Iranian Journal of Fuzzy Systems

Volume 20, Number 3, (2023), pp. 191-203

Original paper

University of Sistan and Baluchestan

Type-2 lattice-valued preorders: A common framework of lattice-valued preorders and various kinds of metrics

W. Yao[1], G. X. Zhang[2] and Y. Shi[3]

[1,2,3] *School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing, China*

yaowei0516@163.com, hbgxzhang@163.com, shiyi7353@126.com

Abstract

The aim of this paper is to introduce a concept of type-2 L-preorders for L being a complete residuated lattice. It can be considered as a common framework of L-preorders and hemimetrics, and also contains various kinds of fuzzy metrics, including Morsi fuzzy metrics, KM-fuzzy metrics and modular metrics, as natural examples. It is shown that the category of L-preordered sets can be reflectively and coreflectively embedded in that of type-2 L-preordered sets. A type-2 L-preorder can be supplied as different models for further study.

Keywords: Fuzzy relation, L-preorder, type-2 L-preorder, Morsi fuzzy metric, GV-fuzzy metric, modular metric, metric-type topology.

1 Introduction

First of all, let us talk about two concepts: *distance function* and *fuzzy equivalence relation*. Mathematically speaking, a distance function on a set X usually means a mapping $d : X \times X \longrightarrow [0, +\infty)$; and a fuzzy equivalence relation on a set X is a mapping $R : X \times X \longrightarrow [0, 1]$. If we have a normalized data set and use some formulas to construct a distance function and a fuzzy equivalence relation, then for $x, y \in X$, $d(x, y)$ means how far y is from x, while $R(x, y)$ means how close y is to x. Temporally ignoring the difference between their codomains $[0, +\infty)$ and $[0, 1]$, a distance function and a fuzzy equivalence relation have opposite meanings, that is, if the distance between two points is relatively smaller, then they are close to each other and should have more similarities between data; dually, if the distance is relatively larger, then they are far away from each other and should be less related. This idea has been used to study enriched category approach to generalized metric spaces [28] and the hemimetric-based fuzzy rough set models [45, 47].

A distance function, under the standard name *metric*, is a function $d : X \times X \longrightarrow [0, \infty)$ satisfying: $\forall x, y, z \in X$, (D1) $d(x, x) = 0$; (D2) $d(x, y) = d(y, x) = 0$ implies $x = y$; (D3) $d(x, y) = d(y, x)$; (D4) $d(x, z) \leq d(x, y) + d(y, z)$. From the theoretical aspect, there are also lots of weak versions of the standard metrics; and also from the viewpoint of applications, we sometimes do not need/have all the axioms (D1–D4). For example, (1) the function of the shortest path of a directed weighted graph is a function only satisfying (D1), (D2) and (D4) [13]; (2) the partial metric in theoretical computer science allows $d(x, y) = 0$ for some distinct points x, y and allows $d(x, x) \neq 0$ for some $x \in X$ [30]; (3) a useful class of metrics in domain theory is the hemimetric [14] (also called generalized metrics [23, 28], or pseudo-quasi-metrics [34]) only satisfying the conditions (D1) and (D4) .

The axioms of a fuzzy equivalence relation [9, 10] (or called a fuzzy similarity [32], or a global fuzzy equality [19]) $R : X \times X \longrightarrow [0, 1]$ are (FR1) $R(x, x) = 1$; (FR2) $R(x, y) = R(y, x)$; (FR3) $R(x, y) \otimes R(y, z) \leq R(x, z)$, where $\otimes$ is a left-continuous t-norm on $[0, 1]$. The symmetry condition (FR2) is a very natural at theoretical level, while for some applications, when we prove something, we just need to make it very clear that which of $R(x, y)$ and $R(y, x)$ is needed in process, and then we can keep going without the help of (FR2) (e.g., [47]). A fuzzy relation with (FR1,FR3) is always called a fuzzy preorder [2].

Corresponding Author: W. Yao

Received: April 2022; Revised: October 2022; Accepted: February 2023.

https://doi.org/10.22111/IJFS.2023.7647

Since both the pairs $([0,\infty]^{op},+)$ and $([0,1],\otimes)$ (if $\otimes$ is a left-continuous t-norm) are complete residuated lattices, we can replace the codomain of hemimetrics and that of fuzzy preorders by a complete residuated lattice $(L,*)$, then get a concept of the so-called L-preorder $e: X\times X\longrightarrow L$. Conditions (D1) and (FR1) are changed to (E1) $e(x,x)=1$; and those Conditions (D4) and (FR3) are changed to (E2) $e(x,y)*e(y,z)\leq e(x,z)$. Thus a complete residuated lattice valued preorder is a common generalization of hemimetrics and fuzzy preorder and have become a hot topic in the community of fuzzy sets [2], fuzzy order and fuzzy domain [4, 27, 29, 44, 46], fuzzy logic [3], and enriched category theory [23, 28].

As for fuzzification of metrics, the first notable breakthrough was done by Menger in 1942 by introducing the notion of statistical metrics using cumulative distribution functions [31]. Later Schweizer and Sklar developed the notion of statistical metrics with a new name probabilistic metrics [35, 36]. In 1975, Kramosil and Michálek [25] proposed the concept of KM-fuzzy metrics by modifying Menger's definition in order to deal with fuzzy quantities instead of probabilistic quantities. In 1994, George and Veeramani [12] modified the original definition of fuzzy metrics in order to make it more appropriate to deal with topological issues. In 1988, Morsi presented a kind of fuzzy metrics under a dual manner of KM-fuzzy metrics by using Hutton's fuzzy real numbers [33]. Morsi's fuzzy metrics are recently generalized to $\oplus$-based fuzzy pseudo-metrics in [18] for $\oplus$ being an s-norm on $[0,1]$. Since then, fruitful achievements have been gotten for different kinds of fuzzy metrics, such as topological structures, completion problems and fixed point properties [15, 16, 17, 38, 39, 43].

In Zadeh's opinion [33], fuzzy sets have different types. The traditional fuzzy set is of type-1, and those having membership functions as type-1 fuzzy sets are called type-2 fuzzy sets. The advantage of type-2 fuzzy sets is that they are helpful in some cases where it is difficult to find the exact membership functions for a fuzzy sets. As we know, fuzzy relations play an important role in the type-1 fuzzy set theory and applications [1, 7, 40].

Accordingly we need a concept of type-2 fuzzy relations in the type-2 fuzzy set theory and applications. The concept of type-2 fuzzy relations were introduced in [22]. The properties of type-2 fuzzy relations, especially the composition problem of type-2 fuzzy relations was discussed in [20]. Type-2 fuzzy rough sets based on extended t-norms with respect to type-2 fuzzy relations from both constructive and axiomatic approaches were studied in [42]. The notion of type-2 fuzzy G-tolerance relations, G-equivalence relations and G-congruence on a groupoid G were introduced and studied in [11, 37]. From the viewpoint of the form, a type-2 L-preorder is a type-2 L-relation satisfying a kind of self-reflectivity and a kind of transitivity.

The aim of this paper is, for a complete residuated lattice L, to introduce a concept of type-2 L-preorders, which can be considered as a common framework of L-preorders and hemimetrics, and also contains various kinds of fuzzy metrics, including Morsi fuzzy metrics, KM-fuzzy metrics and modular metrics, as natural examples.

