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# Grouping fuzzy granular structures based on k-means and fuzzy c-means clustering algorithms in information granulation

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**Abstract**

Fuzzy information granulation theory is based on the way humans granulate and reason about information, and it is essential to the remarkable ability of people to act logically in ambiguous and uncertain situations. In the study of fuzzy information granulation, instead of discussing single fuzzy granules, it is common to consider a fuzzy granular structure arising from a set of fuzzy information granules. Different approaches and perspectives may generate different fuzzy granular structures in the same universe by dividing the object into a number of meaningful fuzzy information granules. However, a specific task usually requires only a selection of representative fuzzy granular structures. Therefore, the main aim of this paper is to group fuzzy granular structures efficiently and accurately. To this end, we first introduce the distances between two fuzzy granular structures and illustrate the relevant properties. Subsequently, k-means and fuzzy c-means clustering algorithms are designed for clustering fuzzy granular structures, and their convergence is demonstrated. In this way, similar fuzzy granular structures can be grouped into the same class. In addition, two evaluation indicators, dispersion and separation, are constructed to evaluate the effect of clustering fuzzy granular structures. Experiments on 12 publicly available datasets demonstrate the feasibility and effectiveness of the proposed algorithms.

*Keywords:* Fuzzy granular structure, distance measure, k-means clustering, fuzzy c-means clustering, granular computing, fuzzy relation.

## 1 Introduction

Granular computing proposed by Zadeh [40] and studied by Lin [18, 19, 20] is an effective approach to dividing complex problems into smaller granules for large amounts of information in computational intelligence. Yao [36] noted that granular computing could be viewed as structured for data processing and problem solving. One of the basic concepts in granular computing is information granulation [35]. Information granulation can be understood as the decomposition of information into a collection of granules, and the granule is considered as a cluster of objects which are similar and indistinguishable. Granulation is an extensively applied idea, for example, divide and conquer, clustering [13], and data compression [21]. Information granules permeate almost all kinds of human activities [25], and human beings have the ability to construct and process information granules [42]. For example, people use seconds, minutes, days, and months to describe time, which can be seen as the granulation of time.

In fact, not all information granules are unambiguous, and fuzzy situations generally exist in the process of human cognition and concept formation. In 1997, Zadeh [41] proposed that the fuzziness of concepts brought about by similarity, indistinguishability, and proximity directly leads to the fuzziness of granules. There is a classic bald paradox with no clear line between being bald and not being bald. Similarly, the line between youth and middle age is not so clear.

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Therefore, extending the information granules to fuzzy information granules is necessary. Inspired by human reasoning and the granulation of information, the theory of fuzzy information granulation [41] is thus generated. The fuzzy granular structure can be regarded as a mathematical structure composed of a group of fuzzy information granules. If people use a fuzzy relation [31] to granulate the information, the objects can be divided into a group of fuzzy information granules, from which a corresponding fuzzy granular structure can be generated. Fuzzy sets are regarded as the basic information granules that support human-centered computing. In the fuzzy granular structure we study, a fuzzy information granule is essentially a fuzzy set. There have been studies on similarity [8, 9] and distance measures [5] between two fuzzy sets, and the uncertainty measure of fuzzy granular structures is also noteworthy. Xie et al. [32] discussed the properties of fuzzy granular structures and the dependence between fuzzy granular structures. Li et al. [14] focused on the granulation measure on the basis of fuzzy information structures in a fully fuzzy information system. Qian et al. discussed the fuzzy granular structure distance [28] and the measure about the fuzzy information granularity of a fuzzy granular structure [29]. In addition, some scholars have studied fuzzy granular structures associated with the characteristics of some specific fuzzy environment, such as granular structures for fuzzy set-valued information systems [15], hesitant fuzzy approximation space [43], and fuzzy $\beta$-covering information systems [22]. Furthermore, Yang et al. [34] proposed the reduction of fuzzy granular structures based on the granular matrix.

To summarize, the previous studies mentioned above mainly focus on a single fuzzy granular structure or the difference between two fuzzy granular structures. However, the group relationship of fuzzy granular structures is not concerned. In fact, different methods and perspectives can produce a variety of fuzzy granular structures in the same universe. People usually only need to select some typical fuzzy granular structures in a specific task. If people can reasonably group the fuzzy granular structures generated in the same universe and collect the fuzzy granular structures with similarity together, they do not need to consider all the fuzzy granular structures when dealing with the subsequent tasks. This is a major motivation for the paper. In previous studies, Qian et al. [27] proposed the grouping of granular structures where granules are boolean. It is undeniable that taking granules as a boolean type has advantages, such as simple calculation and wide application range. However, it ignores the fuzzy information and does not consider the characteristics and properties of fuzzy granular structures. Hence, the aim of the paper is to investigate how to group fuzzy granular structures effectively. The following illustrates the practical significance of fuzzy granular structure clustering through an application scenario of attribute reduction or feature extraction. It is assumed that a set of fuzzy granular structures is obtained by the same method from different attributes in a set of attributes, and this set of fuzzy granular structures can be clustered to estimate which attributes can obtain similar granular structures. Therefore, the attribute reduction can be performed in this way.

As described above, different methods and perspectives may result in different fuzzy granular structures. Therefore, it is necessary to establish a unified representation based on general fuzzy relations that emphasizes the commonality of fuzzy granular structures. Moreover, in order to design an algorithm for clustering a group of fuzzy granular structures in a universe, a distance measure between fuzzy granular structures is proposed referring to methods in literature [28, 33]. The clustering algorithm [7] aims at grouping elements according to their similarity. Cluster-centric algorithms, such as k-means [10] and fuzzy c-means [1] and their variants [11], mostly calculate the center of a set of data points based on distance as a measure of similarity or dissimilarity. The research on the distance of fuzzy granular structures is relatively mature, so the algorithm based on initial clustering center optimization is prioritized when clustering fuzzy granular structures. The clustering in the paper aims to divide the fuzzy granular structures into several clusters to ensure the fuzzy granular structures in the same cluster are as close as possible and ensure the distance between the fuzzy granular structures of different clusters is as far as possible. Considering the fuzziness of fuzzy granular structures, both hard clustering and soft clustering are taken into account to design the algorithm. In the final clustering result, the fuzzy granular structure nearest to the center of the cluster can represent this cluster of fuzzy granular structures. Clustering can be regarded as the compression of fuzzy granular structures to a certain extent. The evaluation method of clustering is divided into two categories. One is to analyze external information, and the other is to analyze internal information. Because the fuzzy granular structures do not have the original label, only the internal evaluation indicators are considered. Two evaluation indexes representing intra-class tightness and inter-class differentiation are designed to evaluate the effectiveness of the proposed clustering method. Finally, a series of experiments are conducted on 12 public datasets to verify the convergence, effectiveness, and scalability of the clustering algorithms for grouping fuzzy granular structures based on the k-means and the fuzzy c-means algorithms.

The contributions and innovation of the paper are shown as follows: (1) A unified representation based on general fuzzy relations is given starting from the commonness of fuzzy granular structures. This makes the clustering algorithms proposed in the paper for fuzzy granular structures more generalizable. (2) A measure for the distance between two fuzzy granular structures is proposed. This distance measure can reflect the similarity and differences between fuzzy granular structures and is the basis of the clustering algorithms in the later paper. (3) The k-means and fuzzy c-means clustering algorithms for clustering fuzzy granular structures are proposed. The algorithms gradually converge

and stabilize during the iterative process, and the objective function can reach the local optimal solution. (4) Two evaluation indexes to evaluate the validity of clustering are designed. They represent the degree of dispersion within and between classes, respectively. The dispersion indicates the compactness of fuzzy granular structures within a cluster. The separation characterizes the disparity between categories of the cluster.

The remaining sections of the paper are structured below. Section 2 briefly reviews some preliminary concepts and fundamental properties of fuzzy relations, fuzzy information granules, and fuzzy granular structures. Section 3 constructs the distance measures between any two fuzzy granular structures and gives some interesting properties. Section 4 designs the k-means clustering algorithm and the fuzzy c-means clustering algorithm for grouping fuzzy granular structures and analyzes the convergence of algorithms. In this section, two internal evaluation indexes of clustering are proposed. In Section 5, the convergence, effectiveness, and scalability of the clustering algorithms are tested through employing 12 real data sets. Discussions on experimental results and applicability are also presented in this section. Finally, Section 6 gives the concluding remarks.

## 2 Preliminaries

For later need, this section recalls some basic concepts, such as fuzzy relation, fuzzy information granules, and fuzzy granular structures.

Throughout the paper, let $U$ be a finite and nonempty universe, and a mapping $A$: $U \longrightarrow [0,1]$ is called a fuzzy set [39] on $U$. For any $x \in U$, the value $A(x)$ is called the membership degree of the object $x$ to the fuzzy set $A$. Pawlak defined the knowledge base as $(U,R) = (U, R_1, R_2, \ldots, R_m)$, where $R_i$ $(i = 1, 2, \ldots, m)$ is an equivalence relation [24]. Correspondingly, $U/R$ is a partition of $U$, and the equivalence classes are called information granules. The set of information granules is a granular structure on $U$. If the relation $R_i$ $(i = 1, 2, \ldots, m)$ is a tolerance relation, tolerance classes induced by the tolerance relation are called tolerance information granules [12]. If the information granules of objects are induced by general binary relation, the set of information granules is a general binary granular structure. For the above situations of information granulation, the information granule is crisp which plays an important role in many fields. However, in the formation of human reasoning and conceptual cognition, information granules are often fuzzy. Therefore, it is necessary to extend the relation and granular structures to the fuzzy case. Let $U \times U$ be the Cartesian product of $U$. Fuzzy relations [26] on $U$ are fuzzy sets of $U \times U$. Objects in a universe can be granulated into a set of fuzzy information granules by a fuzzy relation. The set of fuzzy information granules constitutes a fuzzy granular structure.

**Definition 2.1.** [41] *Let $R$ be a fuzzy relation on $U$. For any $x_i \in U$, the fuzzy information granule of $x_i$ is defined as $g_R(x_i)$, where $g_R(x_i) = \frac{r_{i1}}{x_1} + \frac{r_{i2}}{x_2} + \cdots + \frac{r_{in}}{x_n}$.*

Obviously, a fuzzy information granule $g_R(x_i)$ is a fuzzy set on $U$. It is worth noting that $g_R(x_i)(x_j) = R(x_i, x_j) = r_{ij}$ for any $x_j \in U$. If $r_{ij} = 1$ or $r_{ij} = 0$, it means that $x_j$ certainly belongs to $g_R(x_i)$ or certainly does not belong to $g_R(x_i)$. This is back to the classic situation. The family of fuzzy information granules on $U$ based on the fuzzy relation $R$ can be represented by $FGS_R = \{g_R(x) | x \in U\}$. The cardinality [29] of the fuzzy information granule $g_R(x_i)$ can be calculated with

$$|g_R(x_i)| = \sum_{j=1}^{n} R(x_i, x_j) = \sum_{j=1}^{n} r_{ij}. \tag{1}$$

It has $0 \leq |g_R(x_i)| \leq n$. Suppose that a fuzzy information granule $g_{R_1}(x_1) = \frac{0.6}{x_1} + \frac{0.8}{x_2} + \frac{0.2}{x_3}$. Then $|g_{R_1}(x_1)| = 0.6 + 0.8 + 0.2 = 1.6$. Fuzzy information granules can be represented simply by vector without causing ambiguity. For example, the granule $g_R(x_i) = \frac{r_{i1}}{x_1} + \frac{r_{i2}}{x_2} + \cdots + \frac{r_{in}}{x_n}$ can be denoted by $(r_{i1}, r_{i2}, \ldots, r_{in})$. Corresponding to the above two special fuzzy relations, two special fuzzy information granules can be obtained. If the fuzzy information granule is equel to $(\overbrace{0, 0, \ldots, 0}^{n})$, it is called a fuzzy empty information granule and denoted as $g_R(\emptyset)$. If the fuzzy information granule is equal to $(\overbrace{1, 1, \ldots, 1}^{n})$, it is called a fuzzy universal information granule and denoted as $g_R(\bullet)$. Next, the concept of fuzzy granular structure is given. The fuzzy granular structure induced by general fuzzy relation $R$ on the universe $U$ can be represented as

$$FGS(R) = (g_R(x_1), g_R(x_2), \ldots, g_R(x_n)).$$

The fuzzy granular structure library on the universe $U$ is defined as $\mathcal{FGS}(U) = \{FGS_{R_1}, FGS_{R_2}, \ldots, FGS_{R_m}\}$, where $FGS_{R_t} = (g_{R_t}(x_1), g_{R_t}(x_2), \ldots, g_{R_t}(x_n))$ is a fuzzy granular structure and $g_{R_t}(x_i) = \frac{r_{ti1}}{x_1} + \frac{r_{ti2}}{x_2} + \cdots + \frac{r_{tin}}{x_n}$ induced by the fuzzy relation $R_t$. The fuzzy granular structure library $\mathcal{FGS}(U)$ is a family of fuzzy granular structures on $U$.

**Definition 2.2.** [29] *Assume that* $FGS_{R_1} = (g_{R_1}(x_1), g_{R_1}(x_2), \ldots, g_{R_1}(x_n))$ *and* $FGS_{R_2} = (g_{R_2}(x_1), g_{R_2}(x_2), \ldots, g_{R_2}(x_n))$ *are two fuzzy granular structures with respect to fuzzy relations* $R_1$ *and* $R_2$*. Then there are the following definitions.*

(1) $FGS_{R_1}$ *is finer than* $FGS_{R_2}$*, denoted by* $FGS_{R_1} \preceq FGS_{R_2}$*, if for any* $x_i \in U$*,* $g_{R_1}(x_1) \subseteq g_{R_2}(x_1)$*. We also say that* $FGS_{R_2}$ *is coarser than* $FGS_{R_1}$*, denoted by* $FGS_{R_2} \succeq FGS_{R_1}$*.*

(2) $FGS_{R_1}$ *is strictly finer than* $FGS_{R_2}$*, denoted by* $FGS_{R_1} \prec FGS_{R_2}$*, if* $FGS_{R_1} \neq FGS_{R_2}$ *and* $FGS_{R_1} \preceq FGS_{R_2}$*.*

(3) $FGS_{R_1}$ *is partially finer than* $FGS_{R_2}$*, denoted by* $FGS_{R_1} \sqsubseteq FGS_{R_2}$*, if there exists* $x_i \in U$*, such that* $g_{R_1}(x_1) \subseteq g_{R_2}(x_1)$*.*

*If for any* $x_i \in U$*,* $g_{R_1}(x_i) \nsubseteq g_{R_2}(x_i)$ *and* $g_{R_2}(x_i) \nsubseteq g_{R_1}(x_i)$*, then it says that* $FGS_{R_1}$ *is referred to as independent of* $FGS_{R_2}$*. It is clear that* $FGS_{R_1} = FGS_{R_2}$ *if and only if* $FGS_{R_2} \succeq FGS_{R_1}$ *and* $FGS_{R_2} \preceq FGS_{R_1}$*.*

The above definition describes the refinement-coarsening relations among fuzzy granular structures. It is obvious that $\preceq$ is reflexive, anti-symmetric, and transitive, so it is a partial ordering relation. In this section, a unified knowledge representation method of a fuzzy granular structure induced by general fuzzy relation is given.

