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Lattice expansion of fuzzy concepts

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Abstract

The concept of a fuzzy number is expanded to the case of a finite carrier set of partially ordered elements; the set is a lattice, and the membership function of such an expansion also takes values in a lattice. Zadeh's extension principle for determining the degree of membership of a function of fuzzy numbers is changed for this expansion. An analogue of the concept of a mean value is also suggested. The use of partially ordered values in cognitive maps with multiple expert assessments is considered.

Keywords: Fuzzy numbers, Zadeh extension principle, fuzzy cognitive maps, multi-valued sets, multi-valued cognitive maps, linguistic variable lattice.

1 Introduction

In the theory of fuzzy sets, carrier sets always have some metric, and membership functions take values in the interval $[0, 1]$. Therefore, we can always compare the set elements by their membership degree. However, expert estimates in fuzzy systems do not always have to be linearly ordered: e.g., "yes and no" and "neither yes nor no". The ordinary fuzzy theory does not allow working with such unordered assessments. Moreover, fuzzy systems use numbers as variables and weights [8, 15]. Thus, one should perform an ambiguous fuzzyfication/defuzzyfication process to convert linguistic estimates to numbers and vice versa. Hence, it's desirable to work directly with linguistic varibles for a more adequate expression of experts' assessments. In addition, in complex technical systems, it may be preferable to use functions or matrices as variables instead of numbers or linguistic values.

In this case, lattices [1] may be of interest in the theory of fuzzy sets, since they can serve as scales for expressing expert opinions in a more flexible way than the usual linear scales [12, 13, 14]. Also, matrix and function rings can have lattice structures [1]. In lattices, we cannot always compare the elements of such objects, since the lattice is only a partially ordered set (i.e., a poset) in general. Such lattices are considered in this paper as carrier sets and membership scales, instead of number sets[1].

Concepts that expand membership degrees and operations with them, as well as the concept of mean value, have not yet been elaborated for lattice-valued variables. We need these concepts if we want to take into account *multiple* experts' assessments with some degree of confidence (membership). We cannot simply use the fuzzy-like concepts due to the only partial ordering of our values and the absence of a division operation in general in this case.

Thus, we introduce a lattice change of the concept of fuzzy numbers in Sec. 2. In Sec. 3, we also introduce a change of Zadeh's extension principle of the membership degree for a function of fuzzy numbers in the case of the expansion to lattice values of variables. We consider possible generalizations of the concept of a mean value in Sec. 4.

Cognitive maps with weights and concepts which take values in a lattice have been considered in [12, 13]. However, a unique expert assessment was used there for each weight. In this article, in Sec. 5, we demonstrate this using multiple expert assessments for each weight using the example of a cognitive map from [12, 13] with degrees of confidence for

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[1]We consider here only the case of lattice operations and expert assessments with lattice values. Thus we leave the case of general algebraic structures for the future

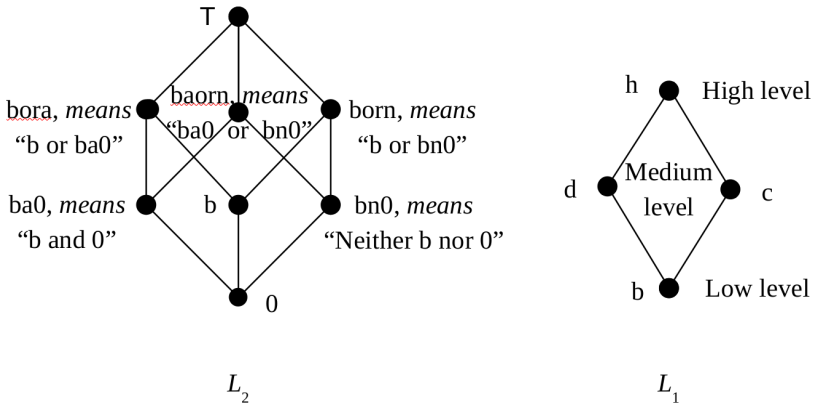


Figure 1: Examples of lattices

weights' and concepts' values. Also, we compare the results with various ways of aggregating estimates as analogues of the mean value concept. We conclude the paper in Sec. 6.

2 Lattice change of the concept of fuzzy numbers

We have used general finite lattices in [13, 14] as scales for expressing expert assessments, instead of the linguistic variable scale of numerical interval subintervals. In such a scale, an element of the lattice corresponds to an expert opinion estimation. Typically, each expert gives an estimate from a very limited set of lattice generators, and the remaining elements of the lattice are obtained as expert estimates as a result of statistical processing: e.g., if different experts give two different assessments, then the resulting assessment will be their join [13]. However, if each expert gives a set of estimates with different confidence degrees for them, we must be able to handle such multiple expert opinions along with degrees of confidence.

In the case of fuzzy numbers, the carrier set is a numerical interval and we can always say that there are some set elements with a full membership degree, and some that only partially belong to the interval following the metric. However, it makes no sense to evaluate the degree of membership in a finite carrier set of partially ordered elements without a metric.

Indeed, let us consider lattices in Fig. 1. We have no a criterion in the set $\{d, c\} \subset L_1$ or the set $\{ba0, b, bn0\} \subset L_2$ to decide whether the element has a higher membership degree. All these elements are incomparable. Thus, it seems reasonable to use a degree of expert confidence instead of the degree of membership.