This paper is organized as follows: In Section 2, we recall some basic concepts and results mainly on complete residuated lattices. In Section 3, we introduce a concept of type-2 L-preorders and study its relation with L-preorders. It is proved that, the category of L-preordered sets can be embedded in that of type-2 L-preordered sets as a reflective and coreflective subcategory. In Section 4, we show that hemimetrics, Morsi fuzzy metrics, KM-fuzzy metrics and modular metrics can be considered as examples of type-2 L-preorders. In the last section, we make some attempts to define the metric-type topology, intrinsic topology, rough approximation operators and convexity structures.

2 Preliminaries

Definition 2.1. [24] *A residuated lattice is an algebra* $(L;\wedge,\vee,*,\rightarrow,0,1)$ (*sometimes written as* $(L,*)$ *of type* $(2,2,2,2,0,0)$ *such that:*

(1) $(L;\wedge,\vee,0,1)$ *is a bounded lattice with* $0,1$ *as the least and the greatest elements, respectively;*

(2) $(L,*)$ *is a commutative monoid with the identity* 1 *and* $*$ *is isotone at both variables;*

(3) $a*b\leq c$ *iff* $a\leq b\rightarrow c$ *holds for all* $a,b,c\in L$.

A residuated lattice is called complete if the underlying lattice is complete. A *Heyting algebra* is a residuated lattice with $*=\wedge$. It is well-known that Heyting algebra is the unique residuated lattice such that $*$ is idempotent (i.e., $a*a=a\ \forall a\in L$).

Properties of complete residuated lattices are familiar with researchers in this field and can be found in lots of papers and monographs.

Example 2.2. [24] (1) *A canonical example of complete residuated lattices is the unit interval* $[0,1]$ *with a left-continuous triangular norm (t-norm for short). A t-norm is a binary operation* $*$ *on the unit interval* $[0,1]$ *such that* $([0,1],*,1)$ *is a commutative monoid and for all* $x,y,z\in[0,1]$, $x*y\leq x*z$ *whenever* $y\leq z$. *A t-norm called left-continuous if* $x*\sup_i y_i=\sup_i(x*y_i)$ *for all* $x\in[0,1]$ *and* $\{y_i\mid i\in I\}\subseteq[0,1]$. *The concept of t-norms can be generalized to bounded lattices.*

There are lots of continuous t-norms, for example,

(i) *The minimum t-norm* $\wedge$*:* $a \wedge b = \min\{a, b\}$*;*

(ii) *The product t-norm* $*_p$*:* $a *_p b = a \cdot b$*;*

(iii) *The Lukasiewicz t-norm* $*_{Lu}$*:* $a *_{Lu} b = \max\{a + b - 1, 0\}$*;*

(iv) *The Hamacher t-norm*

$$a *_H b = \begin{cases} \frac{ab}{a+b-ab}, & (a, b) \neq (0, 0); \\ 0, & (a, b) = (0, 0). \end{cases}$$

(3) *Let* $[0, \infty]$ *be the set of extended non-negative real numbers, where* $a \leq \infty$ *and* $a + \infty = \infty$, $0 * \infty = 0$ *and* $b \cdot \infty = \infty$ *for all* $a \in [0, \infty]$ *and* $b \in (0, \infty]$*. Let* $L = [0, \infty]^{op}$ *be the opposite poset of* $[0, \infty]$*, that is,* $a \leq b$ *iff* $b \leq_L a$ $(\forall a, b \in [0, \infty])$*. The pair* $L = ([0, \infty]^{op}, +)$ *becomes a complete residuated lattice.*

Definition 2.3. [8] *An s-norm on a bounded lattice* L *is an operation* $\oplus$ *such that* $\oplus$ *is a t-norm on* L^{op}*. If there is a negation* $'$ *on* L*, then for an s-norm* $\oplus$*,* $a \otimes b = [a' \oplus b']'$ *defines a t-norm on* L*; vice versa, for a t-norm* $*$*,* $a \oplus b = [a' * b']'$ *defines an s-norm on* L*. For example, the Lukasiewicz s-norm is given by* $a \oplus_{Lu} b = \min\{a + b, 1\}$*; the Hamacher s-norm is given by*

$$a \oplus_H b = \begin{cases} \frac{a+b-2ab}{1-ab}, & (a, b) \neq (1, 1); \\ 1, & (a, b) = (1, 1). \end{cases}$$

The following propositions are easy to be verified, which will be used in the rest of this paper.

Proposition 2.4. *For all* $a, b, c \in [0, 1]$*,*

(1) $a \leq b + c$ *iff* $a \leq b \oplus_{Lu} c$

(2) $a *_H b \leq c$ *iff* $\frac{1-c}{c} \leq \frac{1-a}{a} + \frac{1-b}{b}$ $(a, b, c \neq 0)$.

Proposition 2.5. *For an s-norm* $\oplus$ *on* $[0, 1]$*, the following conditions are equivalent:*

(1) $c \leq a \oplus b$ *implies* $\frac{c}{1-c} \leq \frac{a}{1-a} + \frac{b}{1-b}$ $(\forall a, b, c \in [0, 1))$*;*

(2) $\oplus \leq \oplus_H$.

3 Type-2 L-preorders and their relations with L-preorders

In this section, we will introduce a concept of type-2 L-preorders and study their relation with L-preorders.

Definition 3.1. [2] *Let* X *be a nonempty set. A mapping* $e : X \times X \longrightarrow L$ *is called an* L*-preorder, and the pair* (X, e) *is called an* L*-preordered set, if*

(E1) $e(x, x) = 1$ $(\forall x \in X)$*;*

(E2) $e(x, y) * e(y, z) \leq e(x, z)$ $(\forall x, y, z \in X)$.

The value $e(x, y)$ can be interpreted as the degree of $x \leq y$. In some papers on the theory of many-valued orders, an L-preorder is also called an L-category [28, 23, 27] or an L-metric [21].

Example 3.2. (1) *Define* $R_L, D_L : L \times L \longrightarrow L$ *by:* $\forall a, b \in L$*,*

$$R_L(a, b) = a \to b,$$

$$D_L(a, b) = (a \to b) \wedge (b \to a).$$

Then both R_L, D_L *are* L*-preorders on* L*.*

(2) *Let* X *be a nonempty set and let* L^X *be the family of all* L*-subsets (i.e., mappings from* X *to* L*) of* X*. Define* R_X, $D_X : L^X \times L^X \longrightarrow L$ *by:* $\forall A, B \in L^X$*,*

$$R_X(A, B) = \bigwedge_{x \in X} R_L(A(x), B(x)),$$

$$D_X(A, B) = \bigwedge_{x \in X} D_L(A(x), B(x)).$$

Then both R_X, D_X *are* L*-preorders on* L^X.

Let X be a nonempty set. A type-1 fuzzy subset of X is a mapping $A : X \longrightarrow [0,1]$. Denote by $\mathcal{F}([0,1])$ the family of all type-1 fuzzy subsets of X. A type-2 fuzzy subset of X is a mapping $\mathcal{A} : X \longrightarrow \mathcal{F}([0,1])$, and then it can be rewritten as $\mathcal{A} : X \times [0,1] \longrightarrow [0,1]$. Replace $[0,1]$ by a lattice L, we can get the notions of type-1 L-subsets and type-2 L-subsets.

By an L-relation on a set X, we mean a mapping $R : X \times X \longrightarrow L$, which exactly is a type-1 L-subset of $X \times X$. In this sense, every L-preorder $e : X \times X \longrightarrow L$ is a type-1 L-subset of $X \times X$, and thus a type-2 L-preorder of X has the form of $m : X \times X \times L \longrightarrow L$.

