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Divisible associative aggregation operations on finite chains

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Abstract

There exist several versions of discrete counterpart of continuity in the framework of finite chains, e.g., the smoothness, the divisibility, intermediate-value property and the 1-Lipschitz property. In this paper, we first discuss the relationships among the smoothness, divisibility, intermediate-value property and 1-Lipschitz property. Second, we present complete characterizations of divisible associative aggregation operations on finite chains.

Keywords: Aggregation function, smoothness, divisibility, 1-Lipschitz property, intermediate-value property.

1 Introduction

In many-valued logics, it is widely accepted to have a finite chain as the set of truth degrees [6]. Interesting applications of many-valued logics can be found in Linguistics [16], Hardware Design [2], etc. In human processed AI [22], a finite chain is used to model linguistically expressed information. On the contrary, a numeric scale can lead to an undesirable effect, called the *tyranny of numbers*. It could be avoided by using a finite ordinal scale mathematically represented as a finite chain [21]. Therefore, a growing interest is taken in aggregation operations defined on finite chains, commonly called discrete aggregation operations, for example, discrete uninorms [1], smooth associative discrete operations [3], bisymmetric discrete operators [10, 19], discrete t-norms and t-conorms [8, 14, 15], copula-like discrete operations [13], discrete conjunctive aggregation operations [11], discrete additive generators [12], autodistributive discrete aggregation operations [20].

In the framework of the unit interval $[0, 1]$, the partial continuity is a staple assumption for characterizing aggregation functions, under which describing the structure of aggregation functions can be achieved, for instance, continuous t-norms and t-conorms [7], uninorms in $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ and representable uninorms [4], continuous associative aggregation functions [9]. This can be applied to the cases of discrete aggregation operations. The smoothness as discrete counterpart of continuity was introduced by Godó and Sierra in [5]. Under the smoothness assumption, t-norms [15], associative aggregation operations [3], bisymmetric and commutative operators [10], have been characterized. Su et al. presented the characterization of bisymmetric aggregation operations with some partial smoothness [19]. The structure of discrete uninorms having smooth underlying operators was described in [17]. On the contrary, in [14], Mayor and Torrens introduced the divisibility as another version of discrete counterpart of continuity, and showed that the smoothness, the divisibility, the 1-Lipschitz property, and intermediate-value property are equivalent for any t-norm. In this paper, we will show that such an equivalence is invalid for general operations, and these four properties except the divisibility are equivalent for general operations (see Section 3). Smooth associative discrete aggregation operations were characterized in [3]. In Section 4, we will focus on divisible associative discrete aggregation operations and describe them only as smooth t-norms. Our conclusions are given in Section 5.

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2 Preliminaries

For a finite chain, the number of its elements is the only decisive factor. Therefore, from now on, we set the finite chain $L_n = \{0, 1, \cdots, n-1, n\}$ with natural order and n being an arbitrarily fixed positive integer, of which an advantage is the possible use of the addition $+$ and subtraction $-$ of integers. The subsets $[x, y]$ of L_n consist of all elements between x and y whenever $x, y \in L_n$ are such that $x \leq y$.

First, we review several properties of binary operations defined on L_n.

Definition 2.1. *For an operation $T : L_n^2 \to L_n$, an element $e \in L_n$ is*

- *a neutral element of T if $T(x, e) = T(e, x) = x$ for each $x \in L_n$.*
- *an annihilator of T if $T(x, e) = T(e, x) = e$ for each $x \in L_n$.*
- *an idempotent element of T if $T(e, e) = e$.*

Definition 2.2. *An operation $T : L_n^2 \to L_n$ is*

- *increasing if $T(x_1, y_1) \leq T(x_2, y_2)$ holds for all $x_1, x_2, y_1, y_2 \in L_n$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.*
- *commutative if $T(x, y) = T(y, x)$ for all $x, y \in L_n$.*
- *associative if $T(x, T(y, z)) = T(T(x, y), z)$ for all $x, y, z \in L_n$.*

For any associative operation $T : L_n^2 \to L_n$, the *power* notation $x_T^{[k]}$, where $k \in \mathbb{N}$, is defined by

$$x_T^{[k]} = \begin{cases} x & \text{if } k = 1, \\ T\left(x, x_T^{[k-1]}\right) & \text{if } k > 1. \end{cases}$$

Next, we recall the notions of the smoothness, the divisibility, 1-Lipschitz property and intermediate-value property.