# 3 Distance measures between fuzzy granular structures

The representation method of fuzzy granular structures based on general fuzzy relations are obtained in the previous section. In order to design the clustering algorithm of fuzzy granular structures, we should first consider selecting the appropriate distance formula to measure the similarity and difference between fuzzy granular structures. In this section, the distance measures between two different fuzzy granular structures are proposed to describe the diversity between two fuzzy granular structures. In order to construct a reasonable distance measure, we start from the information granularity measure [3, 16, 17, 37, 44], which can reflect the characteristic of information granules and granular structures.

**Definition 3.1.** *Let* $\mathcal{FGS}(U) = \{FGS_{R_1}, FGS_{R_2}, \ldots, FGS_{R_m}\}$ *be a fuzzy granular structure library of the universe* $U$*. The function* $G : \mathcal{FGS}(U) \longrightarrow \mathbb{R}$ *is defined as the granularity measure of a fuzzy granular structure if the function* $G$ *satisfies the following conditions:*

(1) *(nonnegativity)* $G(FGS_{R_t}) \geq 0$ *for any* $FGS_{R_t} \in \mathcal{FGS}(U)$*;*

(2) *(monotonicity)* $G(FGS_{R_t}) < G(FGS_{R_s})$ *or* $G(FGS_{R_t}) > G(FGS_{R_s})$ *if* $FGS_{R_s} \prec FGS_{R_t}$ *for any* $FGS_{R_t}, FGS_{R_s} \subseteq \mathcal{FGS}(U)$*;*

(3) *(invariability)* $G(FGS_{R_s}) = G(FGS_{R_t})$ *if* $FGS_{R_s} = FGS_{R_t}$ *for any* $FGS_{R_t}, FGS_{R_s} \subseteq \mathcal{FGS}(U)$*.*

*Suppose that* $FGS_{R_t} = (g_{R_t}(x_1), g_{R_t}(x_2), \ldots, g_{R_t}(x_n))$ *is a fuzzy granular structure. The function* $m : g_{R_t}(x_i) \longrightarrow \mathbb{R}$ *is defined as the granularity measure of a fuzzy information granule if the function* $m$ *satisfies the following conditions:*

(1) *(nonnegativity)* $m(g_{R_t}(x_i)) \geq 0$ *for any* $g_{R_t}(x_i) \in FGS_{R_t}$*;*

(2) *(monotonicity)* $m(g_{R_t}(x_i)) < m(g_{R_t}(x_j))$ *or* $m(g_{R_t}(x_j)) > m(g_{R_t}(x_i))$ *if* $g_{R_t}(x_i) \subseteq g_{R_t}(x_j)$ *for any* $g_{R_t}(x_i), g_{R_t}(x_j) \in FGS_{R_t}$*;*

(3) *(invariability)* $m(g_{R_t}(x_i)) = m(g_{R_t}(x_j))$ *if* $g_{R_t}(x_i) = g_{R_t}(x_j)$ *for any* $g_{R_t}(x_i), g_{R_t}(x_j) \in FGS_{R_t}$*.*

The information granularity of granular structures is a measure of the uncertainty of its actual structure. Generally speaking, information granularity represents the identification ability of information in the granular structure. In fuzzy granular structure, fuzzy information granularity also plays the same role and should be used to describe the classification ability of the fuzzy granular structure. The granularity measure is closely related to the size of information granules and granular structures, so the most straightforward measure is constructed from the cardinal number of fuzzy information granules.

**Definition 3.2.** *Let* $FGS(R) = (g_R(x_1), g_R(x_2), \ldots, g_R(x_n))$ *be a fuzzy granular structure of the universe* $U$*, where* $g_R(x_i) = \frac{r_{i1}}{x_1} + \frac{r_{i2}}{x_2} + \cdots + \frac{r_{in}}{x_n}$ *for any* $x_i \in U$*. The granularity of a fuzzy information granule* $g_R(x_i)$ *is defined as:*

$$\mathcal{G}(g_R(x_i)) = \frac{|g_R(x_i)|}{n},$$

*and the granularity of the fuzzy granular structure $FGS(R)$ is defined as*

$$\mathcal{G}(FGS(R)) = \frac{\sum_{i=1}^{n} \mathcal{G}(g_R(x_i))}{n} = \frac{\sum_{i=1}^{n} |g_R(x_i)|}{n^2}.$$

*It has $0 \leq \mathcal{G}(g_R(x_i)) \leq 1$ and $0 \leq \mathcal{G}(FGS(R)) \leq 1$.*

If the fuzzy information granule $g_R(x_i)$ is a fuzzy empty information granule for all $x_i \in U$, then the granularity of the fuzzy granular structure is equal to 0. Similarly, if the fuzzy information granule $g_R(x_i)$ is a fuzzy universal information granule for all $x_i \in U$, then the granularity of the fuzzy granular structure is equal to 1. For two fuzzy granular structures $FGS(R_1)$ and $FGS(R_2)$ of $U$, it has $\mathcal{G}(FGS(R_1)) \geq \mathcal{G}(FGS(R_2))$ if $FGS(R_2) \preceq FGS(R_1)$. Next, consider the operation between two fuzzy information granules. Suppose that $A$ and $B$ are two finite sets. The symmetric different of $A$ and $B$ is defined as $A \oplus B = (A - B) \cup (B - A)$. If $A \subseteq B$, then $|A \oplus B| = |B - A| = |B| - |A|$. This can characterize the difference between two finite sets and be extended to fuzzy sets.

**Definition 3.3.** *Let $g_{R_1}(x_i) = \frac{r_{1i1}}{x_1} + \frac{r_{1i2}}{x_2} + \cdots + \frac{r_{1in}}{x_n}$ and $g_{R_2}(x_i) = \frac{r_{2i1}}{x_1} + \frac{r_{2i2}}{x_2} + \cdots + \frac{r_{2in}}{x_n}$ be any two fuzzy information granules. An operation of fuzzy information granules is defined as follows:*

$$g_{R_1}(x_i) \oplus g_{R_2}(x_i) = g_{R_1}(x_i) \cup g_{R_2}(x_i) - g_{R_1}(x_i) \cap g_{R_2}(x_i),$$

*where $g_{R_1}(x_i) \cup g_{R_2}(x_i) = \frac{r_{1i1} \vee r_{2i1}}{x_1} + \frac{r_{1i2} \vee r_{2i2}}{x_2} + \cdots + \frac{r_{1in} \vee r_{2in}}{x_n}$ and $g_{R_1}(x_i) \cap g_{R_2}(x_i) = \frac{r_{1i1} \wedge r_{2i1}}{x_1} + \frac{r_{1i2} \wedge r_{2i2}}{x_2} + \cdots + \frac{r_{1in} \wedge r_{2in}}{x_n}$. Here we specify a subtraction operator $-$ between two fuzzy information granules. That is $g_{R_1}(x_i) \cup g_{R_2}(x_i) - g_{R_1}(x_i) \cap g_{R_2}(x_i) = \frac{r_{1i1} \vee r_{2i1} - r_{1i1} \wedge r_{2i1}}{x_1} + \frac{r_{1i2} \vee r_{2i2} - r_{1i2} \wedge r_{2i2}}{x_2} + \cdots + \frac{r_{1in} \vee r_{2in} - r_{1in} \wedge r_{2in}}{x_n}$.*

A straightforward example is provided to help clarify the aforementioned concept.

**Example 3.4.** *Assume that there are two fuzzy granular structures, $FGS(R_1) = (\frac{0.6}{x_1} + \frac{0.8}{x_2} + \frac{0.2}{x_3}, \frac{0.5}{x_1} + \frac{0.3}{x_2} + \frac{0.7}{x_3}, \frac{0.3}{x_1} + \frac{0.5}{x_2} + \frac{0.6}{x_3})$ and $FGS(R_2) = (\frac{0.7}{x_1} + \frac{0.2}{x_2} + \frac{0.4}{x_3}, \frac{0.3}{x_1} + \frac{0.5}{x_2} + \frac{0.6}{x_3}, \frac{0.6}{x_1} + \frac{0.1}{x_2} + \frac{0.2}{x_3})$, where $g_{R_1}(x_1) = \frac{0.6}{x_1} + \frac{0.8}{x_2} + \frac{0.2}{x_3}$, $g_{R_1}(x_2) = \frac{0.5}{x_1} + \frac{0.3}{x_2} + \frac{0.7}{x_3}$, $g_{R_1}(x_3) = \frac{0.3}{x_1} + \frac{0.5}{x_2} + \frac{0.6}{x_3}$, $g_{R_2}(x_1) = \frac{0.7}{x_1} + \frac{0.2}{x_2} + \frac{0.4}{x_3}$, $g_{R_2}(x_2) = \frac{0.3}{x_1} + \frac{0.5}{x_2} + \frac{0.6}{x_3}$, and $g_{R_2}(x_3) = \frac{0.6}{x_1} + \frac{0.1}{x_2} + \frac{0.2}{x_3}$. It can calculate that $g_{R_1}(x_1) \oplus g_{R_2}(x_1) = g_{R_1}(x_1) \cup g_{R_2}(x_1) - g_{R_1}(x_1) \cap g_{R_2}(x_1) = \{\frac{0.7}{x_1} + \frac{0.8}{x_2} + \frac{0.4}{x_3}\} - \{\frac{0.6}{x_1} + \frac{0.2}{x_2} + \frac{0.2}{x_3}\} = \frac{0.1}{x_1} + \frac{0.6}{x_2} + \frac{0.2}{x_3}$. Therefore, it can obtain $|g_{R_1}(x_1) \oplus g_{R_2}(x_1)| = 0.1 + 0.6 + 0.2 = 0.9$.*

In order to ensure that the distance measure we proposed satisfies the relevant properties of distance, two lemmas and their proofs are illustrated.

**Lemma 3.5.** *Suppose that $g_{R_1}(x_i)$ and $g_{R_2}(x_i)$ are any two fuzzy information granules. Then $g_{R_1}(x_i) = g_{R_2}(x_i)$ if and only if $|g_{R_1}(x_i) \oplus g_{R_2}(x_i)| = 0$.*

*Proof.* For the "only if" part, suppose that $g_{R_1}(x_i) = g_{R_2}(x_i)$. Then

$$|g_{R_1}(x_i) \oplus g_{R_2}(x_i)| = |g_{R_1}(x_i) \cup g_{R_2}(x_i) - g_{R_1}(x_i) \cap g_{R_2}(x_i)| = |g_{R_1}(x_i) \cup g_{R_1}(x_i) - g_{R_1}(x_i) \cap g_{R_1}(x_i)| = 0.$$

For the converse, suppose that $|g_{R_1}(x_i) \oplus g_{R_2}(x_i)| = 0$. Then it has $|g_{R_1}(x_i) \cup g_{R_2}(x_i) - g_{R_1}(x_i) \cap g_{R_2}(x_i)| = 0$. So $|g_{R_1}(x_i) \cup g_{R_2}(x_i)| = |g_{R_1}(x_i) \cap g_{R_2}(x_i)|$. Therefore, for any $x_j \in U$, $g_{R_1}(x_i)(x_j) \leq g_{R_2}(x_i)(x_j)$ and $g_{R_1}(x_i)(x_j) \geq g_{R_2}(x_i)(x_j)$. Hence, $g_{R_1}(x_i) = g_{R_2}(x_i)$. □

**Lemma 3.6.** *Suppose that $g_{R_1}(x_i)$, $g_{R_2}(x_i)$, and $g_{R_3}(x_i)$ are any three fuzzy information granules. Then*

$$|g_{R_1}(x_i) \oplus g_{R_3}(x_i)| + |g_{R_3}(x_i) \oplus g_{R_2}(x_i)| \geq |g_{R_1}(x_i) \oplus g_{R_2}(x_i)|.$$

*It is worth mentioning that if $g_{R_1}(x_i) \subseteq g_{R_3}(x_i) \subseteq g_{R_2}(x_i)$ or $g_{R_1}(x_i) \supseteq g_{R_3}(x_i) \supseteq g_{R_2}(x_i)$, then $|g_{R_1}(x_i) \oplus g_{R_3}(x_i)| + |g_{R_3}(x_i) \oplus g_{R_2}(x_i)| = |g_{R_1}(x_i) \oplus g_{R_2}(x_i)|$.*

*Proof.* For any $x_j \in U$, there are the following cases about $g_{R_1}(x_i)(x_j)$, $g_{R_2}(x_i)(x_j)$, and $g_{R_3}(x_i)(x_j)$:

(1) $g_{R_1}(x_i)(x_j) \leq g_{R_2}(x_i)(x_j) \leq g_{R_3}(x_i)(x_j)$, (2) $g_{R_2}(x_i)(x_j) \leq g_{R_1}(x_i)(x_j) \leq g_{R_3}(x_i)(x_j)$,
(3) $g_{R_2}(x_i)(x_j) \leq g_{R_3}(x_i)(x_j) \leq g_{R_1}(x_i)(x_j)$, (4) $g_{R_1}(x_i)(x_j) \leq g_{R_3}(x_i)(x_j) \leq g_{R_2}(x_i)(x_j)$,
(5) $g_{R_3}(x_i)(x_j) \leq g_{R_1}(x_i)(x_j) \leq g_{R_2}(x_i)(x_j)$, (6) $g_{R_3}(x_i)(x_j) \leq g_{R_2}(x_i)(x_j) \leq g_{R_1}(x_i)(x_j)$.

We only prove case(1). Given $x_j \in U$, due to $g_{R_1}(x_i)(x_j) \leq g_{R_2}(x_i)(x_j) \leq g_{R_3}(x_i)(x_j)$, it have $|g_{R_1}(x_i)(x_j) \vee g_{R_3}(x_i)(x_j) - g_{R_3}(x_i)(x_j) \wedge g_{R_1}(x_i)(x_j)| + |g_{R_3}(x_i)(x_j) \vee g_{R_2}(x_i)(x_j) - g_{R_3}(x_i)(x_j) \wedge g_{R_2}(x_i)(x_j)| - |g_{R_1}(x_i)(x_j) \vee g_{R_2}(x_i)(x_j) - g_{R_1}(x_i)(x_j) \wedge g_{R_2}(x_i)(x_j)| = (|g_{R_1}(x_i)(x_j) - g_{R_3}(x_i)(x_j)|) + (|g_{R_3}(x_i)(x_j) - g_{R_2}(x_i)(x_j)|) - (|g_{R_1}(x_i)(x_j) - g_{R_2}(x_i)(x_j)|) = 2g_{R_1}(x_i)(x_j) \geq 0$. Other cases can be similarly proved. Thus, $|g_{R_1}(x_i) \oplus g_{R_3}(x_i)| + |g_{R_3}(x_i) \oplus g_{R_2}(x_i)| \geq |g_{R_1}(x_i) \oplus g_{R_2}(x_i)|$.

