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A bilevel linear programming model with interval type-2 triangular fuzzy numbers

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Abstract

In the real world, the parameters of a problem may not be the crisp values. The fuzzy theory among the theories in which uncertainty plays a crucial role. Type-2 fuzzy sets generalize fuzzy sets. We consider a special type of such sets here. In this paper, we consider two issues. First, we review the method proposed by Javanmard and Mishmast Nehi for solving an interval type-2 triangular fuzzy linear programming problem, and improve it. Then, we express a bilevel linear programming problem, that, to the best of our knowledge, has not been investigated so far. We consider the bilevel linear programming problem with uncertainty where all the coefficients in the problem are interval type-2 triangular fuzzy numbers. We convert an interval type-2 triangular fuzzy bilevel linear programming problem into an interval bilevel linear programming problem using Grzegorzewski's nearest interval approximation method. Finally, we obtain five problems, and by solving them, we achieve the solution of interval type-2 triangular fuzzy bilevel linear programming problem as an interval type-2 triangular fuzzy number.

Keywords: Fuzzy programming, bilevel linear programming, interval type-2 fuzzy number.

1 Introduction

As a programming problem, bilevel linear programming problem is very similar to the problem first considered by Stackelberg [71]. In 1973, bilevel linear programming problem was formulated by Bracken and McGill [12]. So far, several methods have been proposed to solve it. These include the Kth-best algorithm [10], the branch and bound algorithm [7, 41, 68], the sequential LCP method [45], the penalty function method [52] and the heuristic algorithms [15, 48].

In the real world, the parameters of a problem may not be the crisp values; instead, they may be expressed as fuzzy values or intervals.

Zadeh [80] developed the theory of fuzzy sets. So far, many studies have been conducted in the fuzzy environment. Agarwal et al. [1] studied fuzzy solutions for multipoint boundary value problems. Chalishajar et al. [17] examined the existence, uniqueness, and controllability for impulsive fuzzy neutral functional integrodifferential equations using the Banach fixed point theorem. Chalishajar and Ramesh [18] studied the fuzzy solution of the second-order three-point boundary value problem with an application for absolute retracts and the Banach fixed point theorem. Also, in [19], they checked the existence of fuzzy solutions for first and second order nonlocal impulsive neutral functional differential equations using the Banach fixed point theorem. For more details on fuzzy differential equations, we refer the reader to [47, 63, 64, 74].

Lai et al. [49, 69] presented a fuzzy method for solving the crisp bilevel linear programming problem. Then, Zhang et al. [83, 84, 85, 86] solved bilevel linear programming problems with fuzzy parameters. In 2013, Ruziyeva and Dempe [65] solved a fuzzy bilevel linear programming by using the Yager ranking index. Budnitzkia [13] proposed a solution approach for fuzzy bilevel linear programming problem by using the α-cut of fuzzy polytopes.

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Another method for the modeling of problems under uncertainty is the interval linear programming problem. Recently, many researchers have studied it. See [2, 3, 4, 5, 59, 75]. Calvete et al. [14] considered bilevel linear programming problem when the coefficients of the leader and the follower objective functions were interval and presented two algorithms for its solution. Mishmast Nehi and Hamidi [60] generalized the algorithm provided by Calvete et al. [14] for bilevel linear programming problem when all coefficients were intervals and provided corrections to improve their algorithm. Li and Fang [50] solved interval bilevel linear programming problem by using a heuristic algorithm.
Zadeh introduced type-2 fuzzy sets [81] in 1975. Daoyuan and Mendel [27] introduced the theory of representing type-2 fuzzy sets based on the α-plane, and presented a strategy for calculating five measures of the definition of the uncertainty of a type-2 fuzzy sets including centroid, cardinality, fuzziness, variance, and skewness.
Interval type-2 fuzzy sets were introduced by Hisdal [42] in 1981. Some examples of the applications of interval type-2 fuzzy sets include the control of robot [87] or vehicle [46, 67], choosing the best suppliers [73], decision-making [62, 78], inventory management [58], wireless communication [51, 70], web shopping [39] and word modeling [77].
Given that ranking is one of the most important topics for interval type-2 fuzzy numbers, it has been considered by many researchers. De et al. [28] proposed a new ranking method for interval type-2 fuzzy numbers based on centroid and the rank index. Gong et al. [37] presented some new distance measures and possibility degree formula to compare interval type-2 fuzzy numbers based on their Mellin mean value and variance. More ranking methods for interval type-2 fuzzy numbers were described in [20, 21, 22, 44, 79].
Figueroa et al. [30, 31, 32, 33, 34, 35] proposed different methods for the modeling and solution of the interval type-2 fuzzy linear programming problem of the kind vagueness. Melgarejo [55], Mendel et al. [56], Zeng and Liu [82], and Javanmard and Mishmast Nehi [43] investigated interval type-2 fuzzy linear programming problem with the uncertainty of the kind ambiguity.
This paper is organized as follows.
In Section 2, the key concepts of bilevel linear programming problem and type-2 fuzzy sets are defined. Then, we review the method used in [43] for interval type-2 triangular fuzzy linear programming problem and improve the proposed method. In the next section, we consider interval type-2 triangular fuzzy bilevel linear programming problem, which has not been investigated to the best of our knowledge, and solve it.

2 Preliminaries

In this section, we present a berief summary of the basic concepts that are required throughout the paper.

2.1 Bilevel linear programming problem

The general form of bilevel linear programming problem is:

$$\begin{aligned} &\max_{\mathbf{x}_1 \in X_1} F_1(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{c}_1\mathbf{x}_1 + \mathbf{d}_1\mathbf{x}_2 \\ &\text{s.t.} \ \max_{\mathbf{x}_2 \in X_2} F_2(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{c}_2\mathbf{x}_1 + \mathbf{d}_2\mathbf{x}_2 \\ &\quad \text{s.t.} \ A\mathbf{x}_1 + B\mathbf{x}_2 \leq \mathbf{b}, \\ &\quad\quad \mathbf{x}_1, \mathbf{x}_2 \geq 0, \end{aligned} \tag{1}$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are the leader and the follower decision variables, respectively. $F_1(\mathbf{x}_1, \mathbf{x}_2)$ and $F_2(\mathbf{x}_1, \mathbf{x}_2)$ are called the leader objective function and the follower objective function, respectively. Also, $\mathbf{x}_1 \in X_1 \subset \mathbb{R}^n$, $\mathbf{x}_2 \in X_2 \subset \mathbb{R}^m$, $F_1, F_2 : X_1 \times X_2 \to \mathbb{R}^1$, $\mathbf{c}_1, \mathbf{c}_2 \in \mathbb{R}^n$, $\mathbf{d}_1, \mathbf{d}_2 \in \mathbb{R}^m$, $\mathbf{b} \in \mathbb{R}^p$, $A \in \mathbb{R}^{p\times n}$ and $B \in \mathbb{R}^{p\times m}$. The bilevel linear programming problem involves two problems; (a) the problem of the leader level and (b) the problem of the follower level. First, the leader tries to optimize $F_1(\mathbf{x}_1, \mathbf{x}_2)$ under some constraints by selecting $\mathbf{x}_1$. Then, the follower tries to optimize $F_2(\mathbf{x}_1, \mathbf{x}_2)$ by choosing $\mathbf{x}_2$ for the specific value of $\mathbf{x}_1$ [11].
Several methods have been proposed to solve the bilevel linear programming problem, such as the Kth-best algorithm [10], the branch and bound algorithm [7, 41, 68], the sequential LCP method [45], the penalty function method [52], and the heuristic algorithms [15, 48].
As an example of bilevel linear programming problem solving methods, we present the flowchart of the extended branch and bound algorithm [68] in Figure 1. This algorithm is based on the following theorem.

Theorem 2.1. [68] *A pair* $(\boldsymbol{x}_1^*, \boldsymbol{x}_2^*)$ *solves the bilevel linear programming problem (1) if there exist (row) vectors* $\boldsymbol{u}^*$ *and* $\boldsymbol{w}^*$ *such that* $(\boldsymbol{x}_1^*, \boldsymbol{x}_2^*, \boldsymbol{u}^*, \boldsymbol{w}^*)$ *solves:*

$$\max_{\boldsymbol{x}_1 \in X_1} F_1(\boldsymbol{x}_1, \boldsymbol{x}_2) = \boldsymbol{c}_1\boldsymbol{x}_1 + \boldsymbol{d}_1\boldsymbol{x}_2 \tag{2a}$$

$$s.t. \quad A\boldsymbol{x}_1 + B\boldsymbol{x}_2 \leq \boldsymbol{b} \tag{2b}$$

$$\boldsymbol{u}B - \boldsymbol{w} = -\boldsymbol{d}_2 \tag{2c}$$

$$\boldsymbol{u}(\boldsymbol{b} - A\boldsymbol{x}_1 - B\boldsymbol{x}_2) + \boldsymbol{w}\boldsymbol{x}_2 = 0 \tag{2d}$$

$$\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u}, \boldsymbol{w} \geq 0. \tag{2e}$$

Remark 2.2. *Suppose that* $I = \{\boldsymbol{d}_2 \in \mathbb{R}^m : d_{2i} \in [\check{d}_{2i}, \hat{d}_{2i}]; i = 1, ..., m\}$ *and* I_E *represents the set of extreme points of* I*. Throughout this paper, it is assumed that the inducible region* [7] *does not change for any* $\boldsymbol{d}_2 \in I_E$*.*

2.2 Interval type-2 triangular fuzzy number

Definition 2.3. [80] *Let* X *be a universal set. A fuzzy set* $\tilde{A}$ *is defined as an ordered pair*

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X, \mu_{\tilde{A}}(x) \in [0, 1]\},$$

where $\mu_{\tilde{A}}(x)$ *is the membership function of* $x \in X$*.*

Definition 2.4. [8] *Let* $\tilde{A}$ *be a fuzzy set in* X*. The support of* $\tilde{A}$ *is the crisp set* $\{x \in X : \mu_{\tilde{A}}(x) > 0\}$*.*

Definition 2.5. [8] *Let* $\tilde{A}$ *be a fuzzy set in* $\mathbb{R}$*.* $\tilde{A}$ *is called a fuzzy number if*
(i) $\exists x \in \mathbb{R}$ *s.t.* $\mu_A(x) = 1$*,*
(ii) $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}$*,*
(iii) μ_A *is upper semicontinuous, and*
(iv) *the set* $\{x \in X : \mu_A(x) > 0\}$ *is bounded.*

Definition 2.6. [8] *A fuzzy number* $\tilde{A} = (a, b, c)$ *is called a triangular fuzzy number if its membership function is defined as follows.*

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a}{b-a} & a \leq x \leq b \\ \dfrac{c-x}{c-b} & b \leq x \leq c \\ 0 & O.W. \end{cases}$$

Definition 2.7. [8] *The* α*-cut of a fuzzy number* $\tilde{A}$ *is a set defined as* $\tilde{A}_\alpha = \{x \in \mathbb{R} | \mu_{\tilde{A}}(x) \geq \alpha\} = [A^l(\alpha), A^u(\alpha)]$*, where* $A^l(\alpha) = inf\{x \in \mathbb{R} | \mu_{\tilde{A}}(x) \geq \alpha\}$ *and* $A^u(\alpha) = sup\{x \in \mathbb{R} | \mu_{\tilde{A}}(x) \geq \alpha\}$*.*

Definition 2.8. [38] *The core of a fuzzy number* $\tilde{A}$ *is defined as* $Core(\tilde{A}) = \{x \in \mathbb{R} | \mu_{\tilde{A}}(x) = 1\}$*.*

Definition 2.9. [61] *A type-2 fuzzy sets* $\tilde{\tilde{A}}$*, is characterized by a type-2 membership function* $\mu_{\tilde{\tilde{A}}}(x, u)$ *where* $x \in X$ *and* $u \in J_x \subset [0, 1]$*:*

$$\tilde{\tilde{A}} = \{((x, u), \mu_{\tilde{\tilde{A}}}(x, u)) | \forall x \in X, \forall u \in J_x \subset [0, 1],\ 0 \leq \mu_{\tilde{\tilde{A}}}(x, u) \leq 1\},$$

where J_x *is called the primary membership of* x *and* $\mu_{\tilde{\tilde{A}}}(x, u)$ *for* $x \in X$ *and* $u \in J_x$ *is called a secondary membership.*

Definition 2.10. [56] *In Definition 2.9, when* $\mu_{\tilde{\tilde{A}}}(x, u) = 1, \forall x \in X, u \in J_x \subset [0, 1]$*,* $\tilde{\tilde{A}}$ *is called interval type-2 fuzzy set.*

