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Weighted approximation of fuzzy numbers by using $m-n-$step type fuzzy number

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Abstract

In this paper, based on the weighted metric, the problem of approximating a general continuous fuzzy number u with derivable $\underline{u}(r)$ and $\overline{u}(r)$ (left and right cut functions of level value r) for $r \in (0,1]$ by using the $m-n-$step type fuzzy number is studied. Two kinds of approximations ($I-$nearest approximation and $II-$nearest approximation which satisfy different conditions) of using $m-n-$step type fuzzy numbers to approximate general continuous fuzzy numbers are defined, and the relationship between the $I-$nearest approximation and $II-$nearest approximation is obtained. Then the methods of the two kinds of approximations are respectively presented by the theorems more specifically. At last, an example is given to show the effectiveness and usability of the methods set up by us.

Keywords: Fuzzy numbers, step type fuzzy numbers, membership function, approximation of fuzzy numbers.

1 Introduction

After the introduction of the concept and operation of fuzzy sets by Zadeh [22] in 1965, Chang and Zadeh put forward the concept of fuzzy numbers in [3]. A fuzzy number is a special kind of fuzzy set on real number field which can be used to express uncertain or imprecise digital information that has a wide scope of applications in many research fields such as expert systems, pattern recognition or data mining [16].

With the development of theories and applications of fuzzy numbers, the fuzzy number approximations as one kind of the defuzzification methods became much more important and were implemented in [2, 5, 7, 8, 9, 20]. Ban proved that the approximate conclusion in [12] may not obtain the fuzzy numbers by simple examples and improved the solution by using Karush-Kuhn-Tucker theorem in [1]. Pedrycz and Grzegorzewski studied the fuzzy number approximation by the shadow set, proposed a new way to simplify fuzzy numbers and further investigated the approximation algorithm in [6, 15]. Nayagam and Murugan presented a new weighted trigonometric approximation method for intuitionistic fuzzy numbers, and discussed some useful properties of triangular approximation of intuitionistic fuzzy numbers in [13]. From the condition of given horizontal values (i.e., $(r_1, r_2, \ldots, r_{m-1})$ and $(s_1, s_2, \ldots, s_{n-1})$ are known), the approximation of general fuzzy numbers with simple fuzzy numbers (i.e., step type fuzzy numbers) is studied by Wang and Li in [17]. Under the background, we attempt to set the corresponding horizontal values of the nodes to be solved as unknown numbers in order to obtain better approximation results.

In addition, we observe that the membership degree of elements belonging to fuzzy sets is proportional to the effect of elements on fuzzy sets. In other words, compared with the unweighted metric (used in the paper [4, 11, 14, 18]), the weighted metric (used in the paper [10, 13, 19, 21]) describes the relationship between two fuzzy numbers more objectively and reasonably. Therefore, in this paper, we further study the approximation of simple fuzzy numbers, using weighted Euclidean metric as an approximation measure,establishing a new approximate method in which the level value as well as the nodes involved are considered to be solved.

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The specific arrangement of this paper is as follows: In Section 2, some basic concepts, definitions and results of fuzzy numbers are briefly reviewed by us. In Section 3, we introduce two concepts of $I-$ and $II-$nearest $m-n-$step type fuzzy number approximations, and study the relationship between the two kinds of nearest approximation, then establish the solving methods of the two kinds of approximations. In Section 4, an example is given to show the effectiveness and usability of the methods. In Section 5, we make a conclusion for this paper.

2 Preliminaries

If a function u is a normal and fuzzy convex fuzzy set of the real line $\mathbb{R}$, the membership function $u(x)$ of u is upper semi-continuous, and the closure of the $supp(u)$, i.e., $[u]^0 = \overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact, then we call u a fuzzy number, and notate the collection of all fuzzy numbers by E.

For any $r \in (0,1]$, let $u \in E$, by $[u]^r = \{x \in \mathbb{R} : u(x) \geq r\}$ we denote the $r-$level set of u. It is known that $[u]^r$ is a closed and bounded interval on $\mathbb{R}$. That is to say, for $u \in E$, we characterise the closed interval as $[u]^r = [\underline{u}(r), \overline{u}(r)]$ for any $r \in [0,1]$ where $\underline{u}(r)$ is nondecreasing, $\overline{u}(r)$ is nonincreasing and $\underline{u}(r) \leq \overline{u}(r)$.

If the mapping $d^* : E \times E \to \mathbb{R}$ is defined by

$$d^*(u,v) = \sqrt{\int_0^1 2r(\underline{u}(r) - \underline{v}(r))^2 dr + \int_0^1 2r(\overline{u}(r) - \overline{v}(r))^2 dr}, \ \ \forall (u,v) \in E \times E,$$

then d^* is a metric on E. We say d^* to be an unweighted metric on E.

Definition 2.1. [17] *Let m, n be two positive integers, $u \in E$ is defined as*

$$u(x) = \begin{cases} r_1, & x \in [a_1, a_2) \\ r_2, & x \in [a_2, a_3) \\ \vdots & \\ r_{m-1}, & x \in [a_{m-1}, a_m) \\ 1, & x \in [a_m, b_n] \\ s_{n-1}, & x \in (b_n, b_{n-1}] \\ \vdots & \\ s_2, & x \in (b_3, b_2] \\ s_1, & x \in (b_2, b_1] \\ 0, & x \notin [a_1, b_1] \end{cases},$$

then we call u as $m-n-$step type fuzzy number, and denote it as

$$u = \left[\begin{pmatrix} a_1 & r_1 \\ a_2 & r_2 \\ \vdots & \vdots \\ a_m & 1 \end{pmatrix}, \begin{pmatrix} b_1 & s_1 \\ b_2 & s_2 \\ \vdots & \vdots \\ b_n & 1 \end{pmatrix} \right] (in\ short, u = [(a_i, r_i)_{i=1}^m, (b_j, s_j)_{j=1}^n]),$$

where $r_i, s_j \in (0,1]$ and $a_i, b_j \in R$ $(i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n)$ with $r_1 < r_2 < \cdots < r_{m-1} < r_m = 1$ and $s_1 < s_2 < \cdots < s_{n-1} < s_n = 1$ and $a_1 < a_2 < \cdots < a_m < b_n < b_{n-1} < \cdots < b_1$. And we denote the collection of all $m-n-$step type fuzzy numbers by ${}_mS_n(E)$.

Then for any $u \in_m S_n(E)$ we have that

$$\underline{u}(r) = \begin{cases} a_1, & r \in [0, r_1] \\ a_2, & r \in (r_1, r_2] \\ \vdots & \\ a_m, & r \in (r_{m-1}, r_m] = (r_{m-1}, 1] \end{cases},$$

and

$$\overline{u}(r) = \begin{cases} b_1, & r \in [0, s_1] \\ b_2, & r \in (s_1, s_2] \\ \vdots & \\ b_n, & r \in (s_{n-1}, s_n] = (s_{n-1}, 1] \end{cases}.$$

We denote matrix $A_{n\times n}$

$$diag(A_1, A_2, \ldots, A_s) \triangleq \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_s \end{pmatrix},$$

where $A_i (i = 1, 2, \ldots, s)$ are all square matrices of n_i-order, and $\sum_{i=1}^{s} n_i = n$.