Definition 3.3. *Let X be a nonempty set and let $m : X \times X \times L \longrightarrow L$ be a mapping. We call m a type-2 L-preorder on X and call the pair (X, m) a type-2 L-preordered set if*

(M1) $m(x, x, r) = 1$ $(\forall x \in X,\ \forall r \in L)$;

(M2) $m(x, y, s) * m(y, z, t) \leq m(x, z, s * t)$ $(\forall x, y, z \in X,\ \forall s, t \in L)$.

Remark 3.4. (1) *The value $m(x, y, r)$ can be interpreted as the degree of the degree of the relation $x \leq y$ being greater than or equal to r. The word type-2 also refers to the phrase degree of degree. For a detailed explanation, suppose we have a preorder e on a set X and $x, y, z \in X$. If $e(x, y) \geq s$ and $e(y, z) \geq t$, then $e(x, z) \geq e(x, y) * e(y, z) \geq s * t$. Therefore, (M2) is a type-2 version of the transitivity condition (E2).*

(2) *If we change to a type-2 L-preorder $m : X \times X \times L \longrightarrow L$ to $M : X \times X \longrightarrow L^L$ by $M(x, y)(r) = m(x, y, r)$ $(\forall x, y \in X,\ \forall r \in L)$, then (M1) and (M2) can be equivalently rewritten as*

(M1') $M(x, x) = \overline{1}$ $(\forall x \in X)$;

(M2') $M(x, y)\tilde{*}M(y, z) \leq M(x, z)$ $(\forall x, y, z \in X)$.

Here $\overline{1}$ is the constant mapping from L to $L$with the value 1 and

$$[M(x, y)\tilde{*}M(y, z)](r) = \bigvee_{s*t=r} M(x, y)(s) * M(y, z)(t).$$

Then (M2') is the same as the transitivity condition of type-2 fuzzy relations in [20, 42], while (M1') is different from the reflectivity condition in [20, 42].

Now we study the relation between type-2 L-preorders and L-preorders.

Theorem 3.5. *Let X be a nonempty set.*

(1) *Let $e : X \times X \longrightarrow L$ be a mapping and define $m^e : X \times X \times L \longrightarrow L$ by*

$$m^e(x, y, r) = r \to e(x, y) \ (\forall x, y \in X,\ \forall r \in L).$$

Then e is an L-preorder iff m^e is a type-2 L-preorder.

(2) *Let $m : X \times X \times L \longrightarrow L$ be a type-2 L-preorder on X. Define*

$$e^m(x, y) = \bigvee_{r \in L} m(x, y, r) * r \ (\forall x, y \in X).$$

Then e^m is an L-preorder on X.

Proof. (1) The necessity: Let $x, y, z \in X$ and $s, t, r \in L$.

(M1) $m^e(x, x, r) = r \to e(x, x) = r \to 1 = 1$.

(M2)

$$\begin{array}{rl} & m^e(x, y, s) * m^e(y, z, t) \\ = & (s \to e(x, y)) * (t \to e(y, z)) \\ \leq & (s * t) \to (e(x, y) * e(y, z)) \\ \leq & (s * t) \to e(x, z) \\ = & m^e(x, z, s * t). \end{array}$$

The sufficiency: Let $x, y, z \in X$.

(E1) $e(x, x) = 1 \to e(x, x) = m^e(x, x, 1) = 1$.

(E2) Put $s = e(x, y)$ and $t = e(y, z)$. Then

$$\begin{array}{rl} & [e(x, y) * e(y, z)] \to e(x, z) \\ = & m^e(x, z, s * t) \\ \geq & m^e(x, y, s) * m^e(y, z, t) \\ = & [e(x, y) \to e(x, y)] * [e(y, z) \to e(y, z)] \\ = & 1 * 1 = 1, \end{array}$$

and thus $e(x,y) * e(y,z) \leq e(x,z)$.

(2) Let $x, y, z \in X$.

(E1) $e^m(x,x) = \bigvee_{r\in L} m(x,x,r) * r = \bigvee_{r\in L} 1 * r = 1$.

(E2)

$$\begin{aligned} & e^m(x,y) * e^m(y,z) \\ = & [\bigvee_{s\in L} m(x,y,s) * s] * [\bigvee_{t\in L} m(y,z,t) * t] \\ = & \bigvee_{s,t\in L} m(x,y,s) * m(y,z,t) * s * t \\ \leq & \bigvee_{s,t\in L} m(x,z,s*t) * s * t \\ = & \bigvee_{r\in L} m(x,z,r) * r \\ = & e^m(x,z). \end{aligned}$$

□

Theorem 3.6. *Suppose that X is a nonempty set. Let $e : X \times X \longrightarrow L$ be an L-preorder and let $m : X \times X \times L \longrightarrow L$ be a type-2 L-preorder. Then*

$$e^{m^e} = e,\ m^{e^m} \geq m.$$

Proof. Firstly, for all $x, y \in X$,

$$e^{m^e}(x,y) = \bigvee_{r\in L} m^e(x,y,r) * r = \bigvee_{r\in L} [r \to e(x,y)] * r = e(x,y).$$

Secondly, for all $x, y \in X$ and all $r \in L$,

$$\begin{aligned} m^{e^m}(x,y,r) &= r \to e_m(x,y) \\ &= r \to \bigvee_{s\in L} m(x,y,s) * s \\ &\geq r \to [m(x,y,r) * r] \\ &\geq m(x,y,r). \end{aligned}$$

Hence, $e^{m^e} = e$ and $m^{e^m} \geq m$. □

Theorem 3.7. *Let X be a nonempty set.*

(1) *Let $e : X \times X \longrightarrow L$ be an L-preorder. Define*

$$m_e(x,y,r) = e(x,y)\ (\forall x, y \in X,\ \forall r \in L);$$

Then e is an L-preorder iff m_e is a type-2 L-preorder.

(2) *Let $m : X \times X \times L \longrightarrow L$ be a type-2 L-preorder. Define*

$$e_m(x,y) = m(x,y,1)\ (\forall x, y \in X).$$

Then e_m is an L-preorder.

Proof. It is straightforward. □

Theorem 3.8. *Let X be a nonempty set, let $e : X \times X \longrightarrow L$ be an L-preorder and let $m : X \times X \times L \longrightarrow L$ be a type-2 L-preorder. Then*

$$e_{m_e} = e,\ m_{e_m} \leq m.$$

Proof. Let $x, y \in X$ and $r \in L$. It is easily seen that $e_{m_e}(x,y) = m_e(x,y,1) = e(x,y)$ and $m_{e_m}(x,y,r) = e_m(x,y) = m(x,y,1) = m(x,y,1) * m(y,y,r) \leq m(x,y,1*r) = m(x,y,r)$. Hence, $e_{m_e} = e$ and $m_{e_m} \leq m$. □

Let (X, e_X) and (Y, e_Y) be two L-preordered sets and let $f : X \longrightarrow Y$ be a mapping. If $e_X(x,y) \leq e_Y(f(x), f(y))$ holds for all $x, y \in X$, then we call f an order-preserving mapping from (X, e_X) to (Y, e_Y). The resulting category is denoted by L-**POrd**. Let (X, m_X) and (Y, m_Y) be two type-2 L-preordered sets and let $g : X \longrightarrow Y$ be a mapping. If $m_X(x,y,r) \leq m_Y(g(x), g(y), r)$ holds for all $x, y \in X$ and $r \in L$, then we call g an order-preserving mapping from X to Y. The resulting category is denoted by L_2-**POrd**.