Definition 2.11. [56] *Uncertainty in the primary membership of an interval type-2 fuzzy set, consists of a bounded region called the footprint of uncertainty (FOU). It is the union of all primary memberships, i.e., FOU=* $\bigcup_{x \in X} J_x$*.*

Definition 2.12. [22] *A general interval type-2 triangular fuzzy number,* $\tilde{\tilde{A}} = (\tilde{A}^l, \tilde{A}^u) = \big((\check{a}^l, \bar{a}, \hat{a}^l, h^l), (\check{a}^u, \bar{\bar{a}}, \hat{a}^u, h^u)\big)$ *is defined on the interval* $[\check{a}^u, \hat{a}^u]$*. Its lower membership function and upper membership function takes the value equal to* $h^l \in [0, 1]$ *in the point* $\bar{a}$ *and* $h^u \in [0, 1]$ *in the point* $\bar{\bar{a}}$*, respectively, where* $\check{a}^u \leq \check{a}^l \leq \bar{a} \leq \hat{a}^l \leq \hat{a}^u$ *and* $\check{a}^u \leq \bar{\bar{a}} \leq \hat{a}^u$*.*

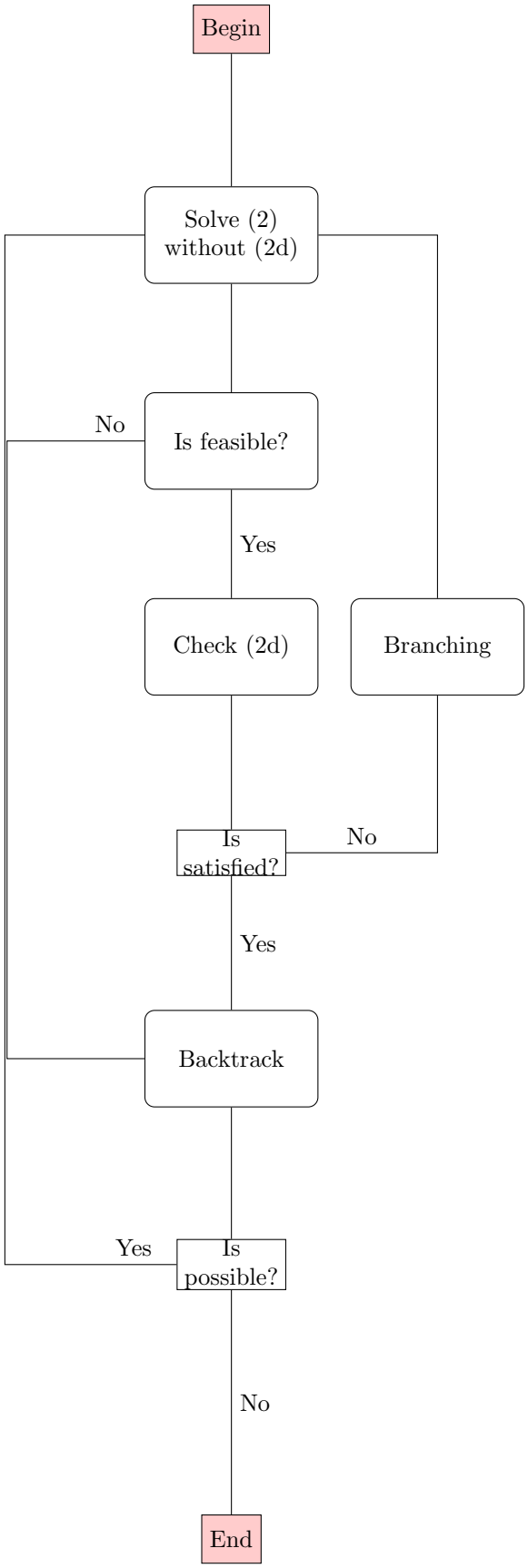


Figure 1: Flowchart of the extended branch and bound algorithm

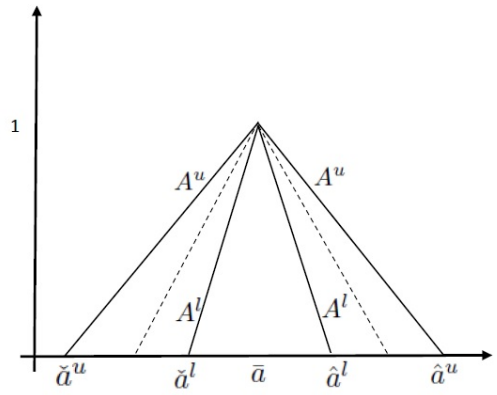


Figure 2: The interval type-2 triangular fuzzy number.

Definition 2.13. [44] *In Definition 2.12, if $h^u = h^l = 1$ and $\bar{a} = \bar{\bar{a}}$, a general interval type-2 triangular fuzzy number is named an interval type-2 triangular fuzzy number where it is shown as $\tilde{\tilde{A}} = (\tilde{A}^l, \tilde{A}^u) = \big((\check{a}^l, \bar{a}, \hat{a}^l), (\check{a}^u, \bar{a}, \hat{a}^u)\big)$. An interval type-2 triangular fuzzy number is shown in Figure 2.*

Definition 2.14. [38] *Suppose $\tilde{A}$ is a fuzzy number. The interval*

$$C_d(\tilde{A}) = [\int_0^1 A^l(\alpha)d\alpha, \int_0^1 A^u(\alpha)d\alpha],$$

is named Grzegorzewski's nearest interval approximation of fuzzy number $\tilde{A}$ with respect to the metric d. Grzegorzewski's nearest interval approximation is a simple and natural continuous approximation operator which this approach might be generalized for one-sided fuzzy numbers as well.

Theorem 2.15. [43] *Grzegorzewski's nearest interval approximation for the triangular fuzzy number $\tilde{A} = (a, b, c)$ is $[\frac{a+b}{2}, \frac{c+b}{2}]$.*

Theorem 2.16. [72] *The objective function value of any arbitrary characteristic model of interval linear programming problem belongs to the interval of the objective function values of the best-worst cases method.*

3 Interval type-2 triangular fuzzy linear programming problem

In this section, we review the method in [43] and correct it.
Consider the fuzzy linear programming problem with the uncertainties of kind ambiguity such that all the coefficients are interval type-2 triangular fuzzy numbers:

$$\begin{aligned} \max \ & \sum_{j=1}^{n} \tilde{\tilde{c}}_j x_j \\ \text{s.t.} \ & \sum_{j=1}^{n} \tilde{\tilde{a}}_{ij} x_j \leq \tilde{\tilde{b}}_i \quad i = 1, ..., m \quad j = 1, 2, ..., n \\ & x_j \geq 0 \qquad\qquad j = 1, 2, ..., n, \end{aligned} \tag{3}$$

where $\tilde{\tilde{c}}_j = (\tilde{c}_j^l, \tilde{c}_j^u)$, $\tilde{\tilde{a}}_{ij} = (\tilde{a}_{ij}^l, \tilde{a}_{ij}^u)$, $\tilde{\tilde{b}}_i = (\tilde{b}_i^l, \tilde{b}_i^u)$.
Since an interval type-2 triangular fuzzy number is composed of the infinite union of the characteristic triangular fuzzy numbers, the traditional embedded triangular fuzzy numbers are used for all the interval type-2 triangular fuzzy numbers of (3). Therefore, it is converted to the following fuzzy linear programming problem:

$$\max \ \sum_{j=1}^{n} \tilde{c}_j^e x_j,$$

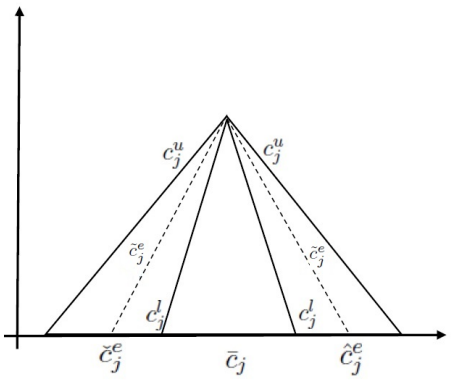


Figure 3: A fuzzy number $\tilde{c}_j^e = (\check{c}_j^e, \bar{c}_j, \hat{c}_j^e)$.

$$\begin{aligned} \text{s.t.} \quad & \sum_{j=1}^{n} \tilde{a}_{ij}^e x_j \leq \tilde{b}_i^e \quad i = 1, ..., m \quad j = 1, 2, ..., n, \\ & x_j \geq 0 \qquad\qquad\quad j = 1, 2, ..., n, \end{aligned} \tag{4}$$

where $\tilde{c}_j^e \in \text{FOU}(\tilde{\tilde{c}}_j)$, $\tilde{a}_{ij}^e \in \text{FOU}(\tilde{\tilde{a}}_{ij})$, $\tilde{b}_i^e \in \text{FOU}(\tilde{\tilde{b}}_i)$ (Figure 3).

Javanmard and Mishmast [43] Nehi used Grzegorzewski's nearest interval approximation to convert interval type-2 triangular fuzzy number into interval linear programming problem. They obtain interval linear programming problem as follows:

$$\begin{aligned} \max \ & \sum_{j=1}^{n} [\check{c}_j^e, \hat{c}_j^e] x_j, \\ \text{s.t.} \ & \sum_{j=1}^{n} [\check{a}_{ij}^e, \hat{a}_{ij}^e] x_j \leq [\check{b}_i^e, \hat{b}_i^e] \quad i = 1, ..., m \quad j = 1, 2, ..., n, \\ & x_j \geq 0 \qquad\qquad\qquad\quad j = 1, 2, ..., n, \end{aligned} \tag{5}$$

where $\check{c}_j^e \in [\check{c}_j^u, \check{c}_j^l]$, $\hat{c}_j^e \in [\hat{c}_j^l, \hat{c}_j^u]$, $\check{a}_{ij}^e \in [\check{a}_{ij}^u, \check{a}_{ij}^l]$, $\hat{a}_{ij}^e \in [\hat{a}_{ij}^l, \hat{a}_{ij}^u]$, $\check{b}_i^e \in [\check{b}_i^u, \check{b}_i^l]$, $\hat{b}_i^e \in [\hat{b}_i^l, \hat{b}_i^u]$.
Then they obtained the following problems by using Theorem 2.16 for (5).
The worst-worst model:

$$\begin{aligned} \max & \sum_{j=1}^{n} \check{c}_j^u x_j, \\ \text{s.t.} \ & \hat{a}_{ij}^u x_j \leq \check{b}_i^u \quad i = 1, ..., m \quad j = 1, ..., n, \\ & x_j \geq 0, \qquad j = 1, ..., n. \end{aligned} \tag{6}$$

The worst-best model:

$$\begin{aligned} \max & \sum_{j=1}^{n} \check{c}_j^l x_j, \\ \text{s.t.} \ & \hat{a}_{ij}^l x_j \leq \check{b}_i^l \quad i = 1, ..., m \quad j = 1, ..., n, \\ & x_j \geq 0, \qquad j = 1, ..., n. \end{aligned} \tag{7}$$

The best-worst model:

$$\begin{aligned} \max & \sum_{j=1}^{n} \hat{c}_j^l x_j, \\ \text{s.t.} \ & \check{a}_{ij}^l x_j \leq \hat{b}_i^l \quad i = 1, ..., m \quad j = 1, ..., n, \\ & x_j \geq 0, \qquad j = 1, ..., n. \end{aligned} \tag{8}$$

The best-best model:

$$\begin{aligned} &\max \sum_{j=1}^{n} \hat{c}_j^u x_j, \\ &\text{s.t.} \quad \check{a}_{ij}^u x_j \le \hat{b}_i^u \qquad i=1,...,m \quad j=1,...,n, \\ &\qquad x_j \ge 0, \qquad\qquad j=1,...,n. \end{aligned} \tag{9}$$