If $g_{R_1}(x_i) \subseteq g_{R_3}(x_i) \subseteq g_{R_2}(x_i)$, then $|g_{R_1}(x_i) \oplus g_{R_3}(x_i)| + |g_{R_3}(x_i) \oplus g_{R_2}(x_i)| = |g_{R_1}(x_i) \vee g_{R_3}(x_i) - g_{R_3}(x_i) \wedge g_{R_1}(x_i)| + |g_{R_3}(x_i) \vee g_{R_2}(x_i) - g_{R_3}(x_i) \wedge g_{R_2}(x_i)| = |g_{R_3}(x_i) - g_{R_1}(x_i)| + |g_{R_2}(x_i) - g_{R_3}(x_i)| = |g_{R_2}(x_i) - g_{R_1}(x_i)| = |g_{R_1}(x_i) \oplus g_{R_2}(x_i)|$. □

The above two lemmas show that the symmetric difference operation of fuzzy information granules has invariance and satisfies the triangle inequality, and these properties will provide theoretical support for designing the distance of fuzzy granular structures.

**Example 3.7.** *Let us revisit Example 3.4 and add* $FGS(R_3) = (\frac{0.8}{x_1} + \frac{0.2}{x_2} + \frac{0.4}{x_3}, \frac{0.7}{x_1} + \frac{0.8}{x_2} + \frac{0.2}{x_3}, \frac{0.5}{x_1} + \frac{0.6}{x_2} + \frac{0.3}{x_3})$, *where* $g_{R_3}(x_1) = \frac{0.8}{x_1} + \frac{0.2}{x_2} + \frac{0.4}{x_3}$, $g_{R_3}(x_2) = \frac{0.7}{x_1} + \frac{0.8}{x_2} + \frac{0.2}{x_3}$, *and* $g_{R_3}(x_3) = \frac{0.5}{x_1} + \frac{0.6}{x_2} + \frac{0.3}{x_3}$. *It can calculate that* $|g_{R_1}(x_1) \oplus g_{R_3}(x_1)| = 0.7$ *and* $|g_{R_3}(x_1) \oplus g_{R_2}(x_1)| = 1$. *Hence* $|g_{R_1}(x_1) \oplus g_{R_3}(x_1)| + |g_{R_3}(x_1) \oplus g_{R_2}(x_1)| = 1.7$. *Furthermore,* $|g_{R_1}(x_1) \oplus g_{R_2}(x_1)| = 0.5$. *Thus* $|g_{R_1}(x_1) \oplus g_{R_3}(x_1)| + |g_{R_3}(x_1) \oplus g_{R_2}(x_1)| \geq |g_{R_1}(x_1) \oplus g_{R_2}(x_{1i})|$.

Next, the definition of distance between any two fuzzy information granules is given.

**Definition 3.8.** *Given any two fuzzy information granules* $g_{R_1}(x_i) = \frac{r_{1i1}}{x_1} + \frac{r_{1i2}}{x_2} + \cdots + \frac{r_{1in}}{x_n}$ *and* $g_{R_2}(x_i) = \frac{r_{2i1}}{x_1} + \frac{r_{2i2}}{x_2} + \cdots + \frac{r_{2in}}{x_n}$, *the distance between the two fuzzy information granules* $g_{R_1}(x_i)$ *and* $g_{R_2}(x_i)$ *is defined as*

$$D(g_{R_1}(x_i), g_{R_2}(x_i)) = \frac{|g_{R_1}(x_i) \oplus g_{R_2}(x_i)|}{n}. \tag{2}$$

*The distance* $D$ *satisfies* $0 \leq D(g_{R_1}(x_i), g_{R_2}(x_i)) \leq 1$.

By definition of the distance above and two lemmas about the operation of fuzzy information granules, the following properties are obtained naturally.

**Proposition 3.9.** *The distance* $D$ *between any two fuzzy information granules is a distance measure, which satisfies the following three properties:*

(1) *(nonnegativity)* $D(g_{R_1}(x_i), g_{R_2}(x_i)) \geq 0$;

(2) *(symmetry)* $D(g_{R_1}(x_i), g_{R_2}(x_i)) = D(g_{R_2}(x_i), g_{R_1}(x_i))$;

(3) *(triangle inequality)* $D(g_{R_1}(x_i), g_{R_3}(x_i)) + D(g_{R_3}(x_i), g_{R_2}(x_i)) \geq D(g_{R_1}(x_i), g_{R_2}(x_i))$.

The distance formula above can be thought of as the normalization of $|g_{R_1}(x_i) \oplus g_{R_2}(x_i)|$. It can be called the absolute distance between two fuzzy information granules. Correspondingly, the relative distance between two fuzzy information granules can be constructed. Given two fuzzy information granules $g_{R_1}(x_i) = \frac{r_{1i1}}{x_1} + \frac{r_{1i2}}{x_2} + \cdots + \frac{r_{1in}}{x_n}$ and $g_{R_2}(x_i) = \frac{r_{2i1}}{x_1} + \frac{r_{2i2}}{x_2} + \cdots + \frac{r_{2in}}{x_n}$, the relative distance between the two fuzzy information granules $g_{R_1}(x_i)$ and $g_{R_2}(x_i)$ is defined as

$$d(g_{R_1}(x_i), g_{R_2}(x_i)) = \frac{|g_{R_1}(x_i) \oplus g_{R_2}(x_i)|}{|g_{R_1}(x_i) \cup g_{R_2}(x_i)|} = \frac{|g_{R_1}(x_i) \cup g_{R_2}(x_i) - g_{R_1}(x_i) \cap g_{R_2}(x_i)|}{|g_{R_1}(x_i) \cup g_{R_2}(x_i)|}. \tag{3}$$

It can be proved that the relative distance also satisfies the above three conditions and is a distance measure by Definition 3.8 and Lemmas 3.5 and 3.6.

As mentioned above, to construct a reasonable distance that can be applied to design the algorithm of clustering fuzzy granular structure, we start with the granularity measure that can reflect the characteristics of fuzzy information granules. The following illustrates the potential connection between granularity measures and distance measures.

**Proposition 3.10.** *For any* $x_i \in U$, *let* $g_R(x_i)$ *be a fuzzy information granule and* $g_R(\bullet)$ *be a fuzzy universal information granule. Then*

$$\mathcal{G}(g_R(x_i)) = 1 - d(g_R(x_i), g_R(\bullet)).$$

*Proof.* According to the definitions of the granularity of fuzzy information granules and the distance between two fuzzy information granules, it has that $\mathcal{G}(g_R(x_i)) = \frac{|g_R(x_i)|}{n}$ and

$$\begin{aligned} d(g_R(x_i), g_R(\bullet)) &= \frac{|g_R(x_i) \oplus g_R(\bullet)|}{|g_R(x_i) \cup g_R(\bullet)|} = \frac{|g_R(x_i) \cup g_R(\bullet) - g_R(x_i) \cap g_R(\bullet)|}{|g_R(x_i) \cup g_R(\bullet)|} \\ &= \frac{|g_R(\bullet) - g_R(x_i)|}{|g_R(\bullet)|} = \frac{|g_R(\bullet)|}{|g_R(\bullet)|} - \frac{|g_R(x_i)|}{|g_R(\bullet)|} \\ &= 1 - \frac{|g_R(x_i)|}{n}. \end{aligned}$$

Hence, $\mathcal{G}(g_R(x_i)) = 1 - d(g_R(x_i), g_R(\bullet))$. □

In terms of the relation between the distance measure and the similarity measure, if by default the sum of the distance and similarity between two elements is 1, it can discover that $1 - d(g_R(x_i), g_R(\bullet))$ is the similarity measure between the fuzzy information granules $g_R(x_i)$ and $g_R(\bullet)$. In other words, the granularity of $g_R(x_i)$ can be understood as the similarity measure between the fuzzy information granules $g_R(x_i)$ and $g_R(\bullet)$.

**Proposition 3.11.** *For any $x_i \in U$, let $g_R(x_i)$ be a fuzzy information granule and $g_R(\emptyset)$ be a fuzzy empty information granule. Then*

$$\mathcal{G}(g_R(x_i)) = D(g_R(x_i), g_R(\emptyset)).$$

*Proof.* According to the definitions of the granularity of fuzzy information granules and the distance between two fuzzy information granules, it has that $\mathcal{G}(g_R(x_i)) = \frac{|g_R(x_i)|}{n}$ and

$$\begin{aligned} D(g_R(x_i), g_R(\emptyset)) &= \frac{|g_R(x_i) \oplus g_R(\emptyset)|}{|g_R(x_i) \cup g_R(\emptyset)|} \\ &= \frac{|g_R(x_i) \cup g_R(\emptyset) - g_R(x_i) \cap g_R(\emptyset)|}{n} \\ &= \frac{|g_R(x_i) - g_R(\emptyset)|}{n} = \frac{|g_R(x_i)|}{n}. \end{aligned}$$

Therefore, $\mathcal{G}(g_R(x_i)) = D(g_R(x_i), g_R(\emptyset))$. □

For the sake of illustrating the above proposition, a simple example is given.

**Example 3.12.** *Let us revisit Example 3.7. It can calculate that $D(g_{R_1}(x_1), g_{R_2}(x_1)) = \frac{|g_{R_1}(x_1) \oplus g_{R_2}(x_1)|}{3} = \frac{0.5}{3} = \frac{1}{6}$ and $d(g_{R_1}(x_1), g_{R_2}(x_1)) = \frac{|g_{R_1}(x_1) \oplus g_{R_2}(x_1)|}{|g_{R_1}(x_1) \cup g_{R_2}(x_1)|} = \frac{0.5}{1.5} = \frac{1}{3}$. Suppose that $g_R(\bullet)$ is a fuzzy universal information granule and $g_R(\emptyset)$ is a fuzzy empty information granule. Further, it has $d(g_{R_1}(x_1), g_R(\bullet)) = \frac{1.4}{3} = \frac{7}{15}$ and $D(g_{R_1}(x_1), g_R(\emptyset)) = \frac{1.6}{3} = \frac{15}{8}$. Therefore,*

$$\mathcal{G}(g_{R_1}(x_1)) = \frac{8}{15} = 1 - d(g_{R_1}(x_1), g_R(\bullet)) = D(g_{R_1}(x_1), g_R(\emptyset)).$$

Correspond to Proposition 3.11, the granularity of $g_R(x_i)$ can represent the distance between $g_R(x_i)$ and $g_R(\emptyset)$. Furthermore, the definition of fuzzy granular structure distance between any two fuzzy granular structures is given.

**Definition 3.13.** *Given any two fuzzy granular structures $FGS(R_1) = (g_{R_1}(x_1), g_{R_1}(x_2), \ldots, g_{R_1}(x_n))$ and $FGS(R_2) = (g_{R_2}(x_1), g_{R_2}(x_2), \ldots, g_{R_2}(x_n))$, where the fuzzy information granules $g_{R_1}(x_i) = \frac{r_{1i1}}{x_1} + \frac{r_{1i2}}{x_2} + \cdots + \frac{r_{1in}}{x_n}$ and $g_{R_2}(x_i) = \frac{r_{2i1}}{x_1} + \frac{r_{2i2}}{x_2} + \cdots + \frac{r_{2in}}{x_n}$, the fuzzy granular structure distance between $FGS(R_1)$ and $FGS(R_2)$ is defined as*

$$D_{sd}(FGS(R_1), FGS(R_2)) = \frac{1}{n} \sum_{i=1}^{n} D(g_{R_1}(x_i), g_{R_2}(x_i)). \tag{4}$$

*It is easy to know $0 \leq D_{sd}(FGS(R_1), FGS(R_2)) \leq 1$.*

We have already proved that the distance between fuzzy information granules satisfies the property of distance. Below we will demonstrate that the fuzzy granular structure distance also satisfies the property of distance.

**Proposition 3.14.** *Let $\mathcal{FGS}(U)$ be a fuzzy granular structure library of the universe $U$. Then $(\mathcal{FGS}(U), D_{sd})$ is a distance space.*

*Proof.* (1) (Nonnegativity) Suppose that $FGS(R_1) = (g_{R_1}(x_1), g_{R_1}(x_2), \ldots, g_{R_1}(x_n))$ and $FGS(R_2) = (g_{R_2}(x_1), g_{R_2}(x_2), \ldots, g_{R_2}(x_n))$ are any two fuzzy granular structures in $\mathcal{FGS}(U)$, where the fuzzy information granules $g_{R_1}(x_i) = \frac{r_{1i1}}{x_1} + \frac{r_{1i2}}{x_2} + \cdots + \frac{r_{1in}}{x_n}$ and $g_{R_2}(x_i) = \frac{r_{2i1}}{x_1} + \frac{r_{2i2}}{x_2} + \cdots + \frac{r_{2in}}{x_n}$. By Definition 3.8 and Lemma 3.5, it has

$$D_{sd}(FGS(R_1), FGS(R_2)) = \frac{1}{n}\sum_{i=1}^{n} D(g_{R_1}(x_i), g_{R_2}(x_i)) = \frac{1}{n}\sum_{i=1}^{n} \frac{|g_{R_1}(x_i) \oplus g_{R_2}(x_i)|}{n} = \frac{1}{n^2}\sum_{i=1}^{n}(|g_{R_1}(x_i) \oplus g_{R_2}(x_i)|) \geq 0.$$

(2) (Symmetry)

$$\begin{aligned} D_{sd}(FGS(R_1), FGS(R_2)) &= \frac{1}{n}\sum_{i=1}^{n} D(g_{R_1}(x_i), g_{R_2}(x_i)) = \frac{1}{n}\sum_{i=1}^{n} \frac{|g_{R_1}(x_i) \oplus g_{R_2}(x_i)|}{n} \\ &= \frac{1}{n^2}\sum_{i=1}^{n}(|g_{R_1}(x_i) \oplus g_{R_2}(x_i)|) = \frac{1}{n^2}\sum_{i=1}^{n}(|g_{R_1}(x_i) \cup g_{R_2}(x_i) - g_{R_1}(x_i) \cap g_{R_2}(x_i)|) \\ &= \frac{1}{n^2}\sum_{i=1}^{n}(|g_{R_2}(x_i) \cup g_{R_1}(x_i) - g_{R_2}(x_i) \cap g_{R_1}(x_i)|) = \frac{1}{n^2}\sum_{i=1}^{n}(|g_{R_2}(x_i) \oplus g_{R_1}(x_i)|) \\ &= \frac{1}{n}\sum_{i=1}^{n} D(g_{R_2}(x_i), g_{R_1}(x_i)) = D_{sd}(FGS(R_2), FGS(R_1)). \end{aligned}$$