The difference between these two concepts is that an expert may not be completely sure in any of his/her assessments. Thus, the confidence function does not have to reach the maximum value in the carrier set, unlike the fuzzy numbers. Then, the confidence degrees of partially-ordered assessments may also be partially-ordered — they are also expert opinions. In this case, the convexity condition of the confidence function (also, unlike the fuzzy number definition) makes no sense.

Thus, we define multi-valued sets as a variation of the fuzzy number concept, in the case of a lattice carrier set and a lattice membership function, in the following general way in the form of a set-to-lattice function.

Definition 2.1. *We refer to a set of pairs*

$$A = \{(x, \mu_A(x)); x \in L\}, \tag{1}$$

where

$$\mu_A : L \to M, \tag{2}$$

is a confidence function of A elements with values in the lattice M, as a multi-valued set A in a lattice L.

Here is an example of such a multi-valued set A of elements of L_2 with values (confidence degrees) in L_1 (Fig. 1): $A = \{(born, c); (bn0, d)\}$.

3 Lattice change of Zadeh's extension principle

Let us remember Zadeh's extension principle.

Definition 3.1. [17] *Let X be a Cartesian product of crisp sets $X_1 \times X_2 \times ... \times X_n$. If there is some crisp mapping*

$$f : X_1 \times X_2 \times ... \times X_n \to Y, \tag{3}$$

and also some crisp sets $A_1 \subseteq X_1, A_2 \subseteq X_2, ...A_n \subseteq X_n$, then the extension principle says that the fuzzy set B formed by the mapping f has the form

$$B = f(A_1, ..., A_n) = \{(y, \mu_B(y)) | y = f(x_1, ..., x_n), (x_1, ..., x_n) \in X\}, \tag{4}$$

while

$$\mu_B(y) = \begin{cases} \sup\limits_{(x_1,...,x_n)\in f^{-1}(y)} min\{\mu_{A_1}(x_1), ..., \mu_{A_n}(x_n)\}, & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{if } f^{-1}(y) = \emptyset. \end{cases} \tag{5}$$

In the lattice-valued case, the max–min operations are replaced by join ($\vee$) – meet ($\wedge$) correspondingly, and this definition becomes unsuitable due to certain points:

- In a lattice, in the definition $\mu(y) = \vee(\mu(a) \wedge \mu(b) \wedge ...)$, meets can lead to zero even when all $\mu(a) \neq 0$. And, in general, the more such $\mu(a)$, the more likely the resulting confidence degree is zero: $\mu(y) = 0$.
- Vice versa, the lattices L and M in Definition 2.1 may have an additional structure of a semi-ring or a ring. Then, the addition operation in these structures may be nilpotent. In this case, the resulting confidence degree of the bottom element may be maximum when $f(x) = \oplus\limits_i x_i$: e.g., $\mu(0) = \vee\limits_{\substack{a+a=0; \\ a\in L}} (\mu(a) \wedge \mu(a)) = \vee\limits_{a\in L} (\mu(a)) = \top$, i.e., the lowest estimation is the most confident.
- Finally, an implication $c = a \to b$ may be defined in lattices, which is not reduced to arithmetic operations, in contrast to the fuzzy case. If $M = L$ and $\mu(a) = a$, and if there is a separate assessment $\mu(c)$ for the value c of the implication, then $\mu(c) \neq \vee(\mu(a) \wedge \mu(b))$. The situation is the same with all functions in this case.

Hence, we propose using the following formula in the lattice-valued case instead of (5) (with the same notation), which gives the correct result in the case of $M = L$ and $\mu(a) = a$; the function $\bar{f}$ denotes the same operation as f, but in the lattice M:

$$\mu_B(y) = \begin{cases} \vee\limits_{(x_1,...,x_n)\in f^{-1}(y)} \bar{f}(\mu_{A_1}(x_1), ..., \mu_{A_n}(x_n)), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{if } f^{-1}(y) = \emptyset. \end{cases} \tag{6}$$

Now, we especially consider an example of the function f that includes an additional algebraic operation. Such operations arise in different algebraic constructions that can be used to determine the monoidal multiplication[2]. Thus, let f be the sum operation in the lattice ring L_2, and $\bar{f}$ is the same operation in L_1. Then, $\mu_{L_1}(bn0) = \vee\limits_{\substack{(x_1,...,x_n)\in L_2: \\ bn0 = x_1+x_2+...+x_n}} [d + c + ... + b]$, $b, c, d... \in L_1$ Here, $\mu_{L_1}(x_1) = d$, $\mu_{L_1}(x_2) = c$, ... $\mu_{L_1}(x_n) = b$.

This definition does not completely resolve the first problem described above. Indeed, the result is the same if the function f has the form $f(\mu_A(x)) = \wedge\limits_i \mu(x_i)$. However, the result can be reasonable when the function includes other lattice and ring operations.

The evident consequence of (6) is that the confidence degree of a function of assessments may be greater than the degree of each of them. Indeed, our confidence in the set of two evaluations is greater than in either of them, if we are not completely sure of either.

4 Lattice generalization of mean value

The concept of a mean value is needed if we want to aggregate different expert assessments in some way. In particular, we may need a mean of the lattice-valued analogues of fuzzy numbers. However, we do not have a division operation in lattices and rings or semi-rings that could be associated with a finite lattice of expert estimates. Thus, there is no mean concept in this case.