Theorem 2.2. [23] *If $A, B, C \in \mathbb{M}^{n\times n}$, then matrix*

$$D = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix},$$

is positive definite $\Leftrightarrow$ matrix A and $C - B^T A^{-1} B$ are both positive definite.

Corollary 2.3. *If $A, C \in \mathbb{M}^{n\times n}$ are both positive definite , then we can know that*

$$D = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix},$$

is positive definite.

3 Approximations of fuzzy numbers

In this section, based on the weighted distance d^*, we are going to establish the methods of approximating a general continuous fuzzy number u with derivable $\underline{u}(r)$ and $\overline{u}(r)$ via the $m-n-$step type fuzzy number that the level values of the nearest approximation are not given. Here we denote the collection of all u in E with derivable $\underline{u}(r)$ and $\overline{u}(r)$ for $r \in (0, 1]$ by $D(E)$, that is, $D(E) = \{u \in E :\ \underline{u}(r)$ and $\overline{u}(r)$ are derivable for $r \in (0, 1]\}$. In the following, we give the definitions of two approximate methods.

Definition 3.1. *Let m, n be two positive integers and $u \in D(E)$. If $\hat{u} \in {}_mS_n(E)$ and $d^*(u, \hat{u}) = \min\{d^*(u, v) : v \in {}_mS_n(E)\}$, then $\hat{u}$ is called the $I-$nearest $m-n-$step type fuzzy number approximation (in short, $I-$nearest $m-n-$STFN approximation) of u. Specially, for m=n, $\hat{u}$ is called the $I-$nearest $m-$step type fuzzy number approximation (in short, $I-$nearest $m-$STFN approximation) of u.*

Definition 3.2. *Let m, n be two positive integers and $u \in D(E)$. If $\hat{u} \in {}_mS_n(E)$ and $d^*(u, \hat{u}) = \min\{d^*(u, v) : v \in {}_mS_n(E)$ with $\underline{v}(1) = \underline{u}(1), \overline{v}(1) = \overline{u}(1)\}$, then $\hat{u}$ is called the $II-$nearest $m-n-$step type fuzzy number approximation (in short, $II-$nearest $m-n-$STFN approximation) of u. Specially, for m=n, $\hat{u}$ is called the $II-$nearest $m-$step type fuzzy number approximation (in short, $II-$nearest $m-$STFN approximation) of u.*

Theorem 3.3. *Let m, n be two positive integers and $u \in D(E)$. If $\hat{u}$ is the $I-$nearest $m-n-$step type fuzzy number approximation of u and $\hat{v}$ is the $II-$nearest $m-n-$step type fuzzy number approximation of u, then $d^*(u, \hat{u}) \leq d^*(u, \hat{v})$.*

Proof. The proof of the theorem can be directly completed from

$$\{d^*(u, v) : v \in {}_mS_n(E) \text{ with } \underline{v}(1) = \underline{u}(1), \overline{v}(1) = \overline{u}(1)\} \subset \{d^*(u, v) : v \in {}_mS_n(E)\}.$$

□

Remark 3.4. *Although the $II-$nearest $m-n-$STFN approximation still has the same kernel and support set as the original fuzzy number (i.e. $II-$nearest $m-n-$STFN approximation preserves the kernel and support set), the theorem tells us that the $I-$nearest $m-n-$STFN approximation is closer to the approximated fuzzy number u than the $II-$nearest $m-n-$STFN approximation based on the weighted distance d^*, that is, the $I-$nearest $m-n-$STFN approximation is better than the $II-$nearest $m-n-$STFN approximation with respect to d^*. However, compare to the $I-$nearest $m-n-$STFN approximation, the $II-$nearest $m-n-$STFN approximation preserving kernel and support set possesses more simplicity in calculation.*

Theorem 3.5. *Let m, n be two positive integers and $u \in D(E)$, $(r_1, r_2, \ldots, r_{m-1}) = (\alpha_1, \alpha_2, \ldots, \alpha_{m-1})$ and $(s_1, s_2, \ldots, s_{n-1}) = (\beta_1, \beta_2, \ldots, \beta_{n-1})$ are respectively unique solutions satisfying $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{m-1} < 1$ and $0 < \beta_1 < \beta_2 < \cdots < \beta_{n-1} < 1$ of the following Equations*

$$\begin{cases} (r_2^2 - r_1^2)\int_0^{r_1} t\underline{u}(t)dt + r_1^2\int_{r_1}^{r_2} t\underline{u}(t)dt - (r_2^2 - r_1^2)r_1^2\underline{u}(r_1) = 0 \\ (r_3^2 - r_2^2)\int_{r_1}^{r_2} t\underline{u}(t)dt + (r_2^2 - r_1^2)\int_{r_2}^{r_3} t\underline{u}(t)dt - (r_3^2 - r_2^2)(r_2^2 - r_1^2)\underline{u}(r_2) = 0 \\ \vdots \\ (1 - r_{m-1}^2)\int_{r_{m-2}}^{r_{m-1}} t\underline{u}(t)dt + (r_{m-1}^2 - r_{m-2}^2)\int_{r_{m-1}}^{1} t\underline{u}(t)dt - (1 - r_{m-1}^2)(r_{m-1}^2 - r_{m-2}^2)\underline{u}(r_{m-1}) = 0 \end{cases} \tag{1}$$

$$\begin{cases} (s_2^2 - s_1^2)\int_0^{s_1} t\overline{u}(t)dt + s_1^2\int_{s_1}^{s_2} t\overline{u}(t)dt - (s_2^2 - s_1^2)s_1^2\overline{u}(s_1) = 0 \\ (s_3^2 - s_2^2)\int_{s_1}^{s_2} t\overline{u}(t)dt + (s_2^2 - s_1^2)\int_{s_2}^{s_3} t\overline{u}(t)dt - (s_3^2 - s_2^2)(s_2^2 - s_1^2)\overline{u}(s_2) = 0 \\ \vdots \\ (1 - s_{n-1}^2)\int_{s_{n-2}}^{s_{n-1}} t\overline{u}(t)dt + (s_{n-1}^2 - s_{n-2}^2)\int_{s_{n-1}}^{1} t\overline{u}(t)dt - (1 - s_{n-1}^2)(s_{n-1}^2 - s_{n-2}^2)\overline{u}(s_{n-1}) = 0 \end{cases} \tag{2}$$

