R *is a binary relation on* U*. For each* $x \in U$*, set*

$R(x) = \{z \in U : R(x,z) = 1\}$, $R^{-1}(x) = \{z \in U : R(z,x) = 1\}$,

$\dot{R}_U(x) = \{z \in U : \dot{R}_U(x,z) = 1\}$, $\ddot{R}_U(x) = \{z \in U : \ddot{R}_U(x,z) = 1\}$,

$\dot{R}_V(y) = \{z \in U : \dot{R}_V(y,z) = 1\}$, $\ddot{R}_V(y) = \{z \in U : \ddot{R}_V(y,z) = 1\}$.

Then for each $X \subseteq U$*, the following equalities hold:*

$\widetilde{P^-}(X) = \{x \in U : R(x) \subseteq X\}$, $\widetilde{P^+}(X) = \{x \in U : R(x) \cap X \neq \emptyset\}$,

$\widetilde{Q^-}(X) = \{x \in U : R^{-1}(x) \subseteq X\}$, $\widetilde{Q^+}(X) = \{x \in U : R^{-1}(x) \cap X \neq \emptyset\}$,

$\widetilde{\dot{P}^-}(X) = \{x \in U : \dot{R}_U(x) \subseteq X\}$, $\widetilde{\dot{P}^+}(X) = \{x \in U : \dot{R}_U(x) \cap X \neq \emptyset\}$,

$\widetilde{\ddot{P}^-}(X) = \{x \in U : \ddot{R}_U(x) \subseteq X\}$, $\widetilde{\ddot{P}^+}(X) = \{x \in U : \ddot{R}_U(x) \cap X \neq \emptyset\}$,

$\widetilde{\dot{Q}^-}(X) = \{x \in U : \dot{R}_V(x) \subseteq X\}$, $\widetilde{\dot{Q}^+}(X) = \{x \in U : \dot{R}_V(x) \cap X \neq \emptyset\}$,

$\widetilde{\ddot{Q}^-}(X) = \{x \in U : \ddot{R}_V(x) \subseteq X\}$, $\widetilde{\ddot{Q}^+}(X) = \{x \in U : \ddot{R}_V(x) \cap X \neq \emptyset\}$.

It is worth noting that in this proposition, in general, there are $\dot{R}_U \neq \dot{R}_V$ and $\ddot{R}_U \neq \ddot{R}_V$. Moreover, the two approximation operator pairs $(\widetilde{P^-}(X), \widetilde{P^-}(X))$ and $(\widetilde{Q^-}(X), \widetilde{Q^-}(X))$ are widely used in the field of rough set theory and we are familiar with them, while the others are new. The properties and simple applications of these approximations are similar to those discussed in Subsection 4.1, so we will not repeat them here. Now let's conclude by discussing some simple results about these approximation operators under some particular binary relation R.

Proposition 4.12. *Let* (U, V, R) *be an FAAS, where* $U = V$ *and* R *is a binary relation on* $U = \{x_1, x_2, \cdots, x_m\}$*. If* R *is a reflexive relation, then* $R(x_i, x_j) = 1$ *implies* $\widetilde{\dot{R}_U}(x_i, x_j) = 1$ *and* $\widetilde{\dot{R}_V}(x_i, x_j) = 1$ *for each* $x_i, x_j \in U$*.*

Proof. For $x_i, x_j \in U$,

$$[R(x_i, x_j) = 1] \wedge [R^{-1}(x_j, x_j) = R(x_j, x_j) = 1] \Rightarrow \widetilde{\dot{R}_U}(x_i, x_j) = \vee_{k=1}^{m}[R(x_i, x_k) \wedge R^{-1}(x_k, x_j)] = 1.$$
$$[R(x_i, x_j) = 1] \wedge [R^{-1}(x_i, x_i) = R(x_i, x_i) = 1] \Rightarrow \widetilde{\dot{R}_V}(x_i, x_j) = \vee_{k=1}^{m}[R^{-1}(x_i, x_k) \wedge R(x_k, x_j)] = 1.$$