It is a routine to show that all the above correspondences between L-preordered sets and type-2 L-preordered sets are functorial. By Theorems 3.6 and 3.8, the category L-**POrd** can be embedded in L_2-**POrd** as a simultaneously reflective and coreflective subcategory.

Besides e^m and e_m, there is another way to generate an L-preorder from a type-2 L-preorder.

Theorem 3.9. *Let (X, m) be a type-2 L-preordered set. Define*

$$e(m)(x,y) = \bigvee\{r \in L \mid r \le m(x,y,r)\} \ (\forall x, y \in X).$$

Then $e(m)$ is an L-preorder on X.

Proof. Let $x, y, z \in X$.

(E1) $e(m)(x,x) = \bigvee\{r \in L \mid r \le m(x,x,r)\} = 1$.

(E2)

$$\begin{aligned} & e(m)(x,y) * e(m)(y,z) \\ = \ & \bigvee\{s \in L \mid s \le m(x,y,s)\} * \bigvee\{t \in L \mid t \le m(y,z,t)\} \\ = \ & \bigvee\{s * t \mid s,t \in L,\ s \le m(x,y,s),\ t \le m(y,z,t)\} \\ \le \ & \bigvee\{s * t \mid s,t \in L,\ s * t \le m(x,y,s) * m(y,z,t)\} \\ \le \ & \bigvee\{s * t \mid s,t \in L,\ s * t \le m(x,z,s * t)\} \\ = \ & \bigvee\{r \in L \mid r \le m(x,z,r)\} \\ = \ & e(m)(x,z). \end{aligned}$$

□

Theorem 3.10. *Let (X, m) be a type-2 L-preordered set. Then $e_m \le e^m \le e(m)$.*

Proof. Let $x, y \in X$. Firstly,

$$e^m(x,y) = \bigvee_{r \in L} m(x,y,r) * r \ge m(x,y,1) * 1 = m(x,y,1) = e_m(x,y).$$

Hence, $e_m \le e^m$. Secondly, for $t \in L$, let $s = m(x,y,t) * t$. Then

$$s \le m(x,y,t) = m(x,x,m(x,y,t)) * m(x,y,t) \le m(x,y,m(x,y,t) * t) = m(x,y,s).$$

Thus, $e^m(x,y) \le e(m)(x,y)$. Hence, $e^m \le e(m)$. □

Theorem 3.11. (1) *Let L be a complete Heyting algebra, that is, $* = \wedge$. Then $e^m = e(m)$ for every type-2 L-preordered set (X, m).*

(2) *Denote the type-2 L-preorder m^{R_L} by m_*, that is, $m_*(x,y,r) = (r * x) \to y$. Then $e^{m_*} = e(m_*)$ iff L is a complete Heyting algebra (i.e., $* = \wedge$).*

Proof. (1) By Theorem 3.10, we only need to show that $e(m) \le e^m$. If $s \le m(x,y,s)$ $(s \in L)$, then $s = m(x,y,s) \wedge s$. Thus, $e(m)(x,y) \le e^m(x,y)$. Hence, $e(m) \le e^m$.

(2) The sufficiency is obvious by 3.10. The necessity: We only need to show that $*$ is idempotent. Let $a \in L$. Then

$$e^m(1, a * a) = \bigvee_{r \in L} m(1, a * a, r) * r = \bigvee_{r \in L} [r \to (a * a)] * r = a * a;$$

$$\begin{aligned} e(m)(1, a * a) &= \bigvee\{r \in L \mid r \le m(1, a * a, r)\} \\ &= \bigvee\{r \in L \mid r \le r \to (a * a)\} \ge a. \end{aligned}$$

Then $a * a \ge a$ $(\forall a \in L)$. Hence, $*$ is idempotent, and then $* = \wedge$. Consequently, L is a complete Heyting algebra. □

Example 3.12. *Let X be a nonempty set and L be a bounded lattice with an s-norm $\oplus$. A mapping $u : X \times X \longrightarrow L$ is called an ultrametric (cf. generalized ultrametric in [26] for $\oplus = \vee$) on X if*

(U0) $u(x,x) = 0$ $(\forall x \in X)$.

(U1) *if $u(x,y) = 0$, then $x = y$ $(\forall x \in X)$;*

(U2) $u(x,y) = u(y,x)$ $(\forall x, y \in X)$;

(U3) $u(x,z) \le u(x,y) \oplus u(y,z)$ $(\forall x, y, z \in X)$.

If there is a negation $' : L \longrightarrow L$, then a mapping $u : X \times X \longrightarrow L$ with (U0) *and* (U3)*, the form $e(x,y) = [u(x,y)]'$ defines an L-preorder on X w.r.t. the dual t-norm of $\oplus$. By Theorems 3.5 and 3.7, every ultrametric can be considered as a type-2 L-preorder via at least two ways.*

4 More examples from various kinds of metrics and fuzzy metrics

In this section, we will see that a lot kinds of metrics and fuzzy metrics are natural examples of type-2 L-preorders.

Remark 4.1. *For the case of $L = ([0,\infty]^{op}, +)$, it is easily seen that a mapping $m : X \times X \times L \longrightarrow L$ is a type-2 L-preorder iff*

(1) $m(x,x,r) = 0$ $(\forall x \in X,\ \forall r \in [0,\infty])$;

(2) $m(x,z,s+t) \leq m(x,y,s) + m(y,z,t)$ $(\forall x,y,z \in X,\ \forall s,t \in [0,\infty])$.

Definition 4.2. *Let X be a nonempty set. A mapping $d : X \times X \longrightarrow [0,\infty)$ is called a hemimetric* [14] *or a pseudo-quasi-metric* [34]*, or a generalized metric* [28] *on X if*

(HD1) $d(x,x) = 0$ $(\forall x \in X)$;

(HD2) $d(x,z) \leq d(x,y) + d(y,z)$ $(\forall x,y,z \in X)$.

The pair (X,d) is called a hemimetric space or a pseudo-quasi-metric space or a generalized metric space.

Theorem 4.3 (Hemimetrics as type-2 L-preorders: Case 1)**.** *Let X be a nonempty set and let $d : X \times X \longrightarrow [0,\infty)$ be a mapping. Define: $\forall x,y \in X,\ \forall t \in [0,\infty]$,*

$$m_d(x,y,r) = \max\{d(x,y) - t, 0\}.$$

Then d is a hemimetric iff m_d is a type-2 L-preorder for $L = ([0,\infty]^{op}, +)$.

Proof. The necessity: Let $x,y,z \in X$ and $s,t,r \in [0,\infty]$.

(M1) $m_d(x,x,r) = \max\{d(x,x) - t, 0\} = 0$.

(M2)

$$\begin{array}{rl} & m_d(x,y,s) + m_d(y,z,r) \\ = & \max\{d(x,y) - s, 0\} + \max\{d(y,z) - t, 0\} \\ \geq & \max\{d(x,y) - s + d(y,z) - t, 0 + 0\} \\ \geq & \max\{d(x,z) - s - t, 0\} \\ = & m_d(x,z,s+t). \end{array}$$

The sufficiency: Let $x,y,z \in X$.

(HD1) By (M1), $0 = m_d(x,x,0) = \max\{d(x,x), 0\}$ and then $d(x,x) = 0$.