and $a_i = \frac{2\int_{\alpha_{i-1}}^{\alpha_i} t\underline{u}(t)dt}{\alpha_i^2 - \alpha_{i-1}^2}$ *($i = 1, 2, \ldots, m$, where $\alpha_0 = 0$ and $\alpha_m = 1$)*, $b_j = \frac{2\int_{\beta_{j-1}}^{\beta_j} t\overline{u}(t)dt}{\beta_j^2 - \beta_{j-1}^2}$ *($j = 1, 2, \ldots, n$, where $\beta_0 = 0$ and $\beta_n = 1$)*, *and* $\underline{\Delta}_1 = diag(2(\alpha_1^2 - \alpha_0^2), 2(\alpha_2^2 - \alpha_1^2), \ldots, 2(\alpha_m^2 - \alpha_{m-1}^2))$, $\overline{\Delta}_1 = diag(2(\beta_1^2 - \beta_0^2), 2(\beta_2^2 - \beta_1^2), \ldots, 2(\beta_n^2 - \beta_{n-1}^2))$, $\underline{\Delta}_2 = diag(c_1, c_2, \ldots, c_{m-1})$ *(where* $c_i = 2(a_i - a_{i+1})\left[a_i + a_{i+1} - 2\left(\underline{u}(\alpha_i) + \alpha_i\underline{u}'(\alpha_i)\right)\right]$, $i = 1, 2, \ldots, m-1$*)*, $\overline{\Delta}_2 = diag(d_1, d_2, \ldots, d_{n-1})$ *(where* $d_j = 2(b_j - b_{j+1})\left[b_j + b_{j+1} - 2\left(\overline{u}(\beta_j) + \beta_j\overline{u}'(\beta_j)\right)\right]$, $j = 1, 2, \ldots, n-1$*)*,

$$\underline{M} = \begin{pmatrix} 4\alpha_1(a_1 - \underline{u}(\alpha_1)) & 0 & \cdots & 0 & 0 \\ 4\alpha_1(\underline{u}(\alpha_1) - a_2) & 4\alpha_2(a_2 - \underline{u}(\alpha_2)) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 4\alpha_{m-2}(\underline{u}(\alpha_{m-2}) - a_{m-1}) & 4\alpha_{m-1}(a_{m-1} - \underline{u}(\alpha_{m-1})) \\ 0 & 0 & \cdots & 0 & 4\alpha_{m-1}(\underline{u}(\alpha_{m-1}) - a_m) \end{pmatrix}_{m\times(m-1)},$$

and

$$\overline{M} = \begin{pmatrix} 4\beta_1(b_1 - \overline{u}(\beta_1)) & 0 & \cdots & 0 & 0 \\ 4\beta_1(\overline{u}(\beta_1) - b_2) & 4\beta_2(b_2 - \overline{u}(\beta_2)) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 4\beta_{n-2}(\overline{u}(\beta_{n-2}) - b_{n-1}) & 4\beta_{n-1}(b_{n-1} - \overline{u}(\beta_{n-1})) \\ 0 & 0 & \cdots & 0 & 4\beta_{n-1}(\overline{u}(\beta_{n-1}) - b_n) \end{pmatrix}_{n\times(n-1)}.$$

If $\underline{\Delta}_2 - \underline{M}^T\underline{\Delta}_1^{-1}\underline{M}$ *and* $\overline{\Delta}_2 - \overline{M}^T\overline{\Delta}_1^{-1}\overline{M}$ *are positive definite, then*

$$\hat{u} = [(a_i, \alpha_i)_{i=1}^m, (b_j, \beta_j)_{j=1}^n],$$

is the $I-$nearest $m-n-$step type fuzzy number approximation of u.

Proof. Let

$$\hat{u} = [(x_i, r_i)_{i=1}^m, (y_j, s_j)_{j=1}^n],$$

be any step type fuzzy number with $0 < r_1 < r_2 < \cdots < r_{m-1} < r_m = 1$ and $0 < s_1 < s_2 < \cdots < s_{n-1} < s_n = 1$. Then we have that

$$\underline{\hat{u}}(t) = \begin{cases} x_1, & r \in [0, r_1] \\ x_2, & r \in (r_1, r_2] \\ \vdots & \\ x_m, & r \in (r_{m-1}, r_m] = (r_{m-1}, 1] \end{cases},$$

and

$$\overline{\hat{u}}(t) = \begin{cases} y_1, & s \in [0, s_1] \\ y_2, & s \in (s_1, s_2] \\ \vdots & \\ y_n, & s \in (s_{n-1}, s_n] = (s_{n-1}, 1] \end{cases}.$$

Denoting $D(x_1, x_2, \ldots, x_m, r_1, r_2, \ldots, r_{m-1}, y_1, y_2, \ldots, y_n, s_1, s_2, \ldots, s_{n-1}) = [d^*(u, \hat{u})]^2$

$$= \int_0^1 2t(\underline{\hat{u}}(t) - \underline{u}(t))^2dt + \int_0^1 2t(\overline{\hat{u}}(t) - \overline{u}(t))^2dt = \sum_{i=1}^m \int_{r_{i-1}}^{r_i} 2t(x_i - \underline{u}(t))^2dt + \sum_{j=1}^n \int_{s_{j-1}}^{s_j} 2t(y_j - \overline{u}(t))^2dt.$$

Then for $i=1,2,\ldots,m$, we have that $\frac{\partial D}{\partial x_i}=2(r_i^2-r_{i-1}^2)x_i-4\int_{r_{i-1}}^{r_i}t\underline{u}(t)dt$, where $r_0=0$ and $r_m=1$, and for $i=1,2,\ldots,m-1$, we have that

$$\frac{\partial D}{\partial r_i}=2r_i(x_i^2-x_{i+1}^2)-4r_i(x_i-x_{i+1})\underline{u}(r_i);$$

for $j=1,2,\ldots,n$, we have that $\frac{\partial D}{\partial y_j}=2(s_j^2-s_{j-1}^2)y_j-4\int_{s_{j-1}}^{s_j}t\overline{u}(t)dt$, where $s_0=0$ and $s_n=1$, and for $j=1,2,\ldots,n-1$, we have that $\frac{\partial D}{\partial s_j}=2s_j(y_j^2-y_{j+1}^2)-4s_j(y_j-y_{j+1})\overline{u}(s_j)$. Let

$$\frac{\partial D}{\partial x_1},\frac{\partial D}{\partial x_2},\ldots,\frac{\partial D}{\partial x_m},\frac{\partial D}{\partial r_1},\frac{\partial D}{\partial r_2},\ldots,\frac{\partial D}{\partial r_{m-1}},\frac{\partial D}{\partial y_1},\frac{\partial D}{\partial y_2},\ldots,\frac{\partial D}{\partial y_{n-1}},\frac{\partial D}{\partial s_1},\frac{\partial D}{\partial s_2},\ldots,\frac{\partial D}{\partial s_{n-1}}=0,$$

we can obtain the following:

$$\begin{cases}
r_1^2x_1-2\int_0^{r_1}t\underline{u}(t)dt=0\\
(r_2^2-r_1^2)x_2-2\int_{r_1}^{r_2}t\underline{u}(t)dt=0\\
\vdots\\
(1-r_{m-1}^2)x_m-2\int_{r_{m-1}}^{1}t\underline{u}(t)dt=0\\
r_1(x_1-x_2)(x_1+x_2-2\underline{u}(r_1))=0\\
r_2(x_2-x_3)(x_2+x_3-2\underline{u}(r_2))=0\\
\vdots\\
r_{m-1}(x_{m-1}-x_m)(x_{m-1}+x_m-2\underline{u}(r_{m-1}))=0\\
s_1^2y_1-2\int_0^{s_1}t\overline{u}(t)dt=0\\
(s_2^2-s_1^2)y_2-2\int_{s_1}^{s_2}t\overline{u}(t)dt=0\\
\vdots\\
(1-s_{n-1}^2)y_n-2\int_{s_{n-1}}^{1}t\overline{u}(t)dt=0\\
s_1(y_1-y_2)(y_1+y_2-2\overline{u}(s_1))=0\\
s_2(y_2-y_3)(y_2+y_3-2\overline{u}(s_2))=0\\
\vdots\\
s_{n-1}(y_{n-1}-y_n)(y_{n-1}+y_n-2\overline{u}(s_{n-1}))=0
\end{cases}\tag{3}$$