They don't used the core of triangular fuzzy number to calculate (5), (6), (7), (8) and (9) while according to Theorem 2.15, they should have considered the core of triangular fuzzy number as well.
Therefore, we correct problem (5), (6), (7), (8) and (9) by using Grzegorzewski's nearest interval approximation.
Using the Theorem 2.15 for (4), we have:

$$\begin{aligned} &\max \ \sum_{j=1}^{n} [\frac{\check{c}_j^e + \bar{c}_j}{2}, \frac{\hat{c}_j^e + \bar{c}_j}{2}] x_j, \\ &\text{s.t.} \ \ \sum_{j=1}^{n} [\frac{\check{a}_{ij}^e + \bar{a}_{ij}}{2}, \frac{\hat{a}_{ij}^e + \bar{a}_{ij}}{2}] x_j \le [\frac{\check{b}_i^e + \bar{b}_i}{2}, \frac{\hat{b}_i^e + \bar{b}_i}{2}] \ i=1,...,m \ j=1,2,...,n, \\ &\quad x_j \ge 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j=1,2,...,n, \end{aligned} \tag{10}$$

where $\check{c}_j^e \in [\check{c}_j^u, \check{c}_j^l]$, $\hat{c}_j^e \in [\hat{c}_j^l, \hat{c}_j^u]$, $\check{a}_{ij}^e \in [\check{a}_{ij}^u, \check{a}_{ij}^l]$, $\hat{a}_{ij}^e \in [\hat{a}_{ij}^l, \hat{a}_{ij}^u]$, $\check{b}_i^e \in [\check{b}_i^u, \check{b}_i^l]$, $\hat{b}_i^e \in [\hat{b}_i^l, \hat{b}_i^u]$.
Problem (10) is an interval linear programming problem.
Now, by using the core of fuzzy number, Theorems 2.15 and 2.16, we present a corrected model for (3).
Using the Theorem 2.16 for (10), we obtain the worst and the best subproblems, respectively:

$$\begin{aligned} &\max \sum_{j=1}^{n} \frac{\check{c}_j^e + \bar{c}_j}{2} x_j, \\ &\text{s.t.} \quad \frac{\hat{a}_{ij}^e + \bar{a}_{ij}}{2} x_j \le \frac{\check{b}_i^e + \bar{b}_i}{2} \qquad i=1,...,m \quad j=1,2,...,n, \\ &\qquad x_j \ge 0, \qquad\qquad\qquad j=1,...,n. \end{aligned} \tag{11}$$

$$\begin{aligned} &\max \sum_{j=1}^{n} \frac{\hat{c}_j^e + \bar{c}_j}{2} x_j, \\ &\text{s.t.} \quad \frac{\check{a}_{ij}^e + \bar{a}_{ij}}{2} x_j \le \frac{\hat{b}_i^e + \bar{b}_i}{2} \qquad i=1,...,m \quad j=1,2,...,n, \\ &\qquad x_j \ge 0, \qquad\qquad\qquad j=1,...,n. \end{aligned} \tag{12}$$

By using Theorem 2.16 again for (11), we have:
The worst-worst model:

$$\begin{aligned} &\max \sum_{j=1}^{n} \frac{\check{c}_j^u + \bar{c}_j}{2} x_j, \\ &\text{s.t.} \quad \frac{\hat{a}_{ij}^u + \bar{a}_{ij}}{2} x_j \le \frac{\check{b}_i^u + \bar{b}_i}{2} \qquad i=1,...,m \quad j=1,2,...,n, \\ &\qquad x_j \ge 0, \qquad\qquad\qquad j=1,...,n, \end{aligned} \tag{13}$$

by solving (13), $\check{\mathbf{x}}_{opt}^u$ and $\check{z}_{opt}^u$ are achieved.
The worst-best model:

$$\begin{aligned} &\max \sum_{j=1}^{n} \frac{\check{c}_j^l + \bar{c}_j}{2} x_j, \\ &\text{s.t.} \quad \frac{\hat{a}_{ij}^l + \bar{a}_{ij}}{2} x_j \le \frac{\check{b}_i^l + \bar{b}_i}{2} \qquad i=1,...,m \quad j=1,2,...,n, \\ &\qquad x_j \ge 0, \qquad\qquad\qquad j=1,...,n. \end{aligned} \tag{14}$$

$\check{\mathbf{x}}_{opt}^l$ and $\check{z}_{opt}^l$ are achieved by solving (14).
Now, by using Theorem 2.16 again, the best-worst and the best-best models are obtained from (12) as follows:

The best-worst model:

$$\begin{aligned} \max & \sum_{j=1}^{n} \frac{\hat{c}_j^l + \bar{c}_j}{2} x_j, \\ \text{s.t.} \quad & \frac{\check{a}_{ij}^l + \bar{a}_{ij}}{2} x_j \le \frac{\hat{b}_i^l + \bar{b}_i}{2} \qquad i = 1, ..., m \quad j = 1, 2, ..., n, \\ & x_j \ge 0, \qquad\qquad\qquad\qquad j = 1, ..., n, \end{aligned} \tag{15}$$

from (15), $\hat{\mathbf{x}}_{opt}^l$ and $\hat{z}_{opt}^l$ are obtained, and
The best-best model:

$$\begin{aligned} \max & \sum_{j=1}^{n} \frac{\hat{c}_j^u + \bar{c}_j}{2} x_j, \\ \text{s.t.} \quad & \frac{\check{a}_{ij}^u + \bar{a}_{ij}}{2} x_j \le \frac{\hat{b}_i^u + \bar{b}_i}{2} \qquad i = 1, ..., m \quad j = 1, 2, ..., n, \\ & x_j \ge 0, \qquad\qquad\qquad\qquad j = 1, ..., n, \end{aligned} \tag{16}$$

by solving (16), $\hat{\mathbf{x}}_{opt}^u$ and $\hat{z}_{opt}^u$ are achieved.
To get $\bar{z}_{opt}$, problem (17) is solved, which is called the middle problem.

$$\begin{aligned} \max & \sum_{j=1}^{n} \bar{c}_j x_j, \\ \text{s.t.} \quad & \bar{a}_{ij} x_j \le \bar{b}_i, \qquad i = 1, ..., m \quad j = 1, 2, ..., n, \\ & x_j \ge 0, \qquad\qquad j = 1, ..., n. \end{aligned} \tag{17}$$

By solving (17), $\bar{\mathbf{x}}_{opt}$ and $\bar{z}_{opt}$ are obtained.
Therefore, by solving (13), (14), (15) (16) and (17), we obtain the optimal solution $\tilde{\tilde{\mathbf{x}}}_{opt} = ((\check{\mathbf{x}}_{opt}^l, \bar{\mathbf{x}}_{opt}, \hat{\mathbf{x}}_{opt}^l), (\check{\mathbf{x}}_{opt}^u, \bar{\mathbf{x}}_{opt}, \hat{\mathbf{x}}_{opt}^u))$ and optimal value $\tilde{\tilde{z}}_{opt} = ((\check{z}_{opt}^l, \bar{z}_{opt}, \hat{z}_{opt}^l), (\check{z}_{opt}^u, \bar{z}_{opt}, \hat{z}_{opt}^u))$ for (3).

4 Interval type-2 triangular fuzzy bilevel linear programming problem

Consider the bilevel linear programming problem wherein all the parameters (the objective function, technological coefficients and the resource values) are expressed as interval type-2 triangular fuzzy numbers:

$$\begin{aligned} \max_{\mathbf{x}_1 \in X_1} \quad & \tilde{\tilde{\mathbf{c}}}_1 \mathbf{x}_1 + \tilde{\tilde{\mathbf{d}}}_1 \mathbf{x}_2 \\ \text{s.t.} \quad & \max_{\mathbf{x}_2 \in X_2} \ \tilde{\tilde{\mathbf{c}}}_2 \mathbf{x}_1 + \tilde{\tilde{\mathbf{d}}}_2 \mathbf{x}_2 \\ & \text{s.t.} \quad \tilde{\tilde{A}} \mathbf{x}_1 + \tilde{\tilde{B}} \mathbf{x}_2 \le \tilde{\tilde{\mathbf{b}}} \\ & \qquad \mathbf{x}_1, \mathbf{x}_2 \ge 0, \end{aligned} \tag{18}$$

where $\tilde{\tilde{\mathbf{c}}}_1 = \big((\check{\mathbf{c}}_1^l, \bar{\mathbf{c}}_1, \hat{\mathbf{c}}_1^l), (\check{\mathbf{c}}_1^u, \bar{\mathbf{c}}_1, \hat{\mathbf{c}}_1^u)\big)$, $\tilde{\tilde{\mathbf{d}}}_1 = \big((\check{\mathbf{d}}_1^l, \bar{\mathbf{d}}_1, \hat{\mathbf{d}}_1^l), (\check{\mathbf{d}}_1^u, \bar{\mathbf{d}}_1, \hat{\mathbf{d}}_1^u)\big)$, $\tilde{\tilde{\mathbf{c}}}_2 = \big((\check{\mathbf{c}}_2^l, \bar{\mathbf{c}}_2, \hat{\mathbf{c}}_2^l), (\check{\mathbf{c}}_{21}^u, \bar{\mathbf{c}}_2, \hat{\mathbf{c}}_2^u)\big)$,
$\tilde{\tilde{\mathbf{d}}}_2 = \big((\check{\mathbf{d}}_2^l, \bar{\mathbf{d}}_2, \hat{\mathbf{d}}_2^l), (\check{\mathbf{d}}_2^u, \bar{\mathbf{d}}_2, \hat{\mathbf{d}}_2^u)\big)$, $\tilde{\tilde{A}} = \big((\check{A}^l, \bar{A}, \hat{A}^l), (\check{A}^u, \bar{A}, \hat{A}^u)\big)$, $\tilde{\tilde{B}} = \big((\check{B}^l, \bar{B}, \hat{B}^l), (\check{B}^u, \bar{B}, \hat{B}^u)\big)$ and
$\tilde{\tilde{\mathbf{b}}} = \big((\check{\mathbf{b}}^l, \bar{\mathbf{b}}, \hat{\mathbf{b}}^l), (\check{\mathbf{b}}^u, \bar{\mathbf{b}}, \hat{\mathbf{b}}^u)\big)$.
In our proposed algorithm, the solution of (18) is obtained by solving five bilevel linear programming problems and finally, we get the optimal solution as interval type-2 triangular fuzzy numbers $\tilde{\tilde{Z}} = \big((\check{Z}^l, \bar{Z}, \hat{Z}^l), (\check{Z}^u, \bar{Z}, \hat{Z}^u)\big)$ for leader objective function and $\tilde{\tilde{z}} = \big((\check{z}^l, \bar{z}, \hat{z}^l), (\check{z}^u, \bar{z}, \hat{z}^u)\big)$ for follower objective function.
According to the proposed method, we first solve problem (19) to obtain $\bar{Z}$ and $\bar{z}$, which we call the middle problem,

$$\begin{aligned} \max_{\mathbf{x}_1 \in X_1} \quad & \bar{\mathbf{c}}_1 \mathbf{x}_1 + \bar{\mathbf{d}}_1 \mathbf{x}_2 \\ \text{s.t.} \quad & \max_{\mathbf{x}_2 \in X_2} \ \bar{\mathbf{c}}_2 \mathbf{x}_1 + \bar{\mathbf{d}}_2 \mathbf{x}_2 \\ & \text{s.t.} \quad \bar{A} \mathbf{x}_1 + \bar{B} \mathbf{x}_2 \le \bar{\mathbf{b}} \\ & \qquad \mathbf{x}_1, \mathbf{x}_2 \ge 0, \end{aligned} \tag{19}$$

(19) is a bilevel linear programming problem that can be easily solved using the K-th best method and $\bar{\mathbf{x}}_{1opt}$, $\bar{\mathbf{x}}_{2opt}$, $\bar{Z}_{opt}$ and $\bar{z}_{opt}$ are obtained.
Since an interval type-2 triangular fuzzy number is formed of the infinite union of the characteristic triangular fuzzy numbers, the traditional embedded triangular fuzzy numbers are used for all the interval type-2 triangular fuzzy number in (18). So (18) can be written as following triangular fuzzy bilevel linear programming problem:

$$\begin{aligned}
&\max_{\mathbf{x}_1\in X_1} \ \mathbf{c}_1^e\mathbf{x}_1+\mathbf{d}_1^e\mathbf{x}_2\\
&\text{s.t.} \ \max_{\mathbf{x}_2\in X_2} \ \mathbf{c}_2^e\mathbf{x}_1+\mathbf{d}_2^e\mathbf{x}_2\\
&\qquad \text{s.t.} \ A^e\mathbf{x}_1+B^e\mathbf{x}_2\le \mathbf{b}^e\\
&\qquad\quad \mathbf{x}_1,\mathbf{x}_2\ge 0,
\end{aligned} \tag{20}$$

that $\tilde{\mathbf{c}}_1^e\in \mathrm{FOU}(\tilde{\tilde{\mathbf{c}}}_1)$, $\tilde{\mathbf{d}}_1^e\in \mathrm{FOU}(\tilde{\tilde{\mathbf{d}}}_1)$, $\tilde{\mathbf{c}}_2^e\in \mathrm{FOU}(\tilde{\tilde{\mathbf{c}}}_2)$, $\tilde{\mathbf{d}}_2^e\in \mathrm{FOU}(\tilde{\tilde{\mathbf{d}}}_2)$, $\tilde{A}^e\in \mathrm{FOU}(\tilde{\tilde{A}})$, $\tilde{B}^e\in \mathrm{FOU}(\tilde{\tilde{B}})$ and $\tilde{\mathbf{b}}^e\in \mathrm{FOU}(\tilde{\tilde{\mathbf{b}}})$ are the triangular fuzzy numbers.
First, we transform problem (20) into a problem where all the coefficients are intervals and then solve it.