(HD2) Let $s = d(x,y)$ and $t = d(y,z)$. Then

$$\begin{array}{rcl} 0 & = & \max\{d(x,y) - s, 0\} + \max\{d(y,z) - t, 0\} \\ & = & m_d(x,y,s) + m_d(y,z,t) \\ & \geq & m_d(x,z,s+t) \\ & = & \max\{d(x,z) - (s+t), 0\}. \end{array}$$

Hence, $d(x,z) \leq s + t = d(x,y) + d(y,z)$. □

Theorem 4.4 (Hemimetrics as type-2 L-preorders: Case 2)**.** *Let X be a nonempty set and let $d : X \times X \longrightarrow [0,\infty)$ be a mapping. Define: $\forall x,y \in X,\ \forall r \in [0,\infty]$,*

$$m^d(x,y,r) = \begin{cases} d(x,y) \div r, & d(x,y) \neq 0; \\ 0, & d(x,y) = 0, \end{cases}$$

If d is a hemimetric, then m^d is a type-2 L-preorder for $L = ([0,\infty]^{op}, +)$.

Proof. The necessity: Let $x,y,z \in X,\ s,t,r \in [0,\infty]$.

(M1) $m^d(x,x,r) = 0$ since $d(x,x) = 0$.

(M2) We need to show that $m^d(x,y,s) + m^d(y,z,t) \geq m^d(x,z,s+t)$.

Case 1. $d(x,y) = 0$. Then $d(x,z) \leq d(x,y) + d(y,z) = d(y,z)$. And further, if $d(x,z) = 0$, then

$$m^d(x,z,s+t) = 0 \leq m^d(x,y,s) + m^d(y,z,t).$$

If $d(x,z) \neq 0$, then $d(y,z) \neq 0$ and then

$$\begin{array}{rcl} m^d(x,y,s) + m^d(y,z,t) & = & 0 + d(y,z) \div t \\ & \geq & d(x,z) \div (s+t) \\ & = & m^d(x,z,s+t). \end{array}$$

Case 2. $d(y,z)=0$. It is similar to Case 1.

Case 3. $d(x,y)\neq 0$ and $d(y,z)\neq 0$. Then

$$\begin{array}{ll} & m^d(x,y,s)+m^d(y,z,t)\\ = & d(x,y)\div s+d(y,z)\div t\\ \geq & d(x,y)\div(s+t)+d(y,z)\div(s+t)\\ \geq & d(x,z)\div(s+t)\\ \geq & m^d(x,z,s+t). \end{array}$$

□

Definition 4.5. [5, 6] *Let X be a nonempty set. A mapping $w: X\times X\times(0,\infty)\longrightarrow[0,\infty]$ is called a modular metric on X or a metric modular over X if*

(MD1) $w(x,y,r)=0$ $(\forall t\in(0,\infty))$ *iff* $x=y$;

(MD2) $w(x,y,r)=w(y,x,r)$ $(\forall x,y\in X,\ \forall r\in(0,\infty))$;

(MD3) $w(x,z,s+t)\leq w(x,y,s)+w(y,z,t)$ $(\forall x,y,z\in X,\ \forall s,t\in(0,\infty))$.

Theorem 4.6 (Modular metrics as type-2 L-preorders)**.** *Let X be a nonempty set and let $w: X\times X\times(0,\infty)\longrightarrow[0,\infty]$ be a mapping. Define: $\forall x,y\in X,\ \forall r\in[0,\infty]$,*

$$w^\sharp(x,y,r)=\begin{cases} w(x,y,r), & r\in(0,\infty);\\ 0, & r=\infty;\\ \infty, & x\neq y,\ r=0;\\ 0, & x=y,\ r=0, \end{cases}$$

Then w satisfies (MD0) *and* (MD3) *iff $w^\sharp$ is a type-2 L-preorder for $L=([0,\infty]^{op},+)$, where*

(MD0) $w(x,x,r)=0$ $(\forall x\in X,\ \forall r\in(0,\infty))$.

Proof. The sufficiency: Let $x,y,z\in X$ and $s,t,r\in(0,\infty)$.

(MD0) Then $w(x,x,r)=w^\sharp(x,x,r)=0$.

(MD3) The equality

$$w^\sharp(x,z,s+t)\leq w^\sharp(x,y,s)+w^\sharp(y,z,t)$$

is the same as

$$w(x,z,s+t)\leq w(x,y,s)+w(y,z,t).$$

The necessity: Let $x,y,z\in X$ and $s,t,r\in[0,\infty]$.

(M1) If $r=0$ or $r=\infty$, then $w^\sharp(x,x,r)=0$ by the definition of $w^\sharp$; if $r>0$, then $w^\sharp(x,x,r)=w(x,x,r)=0$ by (MD0).

(M2) We should prove that

$$w^\sharp(x,z,s+t)\leq w^\sharp(x,y,s)+w^\sharp(y,z,t).$$

Case 1. $s=\infty$ or $t=\infty$. We have $s+t=\infty$ and

$$w^\sharp(x,z,s+t)=0\leq w^\sharp(x,y,s)+w^\sharp(y,z,t).$$

Case 2. $s=t=0$. If $x\neq y$ or $y\neq z$, then $w^\sharp(x,y,s)+w^\sharp(y,z,t)=\infty\geq w^\sharp(x,z,s+t)$. Otherwise $x=y=z$, then $w^\sharp(x,y,s)+w^\sharp(y,z,t)=0=w^\sharp(x,z,s+t)$.

Case 3. $s=0$ and $t\in(0,\infty)$. If $x=y$, then $w^\sharp(x,z,s+t)=w(x,z,t)=0+w(y,z,t)=w^\sharp(x,y,s)+w^\sharp(y,z,t)$. If $x\neq y$, then $w^\sharp(x,y,s)+w^\sharp(y,z,t)=\infty\geq w^\sharp(x,z,t)$.

Case 4. $s\in(0,\infty)$ and $t=0$. This is similar to Case 3.

Case 5. $s,t\in(0,\infty)$. The above inequality is the same as that in (MD3). □

Definition 4.7. [33] *Let $\oplus$ be an s-norm on $[0,1]$. A Morsi fuzzy hemimetric on a nonempty set is a mapping $C: X\times X\times[0,\infty)\longrightarrow[0,1]$ satisfying the following conditions:*

(Mor1) $C(x,x,r)=0$ $(\forall x\in X,\ \forall r\in[0,\infty))$;

(Mor2$_\oplus$) $C(x,z,t+s)\leq C(x,y,s)\oplus C(y,z,t)$ $(\forall x,y,z\in X,\ \forall s,t\in[0,\infty))$.

Remark 4.8. (1) *Morsi defined his fuzzy metrics by using Hutton's fuzzy real numbers w.r.t. the maximum s-norm* $\vee$ *on* $[0,1]$. *We here rewrite it in the form of the mapping in Definition 4.7. A Morsi fuzzy metric on a nonempty set* X *is a mapping* $C : X \times X \times [0,\infty) \longrightarrow [0,1]$ *satisfying* (Mor1) *and*

(Mor2$_\vee$) $C(x,z,s+t) \leq C(x,y,s) \vee C(y,z,t)$ $(\forall s,t \in [0,\infty))$.

(Mor3) $C(x,y,r) = C(y,x,r)$ $(\forall x,y \in X,\ \forall r \in [0,\infty))$;

(Mor4) *for all* $x,y \in X$, $C(x,y,r) = 0$ $(\forall r \in [0,\infty))$ *implies* $x = y$;

(Mor5) *for all* $x,y \in X$, $C(x,y,\cdot)$ *is decreasing and left-continuous.*

(2) *An* $\oplus$*-based fuzzy metric on a nonempty set* X, *as a generalization of Morsi fuzzy metrics, has been introduced in* [18] *by replacing* $\vee$ *by an s-norm* $\oplus$ *in the form of* $C : X \times X \times [0,\infty) \longrightarrow [0,1]$.