[2]These general cases are considered in the next paper

The essence of the mean concept is establishing some common characteristic to a row of values, which can characterize each of them and the whole row at the same time. Thus, we can use a concept of a similarity measure as an analogue of the mean value. There are a lot of such measures between objects that have internal structures [10]. However, there are only the meet and join operations in the partially-ordered sets without an internal structure, whereas, a lattice of finite sets is our main example in applications. In this case, we can approach such a concept from the following consideration.

The distance from a point to a set is expressed by the formula $\rho(x,A) = \underset{a\in A}{inf}(\rho(x,a))$ when there is a metric in the universe. We can introduce a measure of a difference $d(A,B)$ of two sets A and B of lattice elements following this formula when the lattice elements are also arbitrary finite sets, and there is no metric:

$$d(A,B) = \{\underset{a\in A}{\wedge}(b \setminus a)|\forall b \in B\} \cup \{\underset{b\in B}{\wedge}(a \setminus b)|\forall a \in A\}. \tag{7}$$

Here, the operation $\setminus$ denotes a set difference, and a and b are lattice elements which are also considered as sets of generators.

If we take a negation of the measure of difference, we get a definition of a similarity measure as the average of two sets of expert assessments. We can define two types of the mean value: a pessimistic and an optimistic aggregation[3].

Definition 4.1. *We refer to the following union of two sets*

$$S(A,B) = \{\underset{a\in A}{\vee}(b \wedge a)|\forall b \in B\} \cup \{\underset{b\in B}{\vee}(a \wedge b)|\forall a \in A\}, \tag{8}$$

as the pessimistic average of two sets A and B of expert assessments.

Definition 4.2. *We refer to the following union of two sets*

$$Sopt(A,B) = \{\underset{a\in A}{\wedge}(b \vee a)|\forall b \in B\} \cup \{\underset{b\in B}{\wedge}(a \vee b)|\forall a \in A\}, \tag{9}$$

as the optimistic average of two sets A and B of expert assessments.

One can easily see that the mean value of two sets is also a set with the number of elements from one to the total number of elements in both sets.

We get a union of such pairwise aggregations when there are more than two sets in the arguments of functions S or $Sopt$.

Definition 4.3. *We refer to the following union of n sets*

$$S(A_1,...,A_n) = \underset{i,j\in\{1...n\}}{\cup} S(A_i,A_j) = \underset{i,j\in\{1...n\}}{\cup}[\{\underset{a\in A_i}{\vee}(b \wedge a)|\forall b \in A_j\} \cup \{\underset{b\in A_j}{\vee}(a \wedge b)|\forall a \in A_i\}], \tag{10}$$

as the pessimistic average of n sets A_i, $i = 1...n$ of expert assessments. A similar definition is valid for the optimistic average $Sopt$.

Here are some examples to illustrate these concepts in Fig. 2. The simplest examples of the mean values of two one-element sets are depicted in Fig. 2a. Then their optimistic average is the top element singleton, and the pessimistic one is the singleton that contains the bottom vertex. Let us mention that vertex a is not a singleton (though $\{a\}$ is a singleton) since it contains vertex b.

Two different aggregations of a singleton and a two-element set are depicted in Fig. 2b. Here, e.g., in optimistic average, the element p refers to the second member in the union in (9), and elements g and h refer to the first one (here, $A = \{a\}$ and $B = \{b,c\}$ in (9)).

Fig. 2c gives an example of the pessimistic average of a singleton and a two-element set when these sets rearrange their elements. Also, the figure gives an example of different averages of three one-element sets. In this case, both aggregations are unions of pairwise means.

[3]The author is grateful to Dr. O. P. Kuznetsov for the idea of using pessimism and optimism in aggregation