Using Grzegorzewski's nearest interval approximation for (20) we have:

$$\begin{aligned}
&\max_{\mathbf{x}_1\in X_1} [\frac{\check{\mathbf{c}}_1^e+\bar{\mathbf{c}}_1}{2},\frac{\hat{\mathbf{c}}_1^e+\bar{\mathbf{c}}_1}{2}]\mathbf{x}_1+[\frac{\check{\mathbf{d}}_1^e+\bar{\mathbf{d}}_1}{2},\frac{\hat{\mathbf{d}}_1^e+\bar{\mathbf{d}}_1}{2}]\mathbf{x}_2\\
&\text{s.t.} \ \max_{\mathbf{x}_2\in X_2} [\frac{\check{\mathbf{c}}_2^e+\bar{\mathbf{c}}_2}{2},\frac{\hat{\mathbf{c}}_2^e+\bar{\mathbf{c}}_2}{2}]\mathbf{x}_1+[\frac{\check{\mathbf{d}}_2^e+\bar{\mathbf{d}}_2}{2},\frac{\hat{\mathbf{d}}_2^e+\bar{\mathbf{d}}_2}{2}]\mathbf{x}_2\\
&\qquad \text{s.t.} \ [\frac{\check{A}^e+\bar{A}}{2},\frac{\hat{A}^e+\bar{A}}{2}]\mathbf{x}_1+[\frac{\check{B}^e+\bar{B}}{2},\frac{\hat{B}^e+\bar{B}}{2}]\mathbf{x}_2\le [\frac{\check{\mathbf{b}}^e+\bar{\mathbf{b}}}{2},\frac{\hat{\mathbf{b}}^e+\bar{\mathbf{b}}}{2}]\\
&\qquad\quad \mathbf{x}_1,\mathbf{x}_2\ge 0,
\end{aligned} \tag{21}$$

where $\check{\mathbf{c}}_1^e\in[\check{\mathbf{c}}_1^u,\check{\mathbf{c}}_1^l]$, $\hat{\mathbf{c}}_1^e\in[\hat{\mathbf{c}}_1^l,\hat{\mathbf{c}}_1^u]$, $\check{\mathbf{d}}_1^e\in[\check{\mathbf{d}}_1^u,\check{\mathbf{d}}_1^l]$, $\hat{\mathbf{d}}_1^e\in[\hat{\mathbf{d}}_1^l,\hat{\mathbf{d}}_1^u]$, $\check{\mathbf{c}}_2^e\in[\check{\mathbf{c}}_2^u,\check{\mathbf{c}}_2^l]$, $\hat{\mathbf{c}}_2^e\in[\hat{\mathbf{c}}_2^l,\hat{\mathbf{c}}_2^u]$, $\check{\mathbf{d}}_2^e\in[\check{\mathbf{d}}_2^u,\check{\mathbf{d}}_2^l]$, $\hat{\mathbf{d}}_2^e\in[\hat{\mathbf{d}}_2^l,\hat{\mathbf{d}}_2^u]$, $\check{A}^e\in[\check{A}^u,\check{A}^l]$, $\hat{A}^e\in[\hat{A}^l,\hat{A}^u]$, $\check{B}^e\in[\check{B}^u,\check{B}^l]$, $\hat{B}^e\in[\hat{B}^l,\hat{B}^u]$, $\check{\mathbf{b}}^e\in[\check{\mathbf{b}}^u,\check{\mathbf{b}}^l]$ and $\hat{\mathbf{b}}^e\in[\hat{\mathbf{b}}^l,\hat{\mathbf{b}}^u]$.
According to [40], (21) can be divided into the following two the worst and the best subproblems, respectively:

$$\begin{aligned}
&\max_{\mathbf{x}_1\in X_1} \ \frac{\check{\mathbf{c}}_1^e+\bar{\mathbf{c}}_1}{2}\mathbf{x}_1+\frac{\check{\mathbf{d}}_1^e+\bar{\mathbf{d}}_1}{2}\mathbf{x}_2\\
&\text{s.t.} \ \max_{\mathbf{x}_2\in X_2} \ \frac{\check{\mathbf{c}}_2^e+\bar{\mathbf{c}}_2}{2}\mathbf{x}_1+\frac{\check{\mathbf{d}}_2^e+\bar{\mathbf{d}}_2}{2}\mathbf{x}_2\\
&\qquad \text{s.t.} \ \frac{\hat{A}^e+\bar{A}}{2}\mathbf{x}_1+\frac{\hat{B}^e+\bar{B}}{2}\mathbf{x}_2\le \frac{\check{\mathbf{b}}^e+\bar{\mathbf{b}}}{2}\\
&\qquad \mathbf{x}_1,\mathbf{x}_2\ge 0,
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\max_{\mathbf{x}_1\in X_1} \ \frac{\hat{\mathbf{c}}_1^e+\bar{\mathbf{c}}_1}{2}\mathbf{x}_1+\frac{\hat{\mathbf{d}}_1^e+\bar{\mathbf{d}}_1}{2}\mathbf{x}_2\\
&\text{s.t.} \ \max_{\mathbf{x}_2\in X_2} \ \frac{\hat{\mathbf{c}}_2^e+\bar{\mathbf{c}}_2}{2}\mathbf{x}_1+\frac{\hat{\mathbf{d}}_2^e+\bar{\mathbf{d}}_2}{2}\mathbf{x}_2\\
&\qquad \text{s.t.} \ \frac{\check{A}^e+\bar{A}}{2}\mathbf{x}_1+\frac{\check{B}^e+\bar{B}}{2}\mathbf{x}_2\le \frac{\hat{\mathbf{b}}^e+\bar{\mathbf{b}}}{2}\\
&\qquad \mathbf{x}_1,\mathbf{x}_2\ge 0.
\end{aligned} \tag{23}$$

Again, by using the proposed method in [40] for (22), we have:
The worst-worst model:

$$\begin{aligned}
&\max_{\mathbf{x}_1\in X_1} \ \frac{\check{\mathbf{c}}_1^u+\bar{\mathbf{c}}_1}{2}\mathbf{x}_1+\frac{\check{\mathbf{d}}_1^u+\bar{\mathbf{d}}_1}{2}\mathbf{x}_2\\
&\text{s.t.} \ \max_{\mathbf{x}_2\in X_2} \ \frac{\check{\mathbf{c}}_2^u+\bar{\mathbf{c}}_2}{2}\mathbf{x}_1+\frac{\check{\mathbf{d}}_2^u+\bar{\mathbf{d}}_2}{2}\mathbf{x}_2\\
&\qquad \text{s.t.} \ \frac{\hat{A}^u+\bar{A}}{2}\mathbf{x}_1+\frac{\hat{B}^u+\bar{B}}{2}\mathbf{x}_2\le \frac{\check{\mathbf{b}}^u+\bar{\mathbf{b}}}{2}\\
&\qquad \mathbf{x}_1,\mathbf{x}_2\ge 0,
\end{aligned} \tag{24}$$

$\check{\mathbf{x}}_1^u$, $\check{\mathbf{x}}_2^u$, $\check{Z}^u$ and $\check{z}^u$ are achieved by solving bilevel linear programming problem (24).
The worst-best model:

$$\begin{aligned}\max_{\mathbf{x}_1\in X_1}\ & \frac{\check{\mathbf{c}}_1^l x+\bar{\mathbf{c}}_1}{2}\mathbf{x}_1+\frac{\check{\mathbf{d}}_1^l x+\bar{\mathbf{d}}_1}{2}\mathbf{x}_2\\ \text{s.t.}\ & \max_{\mathbf{x}_2\in X_2}\ \frac{\check{\mathbf{c}}_2^l+\bar{\mathbf{c}}_2}{2}\mathbf{x}_1+\frac{\check{\mathbf{d}}_2^l+\bar{\mathbf{d}}_2}{2}\mathbf{x}_2\\ & \text{s.t.}\ \ \frac{\hat{A}^l+\bar{A}}{2}\mathbf{x}_1+\frac{\hat{B}^l+\bar{B}}{2}\mathbf{x}_2\le\frac{\check{\mathbf{b}}^l+\bar{\mathbf{b}}}{2}\\ & \mathbf{x}_1,\mathbf{x}_2\ge 0,\end{aligned}\tag{25}$$

(25) is a bilevel linear programming problem that $\check{\mathbf{x}}_1^l$, $\check{\mathbf{x}}_2^l$, $\check{Z}^l$ and $\check{z}^l$ are achieved by solving it.
Now, by using the method in [40], the best-worst and the best-best models are obtained from (23) as follows:
The best-worst model:

$$\begin{aligned}\max_{\mathbf{x}_1\in X_1}\ & \frac{\hat{\mathbf{c}}_1^l+\bar{\mathbf{c}}_1}{2}\mathbf{x}_1+\frac{\hat{\mathbf{d}}_1^l+\bar{\mathbf{d}}_1}{2}\mathbf{x}_2\\ \text{s.t.}\ & \max_{\mathbf{x}_2\in X_2}\ \frac{\hat{\mathbf{c}}_2^l+\bar{\mathbf{c}}_2}{2}\mathbf{x}_1+\frac{\hat{\mathbf{d}}_2^l+\bar{\mathbf{d}}_2}{2}\mathbf{x}_2\\ & \text{s.t.}\ \ \frac{\check{A}^l+\bar{A}}{2}\mathbf{x}_1+\frac{\check{B}^l+\bar{B}}{2}\mathbf{x}_2\le\frac{\hat{\mathbf{b}}^l+\bar{\mathbf{b}}}{2}\\ & \mathbf{x}_1,\mathbf{x}_2\ge 0,\end{aligned}\tag{26}$$

From solving the bilevel linear programming problem (26), $\hat{\mathbf{x}}_1^l$, $\hat{\mathbf{x}}_2^l$, $\hat{Z}^l$ and $\hat{z}^l$ are obtained.
The best-best model:

$$\begin{aligned}\max_{\mathbf{x}_1\in X_1}\ & \frac{\hat{\mathbf{c}}_1^u+\bar{\mathbf{c}}_1}{2}\mathbf{x}_1+\frac{\hat{\mathbf{d}}_1^u+\bar{\mathbf{d}}_1}{2}\mathbf{x}_2\\ \text{s.t.}\ & \max_{\mathbf{x}_2\in X_2}\ \frac{\hat{\mathbf{c}}_2^u+\bar{\mathbf{c}}_2}{2}\mathbf{x}_1+\frac{\hat{\mathbf{d}}_2^u+\bar{\mathbf{d}}_2}{2}\mathbf{x}_2\\ & \text{s.t.}\ \ \frac{\check{A}^u+\bar{A}}{2}\mathbf{x}_1+\frac{\check{B}^u+\bar{B}}{2}\mathbf{x}_2\le\frac{\hat{\mathbf{b}}^u+\bar{\mathbf{b}}}{2}\\ & \mathbf{x}_1,\mathbf{x}_2\ge 0,\end{aligned}\tag{27}$$

By solving (27), $\hat{\mathbf{x}}_1^u$, $\hat{\mathbf{x}}_2^u$, $\hat{Z}^u$ and $\hat{z}^u$ are achieved.
Therefore, by solving (19), (24), (25), (26) and (27), we obtain the optimal solution $\tilde{\tilde{\mathbf{x}}}_{1opt}=((\check{\mathbf{x}}_{1opt}^l,\bar{\mathbf{x}}_{1opt},\hat{\mathbf{x}}_{1opt}^l),(\check{\mathbf{x}}_{1opt}^u,\bar{\mathbf{x}}_{1opt},\hat{\mathbf{x}}_{1opt}^u))$ and $\tilde{\tilde{\mathbf{x}}}_{2opt}=((\check{\mathbf{x}}_{2opt}^l,\bar{\mathbf{x}}_{2opt},\hat{\mathbf{x}}_{2opt}^l),(\check{\mathbf{x}}_{2opt}^u,\bar{\mathbf{x}}_{2opt},\hat{\mathbf{x}}_{2opt}^u))$ and optimal values
$\tilde{\tilde{Z}}_{opt}=((\check{Z}_{opt}^l,\bar{Z}_{opt},\hat{Z}_{opt}^l),(\check{Z}_{opt}^u,\bar{Z}_{opt},\hat{Z}_{opt}^u))$ and $\tilde{\tilde{z}}_{opt}=((\check{z}_{opt}^l,\bar{z}_{opt},\hat{z}_{opt}^l),(\check{z}_{opt}^u,\bar{z}_{opt},\hat{z}_{opt}^u))$ for (18).
The steps of proposed method are shown in the Figure 4 as flowchart.