Theorem 4.9. [Morsi fuzzy hemimetrics as type-2 L-preorders: Case 1] *Let* $C : X \times X \times [0,\infty) \longrightarrow [0,1]$ *be a mapping with the s-norm* $\oplus$. *Define* $m_C : X \times X \times L \longrightarrow L$ *by*

$$m_C(x,y,r) = \begin{cases} \frac{C(x,y,r)}{1-C(x,y,r)}, & r \in (0,\infty); \\ 0, & r = \infty; \\ \infty, & r = 0,\ x \neq y; \\ 0, & r = 0,\ x = y. \end{cases}$$

Then for $L = ([0,\infty],+)^{op}$,

(1) *If* C *is a Morsi fuzzy hemimetric and* $\oplus \leq \oplus_H$, *then* m_C *is a type-2* L*-preorder;*

(2) *If* m_C *is a type-2* L*-preorder and* $\oplus_H \leq \oplus$, *then* C *is a Morsi fuzzy hemimetric.*

Hence, m_C *is a type-2* L*-preorder iff* C *is a Morsi fuzzy hemimetric for the Hamacher s-norm* $\oplus_H$.

Proof. (1) Let $x,y,z \in X$ and $s,t,r \in [0,\infty]$.

(M1) If $r = 0$ or $r = \infty$, then $m_C(x,x,r) = 0$; if $r \in (0,\infty)$, then $m_C(x,x,r) = \frac{C(x,x,r)}{1-C(x,x,r)} = \frac{0}{1-0} = 0$.

(M2) We need to show that $m_C(x,z,s+t) \leq m_C(x,y,s) + m_C(y,z,t)$. The proofs of **Case 1:** $s = \infty$ or $t = \infty$, **Case 2:** $s = t = 0$, **Case 3:** $s = 0, t \in (0,\infty)$ and **Case 4:** $s \in (0,\infty), t = 0$ are similar to those in Theorem 4.8.

Case 5: $s,t \in (0,\infty)$, By (Mor2), we have $C(x,z,s+t) \leq C(x,y,s) \oplus C(y,z,t)$ and then by Proposition 2.5,

$$\begin{aligned} m_C(x,z,s+t) &= \tfrac{C(x,z,s+t)}{1-C(x,z,s+t)} \\ &\leq \tfrac{C(x,y,s)}{1-C(x,y,s)} + \tfrac{C(y,z,t)}{1-C(y,z,t)} \\ &= m_C(x,y,s) + m_C(y,z,t). \end{aligned}$$

(2) Let $x,y,z \in X$ and $s,t,r \in [0,\infty)$.

(Mor1) $m_C(x,x,r) = \frac{C(x,x,r)}{1-C(x,x,r)} = 0$ and then $C(x,x,r) = 0$.

(Mor2$_\oplus$) By Proposition 2.5, the inequality

$$\frac{C(x,z,s+t)}{1-C(x,z,s+t)} \leq \frac{C(x,y,s)}{1-C(x,y,s)} + \frac{C(y,z,t)}{1-C(y,z,t)},$$

is equivalent to $C(x,z,s+t) \leq C(x,y,s) \oplus_H C(y,z,t)$. If $\oplus_H \leq \oplus$, then $C(x,z,s+t) \leq C(x,y,s) \oplus C(y,z,t)$. □

Theorem 4.10 (Morsi fuzzy hemimetrics as type-2 L-preorders: Case 2)**.** *Let* $M : X \times X \times (0,\infty) \longrightarrow [0,1]$ *be a mapping and let* $L = ([0,\infty]^{op},+)$. *Define* $C^\sharp : X \times X \times L \longrightarrow L$ *by*

$$C^\sharp(x,y,r) = \begin{cases} C(x,y,r), & t \in (0,\infty); \\ 0, & r = \infty; \\ \infty, & r = 0,\ x \neq y; \\ 0, & r = 0,\ x = y. \end{cases}$$

(1) *If* C *is a Morsi fuzzy hemimetric and* $\oplus \leq \oplus_{Lu}$, *then* m_C *is a type-2* L*-preorder.*

(2) *If* $C^\sharp$ *is a type-2* L*-preorder and* $\oplus_{Lu} \leq \oplus$, *then* C *is a Morsi fuzzy hemimetric.*

Hence, $C^\sharp$ *is a type-2* L*-preorder iff* C *is a Morsi fuzzy hemimetric for the Lukasiewicz s-norm* $\oplus_{Lu}$.

Proof. It is similar to that of Theorem 4.9 by using Proposition 2.4(1). □

Definition 4.11. *Let* $*$ *be a t-norm on* $[0,1]$. *A KM-fuzzy hemimetric on a nonempty set* X *is a mapping* $K : X \times X \times [0,\infty) \longrightarrow [0,1]$ *satisfying the following conditions:*

(KM1) $K(x,x,r) = 1$ $(\forall x \in X,\ \forall r \in [0,\infty))$;

(KM2) $K(x,z,s+t) \geq K(x,y,s) * K(y,z,t)$ $(\forall x,y,z \in X,\ \forall s,t \in [0,\infty))$.

Remark 4.12. *Kramosil and Michalek [25] firstly introduced the notion of fuzzy metrics. Let X be a nonempty set and let $*$ be a continuous t-norm on $[0,1]$. A mapping $K : X \times X \times [0,\infty) \longrightarrow [0,1]$ is called a KM-fuzzy metric if it satisfies* (KM2) *and*

(KM0) $K(x,y,0) = 0$ $(\forall x, y \in X)$;

(KM3) *for all* $x, y \in X$, $K(x,y,r) = 1$ $(\forall r \in (0,\infty))$ *if and only if* $x = y$;

(KM4) $K(x,y,t) = K(y,x,t)$ $(\forall x, y \in X,\ \forall r \in (0,\infty))$;

(KM5) *for all* $x, y \in X$, $K(x,y,\cdot) : [0,\infty) \longrightarrow [0,1]$ *is left-continuous.*

Later George and Veeramani [12] modified the definition of KM-fuzzy metric in order to make it more appropriate to deal with topological issues. A mapping $K : X \times X \times (0,\infty) \longrightarrow [0,1]$ is called a GV-fuzzy metric if it satisfies (KM3,KM4) *and*

(GV0) $K(x,y,r) > 0$ $(\forall x, y \in X,\ \forall r \in (0,\infty))$;

(GV2) $K(x,z,s+t) \geq K(x,y,s) * K(y,z,t)$ $(\forall x, y, z \in X,\ \forall s, t \in (0,\infty))$.

(GV5) $K(x,y,\cdot) : (0,\infty) \longrightarrow [0,1]$ *is left-continuous for all* $x, y \in X$.

Clearly, both KM-fuzzy metrics and GV-fuzzy metrics can be considered as special cases of KM-fuzzy hemimetrics.