(3) (Trangle inequality)

$$\begin{aligned} D_{sd}(FGS(R_1), FGS(R_2)) + D_{sd}(FGS(R_2), FGS(R_3)) &= \frac{1}{n}\sum_{i=1}^{n} D(g_{R_1}(x_i), g_{R_2}(x_i)) + \frac{1}{n}\sum_{i=1}^{n} D(g_{R_2}(x_i), g_{R_3}(x_i)) \\ &= \frac{1}{n}\sum_{i=1}^{n} \frac{|g_{R_1}(x_i) \oplus g_{R_2}(x_i)|}{n} + \frac{1}{n}\sum_{i=1}^{n} \frac{|g_{R_2}(x_i) \oplus g_{R_3}(x_i)|}{n} \\ &= \frac{1}{n^2}\sum_{i=1}^{n}(|g_{R_1}(x_i) \oplus g_{R_2}(x_i)| + |g_{R_2}(x_i) \oplus g_{R_3}(x_i)|). \end{aligned}$$

Through Lemma 3.6,

$$\begin{aligned} \frac{1}{n^2}\sum_{i=1}^{n}(|g_{R_1}(x_i) \oplus g_{R_2}(x_i)| + |g_{R_2}(x_i) \oplus g_{R_3}(x_i)|) &\geq \frac{1}{n^2}\sum_{i=1}^{n} |g_{R_1}(x_i) \oplus g_{R_3}(x_i)| \\ &= D_{sd}(FGS(R_1), FGS(R_3)). \end{aligned}$$

Thus, $D_{sd}(FGS(R_1), FGS(R_2)) + D_{sd}(FGS(R_2), FGS(R_3)) \geq D_{sd}(FGS(R_1), FGS(R_3))$. Therefore, $(\mathcal{FGS}(U), D_{sd})$ is a distance space. □

It is worth mentioning that if $FGS(R_1) \preceq FGS(R_2) \preceq FGS(R_3)$ or $FGS(R_1) \succeq FGS(R_2) \succeq FGS(R_3)$, it has $D_{sd}(FGS(R_1), FGS(R_2)) + D_{sd}(FGS(R_2), FGS(R_3)) = D_{sd}(FGS(R_1), FGS(R_3))$. Similar to Proposition 3.10 and Proposition 3.11, it can get the potential connection between the granularity of fuzzy granular structures and the fuzzy granular structure distance. The following are some related properties of fuzzy granular structure distance.

**Proposition 3.15.** *Suppose that $FGS(R_1) = (g_{R_1}(x_1), g_{R_1}(x_2), \ldots, g_{R_1}(x_n))$ and $FGS(R_2) = (g_{R_2}(x_1), g_{R_2}(x_2), \ldots, g_{R_2}(x_n))$ are any two fuzzy granular structures and $FGS(R_1) \preceq FGS(R_2)$. Let $FGS(R) = (g_R(x_1), g_R(x_2), \ldots, g_R(x_n))$ be a special fuzzy granular structure which for any $x_i \in U$, $g_R(x_i) = g_R(\bullet)$. Then*

(1) $\mathcal{G}(FGS(R_1)) = 1 - D_{sd}(FGS(R_1), FGS(R))$.

(2) $D_{sd}(FGS(R_1), FGS(R)) \geq D_{sd}(FGS(R_2), FGS(R))$.

*Proof.* (1) According to the definitions of the granularity of fuzzy information granules and the fuzzy granular structure

distance, it has that $\mathcal{G}(FGS(R_1)) = \frac{\sum_{i=1}^{n}|g_{R_1}(x_i)|}{n^2}$ and

$$\begin{aligned} D_{sd}(FGS(R_1), FGS(R)) &= \frac{1}{n}\sum_{i=1}^{n} D(g_{R_1}(x_i), g_R(x_i)) = \frac{1}{n}\sum_{i=1}^{n}\frac{|g_{R_1}(x_i) \oplus g_R(\bullet)|}{n} \\ &= \frac{1}{n^2}\sum_{i=1}^{n}(|g_{R_1}(x_i) \oplus g_R(\bullet)|) = \frac{1}{n^2}\sum_{i=1}^{n}(|g_{R_1}(x_i) \cup g_R(\bullet) - g_{R_1}(x_i) \cap g_R(\bullet)|) \\ &= \frac{1}{n^2}\sum_{i=1}^{n}(|g_R(\bullet) - g_{R_1}(x_i)|) = \frac{1}{n^2}\sum_{i=1}^{n}(|g_R(\bullet)| - |g_{R_1}(x_i)|) = 1 - \frac{1}{n^2}\sum_{i=1}^{n}|g_{R_1}(x_i)|. \end{aligned}$$

Hence, $\mathcal{G}(FGS(R_1)) = 1 - D_{sd}(FGS(R_1), FGS(R))$.

(2) By (1), we know that $D_{sd}(FGS(R_1), FGS(R)) = \frac{1}{n} - \frac{1}{n^2}\sum_{i=1}^{n}|g_{R_1}(x_i)|$, and $D_{sd}(FGS(R_2), FGS(R)) = \frac{1}{n} - \frac{1}{n^2}\sum_{i=1}^{n}|g_{R_2}(x_i)|$. For any $x_i \in U$, $g_{R_1}(x_i) \leq g_{R_2}(x_i)$ since $FGS(R_1) \preceq FGS(R_2)$. Thus $\frac{1}{n^2}\sum_{i=1}^{n}|g_{R_1}(x_i)| \leq \frac{1}{n^2}\sum_{i=1}^{n}|g_{R_2}(x_i)|$. Hence, $\frac{1}{n} - \frac{1}{n^2}\sum_{i=1}^{n}|g_{R_1}(x_i)| \geq \frac{1}{n} - \frac{1}{n^2}\sum_{i=1}^{n}|g_{R_2}(x_i)|$. Consequently, $D_{sd}(FGS(R_1), FGS(R)) \geq D_{sd}(FGS(R_2), FGS(R))$. □

This proposition explains the relationship between the granularity and the fuzzy granular structure distance. The size relationship between fuzzy granular structure distances is also demonstrated if all the fuzzy information granules of a fuzzy granular structure are fuzzy empty information granules.

**Proposition 3.16.** *Suppose that* $FGS(R_1) = (g_{R_1}(x_1), g_{R_1}(x_2), \ldots, g_{R_1}(x_n))$ *and* $FGS(R_2) = (g_{R_2}(x_1), g_{R_2}(x_2), \ldots, g_{R_2}(x_n))$ *are two fuzzy granular structures and* $FGS(R_1) \preceq FGS(R_2)$. *Let* $FGS(R) = (g_R(x_1), g_R(x_2), \ldots, g_R(x_n))$ *be a special fuzzy granular structure which for any* $x_i \in U$, $g_R(x_i) = g_R(\emptyset)$. *Then*

(1) $\mathcal{G}(FGS(R_1)) = D_{sd}(FGS(R_1), FGS(R))$.

(2) $D_{sd}(FGS(R_1), FGS(R)) \leq D_{sd}(FGS(R_2), FGS(R))$.

*Proof.* (1) By the definitions of the granularity of fuzzy granular structures and the fuzzy granular structure distance, it has that $\mathcal{G}(FGS(R_1)) = \frac{\sum_{i=1}^{n}|g_{R_1}(x_i)|}{n^2}$ and

$$\begin{aligned} D_{sd}(FGS(R_1), FGS(R)) &= \frac{1}{n}\sum_{i=1}^{n} D(g_{R_1}(x_i), g_R(x_i)) = \frac{1}{n}\sum_{i=1}^{n}\frac{|g_{R_1}(x_i) \oplus g_R(\emptyset)|}{n} \\ &= \frac{1}{n^2}\sum_{i=1}^{n}(|g_{R_1}(x_i) \oplus g_R(\emptyset)|) = \frac{1}{n^2}\sum_{i=1}^{n}(|g_{R_1}(x_i) \cup g_R(\emptyset) - g_{R_1}(x_i) \cap g_R(\emptyset)|) \\ &= \frac{1}{n^2}\sum_{i=1}^{n}(|g_{R_1}(x_i) - g_R(\emptyset)|) = \frac{\sum_{i=1}^{n}|g_{R_1}(x_i)|}{n^2}. \end{aligned}$$

Hence, $\mathcal{G}(FGS(R_1)) = D_{sd}(FGS(R_1), FGS(R))$.

(2) By (1), there are $D_{sd}(FGS(R_1), FGS(R)) = \frac{1}{n^2}\sum_{i=1}^{n}|g_{R_1}(x_i)|$ and $D_{sd}(FGS(R_2), FGS(R)) = \frac{1}{n^2}\sum_{i=1}^{n}|g_{R_2}(x_i)|$. For any $x_i \in U$, $g_{R_1}(x_i) \leq g_{R_2}(x_i)$ since $FGS(R_1) \preceq FGS(R_2)$. Thus $\frac{1}{n^2}\sum_{i=1}^{n}|g_{R_1}(x_i)| \leq \frac{1}{n^2}\sum_{i=1}^{n}|g_{R_2}(x_i)|$. Consequently, $D_{sd}(FGS(R_1), FGS(R)) \leq D_{sd}(FGS(R_2), FGS(R))$. □

At this point, the distance definition for fuzzy granular structures is completed and the related properties are illustrated. On the one hand, starting from the granularity measure of fuzzy granular structures, we explain the rationality of the definition of fuzzy granular structure distance. On the other hand, we demonstrate that the fuzzy granular structure distance can reflect the diversity between fuzzy granular structures.

# 4 The clustering algorithms for grouping fuzzy granular structures

In the previous section, the fuzzy granular structure distance is constructed. We prove that it is indeed a distance measure and can reasonably reflect the similarity and difference between fuzzy granular structures and the granularity characteristics of fuzzy granular structures. Next, clustering algorithms for clustering fuzzy granular structures are designed based on the fuzzy granular structure distance.

### 4.1 The k-means fuzzy granular structure clustering algorithm

The basic idea of the k-means algorithm [23] is to divide $N$ objects into $K$ clusters, and the clustering results should make the objects with higher similarity be divided into the same cluster, while the objects with more significant differences are divided into different clusters. Based on the definition of the fuzzy granular structure distance $D_{sd}$ , a corresponding objective function for a clustering algorithm can be constructed for grouping fuzzy granular structures. An objective function is usually the sum of the distances of elements to a set of putative cluster centers. The objective function is optimized step by step until the optimal clustering center is found. The algorithm for clustering fuzzy granular structures aims to divide given fuzzy granular structures into $k$ clusters through minimizing the objective function. Let $\mathcal{FGS}(U) = \{FGS_{R_1}, FGS_{R_2}, \ldots, FGS_{R_N}\}$ be a fuzzy granular library of the universe $U$ and $N = |\mathcal{FGS}(U)|$. For the sake of simplicity, let $\mathcal{FGS}(U) = \{FGS_1, FGS_2, \ldots, FGS_N\}$. The goal is to divide $N$ fuzzy granular structures into $k(k \leq N)$ clusters and to minimize the objective function

$$F_{KM} = \sum_{FGS_j \in G_i} \sum_{i=1}^{k} D_{sd}(FGS_j, C_i), \tag{5}$$

where $C_i = \{g_{C_i}(x_1), g_{C_i}(x_2), \ldots, g_{C_i}(x_n)\}$ is the $i$th cluster center and $G_i$ is the cluster centered on $C_i$. The set $C = \{C_1, C_2, C_3, \ldots, C_k\}$ is the set of $k$ cluster centers that minimize the objective function $F_{KM}$.

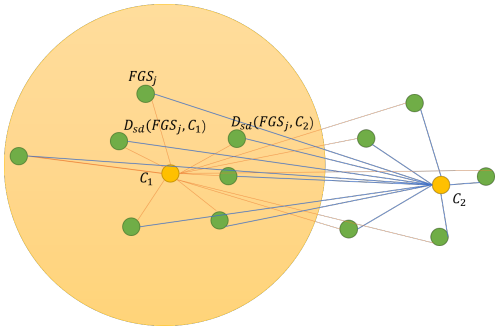



Figure 1: Fuzzy granular structures with two cluster centers.

To illustrate the above objective, consider a simple case where a set of fuzzy granular structures is clustered into two classes. As shown in Figure 1, each green dot represents a fuzzy granular structure, and each yellow dot represents a cluster center. According to the fuzzy granular structure distance defined in Section 3, the distance between each fuzzy granular structure and the cluster center can be calculated by Equation (4). For the fuzzy granular structure $FGS_j$, its distance to $C_1$ is $D_{sd}(FGS_j, C_1)$, and its distance to $C_2$ is $D_{sd}(FGS_j, C_2)$. Since $D_{sd}(FGS_j, C_1)$ is smaller than $D_{sd}(FGS_j, C_2)$, $FGS_j$ is grouped into the cluster centered at $C_1$. The fuzzy granular structure in the yellow area is closer to the cluster center $C_1$ than $C_2$, so they are finally divided into the cluster centered on $C_1$. The objective function $F_{KM}$ is minimal when the other fuzzy granular structures are divided into the cluster centered on $C_2$.

Based on the k-means algorithm, according to the requirement of minimizing the objective function, we design the k-means fuzzy granular structure clustering algorithm for grouping fuzzy granular structures shown in Algorithm 1. The overall idea of the algorithm is to randomly select a fuzzy granular structure as the first cluster center and then select the fuzzy granular structure farthest from the first cluster center as the second cluster center. From this, select the initial $k$ cluster centers. Then, the distance between each fuzzy granular structure and each cluster center is calculated, and each fuzzy granular structure is divided into the cluster with the nearest distance. Then the cluster center is updated, and the calculation method of the new cluster center is to calculate the average value of all fuzzy granular structures in the cluster as the new cluster center. The cluster center obtained through calculation may not be one of the original fuzzy granular structures, but it can represent the cluster of fuzzy granular structures. Next, we analyze the time complexity of this process. In the above algorithm, corresponding to steps 4 to 9, the time complexity of $N$ fuzzy granular structures being assigned cluster labels in one iteration is $O(kN)$, where $k$ is the number of clusters and $N$ is the number of fuzzy granular structures, respectively. Corresponding to Step 3, if the number of iterations is $L$, the time complexity of the algorithm is $O(kNL)$. Algorithm 1 has a linear time complexity. This is similar to the classical k-means clustering algorithm. However, the classical algorithm solves the clustering problem of objects, and Algorithm 1 deals with the clustering problem of fuzzy granular structures. The time complexity is directly related to the scalability of the algorithm, which will be discussed in detail in Section 5. To understand the transformation process of the above algorithm, we take a practical example to show the calculation process of Step 11.

**Algorithm 1:** The k-means fuzzy granular structure clustering algorithm

**Input:** $\mathcal{FGS}(U) = \{FGS_1, FGS_2, \ldots, FGS_N\}$: a fuzzy granular structure library of $U$, $k$: the number of clusters, and $L$: the maximum number of iterations.

