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Jackson-type approximation for fuzzy-valued functions by means of trapezoidal functions

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Abstract

In this paper, we provide new several Jackson-type approximations results for continuous fuzzy-number-valued functions which improve several previous ones. We use alternative techniques adapted from Interval Analysis which rely on the gH-difference (which might not exist) and the generalized difference (which might lack the cancellation law) of fuzzy numbers.

Keywords: Stone-Weierstrass theorem, fuzzy-valued continuous functions, fuzzy approximation, Jackson-type theorem.

1 Introduction

One of the main goals in fuzzy approximation theory is to approximate a continuous fuzzy-valued function defined on a compact space by some simpler functions such as (trigonometric) polynomials, rational functions, piecewise linear functions, etc. (see [1]).

The most common scheme is to find uniform approximation theorems of Weierstrass type [8, 9, 22, 26]), giving, in addition, an estimation of the error by means of the modulus of continuity (see e.g. [3, 10]), the so-called Jackson-type estimates [16].

We will focus on the following scheme: let $\mathbb{E}^1$ denote the set of fuzzy numbers (see Section 2) and let us consider the family of functions

$$\mathfrak{G}_n := \left\{ \sum_{k=0}^{n} \phi_k(x) \cdot u_k \ : \ \phi_k(x) \text{ real-valued continuous functions}, u_k \in \mathbb{E}^1 \right\},$$

and, for a given fuzzy-valued continuous function $f \in C([a,b], \mathbb{E}^1)$, denote the corresponding approximation error

$$E_{n,f} := \inf\{D(f,g) \ : \ g \in \mathfrak{G}_n\},$$

where D is the supremum metric on $C([a,b], \mathbb{E}^1)$.

In this paper we are interested in giving an estimate of this approximation of the form

$$E_{n,f} \leq C\omega^{\mathcal{F}}\left(f, \frac{1}{n}\right),$$

where $\omega^{\mathcal{F}}\left(f, \frac{1}{n}\right)$ stands for the fuzzy modulus of continuity for such functions (defined below), and C is a constant independent of n and f.

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Several such schemes have been already developed. For instance, when the coefficients ϕ_k are polynomials, S. Gal proved in [11] that, given $f \in C([0,1], \mathbb{E}^1)$, then

$$D(f, B_n^{\mathcal{F}}(f)) \leq \frac{3}{2}\omega^{\mathcal{F}}\left(f, \frac{1}{\sqrt{n}}\right),$$

where

$$B_n^{\mathcal{F}}(f) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right),$$

are called fuzzy polynomials of Bernstein type. To improve the rate of convergence from $\frac{1}{\sqrt{n}}$ to $\frac{1}{n}$, Bede and Gal [3] provided a Jackson-type estimate using Szabados-type polynomials but with an unexplicit constant C. Also they gave an estimation for approximating 2π-periodic continuous fuzzy-valued functions by sequences of fuzzy trigonometric polynomials, completing a previous work of Anastassiou and Gal [2]. Also in [3, Lemma 3.4], following the ideas of Szabados [27], they obtained a Jackson-type estimate for rational functions using Shepard-like operators. Other recent papers in the topic give similar estimates of the approximation using some generalizations of Bernstein-type polynomials (see [19, 20, 21]). The technique used in all those papers requires that $\sum_{k=0}^{n} \phi_k(x) = 1$ in order to compute $D(f, \sum \phi_k(x) \cdot u_k)$.

In this paper we will propose another approach which aligns with the ideas developed in [7], where the authors provided a Jackson-type approximation for continuous interval-valued functions defined on an interval $[a, b]$, combining techniques used by Prolla [23] and by D. Chen [5] to give a constructive proof of the Weierstrass-Stone Theorem and to estimate the approximation error by neural networks, respectively. We intend to export such technique to provide similar results for continuous fuzzy-valued functions in a essentially different way from the mentioned papers since it is not required that $\sum_{j=0}^{n} \phi_j(x) = 1$ (in fact, it will be greater than one).

It is well known that a fuzzy number can be identified as a family of nested compact intervals, the so called α-cuts (see, e.g., [17]). So it is expected that our previous research for continuous interval-valued functions can be applied to get a sharp result, that is, given now $f \in C([0,1], \mathbb{E}^1)$, our objective is to find a fuzzy-valued function like

$$g(x) = \sum_{j=0}^{n} \phi_j(x) u_j, \tag{1}$$

where now $u_j \in \mathbb{E}^1$ with

$$D(f, g) \leq C\omega^{\mathcal{F}}\left(f, \frac{1}{n}\right).$$

However, the main drawback in the fuzzy-valued context is that Chen's techniques require some sort of cancellation law, since such techniques use u_j in Equation (1) as a difference between two fuzzy numbers. This fact was not a problem for interval-valued functions because the generalized Hukuhara difference, gH-difference for short, has the cancellation law [24]. So, we would expect to obtain a similar approximation by replacing the compact intervals by fuzzy numbers and using the gH-difference of fuzzy numbers. Nevertheless, such a substraction does not always provide a fuzzy number. So we have to restrict our attention to some kind of fuzzy-valued functions for which our results still work. Namely, in Theorem 3.2, we prove that if $f \in C([0,1], \mathbb{E}^1)$ satisfies some additional conditions, then