Definition 2.3. [5, 15] *An operation $f : L_n \to L_n$ is said to*

- *be smooth if $|f(x+1) - f(x)| \leq 1$ for all $x \in L_n$ such that $x \leq n-1$.*
- *be 1-Lipschitz if $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in L_n$.*
- *have intermediate-value property if $f(x) \leq s \leq f(y)$ implies that there exists some $z \in L_n$ such that z is between x and y, and $f(z) = s$.*

Definition 2.4. [15] *A function $F : L_n^2 \to L_n$ is*

- *divisible if, for all $x, y \in L_n$ such that $x \leq y$ there are $z', z'' \in L_n$ such that $x = F(y, z')$ and $x = F(z'', y)$.*
- *smooth if, for all $x, y \in L_n$, both the vertical section $F(x, \cdot)$ and the horizontal section $F(\cdot, y)$ are smooth.*
- *1-Lipschitz if, for all $x_1, y_1, x_2, y_2 \in L_n$,*

$$|F(x_1, y_1) - F(x_2, y_2)| \leq |x_1 - x_2| + |y_1 - y_2|.$$

- *have intermediate-value property if, all $x, y \in L_n$, both the vertical section $F(x, \cdot)$ and the horizontal section $F(\cdot, y)$ have intermediate-value property.*

Remark 2.5. *In* [3, 10]*, a modified smoothness was defined as follows:*
An operation $f : L_n \to L_n$ is "smooth" if, for all $x \in L_n$ and $x \leq n-1$,

$$0 \leq f(x+1) - f(x) \leq 1.$$

From the definition above, one can deduce that all "smooth" operations are increasing, whereas, the smoothness used in this paper does not imply the increasingness.

Clearly, 1-Lipschitz property of a binary function F is equivalent to the fact that for all $x, y \in L_n$, both the vertical section $F(x, \cdot)$ and the horizontal section $F(\cdot, y)$ have 1-Lipschitz property.

Finally, some basic facts concerning t-norms are recalled.

An increasing operation $A : L_n^2 \to L_n$ is an *aggregation operation* if it fulfils $A(0,0) = 0$ and $A(n,n) = n$. A *t-norm* T on L_n [7] is an associative, commutative aggregation operation that has 1 as its neutral element. The typical examples of t-norms are the Łukasiewicz t-norm T_L given by $T_L(x,y) = \max(0, x+y-n)$, the minimum T_M given by $T_M(x,y) = \min(x,y)$ and the drastic product T_D with $T_D(n,x) = T_D(x,n) = x$, and $T_D(x,y) = 0$ otherwise.

Theorem 2.6. [15] *For a t-norm T on L_n, the following statements are equivalent:*

(i) *T is divisible,*

(ii) *T is smooth,*

(iii) *T is 1-Lipschitz,*

(iv) *T has the intermediate-value property.*

Theorem 2.7. [15] *A t-norm T on L_n is smooth if and only if there exists some subset $\{i_1 < i_2 < \cdots < i_s\}$ of L_n such that*

$$T(x,y) = \begin{cases} \max(x+y-i_{r+1}, i_r) & \text{if } x, y \in [i_r, i_{r+1}], \\ \min(x,y) & \text{otherwise.} \end{cases} \tag{1}$$

Note that (1) is an ordinal sum of the summands $\langle i_r, i_{r+1}, T_L \rangle$ [15].

3 The relationships among the smoothness, the divisibility, intermediate-value property and the 1-Lipschitz property

As stated in Theorem 2.6, the smoothness, the divisibility, intermediate-value property and the 1-Lipschitz property are equivalent for any t-norm. However, this is invalid for general operations. The following examples show that smoothness and divisibility are independent.

Example 3.1. *The following binary operation $T_1 : L_3^2 \to L_3$ is commutative, associative and divisible, and has neutral element 3 and annihilator 0. However, it is neither increasing nor smooth.*

T_1	0	1	2	3
0	0	0	0	0
1	0	2	3	1
2	0	3	1	2
3	0	1	2	3

Example 3.2. *The maximum $S_M : L_n^2 \to L_n$ defined by $S_M(x,y) = \max(x,y)$ is commutative, associative and smooth, and has 0 as neutral element and n as annihilator. However, it is not divisible.*

Next, we will demonstrate the equivalence of the smoothness, 1-Lipschitz property and intermediate-value property for any unary operation because these properties of a binary operation can be reduced to the corresponding properties of its vertical section and horizontal section, i.e., the corresponding properties of an unary operation.