**Output:** The final clustering result $G = \{G_1, G_2, \ldots, G_k\}$ of the fuzzy granular structure library $\mathcal{FGS}(U) = \{FGS_1, FGS_2, \ldots, FGS_N\}$.

0: Initialize the cluster center $C^{(0)} = \{C_1^{(0)}, C_2^{(0)}, \ldots, C_k^{(0)}\}$. Select $k$ fuzzy granular structures as far away as possible as the cluster centers. First, randomly choose a fuzzy granular structure in $\mathcal{FGS}(U)$ as $C_1^{(0)}$. Then the fuzzy granular structure farthest from $C_1^{(0)}$ is chosen as $C_2^{(0)}$; and so on.

0: $G_1 \Leftarrow \emptyset, G_2 \Leftarrow \emptyset, \ldots, G_k \Leftarrow \emptyset$.

0: **for** $l = 1 \to L$ **do**

0: **for** $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, N$ **do**

0: Calculate the fuzzy granular structure distance $D_{sd}(FGS_j, C_i^{(l)})$ between $FGS_j$ and $C_i^{(l)}$.

0: **if** $D_{sd}(FGS_j, C_s^{(l)}) = \min_i\{D_{sd}(FGS_j, C_i^{(l)})\}, i = 1, 2, \ldots, k$ **then**

0: $G_s^{(l+1)} \Leftarrow G_s^{(l+1)} \cup FGS_j$ //$G_s^{(l+1)}$ is the cluster centered on $C_s^{(l)}$. Thereout, new clusters $G_i^{(l+1)}$, $i = 1, 2, \ldots, k$ are obtained.

0: **end if**

0: **end for**

0: Compute the objective function $F_{KM}^{(l)}$: $F_{KM} = \sum_{FGS_j \in G_i} \sum_{i=1}^{k} D_{sd}(FGS_j, C_i)$.

0: Compute the new cluster center $C^{(l+1)} = \{C_1^{(l+1)}, C_2^{(l+1)}, \ldots, C_k^{(0)}\}$ of new clusters $G_i^{(l+1)}$, $i = 1, 2, \ldots, k$: $C_i^{(l+1)} = (C_{i1}^{(l+1)}, C_{i2}^{(l+1)}, \ldots, C_{in}^{(l+1)})$ $(i = 1, 2, \ldots, k)$, and $C_{ip}^{(l+1)} = \{\frac{\frac{1}{n_{G_i^{(l+1)}}} \sum_{FGS_j \in G_i^{(l+1)}} r_{i1}}{x_1} + \frac{\frac{1}{n_{G_i^{(l+1)}}} \sum_{FGS_j \in G_i^{(l+1)}} r_{i2}}{x_2} + \cdots + \frac{\frac{1}{n_{G_i^{(l+1)}}} \sum_{FGS_j \in G_i^{(l+1)}} r_{in}}{x_n}\}$, $p = 1, 2, \ldots, n$, where $n$ is the number of objects in $U$, and $n_{G_i^{(l+1)}}$ represents the number of fuzzy granular structures in cluster $G_i^{(l+1)}$.

0: **if** $(l <= L)$ **then**

0: Go back to step 3 and $l = l + 1$.

0: **end if**

0: **end for**

0: **return** $G_1, G_2, \ldots, G_k$. =0

**Example 4.1.** *Suppose that there are three fuzzy granular structures in the same cluster $G_i^{(l+1)}$, where $FGS(R_1) = (\frac{0.6}{x_1} + \frac{0.8}{x_2} + \frac{0.2}{x_3}, \frac{0.5}{x_1} + \frac{0.3}{x_2} + \frac{0.7}{x_3}, \frac{0.3}{x_1} + \frac{0.5}{x_2} + \frac{0.6}{x_3})$, $FGS(R_2) = (\frac{0.7}{x_1} + \frac{0.2}{x_2} + \frac{0.4}{x_3}, \frac{0.3}{x_1} + \frac{0.5}{x_2} + \frac{0.6}{x_3}, \frac{0.6}{x_1} + \frac{0.1}{x_2} + \frac{0.2}{x_3})$, and $FGS(R_3) = (\frac{0.8}{x_1} + \frac{0.2}{x_2} + \frac{0.4}{x_3}, \frac{0.7}{x_1} + \frac{0.8}{x_2} + \frac{0.2}{x_3}, \frac{0.5}{x_1} + \frac{0.6}{x_2} + \frac{0.3}{x_3})$. Then it can calculate the cluster center $C_i^{(l+1)} = (\frac{\frac{1}{3}(0.6+0.7+0.8)}{x_1} + \frac{\frac{1}{3}(0.8+0.2+0.2)}{x_2} + \frac{\frac{1}{3}(0.2+0.4+0.4)}{x_3}, \frac{\frac{1}{3}(0.5+0.3+0.7)}{x_1} + \frac{\frac{1}{3}(0.3+0.5+0.8)}{x_2} + \frac{\frac{1}{3}(0.7+0.6+0.2)}{x_3}, \frac{\frac{1}{3}(0.3+0.6+0.5)}{x_1} + \frac{\frac{1}{3}(0.5+0.1+0.6)}{x_2} + \frac{\frac{1}{3}(0.6+0.2+0.3)}{x_3})$ $= (\frac{0.6}{x_1} + \frac{0.4}{x_2} + \frac{0.33}{x_3}, \frac{0.5}{x_1} + \frac{0.53}{x_2} + \frac{0.5}{x_3}, \frac{0.47}{x_1} + \frac{0.4}{x_2} + \frac{0.37}{x_3})$.*

Further, consider whether the k-means fuzzy granular structure clustering algorithm has convergence. For any fuzzy granular structure $FGS_j$, it is assumed that it has been divided into the cluster $G_i$ centered on $C_i$. After the next iteration, it is clustered into the cluster $G_i^{'}$ centered on $C_i^{'}$. Suppose $G_i$ and $G_t$ have $m$ and $n$ fuzzy granular structures, respectively. By iteratively updating the clustering center in the algorithm, the distance between all fuzzy granular structures in $G_i$ and the center $C_i$ is $D_{gc_i} = \sum_{FGS_j \in G_i} D_{sd}(FGS_j, C_i)$. We assume that only the fuzzy granular structure $FGS_j$ is changed. After one iteration, the center $C_i$ is updated to

$$C_i^{'} = \left\{ \frac{mg_{C_i(x_1)} - g_{FGS_j}(x_1)}{m-1} + \frac{mg_{C_i(x_2)} - g_{FGS_j}(x_2)}{m-1} + \cdots + \frac{mg_{C_i(x_n)} - g_{FGS_j}(x_n)}{m-1} \right\},$$

and the center $C_i$ is updated to

$$C_t^{'} = \left\{ \frac{ng_{C_t(x_1)} - g_{FGS_j}(x_1)}{n-1} + \frac{ng_{C_t(x_2)} - g_{FGS_j}(x_2)}{n-1} + \cdots + \frac{ng_{C_t(x_n)} - g_{FGS_j}(x_n)}{n-1} \right\}.$$

The distance between all fuzzy granular structures in $G_i^{'}$ and the center $C_i^{'}$ is $D_{gc_i^{'}} = \sum_{FGS_j \in G_i^{'}} D_{sd}(FGS_j, C_t^{'})$. It can calculate that $D_{gc_i} - D_{gc_i^{'}} = \frac{m}{m-1} D_{sd}(FGS_j, C_i) \geq 0$ and $D_{gc_t^{'}} - D_{gc_t} = \frac{n}{n+1} D_{sd}(FGS_j, C_i) \geq 0$. It is obviously that $D_{sd}(FGS_j, C_i) > D_{sd}(FGS_j, C_t)$ and $\frac{m}{m-1} > \frac{n}{n+1}$. Hence, $D_{gc_i} + D_{gc_t} > D_{gc_i^{'}} + D_{gc_t^{'}}$. This indicates that the objective function is decreasing as the number of iterations increases.

**Theorem 4.2.** *The k-means fuzzy granular structure clustering algorithm is convergent.*

*Proof.* We write the objective function as $F_{KM}$. From Proposition 3.9 we know that the fuzzy granular structure distance $D_{sd}$ is always greater than 0, so $F_{KM}$ is also always greater than 0. After the cluster number $k$ and the

initial cluster center are selected, iterations are carried out continuously, and the cluster center changes accordingly. A corresponding sequence of objective function values can be obtained. From the previous description, it can be seen that the sequence of values of the objective function is monotonically decreasing with a lower bound of 0. According to the convergence theorem of monotone bounded sequence, the sequence converges, that is, $\lim_{l\to\infty} F_{KM}$ exists. Therefore, it can be concluded that the k-means fuzzy granular structure clustering algorithm is convergent as the number of iterations of the algorithm increases. □

Although the algorithm can guarantee convergence for a given initial value, the convergence results may differ for different initial value choices. Obviously, the clustering result will be different for different numbers of clusters. Therefore, the initial center and the number of clusters are hyperparameters that need to be adjusted by the algorithm.

## 4.2 The fuzzy c-means fuzzy granular structure clustering algorithm

Fuzzy c-means algorithm [6] is a method of clustering which allows one object to belong to two or more clusters. Because of the fuzziness of fuzzy granular structures, the boundary between two clusters of fuzzy granular structures may be fuzzy and unclear; that is, a fuzzy granular structure may belong to two or more groups with different degrees of membership. Therefore, considering the above situation, the fuzzy c-means clustering is a good choice for clustering fuzzy granular structures. Similar to the previous section, we can construct an objective function for a clustering algorithm for grouping fuzzy granular structures based on fuzzy c-means algorithm. Let $\mathcal{FGS}(U) = \{FGS_{R_1}, FGS_{R_2}, \ldots, FGS_{R_N}\}$ be a fuzzy granular structure library of the universe $U$ and $N = |\mathcal{FGS}(U)|$. For the sake of simplicity, let $\mathcal{FGS}(U) = \{FGS_1, FGS_2, \ldots, FGS_N\}$. Therefore, the goal is to minimize the objective function:

$$F_M = \sum_{j=1}^{N}\sum_{i=1}^{k} u_{ij}^m D(FGS_j, C_i), \tag{6}$$

where $k$ ($k \le N$) is a certain number of clusters, $m$ is a parameter that usually taken to be 2, and $C = \{C_1, C_2, \ldots, C_k\}$ is the family of centers of $k$ clusters, where $C_i = \{g_{C_i}(x_1), g_{C_i}(x_2), \ldots, g_{C_i}(x_n)\}$. The constraint conditions are $\sum_{i=1}^{k} u_{ij} = 1$, where $u_{ij}$ represents the membership degree of fuzzy granular structure $FGS_j$ belonging to cluster center $C_i$. By utilizing Lagrange function, we can calculate that

$$u_{ij} = \frac{1}{\sum_{l=1}^{k} \left(\frac{\sum_{t=1}^{n} |g_j(x_t) - g_{C_i}(x_t)|}{\sum_{t=1}^{n} |g_j(x_t) - g_{C_l}(x_t)|}\right)^{\frac{2}{m-1}}}, \tag{7}$$

and

$$g_{C_i}(x_t) = \frac{\sum_{j=1}^{N} u_{ij}^m g_j(x_t)}{\sum_{j=1}^{N} u_{ij}^m}. \tag{8}$$

On the basis of the fuzzy c-means algorithm, we design the fuzzy c-means fuzzy granular structure clustering algorithm shown in Algorithm 2. Within one iteration, the time complexity of steps 3 to 8 is $O(kN)$, where $k$ is the number of clusters and $N$ is the number of fuzzy granular structures, respectively. If the number of iterations is $L$, the time complexity of the algorithm is $O(kNL)$. Algorithm 2 also has a linear time complexity.

**Remark 4.1.** *Parameter specification in Algorithm 1 and Algorithm 2.*

(1) *The number of clusters k determines how many clusters the fuzzy granular structures will be clustered into.*

(2) *The maximum number of iterations L limits the number of clustering iterations. When the Lth iteration is reached, the iteration is terminated, and the resulting cluster is the final clustering result.*

(3) *N denotes the number of fuzzy granular structures in a fuzzy granular structure library, which is also the number of fuzzy granular structures to be clustered.*

## 4.3 Internal evaluation indicator of clustering algorithms for grouping fuzzy granular structures

Since the modified fuzzy c-means clustering is a soft clustering, the following evaluation indexes are mainly defined from the perspective of the k-means fuzzy granular structure clustering algorithm. The clustering evaluation indicator is used to measure clustering validity and determine the number of clusters. As an unsupervised learning task, evaluating the effect after clustering is necessary. Otherwise, the clustering results will be difficult to be applied. There are many common evaluation indicators for object clustering, such as SSE, Silhouette Coefficient [30], Calinski-Harabaz Index [2], and Davies-Bouldin Index [4]. However, as far as we know, there is no general evaluation index for granular structures,

**Algorithm 2:** The fuzzy c-means fuzzy granular structure clustering algorithm

**Input:** $\mathcal{FGS}(U) = \{FGS_1, FGS_2, \ldots, FGS_N\}$: a fuzzy granular structure library of $U$, $k$: the number of clusters, and $L$: the maximum number of iterations.

**Output:** The final clustering result of the fuzzy granular structure library $\mathcal{FGS}(U) = \{FGS_1, FGS_2, \ldots, FGS_N\}$.

0: Initialize the cluster center $C^{(0)} = \{C_1^{(0)}, C_2^{(0)}, \ldots, C_k^{(0)}\}$. Select $k$ fuzzy granular structures as far away as possible as the cluster centers. First, randomly choose a fuzzy granular structure in $\mathcal{FGS}(U)$ as $C_1^{(0)}$. Then the fuzzy granular structure farthest from $C_1^{(0)}$ is chosen as $C_2^{(0)}$; and so on.

0: Initialize the membership values $u_{ij}^{(0)}$.

0: **for** $l = 1 \to L$ **do**

0: **for** $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, N$ **do**

0: Calculate the centers by $g_{C_i}(x_t) = \frac{\sum_{j=1}^{N} u_{ij}^m g_j(x_t)}{\sum_{j=1}^{N} u_{ij}^m}$, where $C_i = \{g_{C_i}(x_1), g_{C_i}(x_2), \ldots, g_{C_i}(x_n)\}$.

0: Update $u_{ij}^{(l)}$ by $u_{ij} = \frac{1}{\sum_{l=1}^{k} (\frac{\sum_{t=1}^{n} |g_j(x_t) - g_{C_i}(x_t)|}{\sum_{t=1}^{n} |g_j(x_t) - g_{C_l}(x_t)|})^{\frac{2}{m-1}}}$.

0: Compute the value of the objective function $F_M^t = \sum_{i=1}^{k} \sum_{j=1}^{N} u_{ij}^m (\sum_{t=1}^{n} \frac{1}{n^2} |g_j(x_t) \oplus g_{c_i}(x_t)|)^2$.