From the equation (3), we can obtain that:

$$\begin{cases}
(r_2^2-r_1^2)\int_0^{r_1}t\underline{u}(t)dt+r_1^2\int_{r_1}^{r_2}t\underline{u}(t)dt-(r_2^2-r_1^2)r_1^2\underline{u}(r_1)=0\\
(r_3^2-r_2^2)\int_{r_1}^{r_2}t\underline{u}(t)dt+(r_2^2-r_1^2)\int_{r_2}^{r_3}t\underline{u}(t)dt-(r_3^2-r_2^2)(r_2^2-r_1^2)\underline{u}(r_2)=0\\
\vdots\\
(1-r_{m-1}^2)\int_{r_{m-2}}^{r_{m-1}}t\underline{u}(t)dt+(r_{m-1}^2-r_{m-2}^2)\int_{r_{m-1}}^{1}t\underline{u}(t)dt-(1-r_{m-1}^2)(r_{m-1}^2-r_{m-2}^2)\underline{u}(r_{m-1})=0\\
x_1=\frac{2\int_0^{r_1}t\underline{u}(t)dt}{r_1^2}\\
x_2=\frac{2\int_{r_1}^{r_2}t\underline{u}(t)dt}{r_2^2-r_1^2}\\
\vdots\\
x_m=\frac{2\int_{r_{m-1}}^{1}t\underline{u}(t)dt}{1-r_{m-1}^2}\\
(s_2^2-s_1^2)\int_0^{s_1}t\overline{u}(t)dt+s_1^2\int_{s_1}^{s_2}t\overline{u}(t)dt-(s_2^2-s_1^2)s_1^2\overline{u}(s_1)=0\\
(s_3^2-s_2^2)\int_{s_1}^{s_2}t\overline{u}(t)dt+(s_2^2-s_1^2)\int_{s_2}^{s_3}t\overline{u}(t)dt-(s_3^2-s_2^2)(s_2^2-s_1^2)\overline{u}(s_2)=0\\
\vdots\\
(1-s_{n-1}^2)\int_{s_{n-2}}^{s_{n-1}}t\overline{u}(t)dt+(s_{n-1}^2-s_{n-2}^2)\int_{s_{n-1}}^{1}t\overline{u}(t)dt-(1-s_{n-1}^2)(s_{n-1}^2-s_{n-2}^2)\overline{u}(s_{m-1})=0\\
y_1=\frac{2\int_0^{s_1}t\overline{u}(t)dt}{s_1^2}\\
y_2=\frac{2\int_{s_1}^{s_2}t\overline{u}(t)dt}{s_2^2-s_1^2}\\
\vdots\\
y_n=\frac{2\int_{s_{n-1}}^{1}t\overline{u}(t)dt}{1-s_{n-1}^2}
\end{cases}\tag{4}$$

Since $(r_1,r_2,\ldots,r_{m-1})=(\alpha_1,\alpha_2,\ldots,\alpha_{m-1})$ and $(s_1,s_2,\ldots,s_{n-1})=(\beta_1,\beta_2,\ldots,\beta_{n-1})$ are respectively the unique

solutions of equations (1) and (2) on (0 , 1), then we can get that

$$\begin{cases} r_1 = \alpha_1 \\ r_2 = \alpha_2 \\ \vdots \\ r_{m-1} = \alpha_{m-1} \\ x_1 = \frac{2\int_0^{\alpha_1} t\underline{u}(t)dt}{\alpha_1^2} (\text{denoted by } a_1) \\ x_2 = \frac{2\int_{\alpha_1}^{\alpha_2} t\underline{u}(t)dt}{\alpha_2^2-\alpha_1^2} (\text{denoted by } a_2) \\ \vdots \\ x_m = \frac{2\int_{\alpha_{m-1}}^{1} t\underline{u}(t)dt}{1-\alpha_{m-1}^2} (\text{denoted by } a_m) \\ s_1 = \beta_1 \\ s_2 = \beta_2 \\ \vdots \\ s_{n-1} = \beta_{n-1} \\ y_1 = \frac{2\int_0^{\beta_1} t\overline{u}(t)dt}{\beta_1^2} (\text{denoted by } b_1) \\ y_2 = \frac{2\int_{\beta_1}^{\beta_2} t\overline{u}(t)dt}{\beta_2^2-\beta_1^2} (\text{denoted by } b_2) \\ \vdots \\ y_n = \frac{2\int_{\beta_{n-1}}^{1} t\overline{u}(t)dt}{1-\beta_{n-1}^2} (\text{denoted by } b_n) \end{cases}$$

is the solution of the equation(4), i.e., the solution of the equation(3).

From the nondecreasing character of $\underline{u}(r)$ and the nonincreasing character of $\overline{u}(r)$ about r on [0,1] and $\underline{u}(1) < \overline{u}(1)$, the conditions of $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{m-1} < 1$ and $0 < \beta_1 < \beta_2 < \cdots < \beta_{n-1} < 1$, we achieve that $a_1 < a_2 < \cdots < a_m < b_n < b_{n-1} < \cdots < b_1$. Therefore, $\hat{u} = [(a_i, \alpha_i)_{i=1}^m, (b_j, \beta_j)_{j=1}^n]$ is a $m-n-$step type fuzzy number.

In the following, in order to show that $\hat{u}$ is the $I-$nearest $m-n-$step type fuzzy number approximation, we need to show that the Hessian matrix of the function $D(x_1, x_2, \ldots, x_m, r_1, r_2, \ldots, r_{m-1}, y_1, y_2, \ldots, y_n, s_1, s_2, \ldots, s_{n-1})$ at $(a_1, a_2, \ldots, a_m, \alpha_1, \alpha_2, \ldots, \alpha_{m-1}, b_1, b_2, \ldots, b_n, \beta_1, \beta_2, \ldots, \beta_{n-1})$ is positive definite.