□

Proposition 4.13. *Let* (U, V, R) *be an FAAS, where* $U = V$ *and* R *is a binary relation on* $U = \{x_1, x_2, \cdots, x_m\}$*. If* R *is a symmetric relation, then* $\widetilde{\dot{R}_U} = \widetilde{\dot{R}_V}$ *and* $\widetilde{\ddot{R}_U} = \widetilde{\ddot{R}_V}$*.*

Proof. For $x_i, x_j \in U$,

$R(x_i, x_j) = R(x_j, x_i) = R^{-1}(x_i, x_j)$

$\Rightarrow \widetilde{\dot{R}_U}(x_i, x_j) = \vee_{k=1}^{m}[R(x_i, x_k) \wedge R^{-1}(x_k, x_j)] = \vee_{k=1}^{m}[R^{-1}(x_i, x_k) \wedge R(x_k, x_j)] = \widetilde{\dot{R}_V}(x_i, x_j)$.

$R(x_i, x_j) = R(x_j, x_i) = R^{-1}(x_i, x_j)$

$\Rightarrow \widetilde{\ddot{R}_U}(x_i, x_j) = \wedge_{k=1}^{m}[R(x_i, x_k) \vee R^{-1}(x_k, x_j)] = \wedge_{k=1}^{m}[R^{-1}(x_i, x_k) \vee R(x_k, x_j)] = \widetilde{\ddot{R}_V}(x_i, x_j)$. □

Proposition 4.14. *Let (U, V, R) be an FAAS, where $U = V$ and R is a binary relation on $U = \{x_1, x_2, \cdots, x_m\}$. If R is a symmetric and transitive relation, then $\widetilde{\dot{R}_U}(x_i, x_j) = 1$ implies $R(x_i, x_j) = 1$ for each $x_i, x_j \in U$.*

Proof. Since that R is a symmetric and transitive relation on U, then

$R(x_i, x_j) = R^{-1}(x_i, x_j) = R(x_j, x_i)$, and $[R(x_i, x_k) = 1] \wedge [R(x_k, x_j) = 1] \rightarrow R(x_i, x_j) = 1$.

So, $1 = \widetilde{\dot{R}_U}(x_i, x_j) = \vee_{k=1}^{m}[R(x_i, x_k) \wedge R^{-1}(x_k, x_j)] \Rightarrow \exists x_k, (R(x_i, x_k) = 1) \wedge (R(x_k, x_j) = 1) \Rightarrow R(x_i, x_j) = 1$. □

The next proposition is directly derived from Proposition 4.12 and Proposition 4.14.

Proposition 4.15. *Let (U, V, R) be an FAAS, where $U = V$ and R is a symmetric binary relation on U. If R is also a preorder, i.e., R is a reflexive and transitive relation on U, then $R = \widetilde{\dot{R}_U} = \widetilde{\dot{R}_V}$.*

According to Proposition 4.15, the following proposition is obvious.

Proposition 4.16. *Let (U, V, R) be an FAAS, where $U = V$ and R is a binary relation on $U = \{x_1, x_2, \cdots, x_m\}$. If R is a equivalence relation, i.e., R is a reflexive, symmetric and transitive relation on U, then $R = \widetilde{\dot{R}_U} = \widetilde{\dot{R}_V}$.*

5 Conclusions

In this paper, two universes with a fuzzy relation between them are studied as a whole (FAAS) from a new perspective. Two new fuzzy relations such as positive low-value relation and conservative high-value relation are introduced to reflect the interactions between the two spaces in the FAAS, so the fuzzy approximations based on these new relations can reflect the combination of the action and the reaction from one space to another. Moreover, some important special FAAS are studied and the simper applications of the FAAS are explored. The novel concepts and models introduced in this paper may open a new way for the study of rough set theory.

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