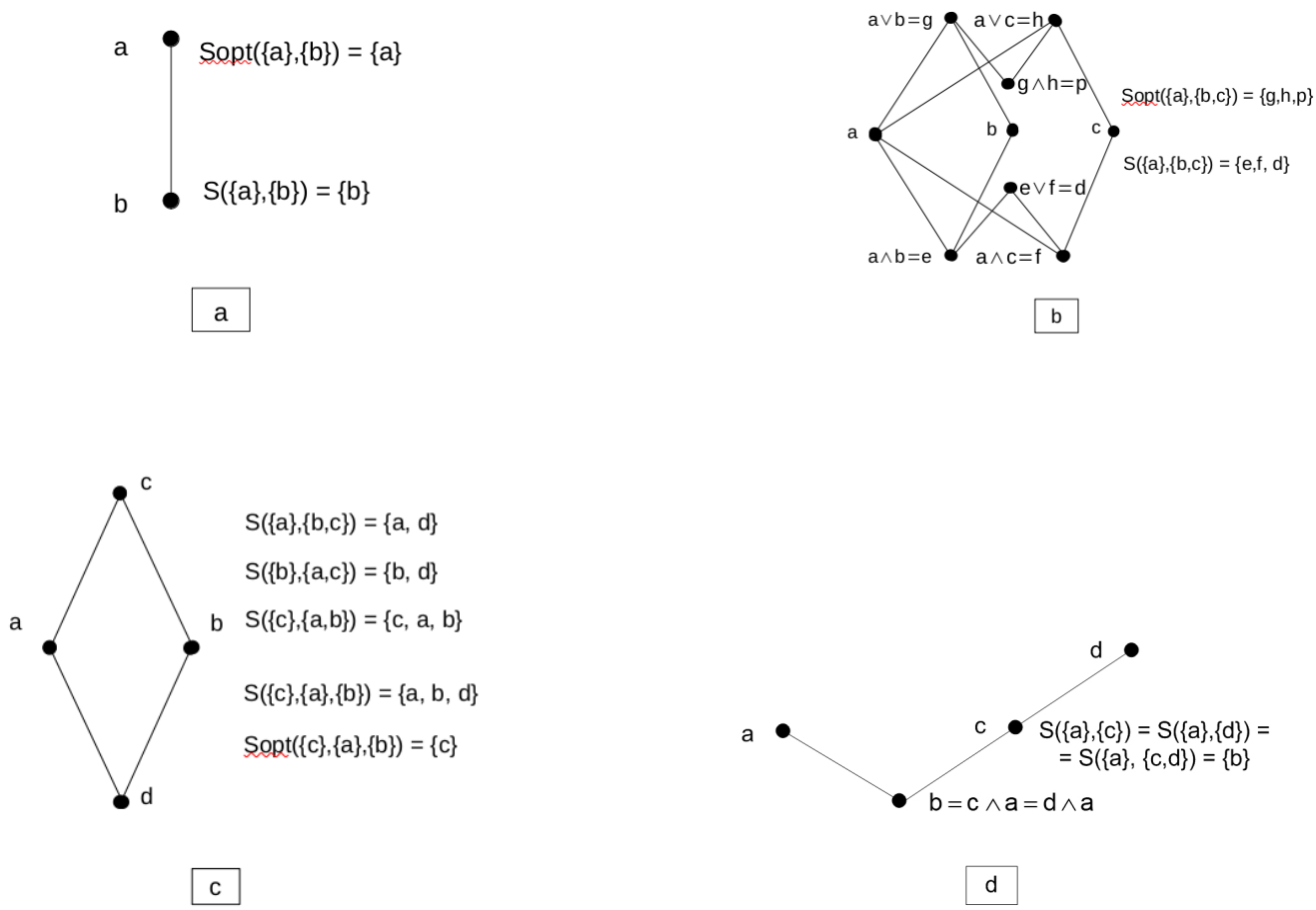


Figure 2: Similarity measure examples

The last example in Fig. 2d shows that different aggregations may have the same mean value. This observation refers to the confidence degree of the mean, and, hence, we can define the following formula for the confidence degree of the average value similarly to (6):

$$\mu(S(a^*)) \quad = \quad \begin{cases} \bigvee\limits_{\substack{a_i \in S^{-1}(a^*), i=1\ldots n: \\ S(a_1 \ldots a_n)=a^*}} \bar{S}(\mu(a_1)\ldots\mu(a_n)), & \text{if } S^{-1}(a^*) \neq \emptyset, \\ 0, & \text{if } S^{-1}(a^*) = \emptyset. \end{cases} . \tag{11}$$

The same formula is suitable for the optimistic aggregation *Sopt*. In (11), function (pessimistic mean) $\bar{S}$ refers to the lattice M of confidence degrees, as does $\bar{f}$ in (6).

5 Using partially ordered plural expert assessments in cognitive maps

Here we consider an illustrative model example of the Multi-Valued Cognitive Map (MVCM) from [13] but with multiple expert assessments; this example was originally considered in [7] in the framework of a Fuzzy Cognitive Map model (FCM). Thus, we consider the example of a Hybrid Energy System combining wind and photovoltaic subsystems; its cognitive map model is depicted in Fig. 3.

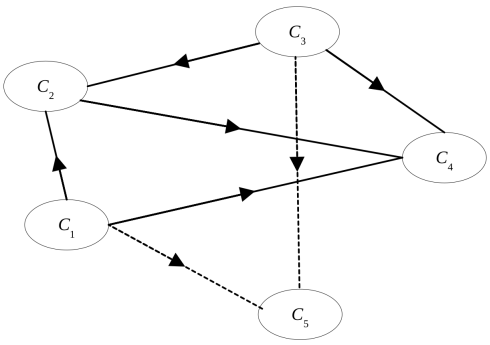


Figure 3: The cognitive map of a Hybrid Energy System

The model includes the following five concepts: C_1: sun insolation; C_2: environment temperature; C_3: wind; C_4: PV-subsystem; C_5: Wind-Turbine-subsystem. In this model, there are two energy source decision concepts (outputs), i.e., two energy sources are considered: the C_4: PV-subsystem and the C_5: Wind-Turbine-subsystem. Concepts C_1–C_3 of nature factors influence the subsystems and determine how each energy source will function in this model. The concept's initial values can be obtained from the experts' assessments of measurements, which take values in a lattice of linguistic variables. The experts' assessments of concept influence also take values in this lattice. The information for hybrid renewable energy systems is given in [3, 6].

The equation that calculates the values of concepts of FCMs with n nodes can be written in its general form as [5]:

$$A_i^k = f(d_1 \sum_{j=1, j\neq i}^{n} w_{ji} A_j^{k-1} + d_{ii} A_i^{k-1}). \tag{12}$$

Here A_i^k is the value of the concept C_i at discrete time k, and d_{ii} is a value of self-feedback to node i. All values belong to the interval $[0, 1]$. The existence and uniqueness of the solutions of (12) in FCMs are proved in [9] for some functions f's. In most applications, d_1 and d_{ii} are set equal to 1 [7].