From the expressions of the first derivative of $D(x_1, x_2, \ldots, x_m, r_1, r_2, \ldots, r_{m-1}, y_1, y_2, \ldots, y_n, s_1, s_2, \ldots, s_{n-1})$, we have that for $i = 1, 2, \ldots, m$

$$\frac{\partial^2 D}{\partial x_i \partial x_k} = \begin{cases} 2(r_i^2 - r_{i-1}^2), & i = k \\ 0, & i \neq k \end{cases},$$

and

$$\frac{\partial^2 D}{\partial x_1 \partial r_k} = \begin{cases} 4r_1(x_1 - \underline{u}(r_1)), & k = 1 \\ 0, & k = 2, 3, \ldots, m-1 \end{cases},$$

$$\frac{\partial^2 D}{\partial x_2 \partial r_k} = \begin{cases} 4r_1(\underline{u}(r_1) - x_2), & k = 1 \\ 4r_2(x_2 - \underline{u}(r_2)), & k = 2 \\ 0, & k = 3, 4, \ldots, m-1 \end{cases},$$

for $i = 3, 4, \ldots, m-1$,

$$\frac{\partial^2 D}{\partial x_i \partial r_k} = \begin{cases} 4r_{i-1}(\underline{u}(r_{i-1}) - x_i), & k = i-1 \\ 4r_i(x_i - \underline{u}(r_i)), & k = i \\ 0, & k = 1, 2, \ldots, i-2, i+1, \ldots, m-1 \end{cases},$$

$$\frac{\partial^2 D}{\partial x_m \partial r_k} = \begin{cases} 4r_{m-1}(\underline{u}(r_{m-1}) - x_m), & k = m-1 \\ 0, & k = 1, 2, \ldots, m-1 \end{cases},$$

and for $i = 1, 2, \cdots, m-1$,

$$\frac{\partial^2 D}{\partial r_i \partial r_k} = \begin{cases} 2(x_i - x_{i+1})[x_i + x_{i+1} - 2(\underline{u}(r_i) + r_i\underline{u}'(r_i))], & i = k \\ 0, & i \neq k \end{cases}.$$

Similarly, we have that for $j = 1, 2, \ldots, n$.

$$\frac{\partial^2 D}{\partial y_j \partial y_k} = \begin{cases} 2(s_j^2 - s_{j-1}^2), & j = k \\ 0, & j \neq k \end{cases},$$

and

$$\frac{\partial^2 D}{\partial y_1 \partial y_k} = \begin{cases} 4s_1(y_1 - \overline{u}(s_1)), & k = 1 \\ 0, & k = 2, 3, \ldots, n-1 \end{cases},$$

$$\frac{\partial^2 D}{\partial y_2 \partial s_k} = \begin{cases} 4s_1(\overline{u}(s_1) - y_2), & k = 1 \\ 4s_2(y_2 - \overline{u}(s_2)), & k = 2 \\ 0, & k = 3, 4, \ldots, n-1 \end{cases},$$

for $j = 3, 4, \cdots, n-1$,

$$\frac{\partial^2 D}{\partial y_j \partial s_k} = \begin{cases} 4s_{j-1}(\overline{u}(s_{j-1}) - y_j), & k = j-1 \\ 4s_j(y_j - \overline{u}(s_j)), & k = j \\ 0, & k = 1, 2, \ldots, j-2, j+1, \ldots, n-1 \end{cases},$$

$$\frac{\partial^2 D}{\partial y_n \partial s_k} = \begin{cases} 4s_{n-1}(\overline{u}(s_{n-1}) - y_n), & k = n-1 \\ 0, & k = 1, 2, \ldots, n-1 \end{cases},$$

and for $j = 1, 2, \ldots, n-1$,

$$\frac{\partial^2 D}{\partial s_j \partial s_k} = \begin{cases} 2(y_j - y_{j+1})[y_j + y_{j+1} - 2(\overline{u}(s_j) + s_j \overline{u}'(s_j))], & j = k \\ 0, & j \neq k \end{cases}.$$

In addition, for $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$,

$$\frac{\partial^2 D}{\partial x_i \partial y_j} = \frac{\partial^2 D}{\partial x_i \partial s_{j-1}} = \frac{\partial^2 D}{\partial r_{i-1} \partial y_j} = \frac{\partial^2 D}{\partial r_{i-1} \partial s_{j-1}} = \frac{\partial^2 D}{\partial y_j \partial x_i} = \frac{\partial^2 D}{\partial y_j \partial r_{i-1}} = \frac{\partial^2 D}{\partial s_{j-1} \partial x_i} = \frac{\partial^2 D}{\partial s_{j-1} \partial r_{i-1}} = 0.$$

So we can see that the Hessian Matrix H of the function $D(x_1, x_2, \ldots, x_m, r_1, r_2, \ldots, r_{m-1}, y_1, y_2, \ldots, y_n, s_1, s_2, \ldots, s_{n-1})$ at $(x_1, x_2, \ldots, x_m, r_1, r_2, \ldots, r_{m-1}, y_1, y_2, \ldots, y_n, s_1, s_2, \ldots, s_{n-1})$ $=(a_1, a_2, \ldots, a_m, \alpha_1, \alpha_2, \ldots, \alpha_{m-1}, b_1, b_2, \ldots, b_n, \beta_1, \beta_2, \ldots, \beta_{n-1})$ as follows:

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}_{2(m+n-1) \times 2(m+n-1)}$$

where H_1 and H_2 are real symmetric matrices:

$$H_1 = \begin{pmatrix} \underline{\Delta}_1 & \underline{M} \\ \underline{M}^T & \underline{\Delta}_2 \end{pmatrix}, \text{ and } H_2 = \begin{pmatrix} \overline{\Delta}_1 & \overline{M} \\ \overline{M}^T & \overline{\Delta}_2 \end{pmatrix},$$

and $\underline{\Delta}_1 = diag(2(\alpha_1^2 - \alpha_0^2), 2(\alpha_2^2 - \alpha_1^2), \ldots, 2(\alpha_m^2 - \alpha_{m-1}^2))$, $\overline{\Delta}_1 = diag(2(\beta_1^2 - \beta_0^2), 2(\beta_2^2 - \beta_1^2), \ldots, 2(\beta_n^2 - \beta_{n-1}^2))$, $\underline{\Delta}_2 = diag(c_1, c_2, \ldots, c_{m-1})$ (where $c_i = 2(a_i - a_{i+1})\,[a_i + a_{i+1} - 2\,(\underline{u}(\alpha_i) + \alpha_i \underline{u}'(\alpha_i))]$, $i = 1, 2, \ldots, m-1$), $\overline{\Delta}_2 = diag(d_1, d_2, \ldots, d_{n-1})$ (where d_j=$2(b_j - b_{j+1})\,[b_j$+b_{j+1}–$2\,(\overline{u}(\beta_j)$+$\beta_j \overline{u}'(\beta_j))]$, $j = 1, 2, \ldots, n-1$),

$$\underline{M} = \begin{pmatrix} 4\alpha_1(a_1 - \underline{u}(\alpha_1)) & 0 & \cdots & 0 & 0 \\ 4\alpha_1(\underline{u}(\alpha_1) - a_2) & 4\alpha_2(a_2 - \underline{u}(\alpha_2)) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 4\alpha_{m-2}(\underline{u}(\alpha_{m-2}) - a_{m-1}) & 4\alpha_{m-1}(a_{m-1} - \underline{u}(\alpha_{m-1})) \\ 0 & 0 & \cdots & 0 & 4\alpha_{m-1}(\underline{u}(\alpha_{m-1}) - a_m) \end{pmatrix}_{m \times (m-1)},$$

and

$$\overline{M} = \begin{pmatrix} 4\beta_1(b_1 - \overline{u}(\beta_1)) & 0 & \cdots & 0 & 0 \\ 4\beta_1(\overline{u}(\beta_1) - b_2) & 4\beta_2(b_2 - \overline{u}(\beta_2)) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 4\beta_{n-2}(\overline{u}(\beta_{n-2}) - b_{n-1}) & 4\beta_{n-1}(b_{n-1} - \overline{u}(\beta_{n-1})) \\ 0 & 0 & \cdots & 0 & 4\beta_{n-1}(\overline{u}(\beta_{n-1}) - b_n) \end{pmatrix}_{n \times (n-1)}.$$

and $\underline{M}^T$ is the transposition of $\underline{M}$, $\overline{M}^T$ is the transposition of $\overline{M}$.