0: **end for**

0: **if** $(l <= L)$ **then**

0: Go back to step 3 and $l = l + 1$.

0: **end if**

0: **end for**

0: **return** The final clustering result $u_{ij}$. =0

especially for fuzzy granular structure clustering. In the paper, we mainly consider the effect of internal clustering and give several evaluation indicators of clustering in detail. Referring to the existing indicators of object clustering and literature [27], the following evaluation indicators are given.

**Definition 4.3.** *Suppose that $\mathcal{FGS}(U) = \{FGS_1, FGS_2, \ldots, FGS_N\}$ is the fuzzy granular structure library of $U$ and the family of final $k$ clustering centers is $C = \{C_1, C_2, C_3, \ldots, C_k\}$. The dispersion of the clustering result is defined as*

$$\mathrm{Dis}(C, k) = \frac{1}{k} \sum_{i=1}^{k} \frac{|G_i|}{N} \sqrt{\frac{1}{|G_i|} \sum_{FGS_j \in G_i}^{|G_i|} D^2 \left(FGS_j, C_i\right)}, \tag{9}$$

*where $G_i$ is the cluster centered on $C_i$ and $|G_i|$ represents the number of fuzzy granular structures in $G_i$.*

Taking Figure 1 as an example, we need the fuzzy granular structures divided into the cluster centered on $C_1$ to be closer to the center or to be sufficiently close to each other. Dispersion is the sum of distances between every fuzzy granular structure and the center of the cluster in which it is located. The smaller the dispersion is, the better the clustering effect is. The clustering aims to divide the fuzzy granular structures into several clusters to ensure the fuzzy granular structure in each cluster as close as possible and ensure the distance between the fuzzy granular structures of different clusters is as far as possible. The dispersion only considers the distance within a cluster, not the relationship between clusters. So the following indicator is defined.

**Definition 4.4.** *Suppose that $\mathcal{FGS}(U) = \{FGS_1, FGS_2, \ldots, FGS_N\}$ is the fuzzy granular structure library of $U$ and the family of final $k$ clustering centers is $C = \{C_1, C_2, C_3, \ldots, C_k\}$. The separation of the clustering result is defined as*

$$SP = \frac{2}{k^2 - k} \sum_{i=1}^{k} \sum_{j=i+1}^{k} D^2(C_i, C_j). \tag{10}$$

According to the previous steps of the k-means fuzzy granular structure clustering algorithm, we know that $C_i$ and $C_j$ are obtained by averaging all the fuzzy granular structures in the cluster centered by $C_i$ and $C_j$. So the distance between two centers can reflect the distance between clusters to some extent. We will also verify this in the next section of experiments.

# 5 Experimental analysis

To evaluate the performance and verify the convergence, effectiveness, and scalability of our proposed two clustering algorithms for grouping fuzzy granular structures, some numerical experiments are carried out in this section. Experiment results show that the two proposed clustering algorithms are reasonable, feasible and effective.

Table 1: Description of twelve public data sets.

| No. | Name | Number of samples | Number of features | Number of fuzzy granular structures | Date donated | Range of k(Algorithm 1) | Range of k(Algorithm 2) |
|---|---|---|---|---|---|---|---|
| 1 | Iris | 150 | 4 | 99 | 1988-07-01 | From 2 to 15 | From 2 to 18 |
| 2 | Wine | 178 | 13 | 153 | 1991-07-01 | From 2 to 15 | From 2 to 20 |
| 3 | Seed | 210 | 7 | 199 | 2012-09-29 | From 2 to 17 | From 2 to 20 |
| 4 | Bupa | 345 | 6 | 285 | 1988 | From 2 to 17 | From 2 to 22 |
| 5 | Wdbc | 569 | 22 | 499 | 1995-11-01 | From 2 to 24 | From 2 to 23 |
| 6 | Sonar | 208 | 60 | 199 | 1988 | From 2 to 17 | From 2 to 20 |
| 7 | Haberman | 306 | 30 | 249 | 1999-03-04 | From 2 to 19 | From 2 to 19 |
| 8 | Appendicitis | 106 | 7 | 99 | 1991 | From 2 to 17 | From 2 to 20 |
| 9 | BreastTissue | 106 | 9 | 99 | 2010-05-10 | From 2 to 16 | From 2 to 20 |
| 10 | Libras | 360 | 90 | 249 | 2009-08-17 | From 2 to 17 | From 2 to 17 |
| 11 | Yeast | 1484 | 8 | 249 | 1996-09-01 | From 2 to 21 | From 2 to 21 |
| 12 | Winequality-red | 1599 | 11 | 199 | 2009-10-07 | From 2 to 18 | From 2 to 18 |

## 5.1 Experimental setup

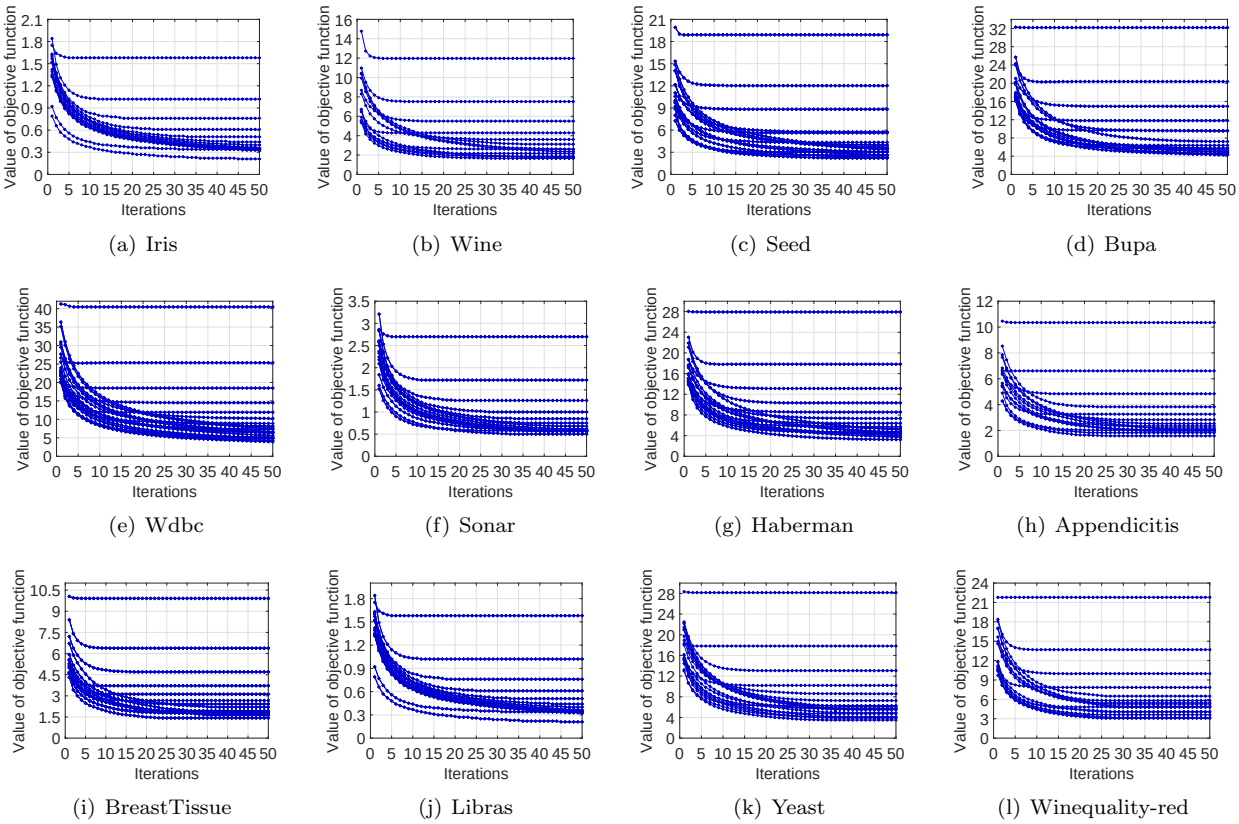
Twelve data sets are downloaded from the UC Irvine machine learning repository (http://archive.ics.uci.edu/ml/index.php) to evaluate our two models. The twelve publicly available data sets used in the experiments are listed in Table 1. All experimental work was performed by a computer (Intel(R) Core(TM) i5-1035G1 CPU @ 1.00GHz 1.19 GHz and 16 GB memory) in the paper.

In order to carry out experimental analysis on the two clustering algorithms, we first need to get the fuzzy granular structure library corresponding to each dataset, that is, to get a cluster of different fuzzy granular structures. Since our study is based on general fuzzy relation, we obtain the relation matrix through Gaussian kernel function without loss of generality. The fuzzy granular structure is obtained, and the specific method is as follows. First, we normalize the conditional attributes of the dataset. Each dataset is viewed as a matrix with $n$ rows and $m$ columns, where $n$ is the number of objects, and $m$ is the number of attributes. Then, each row of the obtained matrix is regarded as a vector, and any two rows are calculated using the RBF kernel to generate an $N * N$ matrix. The element in the first row and the second column results from the operations on the first and second objects. In order to quickly obtain a large number of fuzzy granular structures, we consider all conditional attributes in the data set and use the RBF kernel with parameters as fuzzy relations. The formula of the RBF kernel is $k(x, y) = \exp\left(-\frac{\|x-y\|^2}{2\sigma^2}\right)$. We can generate multiple matrices by varying $\sigma$ in the RBF kernel, for example, from 0 to 1 with a step size of 0.001, generating 1000 matrices. A fuzzy granular structure library corresponding to data sets can be obtained by removing duplicate fuzzy granular structures.This is an efficient way to get multiple fuzzy granular structures quickly. According to the number of objects in different data sets, we get different number of fuzzy granular structures. As shown in Table 1, the number of fuzzy granular structures of 12 data sets are showed.

In [38], Yu pointed out that the possible clustering number is usually within the interval $[2, \sqrt{n}]$, where $n$ is the number of objects to be clustered. The selection of the number $k$ of clustering basically follows this principle and makes appropriate adjustments. The election range of $k$ in experiments on the relationship between the objective function and the number of iterations and clusters are listed in Table 1. The penultimate column in Table 1 is the selection range of $k$ for the first algorithm. The last column in Table 1 is the selection range of $k$ for the second algorithm.

## 5.2 Experimental results

The first part of the experiment tests the convergence of the k-means fuzzy granular structure clustering algorithm. We perform $k-1$ times clustering for each dataset and iterate over the numbers from 1 to 50 times for each clustering. In other words, the maximum number of iterations $L$ is set to 50. The specific value of $k$ is explained in the previous section. The convergence behavior of the algorithm is displayed in Figure 2. Here we show the $k-1$ curves, where each curve refers to the values of the objective function obtained in the successive iterations of the algorithm. As can be seen from the curve changes in Figure 2, the value of the objective function gradually tends to be stable as the number of iterations increases. This shows that the proposed clustering is convergent. If the number of categories in a cluster is fixed, the objective function decreases with the increase of the number of iterations, which is applicable for each value of $k$. We further experimentally verify clustering results with the change of cluster number. The value of the maximum number of iterations is still set to 50. In Figure 3, we draw the curve of the value of the objective function changing with the number of clusters $k$. As shown in the figure, with the increase of the number of clustering $k$, the overall trend of the objective function is declining. However, after a certain $k$, the decline of the objective function becomes smaller and smaller and gradually tends to be relatively stable. The trend is not entirely monotonous, but there are a few twists and turns, and the overall trend is basically monotonous.

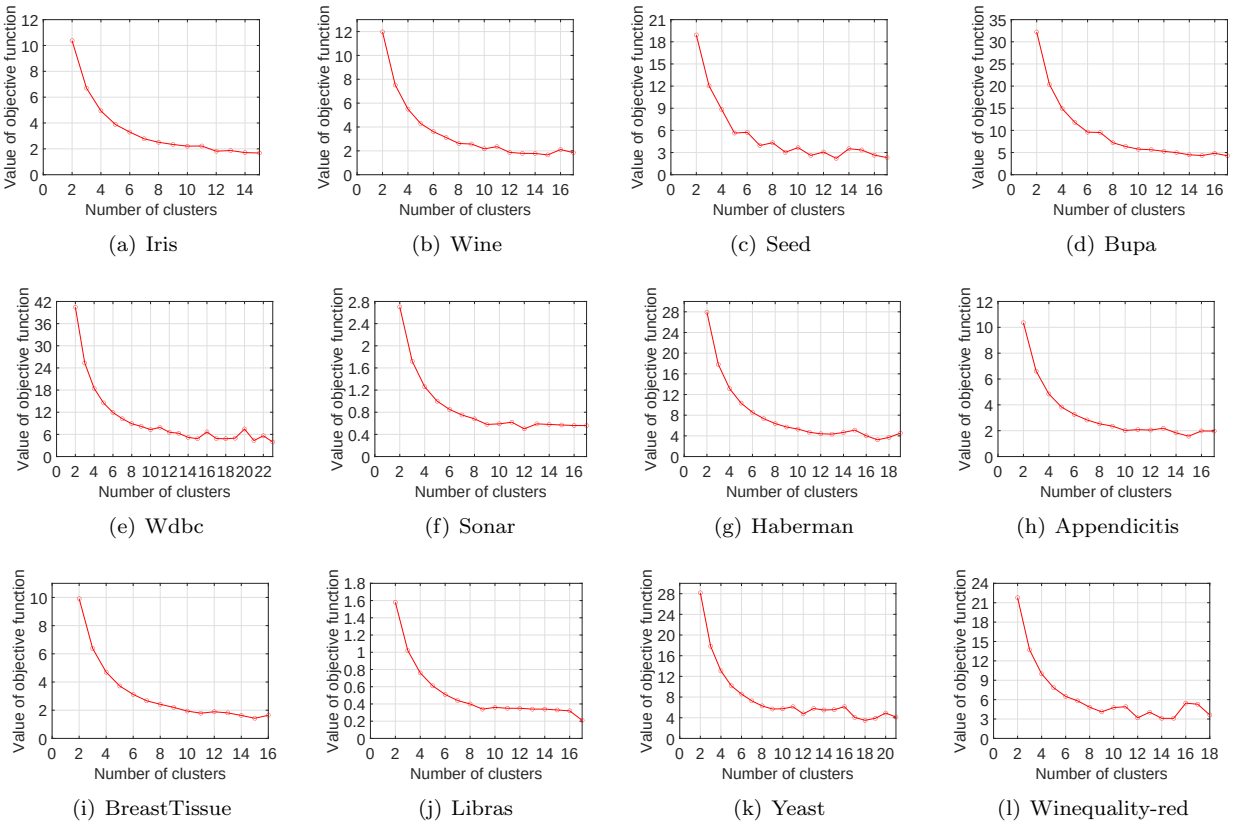



(a) Iris (b) Wine (c) Seed (d) Bupa

(e) Wdbc (f) Sonar (g) Haberman (h) Appendicitis

(i) BreastTissue (j) Libras (k) Yeast (l) Winequality-red

Figure 2: Value of objection function about iterations based on Algorithm 1 .