Proposition 3.3. *Let $f : L_n \to L_n$ be an unary operation. Then the following statements are equivalent:*

(i) *f is smooth.*

(ii) *f is 1-Lipschitz.*

(iii) *f has intermediate-value property.*

Proof. Clearly, item (ii) implies item (i).

(i) ⇒ (ii): Consider, without loss of generality, that $x, y \in L_n$ with $x < y$. Then

$$\begin{aligned} |f(x) - f(y)| &= \left| \sum_{i=0}^{y-x-1} (f(x+i) - f(x+i+1)) \right| \\ &\leq \sum_{i=0}^{y-x-1} |f(x+i) - f(x+i+1)| \\ &\leq |y - x|. \end{aligned}$$

(i) ⇒ (iii): Suppose that there exist some $x, y, s \in L_n$ such that $f(x) \leq s \leq f(y)$ but for all $z \in L_n$ between x and y we have $f(z) \neq s$. Then $f(x) < s < f(y)$.

- If $x < y$, letting $M = \{r \in [x, y] \mid f(r) < s\}$, then $M \neq \emptyset$ since $x \in M$. Setting $r_0 = \max M$, by $f(y) > s$, we then have $f(r_0) \leq s - 1 < s$ and $x \leq r_0 \leq y - 1$. The hypothesis and definition of r_0 yields $f(r_0 + 1) \geq s + 1$, implying $f(r_0 + 1) - f(r_0) \geq (s + 1) - (s - 1) = 2$, contradicting the smoothness.
- The proof of the case $x > y$ is similar to that of the case $x < y$.

Combining all the findings above, we know that f has intermediate-value property.

(iii) ⇒ (i): Suppose that there exists some $x \in L_n$ such that $x \leq n - 1$ and $|f(x) - f(x + 1)| \geq 2$, then we have $f(x) \geq f(x + 1) + 2$ or $f(x) \leq f(x + 1) - 2$. When $f(x) \geq f(x + 1) + 2$, we obtain $r = f(x + 1) < r + 2 \leq f(x)$ if letting $r = f(x + 1)$. Thus, it follows that $f(x) = r + 2$ from the intermediate-value property. So we can obtain $r = f(x + 1) < r + 1 < r + 2 = f(x)$, which contradicts with intermediate-value property. The proof is similar when $f(x) \leq f(x + 1) + 2$. □

4 Divisible associative discrete aggregation operations

In [3], Fodor characterized smooth associative discrete aggregation operations as smooth pseudo-t-operators (Refer to [18] for the definition of the pseudo-t-operators, including the classes of t-norms, of t-conorms and of t-operators as subclasses). In [8], Kouchakinejad et al. discussed discrete I-semigroups and described them as discrete smooth t-norms or discrete smooth t-conorms.

Definition 4.1. [8, Definition 2.11] *A discrete I-semigroup is a set L_n equipped with an associative operation T such that T is 1-Lipschitz, and has one of the elements $\{0, n\}$ as an idempotent element and the other one as annihilator.*

From Proposition 3.3, we can see that all results in [8] are still valid whence the 1-Lipschitz property of Definition 4.1 is replaced by the smoothness. In order to guarantee results in [8], the 1-Lipschitz property of Definition 4.1 cannot be substituted by the divisibility as shown in Example 3.1.

This section is devoted to studying divisible associative discrete aggregation operations.

Lemma 4.2. *Let $T : L_n^2 \to L_n$ be a divisible associative operation. If $x_0 \in L_n$ is an idempotent element of T, then $T(x, x_0) = T(x_0, x) = x$ for all $x \leq x_0$.*

Proof. For all $x \leq x_0$, there exists some $x' \in L_n$ such that $x = T(x_0, x')$ and further

$$T(x_0, x) = T(x_0, T(x_0, x')) = T(T(x_0, x_0), x') = T(x_0, x') = x.$$

A similar argument can show $T(x, x_0) = x$, for all $x \leq x_0$. □

The previous lemma implies the following corollary.

Corollary 4.3. *Let $T : L_n^2 \to L_n$ be a divisible associative operation. Then T has neutral element n if and only if $T(n, n) = n$.*

There exist divisible associative operations $T : L_n^2 \to L_n$ that have neutral element $e \in L_n \setminus \{0, n\}$ as shown in the following example.

Example 4.4. *The operation $T_2 : L_2^2 \to L_2$ given by*

T_2	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

is divisible, commutative, associative and has 1 as neutral element. However, T_2 is not monotone.

The next proposition discusses the monotonicity of a divisible discrete associative operation.