5 Numerical example

In the real world, the parameters of a problem may not be stated crisp. One of the ways to show this uncertainty is to use fuzzy sets. Although one of the most important tools for decision-making in an imprecise environment is the fuzzy programming model, these models are not suitable for solving complex problems with many uncertainties. Fuzzy systems have a limited ability to show the effect of uncertainty due to having crisp membership functions. For this reason, type-2 fuzzy sets were introduced [81]. The interval type-2 fuzzy sets are a special type of type-2 fuzzy sets. Interval type-2 fuzzy sets express more imprecise information than fuzzy sets but compared to type-2 fuzzy sets, it is less complicated and easier to understand. In this paper, we consider a fuzzy bilevel linear programming problem with interval type-2 triangular fuzzy parameters. Although to solve this problem using the proposed method, five bilevel linear programming problem must be solved but given that, the parameters of the problem are interval type-2 triangular fuzzy numbers, they show more uncertainty than the fuzzy sets and have less complexity than the type-2 fuzzy sets.
In this section, we present examples and solve them step by step. In the example 5.1 we express a fuzzy bilevel linear programming problem with interval type-2 symmetric triangular fuzzy numbers, and in the example 5.2, we express the fuzzy bilevel linear programming problem with interval type-2 triangular fuzzy numbers, which are not necessarily

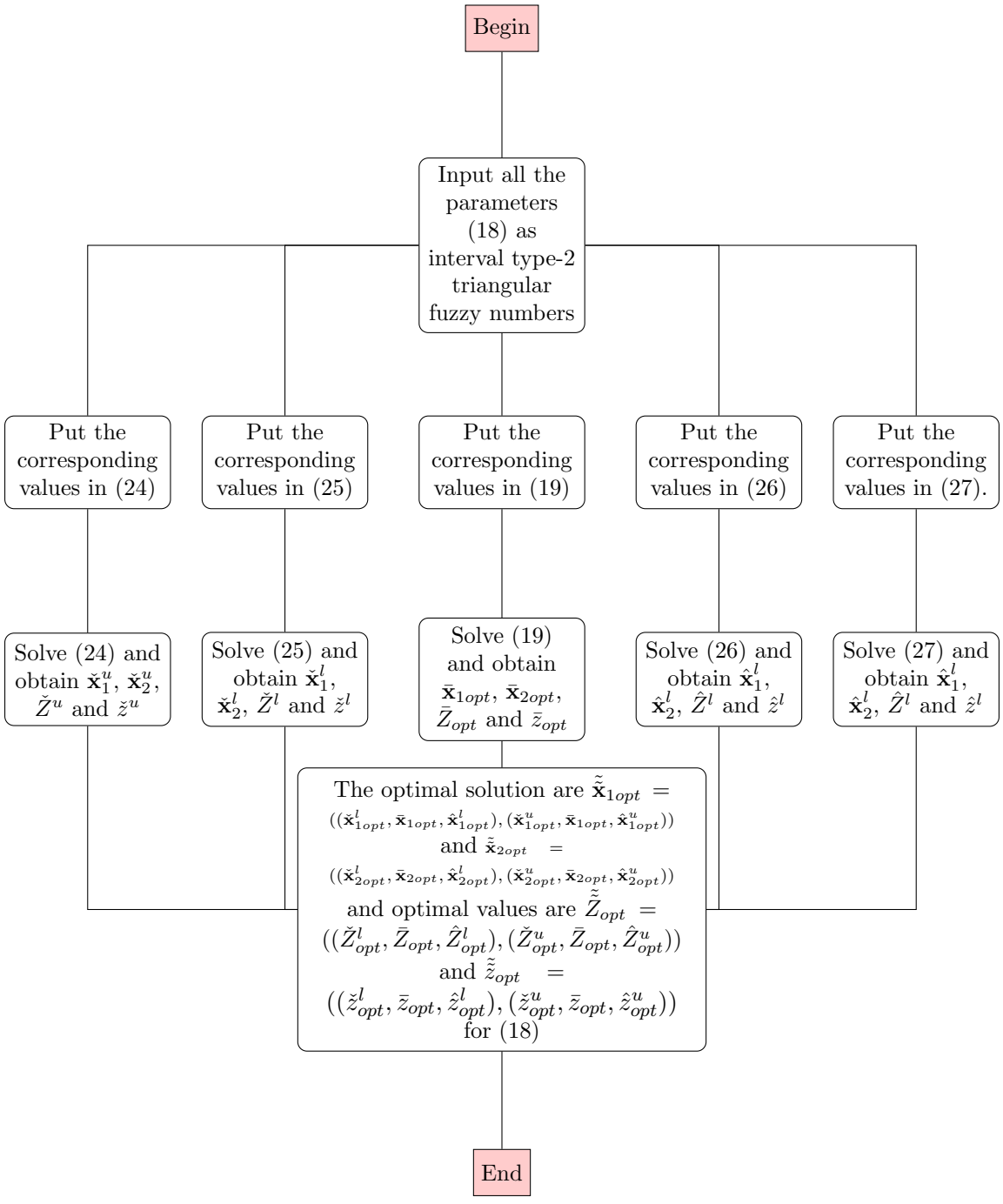


Figure 4: Flowchart of the proposed method

symmetric. In the Example 5.3 an interval type-2 triangular fuzzy bilevel linear programming problem with six variables has been solved. Finally, we solve the application example to show the efficiency of the proposed method in Example 6.1.

In all these examples, to solve each problem, we converted interval type-2 triangular fuzzy bilevel linear programming problem into an interval bilevel linear programming problem using Grzegorzewski's nearest interval approximation. In the next step, we achieved five bilevel linear programming problems, and by solving them, the optimal solution of interval type-2 triangular fuzzy bilevel linear programming problem was obtained as interval type-2 triangular fuzzy number, and this shows the effectiveness of the proposed method under all conditions.

Example 5.1. *Consider the fuzzy bilevel linear programming problem with interval type-2 triangular fuzzy parameters as:*

$$\begin{aligned}\max_x\ & \tilde{\tilde{c}}_1x+\tilde{\tilde{d}}_1y\\ & s.t.\ \max_y\ \tilde{\tilde{c}}_2x+\tilde{\tilde{d}}_2y\\ & \quad s.t.\ \tilde{\tilde{a}}_{i1}x+\tilde{\tilde{a}}_{i2}y\le\tilde{\tilde{b}}_i \quad i=1,2\\ & \quad x,y\ge 0,\end{aligned}\tag{28}$$

where,
$\tilde{\tilde{c}}_1=\big((\check{c}_1^l,\bar{c}_1,\hat{c}_1^l),(\check{c}_1^u,\bar{c}_1,\hat{c}_1^u)\big)=\big((1.5,2,2.5),(1,2,3)\big)$,
$\tilde{\tilde{d}}_1=\big((\check{d}_1^l,\bar{d}_1,\hat{d}_1^l),(\check{d}_1^u,\bar{d}_1,\hat{d}_1^u)\big)=\big((2.5,3,3.5),(2,3,4)\big)$,
$\tilde{\tilde{c}}_2=\big((\check{c}_2^l,\bar{c}_2,\hat{c}_2^l),(\check{c}_2^u,\bar{c}_2,\hat{c}_2^u)\big)=\big((1,1.5,2),(0,1.5,3)\big)$,
$\tilde{\tilde{d}}_2=\big((\check{d}_2^l,\bar{d}_2,\hat{d}_2^l),(\check{d}_2^u,\bar{d}_2,\hat{d}_2^u)\big)=\big((2,3,4),(1,3,5)\big)$,
$\tilde{\tilde{a}}_{11}=\big((\check{a}_{11}^l,\bar{a}_{11},\hat{a}_{11}^l),(\check{a}_{11}^u,\bar{a}_{11},\hat{a}_{11}^u)\big)=\big((0.5,1,1.5),(0,1,2)\big)$,
$\tilde{\tilde{a}}_{12}=\big((\check{a}_{12}^l,\bar{a}_{12},\hat{a}_{12}^l),(\check{a}_{12}^u,\bar{a}_{12},\hat{a}_{12}^u)\big)=\big((3,4,5),(2,4,6)\big)$,
$\tilde{\tilde{b}}_1=\big((\check{b}_1^l,\bar{b}_1,\hat{b}_1^l),(\check{b}_1^u,\bar{b}_1,\hat{b}_1^u)\big)=\big((9,11,13),(7,11,15)\big)$,
$\tilde{\tilde{a}}_{21}=\big((\check{a}_{21}^l,\bar{a}_{21},\hat{a}_{21}^l),(\check{a}_{21}^u,\bar{a}_{21},\hat{a}_{21}^u)\big)=\big((3,4,5),(2,4,6)\big)$,
$\tilde{\tilde{a}}_{22}=\big((\check{a}_{22}^l,\bar{a}_{22},\hat{a}_{22}^l),(\check{a}_{22}^u,\bar{a}_{22},\hat{a}_{22}^u)\big)=\big((1.5,2.5,3.5),(0.5,2.5,4.5)\big)$ *and*
$\tilde{\tilde{b}}_2=\big((\check{b}_2^l,\bar{b}_2,\hat{b}_2^l),(\check{b}_2^u,\bar{b}_2,\hat{b}_2^u)\big)=\big((8,12,16),(4,12,20)\big)$
are interval type-2 triangular fuzzy numbers.
According to the problem data, the middle problem gets as follows:

$$\begin{aligned}\max_x\ & 2x+3y\\ & s.t.\ \max_y\ 1.5x+3y\\ & \quad s.t.\ x+4y\le 11\\ & \quad 4x+2.5y\le 12\\ & \quad x,y\ge 0.\end{aligned}\tag{29}$$

By solving (29), we obtain $\bar{x}_{opt}=1.51851$, $\bar{y}_{opt}=2.37037$ *and optimal values are* $\bar{Z}_{opt}=10.14815$, *and* $\bar{z}_{opt}=9.38888$.
Now, by using Grzegorzewski's nearest interval approximation we have:
1. The worst-worst model:

$$\begin{aligned}\max_x\ & 1.5x+2.5y\\ & s.t.\ \max_y\ 0.75x+2y\\ & \quad s.t.\ 1.5x+5y\le 9\\ & \quad 5x+3.5y\le 8\\ & \quad x,y\ge 0\end{aligned}\tag{30}$$

From (30), $\check{x}_{opt}^u=0.43037$, $\check{y}_{opt}^u=1.67088$, $\check{Z}_{opt}^u=4.82278$, *and* $\check{z}_{opt}^u=3.66455$ *are achieved.*
2. The worst-best model:

$$\begin{aligned}\max_x\ & 1.75x+2.75y\\ & s.t.\ \max_y\ 1.25x+2.5y\\ & \quad s.t.\ 1.25x+4.5y\le 10\end{aligned}$$

$$\begin{aligned} & 4.5x + 3y \leq 10 \\ & x, y \geq 0, \end{aligned} \tag{31}$$

which $\check{x}^l_{opt} = 0.90909$, $\check{y}^l_{opt} = 1.96969$, $\check{Z}^l_{opt} = 7.00757$, *and* $\check{z}^l_{opt} = 6.06060$ *are obtained by solving (31).*
3. The best-worst model:

$$\begin{aligned} \max_x \; & 2.25x + 3.25y \\ & s.t. \; \max_y \; 1.75x + 3.5y \\ & \quad s.t. \; 0.75x + 3.5y \leq 12 \\ & \quad 3.5x + 2y \leq 14 \\ & \quad x, y \geq 0, \end{aligned} \tag{32}$$

by solving problem (32), $\hat{x}^l_{opt} = 2.32558$, $\hat{y}^l_{opt} = 2.93023$, $\hat{Z}^l_{opt} = 14.75581$, *and* $\hat{z}^l_{opt} = 14.32558$ *are obtained.*
4. The best-best model:

$$\begin{aligned} \max_x \; & 2.5x + 3.5y \\ & s.t. \; \max_y \; 2.25x + 4y \\ & \quad s.t. \; 0.5x + 3y \leq 13 \\ & \quad 3x + 1.5y \leq 16 \\ & \quad x, y \geq 0. \end{aligned} \tag{33}$$