Theorem 4.13 (KM-fuzzy hemimetrics as type-2 L-preorders)**.** *Let $K : X \times X \times [0,\infty) \longrightarrow [0,1]$ be a mapping with a t-norm $*$ on $[0,1]$. Define $m_K : X \times X \times L \longrightarrow L$ by*

$$m_K(x,y,r) = \begin{cases} \frac{1-K(x,y,r)}{K(x,y,r)}, & r \in (0,\infty); \\ 0, & r = \infty; \\ \infty, & r = 0,\ x \neq y; \\ 0, & r = 0,\ x = y. \end{cases}$$

Then

(1) *If K is a KM-fuzzy hemimetric and $*_H \leq *$, then m_K is a type-2 L-preorder;*

(2) *If m_K is a type-2 L-preorder and $* \leq *_H$, then K is a KM-fuzzy hemimetric.*

Hence for the Hamacher t-norm $*_H$, m_K is a type-2 L-preorder iff K is a KM-fuzzy hemimetric.

Proof. It is similar to that of Theorem 4.9 by using Proposition 2.4(2). □

5 Conclusions and further study

In this paper, we introduce a concept of type-2 L-preorders for L being a complete residuated lattice. It can be considered as a common framework of L-preorders and hemimetrics, and also contains various kinds of fuzzy metrics, including Morsi fuzzy metrics, KM-fuzzy metrics and modular metrics, as natural examples. It is shown that the category of L-preordered sets can be reflectively and coreflectively embedded in that of type-2 L-preordered sets. A type-2 L-preorder can be supplied as different models for further study.

5.1 The metric-type topology

For a metric space (X,d), the metric topology is generated by the base $\{U(x,r) \mid r > 0\}$, where $U(x,r) = \{y \in X \mid d(x,y) < r\}$. Clearly, an element $y \in U(x,r)$ iff y is near to x w.r.t. to the radius r. In other words, the open ball in a metric space is a structure described by nearness. Let $e : X \times X \longrightarrow [0,1]$ be a fuzzy similarity structure with the conditions

(E1) $e(x,x) = 1$ $(\forall x \in X)$;

(E2) $e(x,y) * e(y,z) \leq e(x,z)$ $(\forall x, y, z \in X)$.

Since, apart from an order relation, $e(x,y)$ can also interpreted as the degree of nearness from x to y, we can also define a kind of open balls as

$$B(x,r) = \{y \in X \mid e(x,y) > r\}\ (\forall s \in X,\ \forall r \in (0,1)).$$

An element $y \in B(x,r)$ iff y is near to x w.r.t. to the degree r. This is an idea for metric-type topology of fuzzy preordered sets. For a type-2 fuzzy preorder $m : X \times X \times [0,1] \longrightarrow [0,1]$, we also can try to define and study its metric-type topology. Let $r, p \in (0,1)$. We call the set

$$B(x,r,p) = \{y \in X \mid m(x,y,r) > p\},$$

the (r,p)-ball of x. Then we can use the family of (r,p)-balls to generate a desired topology

The unit interval $[0,1]$ can be replaced by $L=([0,\infty]^{op},+)$. For a hemimetric $d: X\times X\longrightarrow [0,\infty)$, the metric-type topology on m_d coincides with the standard metric topology on X induced by d, since in (X,m_d), for all $x\in X$ and for all $r,p\in(0,1)$, we have

$$\begin{array}{rcl} B(x,r,p) & = & \{y\in X \mid m_d(x,y,r) >_L p\} \\ & = & \{y\in X \mid \max\{d(x,y)-r,0\} < p\} \\ & = & \{y\in X \mid d(x,y)-r<p\} = U(x,p+r). \end{array}$$

We may say that type-2 L-preordered sets generalize metric spaces from the aspects of the own structure and the induced topological structure simultaneously.

5.2 Some intrinsic topologies

By considering type-2 L-preordered sets as the generalization of L-preordered sets, we can study the related intrinsic topology. For example, by e^m, e_m and $e(m)$ we have three type of upper topology on (X,m):

$\Gamma_1(X)=\{A\in L^X \mid A(x)*m(x,y,r)*r\le A(y), \forall r\in L,\ \forall x,y\in X\}$.

$\Gamma_2(X)=\{A\in L^X \mid A(x)*m(x,y,1)\le A(y),\ \forall x,y\in X\}$.

$\Gamma_3(X)=\{A\in L^X \mid r\le m(x,y,r)\Longrightarrow A(x)*r\le A(y), \forall r\in L,\ \forall x,y\in X\}$.

Clearly, all of $\Gamma_i(X)$ $(i=1,2,3)$ are Alexandrov fuzzy topologies on X and $\Gamma_1(X)\cup\Gamma_3(X)\subseteq\Gamma_2(X)$.

5.3 Rough set structures

Since for a type-2 L-preorder m on X, there are three kinds of induced L-preorders: e^m, e_m and $e(m)$. Hence, we can study the related rough set structures. We here only list the definitions of the corresponding pairs of rough approximation operators: $\overline{\mathrm{Apr}}_j$, $\underline{\mathrm{Apr}}_j: L^X\longrightarrow L^X$ $(j=1,2,3)$.

Pair 1:

$$\overline{\mathrm{Apr}}_1(A)(x)=\bigvee_{y\in X,\ r\in L} A(y)*m(x,y,r)*r;$$
$$\underline{\mathrm{Apr}}_1(A)(x)=\bigvee_{y\in X,\ r\in L} [m(y,x,r)*r]\to A(y).$$

Pair 2:

$$\overline{\mathrm{Apr}}_2(A)(x)=\bigvee_{y\in X} A(y)*m(x,y,1);$$
$$\underline{\mathrm{Apr}}_2(A)(x)=\bigvee_{y\in X} m(y,x,1)\to A(y).$$

Pair 3:

$$\overline{\mathrm{Apr}}_1(A)(x)=\bigvee_{y\in X,\ r\in L} \{A(y)*r \mid r\le m(x,y,r)\};$$
$$\underline{\mathrm{Apr}}_3(A)(x)=\bigvee_{y\in X,\ r\in L} \{r\to A(y) \mid r\le m(y,x,r)\}.$$

5.4 L-convexity structures

We call $A\in L^X$ a *relational convex set* of a type-2 L-preordered set (X,m) if

$$A(x)*A(y)*m(x,z,s)*m(z,y,t)*s*t\le A(z),$$

holds for all $x,y,z\in X$ and all $s,t\in L$. The condition can be interpreted as: if $x,y\in X$ and z lies between x,y, then $z\in A$. This is a description of convex sets in a binary relational set. Denote by $\mathcal{C}(X)$ the family of all relational convex sets of (X,m). We can prove that he family $\mathcal{C}(X)$ is an L-convexity structure on X [41].

Acknowledgement

This paper is supported by National Natural Science Foundation of China (11871189, 12231007, 12111540250), Natural Science Foundation of Hebei Province (A2020208008) and Jiangsu Provincial Innovative and Entrepreneurial Talent Support Plan (JSSCRC202521).

References

[1] R. A. Aliev, W. Pedrycz, B. G. Guirimov, R. R. Aliev, U. Ilhan, M. Babagil, S. Mammadli, *Type-2 fuzzy nueral networks with fuzzy clustering and differential evolution optimization*, Information Sciences, **181**(9) (2011), 1591-1608.

[2] R. Bělohlávek, *Fuzzy relational systems: Foundations and principles*, Kluwer Academic Publishers, Plenum Publishers, New York, 2002.

[3] R. Bělohlávek, J. W. Dauben, G. J. Klir, *Fuzzy logic and mathematics: A historical perspective*, Oxford University Press, Cary, 2017.

[4] I. P. Cabrera, P. Cordero, F. García-Pardo, M. Ojeda-Aciego, B. De Baets, *Galois connections between a fuzzy preordered structure and a general fuzzy structure*, IEEE Transactions on Fuzzy Systems, **26**(3) (2018), 1274-1287.