We use an MVCM model instead of the FCM one, and there is the following general equation for MVCMs in [13]:

$$A_i^k = c_i^{k-1} \cdot f_i^{k-1} \cdot \bigvee_{j=1}^{n} w_{ji} \cdot A_j^{k-1} \tag{13}$$

where all weights and concepts take values in a residuated [2] *atomic* [1] lattice L, and we use the monoid multiplication of the lattice and the lattice join, instead of numerical multiplication and sum correspondingly.

The following convergence theorem is proved in [13]:

Theorem 5.1. *Multi-Valued Cognitive Maps determined by equation (13) where concepts and weights take values in a finite atomic residuated and integrally-closed* [4] *lattice L (hence, L is integral* [4]*), converge under a suitable choice of c_i^k's and f_i^k's.*

The following formulas for c_i^k's and f_i^k's should be used to satisfy the theorem:

$$f_i^k = [A_i^k \vee A_i^{k-1}] \leftarrow (\bigvee_{j=1}^{n} w_{ji} \cdot A_j^k) \tag{14}$$

$$r_i^k \leftarrow f_i^k \leqslant c_i^k \leqslant 1, \tag{15}$$

where

$$r_i^k = [A_i^k \wedge A_i^{k-1}] \leftarrow \bigvee_{i=1}^{n} w_{ji} \cdot A_j^k, \tag{16}$$

and we can take $c_i^k = 1$ until the MVCM loops.

In this paper, we take the operation $\wedge$ (meet) as the multiplication in Heyting algebras, i.e., in Brouwer lattices [1] (as in the example in [13]). Hence, the residuals turn into the lattice implication [1, 2].

However, the lattice of expert assessments may have an additional structure of a ring, and we can use the operations from the structure as the sum and the multiplication in (13)–(16), instead of the join and the monoid multiplication. The proof of Theorem 5.1 does not depend on such a replacement. Here we additionally consider only the sum from the ring of subsets of a finite set (the multiplication in the ring is the lattice meet). In this case, the sum is the operation "excluded or" or 'symmetric difference" $a + b = (a \vee b) \setminus (a \wedge b)$ on the lattice L, where the lattice elements a and b are considered as sets of generators. Let us note that the sum is nilpotent, i.e., $a + a = 0$.

So finally we use the following formulas in our example of MVCM:

$$C_i^k = c_i^{k-1} \wedge f_i^{k-1} \wedge \bigvee_{j=1}^{n} w_{ji} \wedge A_j^{k-1}, \tag{17}$$

or

$$C_i^k = c_i^{k-1} \wedge f_i^{k-1} \wedge \sum_{j=1}^{n} w_{ji} \wedge A_j^{k-1}, \tag{18}$$

for calculating concept values; and similarly for f_i^k and r_i^k with $\bigvee$ replaced by $\sum$ (in the case of (18)) and residuals $\leftarrow$ by lattice implications $\Rightarrow$, e.g.

$$f_i^k = (\sum_{j=1}^{n} w_{ji} \wedge A_j^k) \Rightarrow [A_i^k \vee A_i^{k-1}]. \tag{19}$$

We use the following bi-lattice L as the scale of variables [13], where $L = L_1 \times L_2$ (lattices L_1 and L_2 are depicted in Fig. 1), and the lattice M of L_2 type as the scale of confidence Fig. 4.