By solving (33), we achieve $\hat{x}^u_{opt} = 3.45454$, $\hat{y}^u_{opt} = 3.75757$, $\hat{Z}^u_{opt} = 21.78788$, *and* $\hat{z}^u_{opt} = 22.80303$.
Then for problem (28) by using Grzegorzewski's nearest interval approximation, we have:

$$\tilde{\tilde{x}}_{opt} = \big((0.90909, 1.51851, 2.32558), (0.43037, 1.51851, 3.45454)\big),$$

$$\tilde{\tilde{y}}_{opt} = \big((1.96969, 2.37037, 2.93023), (1.67088, 2.37037, 3.75757)\big),$$

$$\tilde{\tilde{Z}}_{opt} = \big((7.70075, 10.14815, 14.75581), (4.82278, 10.14815, 21.78788)\big),$$

and

$$\tilde{\tilde{z}}_{opt} = \big((6.06060, 9.38888, 14.32558), (3.66455, 9.38888, 22.80303)\big).$$

Example 5.2. *Consider the fuzzy bilevel linear programming problem with interval type-2 triangular fuzzy numbers as:*

$$\begin{aligned} \max_x \; & \tilde{\tilde{c}}_1 x + \tilde{\tilde{d}}_1 y \\ & s.t. \; \max_y \; \tilde{\tilde{c}}_2 x + \tilde{\tilde{d}}_2 y \\ & \quad s.t. \; \tilde{\tilde{a}}_{i1} x + \tilde{\tilde{a}}_{i2} y \leq \tilde{\tilde{b}}_i \quad i = 1, 2 \\ & \quad x, y \geq 0, \end{aligned} \tag{34}$$

where,
$\tilde{\tilde{c}}_1 = \big((\check{c}^l_1, \bar{c}_1, \hat{c}^l_1), (\check{c}^u_1, \bar{c}_1, \hat{c}^u_1)\big) = \big((2.5, 3, 4), (2, 3, 5)\big)$,
$\tilde{\tilde{d}}_1 = \big((\check{d}^l_1, \bar{d}_1, \hat{d}^l_1), (\check{d}^u_1, \bar{d}_1, \hat{d}^u_1)\big) = \big((2, 4, 5), (1, 4, 6)\big)$,
$\tilde{\tilde{c}}_2 = \big((\check{c}^l_2, \bar{c}_2, \hat{c}^l_2), (\check{c}^u_2, \bar{c}_2, \hat{c}^u_2)\big) = \big((0.5, 1, 2), (0.25, 1, 3)\big)$,
$\tilde{\tilde{d}}_2 = \big((\check{d}^l_2, \bar{d}_2, \hat{d}^l_2), (\check{d}^u_2, \bar{d}_2, \hat{d}^u_2)\big) = \big((3, 5, 6), (2, 5, 7)\big)$,
$\tilde{\tilde{a}}_{11} = \big((\check{a}^l_{11}, \bar{a}_{11}, \hat{a}^l_{11}), (\check{a}^u_{11}, \bar{a}_{11}, \hat{a}^u_{11})\big) = \big((1, 2, 3), (0.5, 2, 4)\big)$,
$\tilde{\tilde{a}}_{12} = \big((\check{a}^l_{12}, \bar{a}_{12}, \hat{a}^l_{12}), (\check{a}^u_{12}, \bar{a}_{12}, \hat{a}^u_{12})\big) = \big((5, 6, 7), (4, 6, 8)\big)$,
$\tilde{\tilde{b}}_1 = \big((\check{b}^l_1, \bar{b}_1, \hat{b}^l_1), (\check{b}^u_1, \bar{b}_1, \hat{b}^u_1)\big) = \big((6, 8, 9), (5, 8, 10)\big)$,
$\tilde{\tilde{a}}_{21} = \big((\check{a}^l_{21}, \bar{a}_{21}, \hat{a}^l_{21}), (\check{a}^u_{21}, \bar{a}_{21}, \hat{a}^u_{21})\big) = \big((3, 5, 6), (2, 5, 7)\big)$,
$\tilde{\tilde{a}}_{22} = \big((\check{a}^l_{22}, \bar{a}_{22}, \hat{a}^l_{22}), (\check{a}^u_{22}, \bar{a}_{22}, \hat{a}^u_{22})\big) = \big((3, 4, 6), (2, 4, 8)\big)$ *and*
$\tilde{\tilde{b}}_2 = \big((\check{b}^l_2, \bar{b}_2, \hat{b}^l_2)\big), (\check{b}^u_2, \bar{b}_2, \hat{b}^u_2) = \big((5, 6, 8), (4, 6, 9)\big)$
are interval type-2 triangular fuzzy numbers.
According to the problem data, the middle problem gets as follows:

$$\max_x 3x + 4y$$

$$\begin{aligned} s.t.\ \max_{y}\ & x+5y \\ & s.t.\ 2x+6y\le 8 \\ & 5x+4y\le 6 \\ & x,y\ge 0. \end{aligned} \tag{35}$$

By solving (35), we obtain $\bar{x}_{opt}=0.1818$, $\bar{y}_{opt}=1.2727$ *and optimal values are* $\bar{Z}_{opt}=5.6363$, *and* $\bar{z}_{opt}=6.5454$. *Now, by using proposed method we have:*

1. *The worst-worst model:*

$$\begin{aligned} \max_{x}\ & 2.5x+2.5y \\ & s.t.\ \max_{y}\ 0.625x+3.5y \\ & \quad s.t.\ 3x+7y\le 6.5 \\ & \quad 6x+6y\le 5 \\ & \quad x,y\ge 0 \end{aligned} \tag{36}$$

From (36), $\check{x}^u_{opt}=0$, $\check{y}^u_{opt}=0.8333$, $\check{Z}^u_{opt}=2.0833$, *and* $\check{z}^u_{opt}=2.9166$ *are achieved.*

2. *The worst-best model:*

$$\begin{aligned} \max_{x}\ & 2.75x+3y \\ & s.t.\ \max_{y}\ 0.75x+4y \\ & \quad s.t.\ 2.5x+6.5y\le 7 \\ & \quad 5.5x+5y\le 5.5 \\ & \quad x,y\ge 0, \end{aligned} \tag{37}$$

which $\check{x}^l_{opt}=0.3225$, $\check{y}^l_{opt}=1.0645$, $\check{Z}^l_{opt}=3.2822$, *and* $\check{z}^l_{opt}=4.2822$ *are obtained by solving (37).*

3. *The best-worst model:*

$$\begin{aligned} \max_{x}\ & 3.5x+4.5y \\ & s.t.\ \max_{y}\ 1.5x+5.5y \\ & \quad s.t.\ 1.5x+5.5y\le 8.5 \\ & \quad 4x+3.5y\le 7 \\ & \quad x,y\ge 0, \end{aligned} \tag{38}$$

by solving problem (38), $\hat{x}^l_{opt}=0.5223$, $\hat{y}^l_{opt}=1.4029$, $\hat{Z}^l_{opt}=8.1417$, *and* $\hat{z}^l_{opt}=8.4996$ *are obtained.*

4. *The best-best model:*

$$\begin{aligned} \max_{x}\ & 4x+5y \\ & s.t.\ \max_{y}\ 2x+6y \\ & \quad s.t.\ 1.25x+5y\le 9 \\ & \quad 3.5x+3y\le 7.5 \\ & \quad x,y\ge 0. \end{aligned} \tag{39}$$

By solving (39), we achieve $\hat{x}^u_{opt}=0.7636$, $\hat{y}^u_{opt}=1.6090$, $\hat{Z}^u_{opt}=11.1$, *and* $\hat{z}^u_{opt}=11.1818$.
Then for problem (34) by using Grzegorzewski's nearest interval approximation, we have:

$$\tilde{\bar{x}}_{opt}=\big((0.3225,0.1818,0.5223),(0,0.1818,0.7636)\big),$$

$$\tilde{\bar{y}}_{opt}=\big((1.0645,1.2727,1.4029),(0.8333,1.2727,1.6090)\big),$$

$$\tilde{\bar{Z}}_{opt}=\big((3.2822,5.6363,8.1417),(2.0833,5.6363,11.1)\big),$$

and

$$\tilde{\bar{z}}_{opt}=\big((4.2822,6.5454,8.4996),(2.9166,6.5454,11.1818)\big).$$

Example 5.3. *Consider the fuzzy bilevel linear programming problem with interval type-2 triangular fuzzy numbers as:*

$$\max_{x_1,x_2,x_3}\ \tilde{\bar{c}}_{11}x_1+\tilde{\bar{c}}_{12}x_2+\tilde{\bar{c}}_{13}x_3+\tilde{\bar{d}}_{11}y_1+\tilde{\bar{d}}_{12}y_2+\tilde{\bar{d}}_{13}y_3$$

$$\begin{aligned} s.t. \max_{y_1,y_2,y_3} \ & \tilde{\tilde{c}}_{21}x_1 + \tilde{\tilde{c}}_{22}x_2 + \tilde{\tilde{c}}_{23}x_3 + \tilde{\tilde{d}}_{21}y_1 + \tilde{\tilde{d}}_{22}y_2 + \tilde{\tilde{d}}_{23}y_3 \\ & s.t.\ \tilde{\tilde{a}}_{i1}x_1 + \tilde{\tilde{a}}_{i2}x_2 + \tilde{\tilde{a}}_{i3}x_3 + \tilde{\tilde{a}}_{i4}y_1 + \tilde{\tilde{a}}_{i5}y_2 + \tilde{\tilde{a}}_{i6}y_3 \le \tilde{\tilde{b}}_i \quad i = 1,2 \\ & x_1, x_2, x_3, y_1, y_2, y_3 \ge 0, \end{aligned} \tag{40}$$