[5] V. V. Chistyakov, *Metric modulars and their application*, Doklady Mathematics, **73** (2006), 32-35.

[6] V. V. Chistyakov, *Modular metric spaces, I: Basic concepts*, Nonlinear Analysis: Theory, Methods and Applications, **72**(1) (2010), 1-14.

[7] B. I. Choi, F. C. H. Rhee, *Interval type-2 fuzzy membership function generation methods for pattern recognition*, Information Sciences, **179**(13) (2009), 2102-2122.

[8] G. De Cooman, E. E. Kerre, *Order norms on bounded partially ordered sets*, Journal of Fuzzy Mathematics, **2**(2) (1994), 281-310.

[9] M. Demirci, *On many-valued partitions and many-valued equivalence relations*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, **11**(02) (2003), 235-253.

[10] M. Demirci, *Representations of the extensions of many-valued equivalence relations*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, **11**(03) (2003), 319-341.

[11] D. Dutta, M. Sen, *Type-2 fuzzy equivalence relation on a groupoid under balanced and semibalanced maps*, Journal of Informatics and Mathematical Sciences, **10** (2018), 133-151.

[12] A. George, P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems, **64**(3) (1994), 395-399.

[13] M. Gondran, M. Minoux, *Graphs, dioids and semirings: New models and algorithms*, Springer Science and Business Media, 2008.

[14] J. Goubault-Larrecq, *Non-Hausdorff topology and domain theory*, Cambridge University Press, Cambridge, 2013.

[15] V. Gregori, A. López-Crevillén, S. Morillas, A. Sapena, *On convergence in fuzzy metric spaces*, Topology and its Applications, **156**(18) (2009), 3002-3006.

[16] V. Gregori, J. J. Miñana, S. Morillas, A. Sapena, *Characterizing a class of completable fuzzy metric spaces*, Topology and its Applications, **203** (2016), 3-11.

[17] V. Gregori, S. Romaguera, *Some properties of fuzzy metric spaces*, Fuzzy Sets and Systems, **115**(3) (2000), 485-489.

[18] O. Grigorenko, J. J. Miñana, A. Šostak, O. Valero, *On t-conorm based fuzzy (pseudo) metrics*, Axioms, **9**(3) (2020), 78.

[19] U. Höhle, *Classification of subsheaves over GL-algebras*, Logic Colloquium, **98** (2000), 238-261.

[20] B. Q. Hu, C. Y. Wang, *On type-2 fuzzy relations and interval-valued type-2 fuzzy sets*, Fuzzy Sets and Systems, **236** (2014), 1-32.

[21] G. Jäger, W. Yao, *Quantale-valued gauge spaces*, Iranian Journal of Fuzzy Systems, **15**(1) (2018), 103-122.

[22] N. N. Karnik, J. M. Mendel, *Operations on type-2 fuzzy sets*, Fuzzy Sets and Systems, **122**(2) (2001), 327-348.

[23] G. M. Kelly, *Basic concepts of enriched category theory*, In: London Mathematical Society Lecture Notes Series, **64**, Cambridge University Press, Cambridge, 1982.

[24] E. P. Klement, R. Mesiar, E. Pap, *Triangular norms*, Kluwer Academic Publishers, Dordrecht, 2000.

[25] I. Kramosil, J. Michálek, *Fuzzy metrics and statistical metric spaces*, Kybernetika, **11**(5) (1975), 336-344.

[26] M. Krötzsch, *Generalized ultrametric spaces in quantitative domain theory*, Theoretical Computer Science, **368**(1-2) (2006), 30-49.

[27] H. Lai, D. Zhang, *Complete and directed complete Ω-categories*, Theoretical Computer Science, **388**(1-3) (2007), 1-25.

[28] F. W. Lawvere, *Metric spaces, generalized logic, and closed categories*, Rendiconti del Seminario Matématico e Fisico di Milano, **43** (1973), 135-166.

[29] F. Liu, J. Lu, G. Zhang, *Unsupervised heterogeneous domain adaptation via shared fuzzy equivalence relations*, IEEE Transactions on Fuzzy Systems, **26**(6) (2018), 3555-3568.

[30] S. G. Matthews, *Partial metric topology*, Annals of the New York Academy of Sciences, **728**(1) (1994), 183-197.

[31] K. Menger, *Statistical metrics*, Proceedings of the National Academy of Sciences of the United States of America, **28**(12) (1942), 535-537.

[32] J. Močkoř, P. Hurtik, *Lattice-valued F-transforms and similarity relations*, Fuzzy Sets and Systems, **342** (2018), 67-89.

[33] N. N. Morsi, *On fuzzy pseudo-normed vector spaces*, Fuzzy Sets and Systems, **27**(3) (1988), 351-372.

[34] I. L. Reilly, P. V. Subrahmanyam, M. K. Vamanamurthy, *Cauchy sequences in quasi-pseudo-metric spaces*, Monatshefte für Mathematik, **93** (1982), 127-140.

[35] B. Schweizer, A. Sklar, *Statistical metric spaces*, Pacific Journal of Mathematics, **10**(1) (1960), 313-334.

[36] B. Schweizer, A. Sklar, *Probabilistic metric spaces*, North-Holland, New York, 1983.

[37] M. Sen, D. Dutta, A. Deshpande, *Type-2 fuzzy G-tolerance relation and its properties*, International Journal of Analysis and Applications, **15**(2) (2017), 172-178.

[38] C. Shen, Y. Shi, F. G. Shi, H. Andradi, *Characterizations of pointwise pseudo-metrics via pointwise closed-ball systems*, IEEE Transactions on Fuzzy Systems, **30**(5) (2021), 1212-1223.

[39] S. Shukla, D. Gopal, W. Sintunavarat, *A new class of fuzzy contractive mappings and fixed point theorems*, Fuzzy Sets and Systems, **350** (2018), 85-94.

[40] D. W. W. W. Tan, *A simplified type-2 fuzzy logic controller for real-time control*, ISA Transactions, **45**(4) (2006), 503-516.

[41] M. L. J. Van De Vel, *Theory of convex structures*, North-Holland, Elsevier Science Publishers B.V., Amsterdam, 1993.

[42] C. Y. Wang, *Type-2 fuzzy rough sets based on extended t-norms*, Information Sciences, **305** (2015), 165-183.

[43] J. Z. Xiao, X. H. Zhu, H. Zhou, *On the topological structure of KM fuzzy metric spaces and normed spaces*, IEEE Transactions on Fuzzy Systems, **28**(8) (2019), 1575-1584.

[44] W. Yao, *Quantitative domains via fuzzy sets: Part I: Continuity of fuzzy directed complete posets*, Fuzzy Sets and Systems, **161**(7) (2010), 973-987.

[45] W. Yao, Y. She, L. X. Lu, *Metric-based L-fuzzy rough sets: Approximation operators and definable sets*, Knowledge-Based Systems, **163** (2019), 91-102.

[46] W. Yao, F. G. Shi, *Quantitative domains via fuzzy sets: Part II: Fuzzy Scott topology on fuzzy directed-complete posets*, Fuzzy Sets and Systems, **173**(1) (2011), 60-80.

[47] W. Yao, G. Zhang, C. J. Zhou, *Real-valued hemimetric-based fuzzy rough sets and an application to contour extraction of digital surfaces*, Fuzzy Sets and Systems, **459** (2023), 201-219.