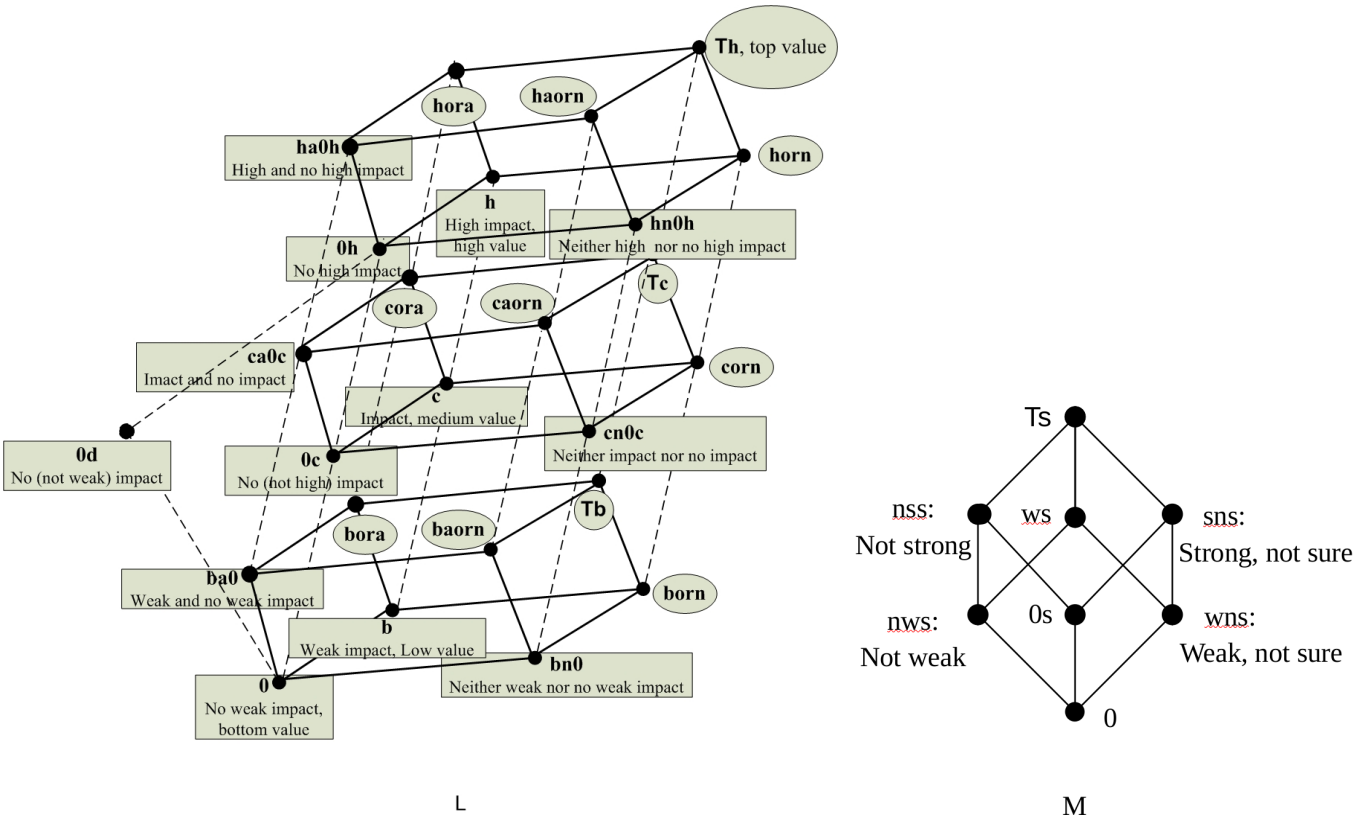


Figure 4: The lattice scale of weights' and concepts' assessments — L and the lattice scale of their confidence estimates — M

We consider the bi-lattice L as the lattice where the unique partial order is generated by atoms $ba0, b, bn0, 0c$, and $0d$. We use two linearly-ordered branches in the lattice L_1 to regularly obtain the distributive and atomic lattice L. Hence, we may use the meet $\wedge$ as the multiplication. The nodes c and d in the L_1 Fig. 1, corresponding to the medium value, may be interpreted as "not high" and "not weak" uncertainty degrees.

The lattice M of confidence degrees is the lattice of L_2 type. Its nodes can be interpreted as follows: "nws" as "sure in a not weak confidence degree", "wns" as "not sure in a weak one"; "nss" as "sure in a not strong confidence degree", "sns" as "not sure in a strong one" and "0s" as ambivalent value: "nss" and "sns".

"A variant of the lattice L_2 was proposed in [11] as the interpretation of N. A. Vasil'ev's logic ideas. Vasil'ev has suggested three types of statements: positive, negative, and indifferent, instead of only positive and negative. He considered also intermediate types as a hesitation between these main ones. Similarly, we consider here three main uncertainty degrees: $ba0, b, bn0$ (Fig. 3) — some assessment "b", the estimation "$bn0$: Neither b nor 0", and the estimate "$ba0$: b and 0". Similarly, we interpret the generators of sublattices at the levels $c - d$ and h Fig. 4. The assessment, e.g., "*born*: b or $bn0$" is the join of b and $bn0$ and can be considered as the hesitation between b and "Neither b nor 0". Similarly, "*bora*: b or $ba0$" is the join of b and $ba0$ and can be interpreted as the hesitation between "b" and "b and 0" and so on. Then, such hesitations refer to sets of expert assessments, which are ambiguous and contain different subsets of estimates. Thus, we obtain many different variants of uncertainty degrees in assessments, usually obtained statistically when experts use the generators as estimates" [13].

The semantic of the lattice elements is not unique: e.g., the review [16] contains another semantic of L_2 elements that is used in *causal* cognitive maps. There, the lattice of the type L_2 contains eight elements: $+, 0, -$ are generators (they correspond to our $ba0$, b and $bn0$) and $\oplus, \ominus, \pm, a, ?$ are obtained from them by the following connectives: $\oplus = 0 \vee +; \ominus = 0 \vee -; \pm = \oplus \vee \ominus; ? = + \vee - \vee 0; a = + \wedge - \wedge 0$ that correspond to *bora*, *born*, etc (our 0 is a in this notation). Such assessments valuate connections between concepts, e.g., + means " excites ", − means " hurts ", 0 means "is neutral"; the meaning of connective $\vee$ as in this paper, and $\wedge$ means "ambivalent". Perhaps such an intuition, like Vasili'ev's, could be more appropriate for the reader, when + means "positive", − means "negative", 0 means "neutral" and a means "ambivalent". Thus, various interpretations of the lattice elements are possible, and the reader may be guided by the most suitable intuition for him.

In general, all variables in formulas (17)–(19) may be multi-valued sets, i.e., they may be sets with confidence degrees of their elements. However, for simplicity, we will use only a set of two assessments for weights and one-element sets for concept values. Thus, we use two following weight matrices for these assessments: one for the first expert assessments, and the other one for the second expert estimates Fig. 5. In the figure, in each matrix element, the first element in parentheses is the weight, and the second is the degree of confidence of this weight element.