(a) Iris (b) Wine (c) Seed (d) Bupa

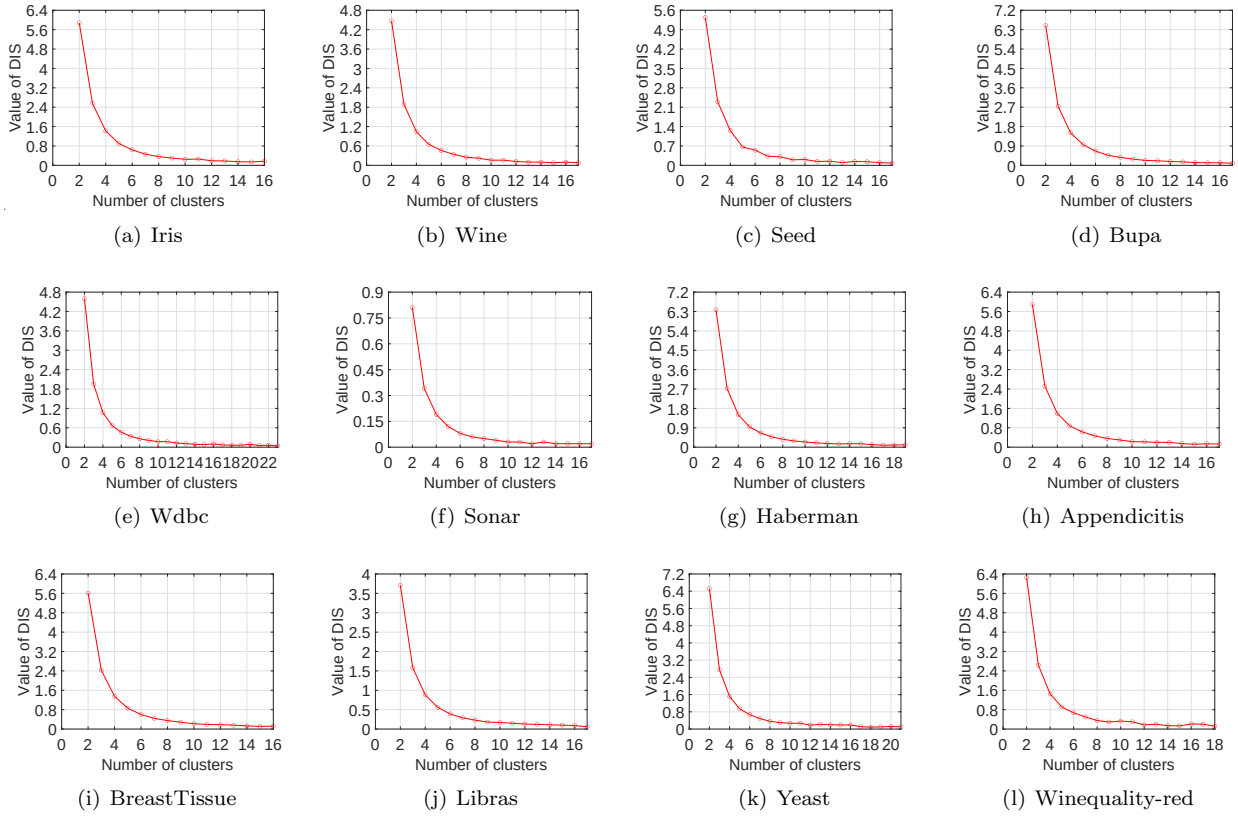
(e) Wdbc (f) Sonar (g) Haberman (h) Appendicitis

(i) BreastTissue (j) Libras (k) Yeast (l) Winequality-red

Figure 3: Value of objection function about number of clusters based on Algorithm 1 .

The second part of the experiment tests effectiveness of the algorithm. We evaluate the clustering performance using the two internal evaluation indicators presented in Section 4.3. The values of $k$ and $L$ are the same as the previous values. Figures 4 and 5 show the trend of dispersion and separation as the number of clusters changes, respectively. In different data sets, the change trend of dispersion is basically the same. As the number of clusters increased, the value of dispersion gradually decreased and gradually stabilized. The result is in line with what we expected. The change trend is basically consistent with Figure 3. The smaller the objective function is, the more the clustering results are in line with our expectations. Meanwhile, the smaller dispersion is, the closer the fuzzy granular structures in each cluster is.

The third part of the experiment tests the convergence of the fuzzy c-means fuzzy granular structure clustering algorithm. For each dataset, we perform $k-1$ clustering and iterate 50 times for each clustering. The specific value of $k$ is explained in the previous section. The convergence of the algorithm is displayed in Figure 6. Here we show the $k-1$ curves, where each curve refers to the values of the objective function obtained in the successive iterations of the algorithm. As can be seen from the curve changes in Figure 6, the value of the objective function gradually tends to be stable as the number of iterations increases. This shows that the proposed clustering is convergent. If the number of categories in the cluster is fixed, the objective function decreases with the increase of the number of iterations, which is applicable for each value of $k$. In Figure 7, we draw the curve of the value of the objective function changing with the number of clusters $k$. As shown in the figure, with the increase in the number of clustering $k$, the overall trend of the objective function is declining. However, after a certain $k$, the decline of the objective function becomes smaller and smaller and gradually tends to be relatively stable. Compared with the first algorithm, the monotonicity is stronger. Except for a few points in the data sets Sonar and Winequality-red, the curves are monotonous.




(a) Iris (b) Wine (c) Seed (d) Bupa

(e) Wdbc (f) Sonar (g) Haberman (h) Appendicitis

(i) BreastTissue (j) Libras (k) Yeast (l) Winequality-red

Figure 4: Dispersion (DIS) about number of clusters based on Algorithm 1.

The last part of the experiment evaluates the scalability of the algorithms. On the one hand is the scalability of the number of clusters for a given number of fuzzy granular structures, and on the other hand is the scalability of the number of fuzzy granular structures for a given number of clusters. We first evaluate the running time of the algorithms as the number of clusters changed on the three data sets of Iris, Appendicitis, and BreastTissue. Figures 8 and 9 depict the trend of the running time of the three datasets as the number of clusters changed. As the number of clusters increases, the execution time of the algorithm increases linearly. It takes more time to divide a given number of fuzzy granular structures into more clusters than fewer clusters. The curve in Figure 9 is smoother, with a linear relationship between the running time and the number of clusters. In Section 4, we analyze the time complexity of the

two algorithms. Both of them are linear, which is well illustrated by the experimental results. The experimental results show that if the number of fuzzy granular structures within a certain size, the time cost is very low, even if the number of cluster categories is large. Next, we test the running time as the number of fuzzy granular structures changed on the three data sets of Iris, Appendicitis, and BreastTissue. The number of fuzzy granular structures varies from 0 to 150 when the values of $k$ are fixed at 3, 5, and 7. Figures 10-11 show that with the increase of the number of fuzzy granular structures, the execution time of the two algorithms increases linearly. Under the same number of clusters, the more fuzzy granular structures need to be clustered, the longer the execution time of the two algorithms will be. The experimental results are consistent with the time complexity of the two algorithms. As shown in Figure 11, when the number of clusters is 7, and the number of fuzzy granular structures is 150, the execution time of Algorithm 1 is less than 200 seconds. Under the same conditions, the execution time of Algorithm 2 is less than 120 seconds under the less excellent device conditions in the paper. It can be seen that the two algorithms have good scalability in increasing the number of fuzzy granular structures and the number of clusters.

## 5.3 Discussion

In the paper, we have proposed a model to reasonably measure the relationship between fuzzy granular structures and effectively cluster fuzzy granular structures. This model can be used to simulate the thinking mode and intelligent behaviour of the human brain to quickly select information granularity. In this way the expected goal of studying how to effectively group fuzzy granular structures is achieved.

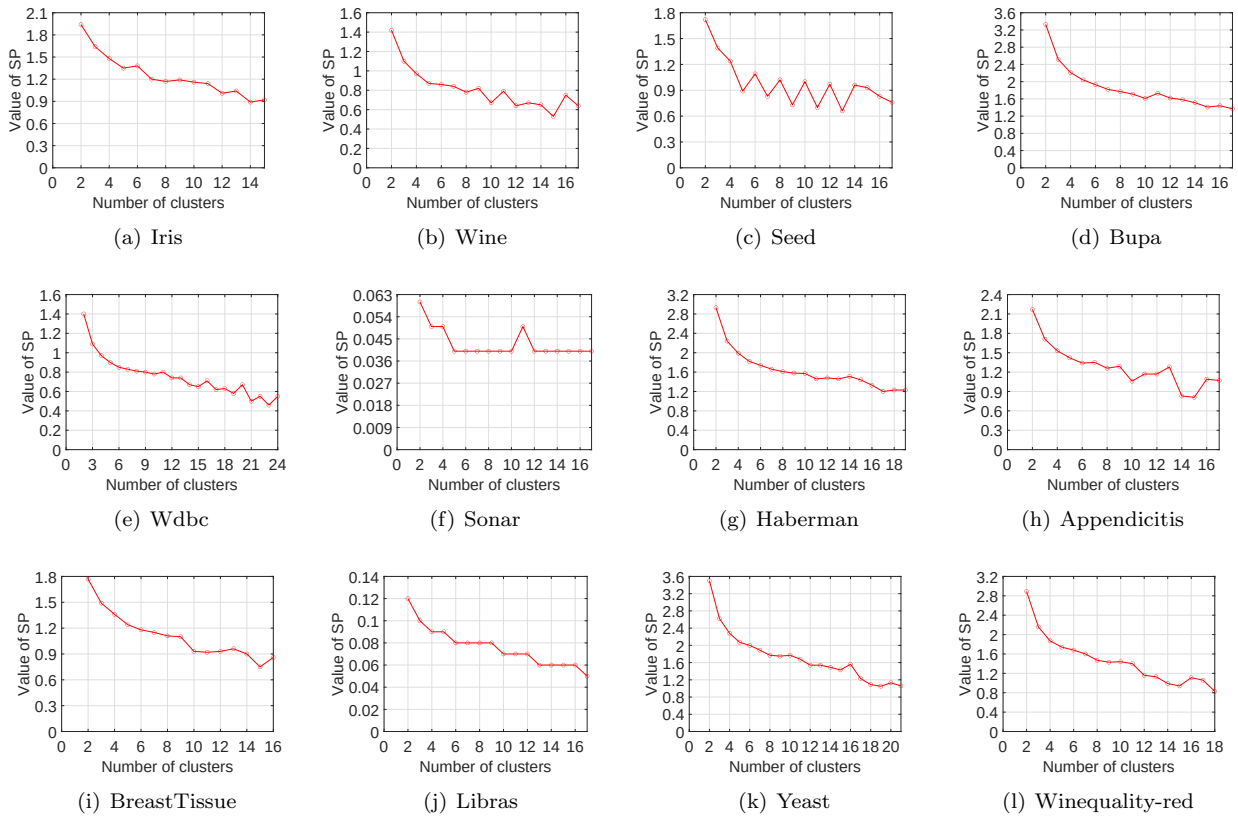
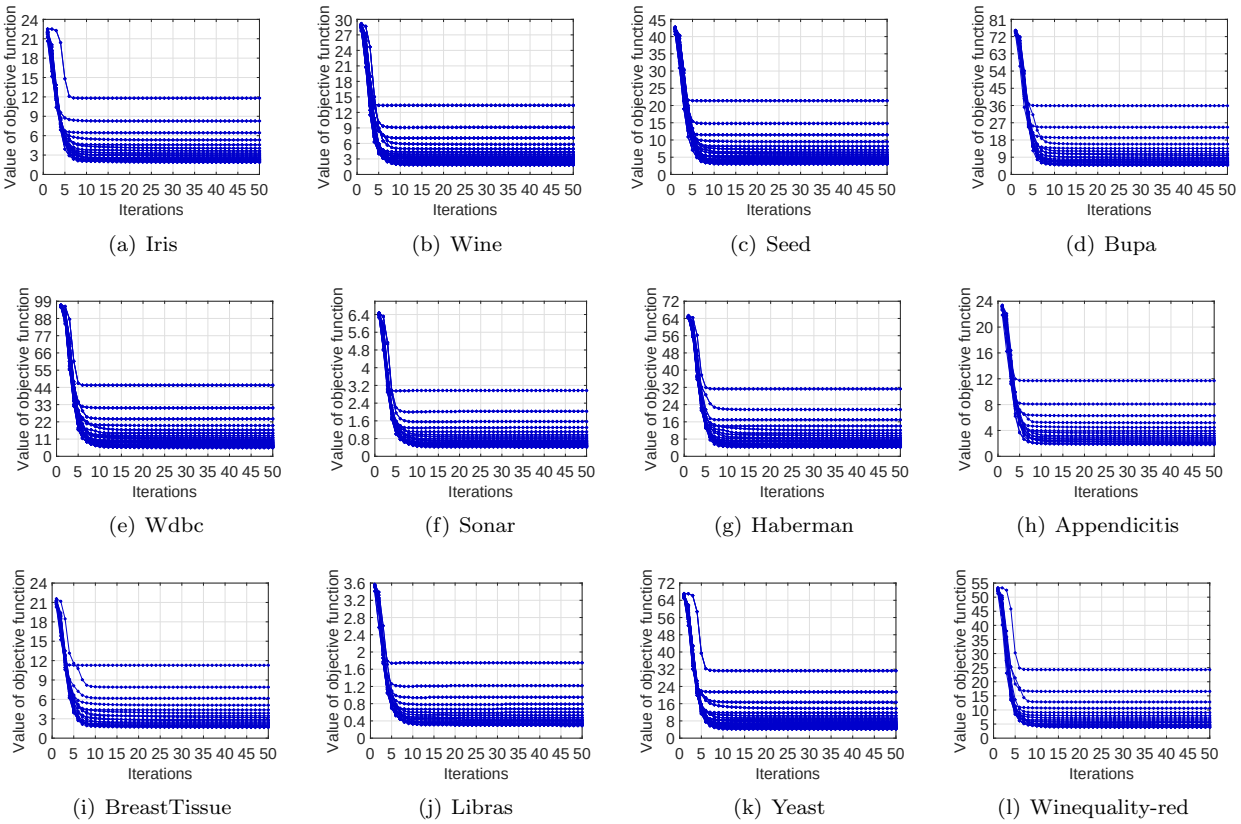



Figure 5: Separation (SP) about number of clusters based on Algorithm 1.