where,
$\tilde{\tilde{c}}_{11} = \big((\check{c}^l_{11}, \bar{c}_{11}, \hat{c}^l_{11}), (\check{c}^u_{11}, \bar{c}_{11}, \hat{c}^u_{11})\big) = \big((2,3,4),(1,3,5)\big)$,
$\tilde{\tilde{c}}_{12} = \big((\check{c}^l_{12}, \bar{c}_{12}, \hat{c}^l_{12}), (\check{c}^u_{12}, \bar{c}_{12}, \hat{c}^u_{12})\big) = \big((1,1.5,2),(0,1.5,3)\big)$,
$\tilde{\tilde{c}}_{13} = \big((\check{c}^l_{13}, \bar{c}_{13}, \hat{c}^l_{13}), (\check{c}^u_{13}, \bar{c}_{13}, \hat{c}^u_{13})\big) = \big((3,4,6),(2,4,7)\big)$,
$\tilde{\tilde{d}}_{11} = \big((\check{d}^l_{11}, \bar{d}_{11}, \hat{d}^l_{11}), (\check{d}^u_{11}, \bar{d}_{11}, \hat{d}^u_{11})\big) = \big((0.5,1,2),(0,1,3)\big)$,
$\tilde{\tilde{d}}_{12} = \big((\check{d}^l_{12}, \bar{d}_{12}, \hat{d}^l_{12}), (\check{d}^u_{12}, \bar{d}_{12}, \hat{d}^u_{12})\big) = \big((3,4,5),(2,4,6)\big)$,
$\tilde{\tilde{d}}_{13} = \big((\check{d}^l_{13}, \bar{d}_{13}, \hat{d}^l_{13}), (\check{d}^u_{13}, \bar{d}_{13}, \hat{d}^u_{13})\big) = \big((2,3,4),(1,3,5)\big)$,
$\tilde{\tilde{c}}_{21} = \big((\check{c}^l_{21}, \bar{c}_{21}, \hat{c}^l_{21}), (\check{c}^u_{21}, \bar{c}_{21}, \hat{c}^u_{21})\big) = \big((3,4,5),(2,4,6)\big)$,
$\tilde{\tilde{c}}_{22} = \big((\check{c}^l_{22}, \bar{c}_{22}, \hat{c}^l_{22}), (\check{c}^u_{22}, \bar{c}_{22}, \hat{c}^u_{22})\big) = \big((1.5,2,2.5),(1,2,3)\big)$,
$\tilde{\tilde{c}}_{23} = \big((\check{c}^l_{23}, \bar{c}_{23}, \hat{c}^l_{23}), (\check{c}^u_{23}, \bar{c}_{23}, \hat{c}^u_{23})\big) = \big((4,5,7),(3,5,8)\big)$,
$\tilde{\tilde{d}}_{21} = \big((\check{d}^l_{21}, \bar{d}_{21}, \hat{d}^l_{21}), (\check{d}^u_{21}, \bar{d}_{21}, \hat{d}^u_{21})\big) = \big((1,1.5,2.5),(0,1.5,4)\big)$,
$\tilde{\tilde{d}}_{22} = \big((\check{d}^l_{22}, \bar{d}_{22}, \hat{d}^l_{22}), (\check{d}^u_{22}, \bar{d}_{22}, \hat{d}^u_{22})\big) = \big((5,6,7),(3,6,9)\big)$,
$\tilde{\tilde{d}}_{23} = \big((\check{d}^l_{23}, \bar{d}_{23}, \hat{d}^l_{23}), (\check{d}^u_{23}, \bar{d}_{23}, \hat{d}^u_{23})\big) = \big((2,4,5),(1,4,6)\big)$,
$\tilde{\tilde{a}}_{11} = \big((\check{a}^l_{11}, \bar{a}_{11}, \hat{a}^l_{11}), (\check{a}^u_{11}, \bar{a}_{11}, \hat{a}^u_{11})\big) = \big((3,4,5),(2,4,6)\big)$,
$\tilde{\tilde{a}}_{12} = \big((\check{a}^l_{12}, \bar{a}_{12}, \hat{a}^l_{12}), (\check{a}^u_{12}, \bar{a}_{12}, \hat{a}^u_{12})\big) = \big((0,0,0),(0,0,0)\big)$,
$\tilde{\tilde{a}}_{13} = \big((\check{a}^l_{13}, \bar{a}_{13}, \hat{a}^l_{13}), (\check{a}^u_{13}, \bar{a}_{13}, \hat{a}^u_{13})\big) = \big((2,3,5),(1,3,6)\big)$,
$\tilde{\tilde{a}}_{14} = \big((\check{a}^l_{14}, \bar{a}_{14}, \hat{a}^l_{14}), (\check{a}^u_{14}, \bar{a}_{14}, \hat{a}^u_{14})\big) = \big((0.5,1,2),(0,1,3)\big)$,
$\tilde{\tilde{a}}_{15} = \big((\check{a}^l_{15}, \bar{a}_{15}, \hat{a}^l_{15}), (\check{a}^u_{15}, \bar{a}_{15}, \hat{a}^u_{15})\big) = \big((1,2,4),(0,2,5)\big)$,
$\tilde{\tilde{a}}_{16} = \big((\check{a}^l_{16}, \bar{a}_{16}, \hat{a}^l_{16}), (\check{a}^u_{16}, \bar{a}_{16}, \hat{a}^u_{16})\big) = \big((2,3,5),(1,3,6)\big)$,
$\tilde{\tilde{b}}_1 = \big((\check{b}^l_1, \bar{b}_1, \hat{b}^l_1), (\check{b}^u_1, \bar{b}_1, \hat{b}^u_1)\big) = \big((9,10,12),(6,10,14)\big)$,
$\tilde{\tilde{a}}_{21} = \big((\check{a}^l_{21}, \bar{a}_{21}, \hat{a}^l_{21}), (\check{a}^u_{21}, \bar{a}_{21}, \hat{a}^u_{21})\big) = \big((1,1.5,2.5),(0,1.5,3)\big)$,
$\tilde{\tilde{a}}_{22} = \big((\check{a}^l_{22}, \bar{a}_{22}, \hat{a}^l_{22}), (\check{a}^u_{22}, \bar{a}_{22}, \hat{a}^u_{22})\big) = \big((2,3,5),(1,3,6)\big)$,
$\tilde{\tilde{a}}_{23} = \big((\check{a}^l_{23}, \bar{a}_{23}, \hat{a}^l_{23}), (\check{a}^u_{23}, \bar{a}_{23}, \hat{a}^u_{23})\big) = \big((1,2,4),(0,2,5)\big)$,
$\tilde{\tilde{a}}_{24} = \big((\check{a}^l_{24}, \bar{a}_{24}, \hat{a}^l_{24}), (\check{a}^u_{24}, \bar{a}_{24}, \hat{a}^u_{24})\big) = \big((3,4,5),(1,4,6)\big)$,
$\tilde{\tilde{a}}_{25} = \big((\check{a}^l_{25}, \bar{a}_{25}, \hat{a}^l_{25}), (\check{a}^u_{25}, \bar{a}_{25}, \hat{a}^u_{25})\big) = \big((1,2,3),(0,2,4)\big)$,
$\tilde{\tilde{a}}_{26} = \big((\check{a}^l_{26}, \bar{a}_{26}, \hat{a}^l_{26}), (\check{a}^u_{26}, \bar{a}_{26}, \hat{a}^u_{26})\big) = \big((0.5,1,3),(0,1,5)\big)$, *and*
$\tilde{\tilde{b}}_2 = \big((\check{b}^l_2, \bar{b}_2, \hat{b}^l_2), (\check{b}^u_2, \bar{b}_2, \hat{b}^u_2)\big) = \big((14,16,18),(12,16,20)\big)$
are interval type-2 triangular fuzzy numbers.
According to the problem data, the middle problem gets as follows:

$$\begin{aligned} \max_{x_1,x_2,x_3} \ & 3x_1 + 1.5x_2 + 4x_3 + y_1 + 4y_2 + 3y_3 \\ & s.t. \max_{y_1,y_2,y_3} \ 4x_1 + 2x_2 + 5x_3 + 1.5y_1 + 6y_2 + 4y_3 \\ & \quad s.t.\ 4x_1 + 3x_3 + y_1 + 2y_2 + 3y_3 \le 10 \\ & \quad 1.5x_1 + 3x_2 + 2x_3 + 4y_1 + 2y_2 + y_3 \le 16 \\ & \quad x_1, x_2, x_3, y_1, y_2, y_3 \ge 0. \end{aligned} \tag{41}$$

By solving (41), we obtain $\bar{x}_{1opt} = 0$, $\bar{x}_{2opt} = 2$, $\bar{x}_{3opt} = 0$, $\bar{y}_{1opt} = 0$, $\bar{y}_{2opt} = 5$, *and* $\bar{y}_{3opt} = 0$ *and optimal values are* $\bar{Z}_{opt} = 23$, *and* $\bar{z}_{opt} = 34$.
Now, by using Grzegorzewski's nearest interval approximation we have:
1. The worst-worst model:

$$\begin{aligned} \max_{x_1,x_2,x_3} \ & 2x_1 + 0.75x_2 + 3x_3 + 0.5y_1 + 3y_2 + 2y_3 \\ & s.t. \max_{y_1,y_2,y_3} \ 3x_1 + 1.5x_2 + 4x_3 + 0.75y_1 + 4.5y_2 + 2.5y_3 \\ & \quad s.t.\ 5x_1 + 4.5x_3 + 2y_1 + 3.5y_2 + 4.5y_3 \le 8 \\ & \quad 2.25x_1 + 4.5x_2 + 3.5x_3 + 5y_1 + 3y_2 + 3y_3 \le 14 \\ & \quad x_1, x_2, x_3, y_1, y_2, y_3 \ge 0. \end{aligned} \tag{42}$$

From (42), $\check{x}^u_{1opt} = 0$, $\check{x}^u_{2opt} = 1.587302$, $\check{x}^u_{3opt} = 0$, $\check{y}^u_{1opt} = 0$, $\check{y}^u_{2opt} = 2.285714$, $\check{y}^u_{3opt} = 0$, $\check{Z}^u_{opt} = 8.047619$, *and* $\check{z}^u_{opt} = 12.66666$ *are achieved.*
2. The worst-best model:

$$\begin{aligned}\max_{x_1,x_2,x_3}\ & 2.5x_1 + 1.25x_2 + 3.5x_3 + 0.75y_1 + 3.5y_2 + 2.5y_3\\ s.t.\ & \max_{y_1,y_2,y_3}\ 3.5x_1 + 1.75x_2 + 4.5x_3 + 1.25y_1 + 5.5y_2 + 3y_3\\ & \quad s.t.\ 4.5x_1 + 4x_3 + 1.5y_1 + 3y_2 + 4y_3 \le 9.5\\ & \quad 2x_1 + 4x_2 + 3x_3 + 4.5y_1 + 2.5y_2 + 2y_3 \le 15\\ & \quad x_1, x_2, x_3, y_1, y_2, y_3 \ge 0,\end{aligned} \tag{43}$$

which $\check{x}^l_{1opt} = 0$, $\check{x}^l_{2opt} = 1.770833$, $\check{x}^l_{3opt} = 0$, $\check{y}^l_{1opt} = 0$, $\check{y}^l_{2opt} = 3.166667$, $\check{y}^l_{3opt} = 0$, $\check{Z}^l_{opt} = 13.29688$, *and* $\check{z}^l_{opt} = 20.51562$ *are obtained by solving (43).*
3. The best-worst model:

$$\begin{aligned}\max_{x_1,x_2,x_3}\ & 3.5x_1 + 1.75x_2 + 5x_3 + 1.5y_1 + 4.5y_2 + 3.5y_3\\ s.t.\ & \max_{y_1,y_2,y_3}\ 4.5x_1 + 2.25x_2 + 6x_3 + 2y_1 + 6.5y_2 + 4.5y_3\\ & \quad s.t.\ 3.5x_1 + 2.5x_3 + 0.75y_1 + 1.5y_2 + 2.5y_3 \le 11\\ & \quad 1.25x_1 + 2.5x_2 + 1.5x_3 + 3.5y_1 + 1.5y_2 + 0.75y_3 \le 17\\ & \quad x_1, x_2, x_3, y_1, y_2, y_3 \ge 0,\end{aligned} \tag{44}$$

by solving problem (44), $\hat{x}^l_{1opt} = 0$, $\hat{x}^l_{2opt} = 2.4$, $\hat{x}^l_{3opt} = 0$, $\hat{y}^l_{1opt} = 0$, $\hat{y}^l_{2opt} = 7.333333$, $\hat{y}^l_{3opt} = 0$, $\hat{Z}^l_{opt} = 37.2$, *and* $\hat{z}^l_{opt} = 53.06667$ *are obtained.*
4. The best-best model:

$$\begin{aligned}\max_{x_1,x_2,x_3}\ & 4x_1 + 2.25x_2 + 5.5x_3 + 2y_1 + 5y_2 + 4y_3\\ s.t.\ & \max_{y_1,y_2,y_3}\ 5x_1 + 2.5x_2 + 6.5x_3 + 2.75y_1 + 7.5y_2 + 5y_3\\ & \quad s.t.\ 3x_1 + 2x_3 + 0.5y_1 + y_2 + 2y_3 \le 12\\ & \quad 0.75x_1 + 2x_2 + x_3 + 2.5y_1 + y_2 + 0.5y_3 \le 18\\ & \quad x_1, x_2, x_3, y_1, y_2, y_3 \ge 0.\end{aligned} \tag{45}$$

By solving (45), we achieve $\hat{x}^u_{1opt} = 0$, $\hat{x}^u_{2opt} = 3$, $\hat{x}^u_{opt} = 0$, $\hat{y}^u_{1opt} = 0$, $\hat{y}^u_{2opt} = 12$, $\hat{y}^u_{3opt} = 0$, $\hat{Z}^u_{opt} = 66.75$, *and* $\hat{z}^u_{opt} = 97.5$.
Then for problem (40) by using Grzegorzewski's nearest interval approximation, we have:

$$\tilde{\check{x}}_{1opt} = \big((0,0,0),(0,0,0)\big),$$
$$\tilde{\check{x}}_{2opt} = \big((1.770833,2,2.4),(1.587302,2,3)\big),$$
$$\tilde{\check{x}}_{3opt} = \big((0,0,0),(0,0,0)\big),$$
$$\tilde{\check{y}}_{1opt} = \big((0,0,0),(0,0,0)\big),$$
$$\tilde{\check{y}}_{2opt} = \big((3.166667,5,7.333333),(2.285714,5,12)\big),$$
$$\tilde{\check{y}}_{3opt} = \big((0,0,0),(0,0,0)\big),$$
$$\tilde{\check{Z}}_{opt} = \big((13.29688,23,37.2),(8.047619,23,66.75)\big),$$

and

$$\tilde{\check{z}}_{opt} = \big((20.51562,34,53.06667),(12.66666,34,97.5)\big).$$

The above examples show that the optimal solution and the optimal values of problem interval type-2 triangular fuzzy bilevel linear programming problem can be easily obtained by solving five bilevel linear programming problems and this expresses the efficiency of the proposed method.