	C1	C2	C3	C4	C5
C1	('Th', 'Ts')	('born', 'ws')	0	('hora', 'Ts')	('c', 'nss')
C2	0	0	0	('-Tb', 'Ts')	0
C3	0	('-ca0c', 'ws')	('Th', 'Ts')	('bora', 'Ts')	('horn', 'nss')
C4	0	0	0	0	0
C5	0	0	0	0	0

	C1	C2	C3	C4	C5
C1	('Th', 'Ts')	('bora', '0s')	0	('horn', 'ws')	('d', 'nws')
C2	0	0	0	('-c', 'nss')	0
C3	0	('-da0d', '0s')	('Th', 'Ts')	('born', 'ws')	('haorn', 'nws')
C4	0	0	0	0	0
C5	0	0	0	0	0

Figure 5: Weight matrices of two expert opinions with confidence degrees

During the process of the map iteration, the weights from both these matrices may arise in formulas (17) and (18) Fig. 6.

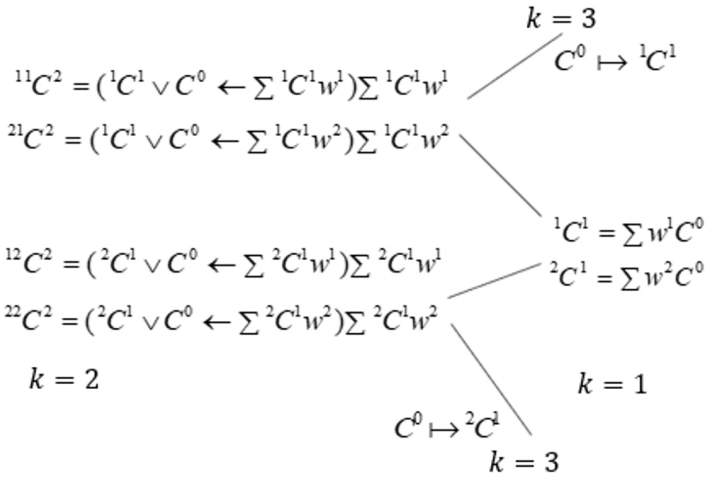


Figure 6: The map iterations; at the third step, C^0 is changed to the corresponding C^1, and so on

The same process is going for the confidence degrees of the weights and concepts due to (6); e.g., the variable ${}^{11}C^2$ means the second iteration ($k = 2$) where w is taken from the first matrix, in which the first iteration ${}^1C^1$ is also with w from the first matrix.

In the beginning, we present the results of calculations for two matrices separately, i.e., when all variables are lattice elements, not sets Tab. 1^4.

Table 1: The map iterations of separate calculations with joins in formulas; insolation decreases, wind increases

k	C1	C2	C3	C4	C5
0	[('horn', 'sns')]	[('b', 'ws')]	[('d', '0s')]	[('h', 'nss')]	[('d', 'sns')]
5	[('dn0d', 'wns')]	[('bn0', 'Ts'), ('0d', 'Ts')]	[('hora', 'nss')]	[('0d', 'Ts'), ('dn0d', 'Ts')]	[('h', 'Ts'), ('0d', 'sns')]

The first clause in square brackets in each cell corresponds to the weight assessment of the first expert and behaves just as well as in [13] — the wind turbine output rises and the photovoltaic system output drops as insolation decreases and wind increases. However, the second one, corresponding to the second weight matrix, shows the non-physical final result C_5 — the output of the wind turbine decreases as the wind increases. Thus, the second expert gives inappropriate estimates.

In the case of sums in formulas (see (18), etc.), the last row in Tab. 2 is similar to Tab. 1, though nilpotency influences the result.

Table 2: The final map iterations of separate calculations with sums in formulas; insolation decreases, wind increases

k	C1	C2	C3	C4	C5
5	[('dn0d', 'wns')]	[('bn0', '0s'), ('0d', 'wns')]	[('hora', 'nss')]	[('d', '0'), ('dorn', '0')]	[('0h', '0'), ('0', 'wns')]

5.1 Results with multi-valued sets as weights

We obtain the following results for multi-valued sets as weights and (17) in Fig. 7.

k	0	1	2
C1	[('horn', 'sns')]	[('dorn', 'sns')]	[('dn0d', 'wns')]
C2	[('b', 'ws')]	[('born', 'Ts'), ('d', 'Ts')]	[('born', 'Ts'), ('b', 'Ts'), ('d', 'Ts')]
C3	[('d', '0s')]	[('h', '0s')]	[('hora', 'nss')]
C4	[('h', 'nss')]	[('0h', 'Ts'), ('hn0h', 'Ts')]	[('0d', 'Ts'), ('dn0d', 'Ts')]
C5	[('d', 'sns')]	[('h', 'Ts'), ('d', 'sns')]	[('h', 'Ts'), ('d', 'sns')]

k	3	4	5
C1	[('dn0d', 'wns')]	[('dn0d', 'wns')]	[('dn0d', 'wns')]
C2	[('bn0', 'Ts'), ('0', 'Ts'), ('0d', 'Ts')]	[('bn0', 'Ts'), ('0', 'Ts'), ('0d', 'Ts')]	[('bn0', 'Ts'), ('0', 'Ts'), ('0d', 'Ts')]
C3	[('hora', 'nss')]	[('hora', 'nss')]	[('hora', 'nss')]
C4	[('0d', 'Ts'), ('dn0d', 'Ts')]	[('0d', 'Ts'), ('dn0d', 'Ts')]	[('0d', 'Ts'), ('dn0d', 'Ts')]
C5	[('h', 'Ts'), ('d', 'sns'), ('0h', 'Ts'), ('0d', 'sns')]	[('h', 'Ts'), ('d', 'sns'), ('0h', 'Ts'), ('0d', 'sns')]	[('h', 'Ts'), ('d', 'sns'), ('0h', 'Ts'), ('0d', 'sns')]

Figure 7: Full map iterations with joins in formulas; insolation decreases, wind increases

One can see that confidence increases following Sec. 3.