Since $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{m-1} < 1$, then for $i = 1, 2, \ldots, m$, we know that $\alpha_i^2 - \alpha_{i-1}^2 > 0$. In the same way, since $0 < \beta_1 < \beta_2 < \cdots < \beta_{n-1} < 1$, then for $j = 1, 2, \ldots, n$, we can know that $\beta_j^2 - \beta_{j-1}^2 > 0$. Therefore $\underline{\Delta}_1$ and $\overline{\Delta}_1$ are positive definite.

From the positive definition of $\underline{\Delta}_1$, $\overline{\Delta}_1$, $\underline{\Delta}_2 - \underline{M}^T\underline{\Delta}_1^{-1}\underline{M}$ and $\overline{\Delta}_2 - \overline{M}^T\overline{\Delta}_1^{-1}\overline{M}$, we can know that H_1 and H_2 are positive definite by the theorem 1. And then we can obtain that the block diagonal matrix H is positive definite by the corollary 1.

So the unique stationary point $(a_1, a_2, \ldots, a_m, \alpha_1, \alpha_2, \ldots, \alpha_{m-1}, b_1, b_2, \ldots, b_n, \beta_1, \beta_2, \ldots, \beta_{n-1})$ of function $D(x_1, x_2, \ldots, x_m, r_1, r_2, \ldots, r_{m-1}, y_1, y_2, \ldots, y_n, s_1, s_2, \ldots, s_{n-1})$ is the minimum of

$$D(x_1, x_2, \ldots, x_m, r_1, r_2, \ldots, r_{m-1}, y_1, y_2, \ldots, y_n, s_1, s_2, \ldots, s_{n-1}).$$

Thus, $\hat{u} = [(a_i, \alpha_i)_{i=1}^m, (b_j, \beta_j)_{j=1}^n]$, is the $I-$nearest $m-n-$step type fuzzy number approximation of u based on the weighted distance d^*. □

Remark 3.6. *Let m, n be two positive integers and $u \in D(E)$, if $\hat{u}_m = [(a_i, \alpha_i)_{i=1}^m, (b_j, \beta_j)_{j=1}^m]$ and $\hat{u}_n = [(p_i, \gamma_i)_{i=1}^n, (q_j, \lambda_j)_{j=1}^n]$ are respectively the $I-$nearest $m-$STFN approximation and $I-$nearest $n-$STFN approximation of u, then we can directly obtain the $I-$nearest $m-n-$STFN approximation $\hat{u} = [(a_i, \alpha_i)_{i=1}^m, (q_j, \lambda_j)_{j=1}^n]$ of u. Likewise, the $I-$nearest $n-m-$STFN approximation $\hat{u} = [(p_i, \gamma_i)_{i=1}^n, (b_j, \beta_j)_{j=1}^m]$ of u.*

In the following, a recursion theorem will be established to obtain the $II-$nearest $m-n-$step type fuzzy number approximation maintaining the kernel and support set with respect to d^*.

Theorem 3.7. *Let m, n be two positive integers and $u \in D(E)$, $(r_1, r_2, \ldots, r_{m-1}) = (\alpha_1, \alpha_2, \ldots, \alpha_{m-1})$ and $(s_1, s_2, \ldots, s_{n-1}) = (\beta_1, \beta_2, \ldots, \beta_{n-1})$ are respectively unique solutions satisfying $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{m-1} < 1$ and $0 < \beta_1 < \beta_2 < \cdots < \beta_{n-1} < 1$ of the following Equations*

$$\begin{cases} (r_2^2 - r_1^2)\int_0^{r_1} t\underline{u}(t)dt + r_1^2\int_{r_1}^{r_2} t\underline{u}(t)dt - (r_2^2 - r_1^2)r_1^2\underline{u}(r_1) = 0 \\ (r_3^2 - r_2^2)\int_{r_1}^{r_2} t\underline{u}(t)dt + (r_2^2 - r_1^2)\int_{r_2}^{r_3} t\underline{u}(t)dt - (r_3^2 - r_2^2)(r_2^2 - r_1^2)\underline{u}(r_2) = 0 \\ \vdots \\ 2\int_{r_{m-2}}^{r_{m-1}} t\underline{u}(t)dt + (r_{m-1}^2 - r_{m-2}^2)(\underline{u}(1) - 2\underline{u}(r_{m-1})) = 0 \end{cases} \tag{5}$$

$$\begin{cases} (s_2^2 - s_1^2)\int_0^{s_1} t\overline{u}(t)dt + s_1^2\int_{s_1}^{s_2} t\overline{u}(t)dt - (s_2^2 - s_1^2)s_1^2\overline{u}(s_1) = 0 \\ (s_3^2 - s_2^2)\int_{s_1}^{s_2} t\overline{u}(t)dt + (s_2^2 - s_1^2)\int_{s_2}^{s_3} t\overline{u}(t)dt - (s_3^2 - s_2^2)(s_2^2 - s_1^2)\overline{u}(s_2) = 0 \\ \vdots \\ 2\int_{s_{n-2}}^{s_{n-1}} t\overline{u}(t)dt + (s_{n-1}^2 - s_{n-2}^2)(\overline{u}(1) - 2\overline{u}(s_{n-1})) = 0 \end{cases} \tag{6}$$

and $a_i = \frac{2\int_{\alpha_{i-1}}^{\alpha_i} t\underline{u}(t)dt}{\alpha_i^2 - \alpha_{i-1}^2}$ $(i = 1, 2, \ldots, m-1$, where $\alpha_0 = 0$ $)$, $b_j = \frac{2\int_{\beta_{j-1}}^{\beta_j} t\overline{u}(t)dt}{\beta_j^2 - \beta_{j-1}^2}$ $(j = 1, 2, \ldots, n-1$, where $\beta_0 = 0$ $)$, and $\underline{\Delta}_1 = diag(2(\alpha_1^2 - \alpha_0^2), 2(\alpha_2^2 - \alpha_1^2), \ldots, 2(\alpha_{m-1}^2 - \alpha_{m-2}^2))$, $\overline{\Delta}_1 = diag(2(\beta_1^2 - \beta_0^2), 2(\beta_2^2 - \beta_1^2), \ldots, 2(\beta_{n-1}^2 - \beta_{n-2}^2))$, $\underline{\Delta}_2 = diag(c_1, c_2, \ldots, c_{m-1})$ (where $c_i = 2(a_i - a_{i+1})[a_i + a_{i+1} - 2(\underline{u}(\alpha_i) + \alpha_i\underline{u}'(\alpha_i))]$, $i = 1, 2, \ldots, m-2$, and $c_{m-1} = 2(a_{m-1} - \underline{u}(1))[a_{m-1} + \underline{u}(1) - 2(\underline{u}(\alpha_{m-1}) + \alpha_{m-1}\underline{u}'(\alpha_{m-1}))]$), $\overline{\Delta}_2 = diag(d_1, d_2, \ldots, d_{n-1})$ (where $d_j = 2(b_j - b_{j+1})[b_j + b_{j+1} - 2(\overline{u}(\beta_j) + \beta_j\overline{u}'(\beta_j))]$, $j = 1, 2, \ldots, n-2$, and $d_{n-1} = 2(b_{n-1} - \overline{u}(1))[b_{n-1} + \overline{u}(1) - 2(\underline{u}(\beta_{n-1}) + \beta_{n-1}\underline{u}'(\beta_{n-1}))]$),