Proposition 4.5. *Let $T : L_n^2 \to L_n$ be a divisible associative operation. Then T is bounded from above by the minimum of its arguments, i.e., $T(x, y) \leq \min(x, y)$, for all $x, y \in L_n$ if and only if T is increasing.*

Proof. Suppose that T is increasing. For any $x, y \in L_n$, there exists some $y' \in L_n$ such that $y = T(n, y')$ for $y \leq n$ and hence

$$T(x, y) = T(x, T(n, y')) = T(T(x, n), y') \leq T(n, y') = y.$$

Similarly, $T(x, y) \leq x$. Thus, $T \leq \min$.

Suppose that $T \leq \min$. Consider $x_1, x_2, y \in L_n$ with $x_1 \leq x_2$. Since T is a divisible discrete associative operation, we then have that there exists some $z \in L_n$ such that $x_1 = T(z, x_2)$. Consequently,

$$T(x_1, y) = T(T(z, x_2), y) = T(z, T(x_2, y)) \leq T(x_2, y).$$

An analogous argument shows that $T(y, x_1) \leq T(y, x_2)$. Thus, T is increasing. □

Lemma 3.3 in [8] shows that for a discrete I-semigroup (L, T), T is bounded from above by the minimum. However, by Proposition 2, we know that T_2 in Example 3 is a divisible associative operation but not bounded from above by the minimum. The divisibility in Proposition 4.5 cannot be replaced by the smoothness, for instance, the maximum S_M from Example 3.2 is smooth and increasing but not bounded from above by the minimum.

The next proposition shows that a divisible discrete associative operation that is bounded from above by the minimum is smooth and has n as neutral element.

Proposition 4.6. *Let $T : L_n^2 \to L_n$ be a divisible associative operation that is bounded from above by the minimum. Then T is a smooth operation with neutral element n.*

Proof. (i) From $n \leq n$, it follows that there exists some $z \in L_n$ such that $n = T(n, z)$, which, together with the monotonicity of T, yields $n = T(n, z) \leq T(n, n) \leq n$, i.e., $T(n, n) = n$. Consider $x \in L_n$. Then there exists some $z' \in L_n$ such that $x = T(n, z')$ for $x \leq n$. Consequently,

$$T(n, x) = T(n, T(n, z')) = T(T(n, n), z') = T(n, z') = x.$$

Similarly, we can show that $T(x, n) = x$ for all $x \in L_n$.

(ii) Due to Proposition 4.5 the operation T is increasing. Suppose, to the contrary, that there exist some x_0, y_0 such that $T(x_0, y_0+1) - T(x_0, y_0) \geq 2$. Then there exists some $r \in L_n$ such that $T(x_0, y_0) < r < T(x_0, y_0+1)$. On one hand, since T is divisible, we then have that there exists some $z \in L_n$ such that $r = T(T(x_0, y_0 + 1), z) = T(x_0, T(y_0 + 1, z))$. On the other hand, the monotonicity of T yields that $T(x_0, y) \neq r$ for any $y \in L_n$, which is a contradiction. Thus, for any $x \in L_n$, the vertical section, $T(x, \cdot)$, is smooth. Similarly, for any $y \in L_n$, the horizontal section $T(\cdot, y)$ is smooth. □

The next proposition completely characterizes the divisible discrete associative operation that is bounded from above by the minimum and satisfies the diagonal inequality, i.e., $T(x, x) < x$ for all $x \in L_n \backslash \{0, n\}$.

Proposition 4.7. *Let $T : L_n^2 \to L_n$ be a divisible associative operation bounded from above by the minimum. If $T(x, x) < x$ for all $x \in L_n \backslash \{0, n\}$, then*

(i) *$T(x, y) < \min(x, y)$ for any $x, y \in L_n \backslash \{0, n\}$.*

(ii) *$(n-1)_T^{[k]} = n - k$ for $k \in \{1, 2, \cdots, n\}$.*

(iii) *$T = T_L$ is the Łukasiewicz t-norm.*

Proof. (i) Suppose that there exist some $x_0, y_0 \in L_n\backslash\{0,n\}$ such that $x_0 < y_0$ and $T(x_0,y_0) = x_0$. Then the set $\{y \in L_n \mid T(x_0,y) = x_0\}$ is not empty. Let $y^* = \min\{y \in L_n \mid T(x_0,y) = x_0\}$. We then have $0 < y^* < n$ and $T(x_0,y^*) = x_0$. Hence

$$T(x_0, T(y^*,y^*)) = T(T(x_0,y^*),y^*) = T(x_0,y^*) = x_0,$$

implying, $T(y^*,y^*) = y^*$, a contradiction.