There are several contributions that the current research makes to existing theories in the information granulation field. First and foremost, a unified representation based on general fuzzy relations is given, starting from the commonness of fuzzy granular structures. In human thinking and decision-making, there is no clear boundary for some concepts, and a thing may belong to multiple categories simultaneously. Therefore, the fuzzy granular structure is consistent with human cognition. Next, previous studies focused more on the relationship between two fuzzy granular structures. They did not pay more attention to the overall relationship of multiple fuzzy granular structures from the grouping perspective. For this, a distance measure that can reflect the similarity and differences of fuzzy granular structures



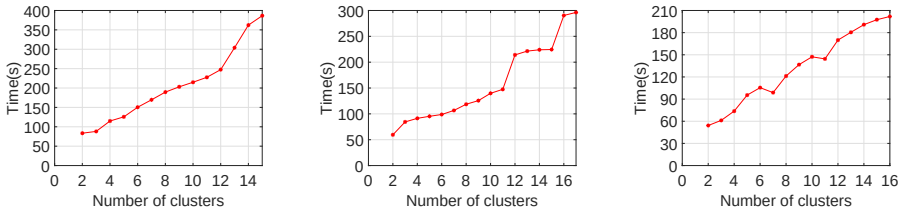

(a) Iris (b) Wine (c) Seed (d) Bupa

(e) Wdbc (f) Sonar (g) Haberman (h) Appendicitis

(i) BreastTissue (j) Libras (k) Yeast (l) Winequality-red

Figure 6: Value of objection function about iterations based on Algorithm 2.

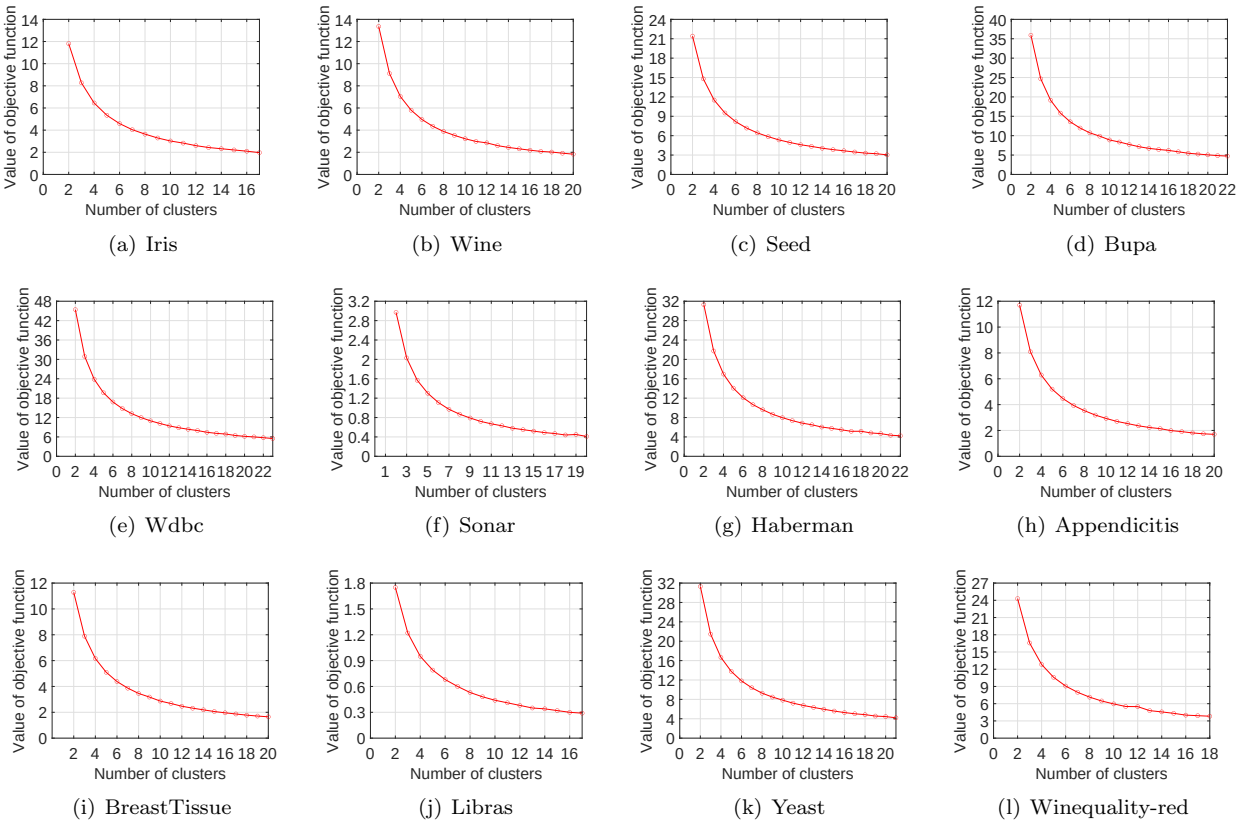
based on the property and meanings of fuzzy granular structures is proposed. This is also the basis of clustering and grouping fuzzy granular structures. To effectively group fuzzy granular structures, the k-means and fuzzy c-means clustering algorithms for clustering fuzzy granular structures are proposed.




(a) Iris (b) Appendicitis (c) BreastTissue

Figure 8: Time about number of clusters based on Algorithm 1.

Numerical experiments verify the convergence, effectiveness, and scalability of the proposed algorithms. As for the convergence of the algorithm, it is verified by adjusting the number of iterations and the number of clusters when the number of fuzzy granular structures is fixed. The results depict that the two algorithms show convergence no matter whether the number of iterations changes under a certain number of clusters or the number of clusters changes under a certain number of iterations. The purpose of fuzzy granular structure clustering is to group the more similar fuzzy granular structures together and effectively distinguish those with significant differences. Based on this purpose, two evaluation indexes are designed to evaluate the effectiveness of the algorithm. The dispersion indicates the compactness of fuzzy granular structures within a cluster. The separation characterizes the disparity between different clusters. The results show that the algorithm is effective. In addition, the scalability of the algorithm is verified by increasing the number of fuzzy granular structures and clusters. The experimental results show that the execution time increases linearly with the increase of the number of fuzzy granular structures and the number of clusters. The low time loss also proves the scalability of the two algorithms.

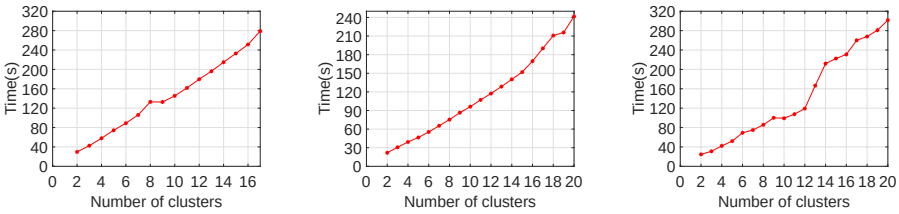



(a) Iris (b) Wine (c) Seed (d) Bupa

(e) Wdbc (f) Sonar (g) Haberman (h) Appendicitis

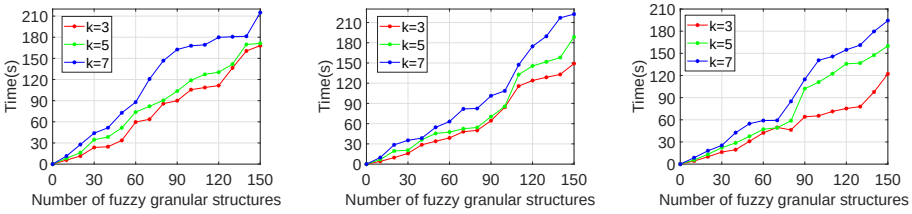
(i) BreastTissue (j) Libras (k) Yeast (l) Winequality-red

Figure 7: Value of objection function about number of clusters based on Algorithm 2.



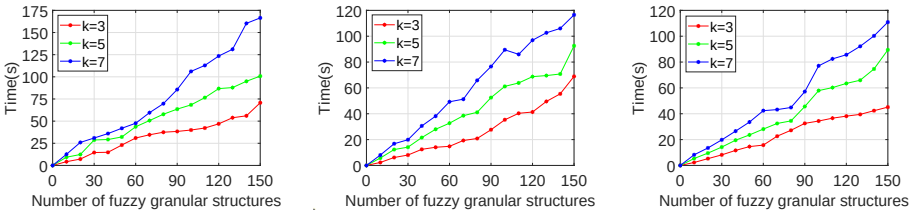

(a) Iris (b) Appendicitis (c) BreastTissue

Figure 9: Time about iterations based on Algorithm 2.




(a) Iris (b) Appendicitis (c) BreastTissue

Figure 10: Time about number of fuzzy granular structures based on Algorithm 1.




(a) Iris (b) Appendicitis (c) BreastTissue

Figure 11: Time about number of fuzzy granular structures based on Algorithm 2.

Compared with previous studies on fuzzy granular structures, the innovation of the paper is to consider the relevance of multiple fuzzy granular structures and cluster them. Compared with the study of granular structure clustering, we extend the case of classical granular structures to fuzzy granular structures, which makes the application more extensive. We compare the k-means fuzzy granular structure clustering algorithm with the clustering algorithm for Boolean granular structures proposed in [27]. We conducted contrast experiments on three data sets, namely Iris data set, Appendicitis data set, and BreastTissue data set. The number of clusters is changed from 10 to 15, and the execution time of the two algorithms is listed in Table 2. $B$ stands for the clustering algorithm for Boolean granular structures, and $F$ stands for the k-means fuzzy granular structure clustering algorithm. Because the time complexity of the two algorithms is linear time complexity, the variation trend of execution time is basically the same. The k-means fuzzy granular structure clustering algorithm deals with fuzzy granular structures, so the execution time is slightly increased. The comparison of the objective function values of the two algorithms is listed in Table 3, and the number of clusters $k$ ranges from 2 to 17. Experimental results show that the k-means fuzzy granular structure clustering algorithm has better convergence, and the objective function value can be lower under the same number of clusters.

Overall, the experimental results show that the model proposed in the paper works well, but there are some minor limitations. As mentioned in the experiment, the RBF kernel with parameters is used to generate fuzzy relations to quickly obtain a large number of fuzzy granular structures. Due to the limitations of the experimental equipment, we do not select a huge number of fuzzy granular structures. The above two points may limit the diversity and quantity of fuzzy granular structures in the experiment. The paper starts with the general fuzzy granular structure. Therefore, it also applies to clustering more special fuzzy granular structures. The distance measure and the clustering algorithm can be adjusted according to the specific fuzzy granular structure and its application.

Table 2: The time comparison of algorithms.

| k | Iris(B) | Iris(F) | Appendicitis(B) | Appendicitis(F) | BreastTissue(B) | BreastTissue(F) |
|---|---|---|---|---|---|---|
| 10 | 141.59 | 228.73 | 102.01 | 162.48 | 132.76 | 179.04 |
| 11 | 152.37 | 249.38 | 109.81 | 173.12 | 148.08 | 193.39 |
| 12 | 163.35 | 275.61 | 118.1 | 181.49 | 160.29 | 207.72 |
| 13 | 185.9 | 282.01 | 125.96 | 197.33 | 171 | 220.61 |
| 14 | 216.57 | 284.56 | 134.12 | 207.03 | 183.42 | 233.86 |
| 15 | 294.33 | 321.93 | 141.33 | 222.54 | 194.33 | 251.43 |

The clustering model of fuzzy granular structures established in the paper is of practical significance. For example, when the government or enterprises make classification strategies, they often encounter the situation of group decision-making. Different experts or organizations will come up with different schemes. A fuzzy granular structure represents a decision scheme. When there are many schemes, it is often impossible to consider them all. In this case, similar schemes can be clustered into the same class by clustering all schemes. At this time, it is only necessary to consider the typical representatives of each cluster, such as the clustering center in the clustering process. In the task of attribute reduction or feature selection of more specific data sets, people can reduce the attributes according to fuzzy granular structure clustering results. Specifically, each fuzzy relation is assumed to be generated by a single conditional attribute, and each fuzzy granular structure corresponds to the fuzzy relation. Therefore, there is a corresponding relationship between conditional attributes and fuzzy granular structures, and the clustering of fuzzy granular structures can also reflect the relationship between attributes to a certain extent. It can be used as a follow-up application of fuzzy granular structures clustering and a new idea of attribute reduction.

Table 3: The objective function value comparison of algorithms.

| k | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Iris(B) | 13.87 | 8.71 | 6.55 | 5.24 | 4.37 | 3.97 | 3.26 | 3.03 | 2.72 | 2.46 | 2.82 | 2.25 | 2.3 | 2.7 | 2.04 | 2.51 |
| Iris(F) | 10.36 | 6.68 | 4.92 | 3.9 | 3.28 | 2.8 | 2.51 | 2.32 | 2.05 | 2.16 | 1.99 | 1.62 | 1.85 | 1.9 | 1.89 | 1.61 |
| Appendicitis(B) | 13.62 | 8.75 | 6.56 | 5.19 | 4.5 | 4.3 | 3.67 | 3.35 | 3.13 | 2.68 | 2.59 | 2.7 | 2.44 | 2.54 | 2.45 | 2.27 |
| Appendicitis(F) | 10.35 | 6.61 | 4.82 | 3.83 | 3.18 | 2.8 | 2.53 | 2.3 | 2.13 | 2.06 | 2.04 | 2.02 | 1.62 | 1.61 | 1.95 | 1.99 |
| BreastTissue(B) | 12.52 | 8.36 | 6.11 | 4.9 | 4.09 | 3.64 | 3.19 | 3.05 | 2.75 | 2.7 | 2.39 | 2.16 | 2.55 | 1.91 | 2 | 2.35 |
| BreastTissue(F) | 9.91 | 6.38 | 4.71 | 3.75 | 3.11 | 2.67 | 2.41 | 2.16 | 2.09 | 1.93 | 1.67 | 1.89 | 1.55 | 1.58 | 1.51 | 1.49 |

# 6 Conclusions

This paper proposes the clustering algorithms for grouping fuzzy granular structures based on k-means and fuzzy c-means clustering algorithms. A unified representation of fuzzy granular structures derived from general fuzzy relations has been given. On this basis, the fuzzy granular structure distance has been proposed to measure the difference between fuzzy granular structures. Moreover, we have analyzed the relationship between granularity and distance measure from the granulation perspective and have further illustrated the rationality of the fuzzy granular structure distance. The significant contribution of the paper is that the k-means and fuzzy c-means fuzzy granular structure clustering algorithms are designed to group fuzzy granular structures. Besides the objective function, two indicators have been designed to evaluate the performance of the algorithm. Finally, we have conducted experiments on 12 real data sets to verify the performance of the proposed algorithms. The experimental results show that the proposed algorithms are convergent, effective, and scalable. In the paper, the general fuzzy granular structure is considered. Future research can adjust the clustering algorithm by considering the clustering of more special fuzzy granular structures and their application. Subsequently, we will also consider the time loss as the number of datasets and fuzzy granular structures increases in order to optimize the algorithms. Furthermore, as mentioned in the discussion, future research could consider attribute reduction or feature selection in terms of fuzzy granular structure clustering as a practical application of this study.

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