In the case of sums in formulas ((18), etc.), we get results for multi-valued sets as weights in Tab. 3.

Table 3: The final full map iterations with sums in formulas; insolation decreases, wind increases

k	C1	C2	C3	C4	C5
15	[('dn0d', 'wns')]	[('bn0', '0'), ('0', 'wns'), ('0d', 'wns')]	[('hora', 'nss')]	[('d', '0'), ('dorn', '0')]	[('0d', '0'), ('h', 'wns'), ('d', '0'), ('c', 'wns'), ('b', '0'), ('0', 'wns'), ('0c', '0')]

Such lengthy calculations (k = 15) are needed to avoid looping due to the nilpotency of the sum.

Comparing Fig. 7 and Tab. 3 with separate calculations in Tab. 1 and Tab. 2 one can see that the initial certain concept values blur into a set of several ones, i.e., the initial one-element multi-valued sets blur into real multi-valued sets. Moreover, they blur more strongly with the sum in the formulas and they may not include the separate values in this case (C_5 drops out)[5]. Also, confidence degrees are less in the sum case, since in this case meets are subtracted from joins.

5.2 Results with mean values

In Tab. 4 we obtain results for the pessimistic aggregation of weights and in Tab. 5, the optimistic aggregation of weights.

[4]Here and below, the confidence degrees are calculated according to (6), where $M \neq L$, and initial concept confidences are determined a priori

[5]In the case of the join in the formulas, both separate paths are present in the blurred sets

Table 4: The final map iterations with the pessimistic mean of weights; insolation decreases, wind increases

k	C1	C2	C3	C4	C5
5	[('dn0d', 'wns')]	[('0', 'ws')]	[('hora', 'nss')]	[('0d', 'Ts')]	[('0d', 'sns')]

Table 5: The final map iterations with the optimistic mean of weights; insolation decreases, wind increases

k	C1	C2	C3	C4	C5
5	[('dn0d', 'wns')]	[('dn0d', 'Ts')]	[('hora', 'nss')]	[('0d', 'Ts')]	[('h', 'Ts')]

Comparing Tab. 4 for the pessimistic mean and Fig. 7, you can see that the resulting concepts in Tab. 4 are the averages of those in Fig. 7. However, this is not always true for confidence degrees, since the routes for obtaining the final values in Tab. 4 and Fig. 7 are different.

For the optimistic mean, we cannot even say that the final concepts in Tab. 5 are the averages of those in Fig. 7 (C_4 drops out). Thus, means are not the averages of final values in general, due to the difference in the paths for full and mean calculations.

In the case of sums in formulas ((18), etc.), in Tab. 6 we obtain results for the final calculation of pessimistic weights aggregation and in Tab. 7, the optimistic aggregation of weights.

Table 6: The final map iterations with the pessimistic mean of weights and the sum in formulas; insolation decreases, wind increases

k	C1	C2	C3	C4	C5
6	[('dn0d', 'wns')]	[('0', 'wns')]	[('hora', 'nss')]	[('d', '0s')]	[('0d', 'sns')]

Table 7: The final map iterations with the optimistic mean of weights and the sum in formulas; insolation decreases, wind increases

k	C1	C2	C3	C4	C5
8	[('dn0d', 'wns')]	[('dn0d', '0')]	[('hora', 'nss')]	[('d', '0')]	[('0c', '0')]

We see that everything gets worse in this case: comparing Tab. 6 and Tab. 7 for means and Tab. 3, we can also obtain a mean that is not the average of the set in Tab. 3. However, the mean may not even be included in this set, though it be of the same level (e.g., final values of C_5 Tab. 6).

Thus, these examples show that the formulas with join are preferable as more adequate. Other variants of a ring structure in lattices should be investigated to confirm the conclusion about the sum operation in these formulas.

6 Conclusion

In this paper, the multi-valued expansion of the concept of fuzzy numbers has been considered when the carrier set and the scale of confidence are lattices of a certain non-chain type. We used confidence degrees here instead of membership ones as more suitable in this case. The ordinary form of Zadeh's extension principle does not work for such a generalization, hence the principle has also been adjusted. We also considered various options for possible definitions of the notion of mean as different similarity measures.

The illustrative example of using such multi-valued assessments in cognitive maps shows that means are not the averages of final values in full calculations in general, due to the difference in the paths for full and mean calculations. Also, we have seen that using the join operation in cognitive map formulas is preferable to the summation operation.

However, this conclusion is made only on the example of a specific sum definition in the ring of subsets of a finite set. Thus, further studies with other rings are needed. Nevertheless, the example has shown that multi-valued sets as estimators can be handled in the same way as single-valued values in the case of lattice variables, and the results are reasonable.

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