$$\underline{M} = \begin{pmatrix} 4\alpha_1(a_1 - \underline{u}(\alpha_1)) & 0 & \cdots & 0 & 0 \\ 4\alpha_1(\underline{u}(\alpha_1) - a_2) & 4\alpha_2(a_2 - \underline{u}(\alpha_2)) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 4\alpha_{m-2}(\underline{u}(\alpha_{m-2}) - a_{m-1}) & 4\alpha_{m-1}(a_{m-1} - \underline{u}(\alpha_{m-1})) \end{pmatrix}_{(m-1)\times(m-1)},$$

and

$$\overline{M} = \begin{pmatrix} 4\beta_1(b_1 - \overline{u}(\beta_1)) & 0 & \cdots & 0 & 0 \\ 4\beta_1(\overline{u}(\beta_1) - b_2) & 4\beta_2(b_2 - \overline{u}(\beta_2)) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 4\beta_{n-2}(\overline{u}(\beta_{n-2}) - b_{n-1}) & 4\beta_{n-1}(b_{n-1} - \overline{u}(\beta_{n-1})) \end{pmatrix}_{(n-1)\times(n-1)}.$$

If $\underline{\Delta}_2 - \underline{M}^T\underline{\Delta}_1^{-1}\underline{M}$ *and* $\overline{\Delta}_2 - \overline{M}^T\overline{\Delta}_1^{-1}\overline{M}$ *are positive definite, then*

$$\hat{v} = \left[\begin{pmatrix} a_1 & \alpha_1 \\ a_2 & \alpha_2 \\ \vdots & \vdots \\ a_{m-1} & \alpha_{m-1} \\ \underline{u}(1) & 1 \end{pmatrix}, \begin{pmatrix} b_1 & \beta_1 \\ b_2 & \beta_2 \\ \vdots & \vdots \\ b_{n-1} & \beta_{n-1} \\ \overline{u}(1) & 1 \end{pmatrix}\right],$$

is the $II-$*nearest* $m-n-$*step type fuzzy number approximation of* u*.*

Proof. The proof process of the theorem will be omitted laconically due to the methodological similarity with the proof of Theorem 3.7. □

Remark 3.8. *Let* m*,* n *be two positive integers and* $u \in D(E)$*, if* $\hat{v}_m = [(a_i,\alpha_i)_{i=1}^m,(b_j,\beta_j)_{j=1}^m]$ *(where* $a_m = \underline{u}(1)$ *and* $b_m = \overline{u}(1)$*) and* $\hat{v}_n = [(p_i,\gamma_i)_{i=1}^n,(q_j,\lambda_j)_{j=1}^n]$ *(where* $p_n = \underline{u}(1)$ *and* $q_n = \overline{u}(1)$*) are respectively the* $II-$*nearest* $m-$*STFN approximation and* $II-$*nearest* $n-$*STFN approximation of* u*, then we can directly obtain the* $II-$*nearest* $m-n-$*STFN approximation* $\hat{v} = [(a_i,\alpha_i)_{i=1}^m,(q_j,\lambda_j)_{j=1}^n]$ *(where* $a_m = \underline{u}(1)$ *and* $q_n = \overline{u}(1)$*) of* u*. Likewise, the* $II-$*nearest* $n-m-$*STFN approximation* $\hat{v} = [(p_i,\gamma_i)_{i=1}^n,(b_j,\beta_j)_{j=1}^m]$ *(where* $p_n = \underline{u}(1)$ *and* $b_m = \overline{u}(1)$*) of* u*.*

4 Application example

In the following, a specific example is given to show the application of the methods proposed in this paper. In this example, the approximation results of $I-$nearest and $II-$nearest Multi-step type fuzzy number are separately presented, and comparisons between the approximation results are acquired from the images.

Example 4.1. *Suppose the fuzzy number* u *is defined as follows:*

$$u(x) = \begin{cases} (x-1)^2, & x \in [1,2) \\ 1, & x \in [2,3] \\ \sqrt{4-x}, & x \in (3,4] \\ 0, & x \notin [1,4] \end{cases}.$$

then we have $\underline{u}(t) = 1+\sqrt{t}$*,* $\underline{u}'(t) = \frac{1}{2\sqrt{t}}$*,* $\overline{u}(t) = 4-t^2$*,* $\overline{u}'(t) = -2t$*,* $t \in (0,1]$*.*

(1) • *Consider the* $I-$*nearest* $3-$*STFN approximation of* u*. By the equations (1) and (2), we can get that* $(r_1,r_2) = (0.380,0.690)$*,* $(s_1,s_2) = (0.577,0.816)$*. Then we can calculate that* $a_1 = 1.493$*,* $a_2 = 1.739$*,* $a_3 = 1.923$*;* $b_1 = 3.834$*,* $b_2 = 3.501$*,* $b_3 = 3.167$*. Thus by Theorem 3, we see that*

$$\hat{u} = \left[\begin{pmatrix} 1.493 & 0.38 \\ 1.739 & 0.69 \\ 1.923 & 1 \end{pmatrix}, \begin{pmatrix} 3.834 & 0.577 \\ 3.501 & 0.816 \\ 3.167 & 1 \end{pmatrix}\right],$$

is the $I-$*nearest* $3-$*step type fuzzy number approximation of* u*.*

• *Consider the* $II-$*nearest* $3-$*STFN approximation of* u*. By the equations (5) and (6), we can get that* $(r_1,r_2) = (0.446,0.811)$*,* $(s_1,s_2) = (0.632,0.894)$*. Then we can calculate that* $a_1 = 1.534$*,* $a_2 = 1.801$*,* $a_3 = \underline{u}(1) = 2.000$*;* $b_1 = 3.800$*,* $b_2 = 3.401$*,* $b_3 = \overline{u}(1) = 3.000$*. Thus by Theorem 4, we see that*

$$\hat{v} = \left[\begin{pmatrix} 1.534 & 0.446 \\ 1.801 & 0.811 \\ 2.000 & 1 \end{pmatrix}, \begin{pmatrix} 3.800 & 0.632 \\ 3.401 & 0.894 \\ 3.000 & 1 \end{pmatrix}\right],$$

is the $II-$*nearest* $3-$*step type fuzzy number approximation of* u*.*

(2) • *Consider the* $I-$*nearest* $4-$*STFN approximation of* u*. By the equations (1) and (2), we can get that* $(r_1,r_2,r_3) = (0.291,0.529,0.765)$*,* $(s_1,s_2,s_3) = (0.500,0.707,0.866)$*. Then we can calculate that* $a_1 = 1.432$*,* $a_2 = 1.647$*,* $a_3 = 1.808$*,* $a_4 = 1.941$*;* $b_1 = 3.875$*,* $b_2 = 3.625$*,* $b_3 = 3.375$*,* $b_4 = 3.125$*. Thus by Theorem 3.7, we see that*

$$\hat{u} = \left[\begin{pmatrix} 1.432 & 0.291 \\ 1.647 & 0.529 \\ 1.808 & 0.765 \\ 1.941 & 1 \end{pmatrix}, \begin{pmatrix} 3.875 & 0.500 \\ 3.625 & 0.707 \\ 3.375 & 0.866 \\ 3.125 & 1 \end{pmatrix}\right],$$