6 Application example

The bilevel linear programming problem has been used to formulate many real-world hierarchical problems in the field of production and capacity planning [53], traffic and transportation [9, 16, 54, 57], chemistry [23, 24, 29], management science [6, 25, 26, 66], as well as energy networks and market [36, 76]. But many bilevel linear programming problems have not been crisp stated. They may be expressed as fuzzy or interval.
In the following example, we present an applied bilevel linear programming problem with interval type-2 triangular fuzzy parameters. Then we solve it using the proposed method.

Example 6.1. *A country's exports are based on two products, A and B, which are produced by a factory. There are two decision-makers in this example. The government wants to have the highest amount of exports, and the factory manager wants to maximize profits. Also, the problem has three constraints of labor hours, capacity, and material. Based on past experiences and opinions of other members, the following parameters are estimated as interval type-2 triangular fuzzy numbers:*
$\tilde{\tilde{c}}_1 = \big((\check{c}_1^l, \bar{c}_1, \hat{c}_1^l), (\check{c}_1^u, \bar{c}_1, \hat{c}_1^u)\big) = \big((4,5,6),(3,5,8)\big)$,
$\tilde{\tilde{d}}_1 = \big((\check{d}_1^l, \bar{d}_1, \hat{d}_1^l), (\check{d}_1^u, \bar{d}_1, \hat{d}_1^u)\big) = \big((2.5,3,4),(2,3,5)\big)$,
$\tilde{\tilde{c}}_2 = \big((\check{c}_2^l, \bar{c}_2, \hat{c}_2^l), (\check{c}_2^u, \bar{c}_2, \hat{c}_2^u)\big) = \big((5,6,8),(4,6,9)\big)$,
$\tilde{\tilde{d}}_2 = \big((\check{d}_2^l, \bar{d}_2, \hat{d}_2^l), (\check{d}_2^u, \bar{d}_2, \hat{d}_2^u)\big) = \big((2,4,5),(1,4,6)\big)$,
$\tilde{\tilde{a}}_{11} = \big((\check{a}_{11}^l, \bar{a}_{11}, \hat{a}_{11}^l), (\check{a}_{11}^u, \bar{a}_{11}, \hat{a}_{11}^u)\big) = \big((3,4,5),(2,4,7)\big)$,
$\tilde{\tilde{a}}_{12} = \big((\check{a}_{12}^l, \bar{a}_{12}, \hat{a}_{12}^l), (\check{a}_{12}^u, \bar{a}_{12}, \hat{a}_{12}^u)\big) = \big((1,2,4),(0.5,2,5)\big)$,
$\tilde{\tilde{b}}_1 = \big((\check{b}_1^l, \bar{b}_1, \hat{b}_1^l), (\check{b}_1^u, \bar{b}_1, \hat{b}_1^u)\big) = \big((6,8,10),(5,8,12)\big)$,
$\tilde{\tilde{a}}_{21} = \big((\check{a}_{21}^l, \bar{a}_{21}, \hat{a}_{21}^l), (\check{a}_{21}^u, \bar{a}_{21}, \hat{a}_{21}^u)\big) = \big((0.5,1,1.5),(0,1,3)\big)$,
$\tilde{\tilde{a}}_{22} = \big((\check{a}_{22}^l, \bar{a}_{22}, \hat{a}_{22}^l), (\check{a}_{22}^u, \bar{a}_{22}, \hat{a}_{22}^u)\big) = \big((1,2,3),(0,2,4)\big)$,
$\tilde{\tilde{b}}_2 = \big((\check{b}_2^l, \bar{b}_2, \hat{b}_2^l), (\check{b}_2^u, \bar{b}_2, \hat{b}_2^u)\big) = \big((6,7,8),(5,7,10)\big)$
$\tilde{\tilde{a}}_{31} = \big((\check{a}_{21}^l, \bar{a}_{21}, \hat{a}_{21}^l), (\check{a}_{21}^u, \bar{a}_{21}, \hat{a}_{21}^u)\big) = \big((2,3,4),(1,3,6)\big)$,
$\tilde{\tilde{a}}_{32} = \big((\check{a}_{22}^l, \bar{a}_{22}, \hat{a}_{22}^l), (\check{a}_{22}^u, \bar{a}_{22}, \hat{a}_{22}^u)\big) = \big((4,5,6),(3,5,7)\big)$ *and*
$\tilde{\tilde{b}}_3 = \big((\check{b}_2^l, \bar{b}_2, \hat{b}_2^l), (\check{b}_2^u, \bar{b}_2, \hat{b}_2^u)\big) = \big((13,14,15),(12,14,18)\big)$.
According to the above, we will have the problem (46) with interval type-2 triangular fuzzy parameters:

$$\begin{aligned}
\max_x\ & \tilde{\tilde{c}}_1 x + \tilde{\tilde{d}}_1 y \\
& s.t.\ \max_y\ \tilde{\tilde{c}}_2 x + \tilde{\tilde{d}}_2 y \\
& \quad s.t.\ \tilde{\tilde{a}}_{11} x + \tilde{\tilde{a}}_{12} y \le \tilde{\tilde{b}}_1 \quad (labor\ hours), \\
& \qquad \tilde{\tilde{a}}_{21} x + \tilde{\tilde{a}}_{22} y \le \tilde{\tilde{b}}_2 \quad (capacity), \\
& \qquad \tilde{\tilde{a}}_{31} x + \tilde{\tilde{a}}_{32} y \le \tilde{\tilde{b}}_3 \quad (material) \\
& \qquad x, y \ge 0,
\end{aligned} \tag{46}$$

Using the proposed method, the middle problem gets as follows:

$$\begin{aligned}
\max_x\ & 5x + 3y \\
& s.t.\ \max_y\ 6x + 4y \\
& \quad s.t.\ 4x + 2y \le 8 \\
& \qquad x + 2y \le 7 \\
& \qquad 3x + 5y \le 14 \\
& \qquad x, y \ge 0.
\end{aligned} \tag{47}$$

By solving (47), we obtain $\bar{x}_{opt} = 0.8571429$, $\bar{y}_{opt} = 2.285714$ *and optimal values are* $\bar{Z}_{opt} = 11.14286$, *and* $\bar{z}_{opt} = 14.28571$.
Now, by using Grzegorzewski's nearest interval approximation we have:
1. The worst-worst model:

$$\begin{aligned}
\max_x\ & 4x + 2.5y \\
& s.t.\ \max_y\ 5x + 2.5y
\end{aligned}$$

$$\begin{aligned} & s.t.\ 5.5x + 3.5y \leq 6.5 \\ & 2x + 3y \leq 6 \\ & 4.5x + 6y \leq 13 \\ & x, y \geq 0, \end{aligned} \tag{48}$$

which $\check{x}^u_{opt} = 0.181818$, $\check{y}^u_{opt} = 0$, $\check{Z}^u_{opt} = 4.727273$, *and* $\check{z}^u_{opt} = 5.909091$ *are obtained by solving (48).*
2. The worst-best model:

$$\begin{aligned} \max_x\ & 4.5x + 2.75y \\ & s.t.\ \max_y\ 5.5x + 3y \\ & \quad s.t.\ 4.5x + 3y \leq 7 \\ & \quad 1.25x + 2.5y \leq 6.5 \\ & \quad 3.5x + 5.5y \leq 13.5 \\ & \quad x, y \geq 0, \end{aligned} \tag{49}$$

by solving problem (49), $\check{x}^l_{opt} = 1.555556$, $\check{y}^l_{opt} = 0$, $\check{Z}^l_{opt} = 7$, *and* $\check{z}^l_{opt} = 8.555556$ *are obtained.*
3. The best-worst model:

$$\begin{aligned} \max_x\ & 5.5x + 3.5y \\ & s.t.\ \max_y\ 7x + 4.5y \\ & \quad s.t.\ 3.5x + 1.5y \leq 9 \\ & \quad 0.75x + 1.5y \leq 7.5 \\ & \quad 2.5x + 4.5y \leq 14.5 \\ & \quad x, y \geq 0, \end{aligned} \tag{50}$$

From (50), $\hat{x}^l_{opt} = 1.5625$, $\hat{y}^l_{opt} = 2.354167$, $\hat{Z}^l_{opt} = 16.83333$, *and* $\hat{z}^l_{opt} = 21.53125$ *are achieved.*
4. The best-best model:

$$\begin{aligned} \max_x\ & 6.5x + 4y \\ & s.t.\ \max_y\ 7.5x + 5y \\ & \quad s.t.\ 3x + 1.75y \leq 10 \\ & \quad 0.5x + y \leq 8.5 \\ & \quad 2x + 4y \leq 16 \\ & \quad x, y \geq 0. \end{aligned} \tag{51}$$

By solving (51), we achieve $\hat{x}^u_{opt} = 1.411765$, $\hat{y}^u_{opt} = 3.294118$, $\hat{Z}^u_{opt} = 22.35294$, *and* $\hat{z}^u_{opt} = 27.05882$.
Then for problem (34) by using Grzegorzewski's nearest interval approximation, we have:

$$\tilde{\check{x}}_{opt} = \big((1.555556, 0.8571429, 1.5625), (0.181818, 0.8571429, 1.411765)\big),$$

$$\tilde{\check{y}}_{opt} = \big((0, 2.285714, 2.354167), (0, 2.285714, 3.294118)\big),$$

$$\tilde{\check{Z}}_{opt} = \big((7, 11.14286, 16.83333), (4.727273, 11.14286, 22.35294)\big),$$

and

$$\tilde{\check{z}}_{opt} = \big((8.555556, 14.28571, 21.53125), (5.909091, 14.28571, 27.05882)\big).$$

7 Conclusions

Decision-making is a very important process for leadership and management. There are processes and techniques available for decision-making that improve the quality of the decision as well. As real-world problems contain uncertainty, we need to use the concept of being fuzzy and its extensions.
In this paper, we first reviewed the method proposed by Javanmard and Mishmast Nehi [43] for solving an interval type-2 triangular fuzzy linear programming problem. Then, we improved their method.
Then we considered a bilevel linear programming problem with the uncertainty of the kind ambiguity where all the parameters were interval type-2 triangular fuzzy numbers. To the best of our knowledge, this problem had not been investigated. To solve a bilevel linear programming problem in which all the parameters (objective function coefficients,

technical coefficients, and right-hand-side values) are interval type-2 triangular fuzzy numbers, it is enough to introduce the equivalent bilevel linear programming problem or bilevel linear programming problems. The use of ranking methods, in which each interval type-2 triangular fuzzy number is approximated with a crisp number, is relatively efficient and appropriate. But there is always a general defect of losing problem information due to ranking in all ranking methods. A well-defined, approximate, closed interval for an interval type-2 triangular fuzzy number is more acceptable and logical than replacing it with a crisp number.

We converted interval type-2 triangular fuzzy bilevel linear programming problem into an interval bilevel linear programming problem using Grzegorzewski's nearest interval approximation. In the next step, we achieved five bilevel linear programming problems, and by solving them, the optimal solution of interval type-2 triangular fuzzy bilevel linear programming problem was obtained as interval type-2 triangular fuzzy number. Finally, we solved some numerical examples with different features using the proposed method which showed the effectiveness of the proposed method in different conditions.

In general, the advantages of the proposed method can be summarized as follows:

1. Using a simple interval to approximate the triangular fuzzy number instead of approximating it with a crisp number.
2. The approximation used to convert the triangular fuzzy number to the interval can be easily calculated.
3. Converting each interval type-2 triangular fuzzy bilevel linear programming problem into bilevel linear programming problems that can be easily solved by bilevel linear programming problem solution methods.
4. Obtaining the optimal values for leader and follower as interval type-2 triangular fuzzy numbers.

On the other hand, using the proposed method to solve a interval type-2 triangular fuzzy bilevel linear programming problem, five bilevel linear programming problems must be solved, which is time-consuming.

As some new investigation ideas for researchers, we recommend solving bilevel linear programming problem with uncertainty of the kinds ambiguity, vagueness, and even both of them. Also, we recommend carrying out the study on interval type-2 non-triangular fuzzy bilevel linear programming problem.

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