(ii) The case $k = 1$ is trivial. Assume that the theorem holds for k, i.e., $(n-1)_T^{[k]} = n-k$. We want to prove that also $(n-1)_T^{[k+1]} = n-k-1$ holds, if $k < n$. From (i) it follows

$$(n-1)_T^{[k+1]} = T(n-1,(n-1)_T^{[k]}) = T(n-1,n-k) < n-k,$$

which, together with $T(n,n-k) = n-k$ and the smoothness of T (implied by Propositions 4.5 and 4.6), yields $(n-1)_T^{[k+1]} = n-k-1$.

(iii) The cases $x \in \{0,n\}$ or $y \in \{0,n\}$ are trivial. For any $x,y \in L_n\backslash\{0,n\}$, we then have $x = (n-1)_T^{[n-x]}$ and $y = (n-1)_T^{[n-y]}$.

- If $x+y \geq n$, then

$$T(x,y) = T\left((n-1)_T^{[n-x]},(n-1)_T^{[n-y]}\right) = (n-1)_T^{[2n-x-y]} = x+y-n.$$

- If $x+y < n$, then $y < n-x$. By (ii) and the monotonicity of T (implied by Proposition 4.5), we have

$$T(x,y) \leq T(x,n-x) = T\left((n-1)_T^{[n-x]},(n-1)_T^{[x]}\right) = (n-1)_T^{[n]} = 0,$$

  implying $T(x,y) = 0$.

Thus, $T(x,y) = \max(x+y-n,0)$ for all $x,y \in L_n$, i.e., $T = T_L$. □

Finally, we completely characterize the divisible discrete associative operation that is bounded from above by the minimum.

Theorem 4.8. *An operation $T : L_n^2 \to L_n$ is a divisible associative operation bounded from above by the minimum if and only if there exists some subset $\{i_1 < i_2 < \cdots < i_s\} \subseteq L_n$ such that T is given by* (1).

Proof. Suppose that $T : L_n^2 \to L_n$ is a divisible associative operation bounded from above by the minimum. Let $I = \{i_1,i_2,\cdots,i_s\} \subseteq L_n$ with $0 = i_1 < i_2 < \cdots < i_s = n$ be the set of idempotent elements of T. For the operation $T|_{[i_r,i_{r+1}]}$, $r \in \{1,2,\cdots,s\}$, a proof like that of Proposition 4.7 shows that, for all $x,y \in [i_r,i_{r+1}]$, $T|_{[i_r,i_{r+1}]}(x,y) = \max(x+y-i_{r+1},i_r)$. Consider $x \in [i_\alpha,i_{\alpha+1}]$ and $y \in [i_\beta,i_{\beta+1}]$ with $\alpha \neq \beta$. If $\alpha < \beta$, then

$$x = T(x,i_{\alpha+1}) \leq T(x,y) \leq \min(x,y) = x.$$

Similarly, if $\beta < \alpha$, then $T(x,y) = y$. Therefore, T has a form of (1).

The reverse implication is guaranteed from a direct calculation. □

From Proposition 4.5 and Theorems 2.6, 2.7 and 4.8, the following corollary is easily obtained.

Corollary 4.9. *Let $T : L_n^2 \to L_n$ be a binary operation. Then T is a divisible associative operation bounded from above by the minimum if and only if T is a smooth t-norm.*

Theorems 3.8 and 3.9 in [8] characterize a discrete I-semigroup operation only as a discrete divisible t-norm or a discrete divisible t-conorm. However, a divisible associative operation bounded from above by the minimum must be a smooth t-norm. The following examples illustrate that such three properties are independent.

Example 4.10. (i) *The drastic product T_D is associative and bounded from above by the minimum but not divisible.*

(ii) *The operation T_2 from Example 4.4 is divisible and associative, but not bounded from above by the minimum.*

(iii) *The operation $T_3 : L_3^2 \to L_3$ given by*

T_3	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	0	2
3	0	1	2	3

is divisible and bounded from above by the minimum, but not associative.

5 Conclusions

In this paper, we have discussed the relationships among several versions of discrete counterpart of continuity in the framework of finite chains, e.g., the smoothness, the divisibility, intermediate-value property and the 1-Lipschitz property (see Section 3), and characterized divisible discrete associative operations bounded from above by the minimum only as smooth t-norms (see Corollary 4.9).

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