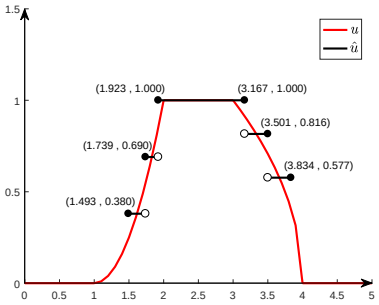


Figure 1: I-nearest 3−STFN approximation of u

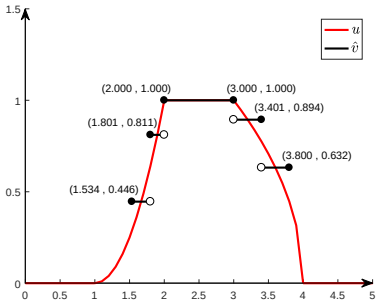


Figure 2: II-nearest 3−STFN approximation of u

is the $I-$nearest $4-$step type fuzzy number approximation of u.

• *Consider the $II-$nearest $4-$STFN approximation of u. By the equations (5) and (6), we can get that* $(r_1, r_2, r_3) = (0.328, 0.597, 0.863)$, $(s_1, s_2, s_3) = (0.535, 0.756, 0.926)$. *Then we can calculate that* $a_1 = 1.458$, $a_2 = 1.687$, $a_3 = 1.858$, $a_4 = \underline{u}(1) = 2.000$; $b_1 = 3.857$, $b_2 = 3.571$, $b_3 = 3.285$, $b_4 = \overline{u}(1) = 3.000$. *Thus by Theorem 4, we see that*

$$\hat{v} = \left[\begin{pmatrix} 1.458 & 0.328 \\ 1.687 & 0.597 \\ 1.858 & 0.863 \\ 2.000 & 1 \end{pmatrix}, \begin{pmatrix} 3.857 & 0.535 \\ 3.571 & 0.756 \\ 3.285 & 0.926 \\ 3.000 & 1 \end{pmatrix} \right],$$

is the $II-$nearest $4-$step type fuzzy number approximation of u.

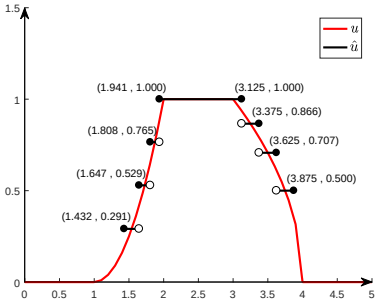


Figure 3: I-nearest 4−STFN approximation of u

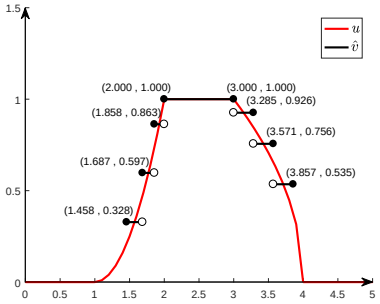


Figure 4: II-nearest 4−STFN approximation of u

(3) • *Consider the $I-$nearest $3-4-$STFN approximation of u. Thus by Theorem 3 and the outcome of example, we directly obtain*

$$\hat{u} = \left[\begin{pmatrix} 1.493 & 0.38 \\ 1.739 & 0.69 \\ 1.923 & 1 \end{pmatrix}, \begin{pmatrix} 3.875 & 0.500 \\ 3.625 & 0.707 \\ 3.375 & 0.866 \\ 3.125 & 1 \end{pmatrix} \right],$$

is the $I-$nearest $3-4-$step type fuzzy number approximation of u.

• *Consider the $II-$nearest $3-4-$STFN approximation of u. Thus by Theorem 4 and the outcome of example, we directly obtain*

$$\hat{v} = \left[\begin{pmatrix} 1.534 & 0.446 \\ 1.801 & 0.811 \\ 2.000 & 1 \end{pmatrix}, \begin{pmatrix} 3.857 & 0.535 \\ 3.571 & 0.756 \\ 3.285 & 0.926 \\ 3.000 & 1 \end{pmatrix} \right],$$

is the $II-$nearest $3-4-$step type fuzzy number approximation of u.

Remark 4.2. *From the approximation results (see the comparison between Figure 1 and Figure 2, Figure 3 and Figure 4) of this example, when using Multi-step type fuzzy numbers to approximate general continuous fuzzy numbers under weighted distance, $I-$nearest $m-n-$STFN approximation is better than $II-$nearest $m-n-$STFN approximation.*

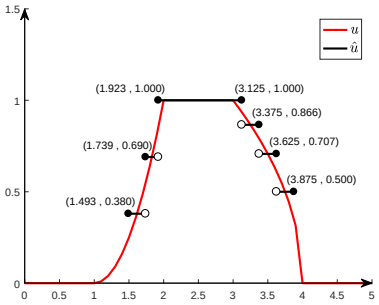


Figure 5: I-nearest $3-4-$STFN approximation of u

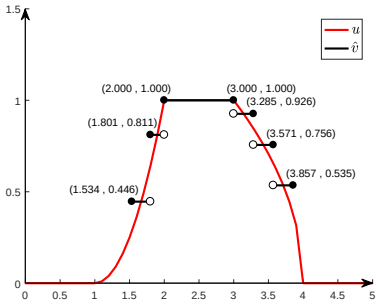


Figure 6: II-nearest $3-4-$STFN approximation of u

Remark 4.3. *In the example, we respectively received the $I-$nearest and $II-$nearest $3-4-$STFN approximation(see the figure 5 and figure 6). Similarly, we can also get $4-3-$STFN approximations of $I-$nearest and $II-$nearest which not intend to list them in detail here. Besides, from the figure 1 and figure 3, we can find that the more unknown nodes of left and right horizontal values, the more obvious the $I-$nearest STFN approximation effect. The same goes for $II-$nearest STFN approximation from the figure 2 and figure 4. That is to say, the results of theorem 3 and theorem 4 can be used successively so as to obtain the best approximation function in the application field of fuzzy number approximation.*

5 Conclusions

In this paper, for using $m-n-$step type fuzzy number to approximate a given continue fuzzy number, we introduced the concepts of $I-$nearest approximation (Definition 2) and $II-$nearest approximation (Definition 3), and pointed out the relationship (Theorem 2) between them. Then we established the methods (Theorems 3 and 4) of solving its $I-$nearest $m-n-$step type fuzzy number approximation and $II-$nearest $m-n-$step type fuzzy number approximation, respectively. At last, we used a specific example (Example 1) to show the effectiveness and usability of the methods set up in this paper. In the future, we can study the problem of using Multi-steps type fuzzy number (i.e., $m-n-$step type fuzzy number) to approximate a given continue fuzzy number in high dimensions.

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