$$E_{n,f} \leq 2\omega^{\mathcal{F}}\left(f, \frac{1}{n}\right).$$

In order to broaden the types of functions to which we can apply our technique, we replace the gH-difference of fuzzy numbers with their generalized difference [4]. In this case, our scope is larger than in Theorem 3.2, but our bound is not sharp as it grows with n due to the fact that the generalized difference of fuzzy number lacks the cancellation law. Indeed we prove (Theorem 3.7) that, for all $n \in \mathbb{N}$,

$$E_{n,f} \leq (2n+2)\omega^{\mathcal{F}}\left(f, \frac{1}{n}\right).$$

So it seems that Chen's technique is not the most adequate one for fuzzy-valued functions.

Finally, Prolla's ideas still allow us to construct a family of piecewise linear functions which provides a good approximation bound, although a bit worse than that of Theorem 3.2, but which is valid for any continuous fuzzy-valued function. Thus, in Theorem 3.10, we obtain, for all $n \in \mathbb{N}$ and $0 < \delta < \frac{1}{2n}$,

$$D\left(f, T_{n,\delta}^{f}\right) \leq 3\omega^{\mathcal{F}}\left(f, \frac{1}{n}\right),$$

where $T^f_{n,\delta}(x) := \sum_{k=0}^{n} \phi_k(x) \cdot f\left(\frac{j}{n}\right)$ and the coefficients $\phi_j(x)$ are piecewise linear functions.

2 Preliminaries

Let $F(\mathbb{R})$ denote the family of all fuzzy subsets on the real numbers $\mathbb{R}$, that is, mappings $u : \mathbb{R} \to [0,1]$. For $u \in F(\mathbb{R})$ and $\lambda \in [0,1]$, the λ-level set of u is defined by

$$[u]_\lambda := \{\, x \in \mathbb{R} \,:\, u(x) \geq \lambda \,\}, \quad \lambda \in]0,1], \quad [u]_0 := \mathrm{cl}_{\mathbb{R}} \{\, x \in \mathbb{R} \,:\, u(x) > 0 \,\}.$$

The fuzzy number space $\mathbb{E}^1$ is the set of elements u of $F(\mathbb{R})$ satisfying the following properties:

(1) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ with $u(x_0) = 1$;

(2) u is convex, i.e., $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}, \lambda \in [0,1]$;

(3) u is upper-semicontinuous;

(4) $[u]_0$ is a compact set in $\mathbb{R}$.

Notice that if $u \in \mathbb{E}^1$, then the λ-level set $[u]_\lambda$ of u is a compact interval for each $\lambda \in [0,1]$. We denote $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$. Every real number r can be identified with the fuzzy number $\tilde{r}$ defined as

$$\tilde{r}(t) := \begin{cases} 1 & \text{if } t = r, \\ 0 & \text{if } t \neq r. \end{cases}$$

We can now state the characterization of fuzzy numbers provided by Goetschel and Voxman [12]:

Theorem 2.1. *Let $u \in \mathbb{E}^1$ and $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$, $\lambda \in [0,1]$. Then the pair of functions $u^-(\lambda)$ and $u^+(\lambda)$ has the following properties:*

(i) *$u^-(\lambda)$ is a bounded left continuous nondecreasing function on $(0,1]$;*

(ii) *$u^+(\lambda)$ is a bounded left continuous nonincreasing function on $(0,1]$;*

(iii) *$u^-(\lambda)$ and $u^+(\lambda)$ are right continuous at $\lambda = 0$;*

(iv) *$u^-(1) \leq u^+(1)$.*

Conversely, if a pair of functions $\alpha(\lambda)$ and $\beta(\lambda)$ satisfy the above conditions (i)-(iv), then there exists a unique $u \in \mathbb{E}^1$ such that $[u]_\lambda = [\alpha(\lambda), \beta(\lambda)]$ for each $\lambda \in [0,1]$.

Given $u, v \in \mathbb{E}^1$ and $k \in \mathbb{R}$, we can define $u + v$ and ku level-wise as $[u+v]_\lambda := [u^-(\lambda), u^+(\lambda)] + [v^-(\lambda), v^+(\lambda)]$ and $[ku]_\lambda := k[u^-(\lambda), u^+(\lambda)]$. It is well-known that $\mathbb{E}^1$ endowed with these two natural operations is not a vector space. In fact, there is no inverse for the sum operation. On the other hand, we can endow $\mathbb{E}^1$ with the following metric:

Definition 2.2. [6, 12] *For $u, v \in \mathbb{E}^1$, we can define*

$$d_\infty(u,v) := \sup_{\lambda \in [0,1]} \max\left\{ |u^-(\lambda) - v^-(\lambda)|, |u^+(\lambda) - v^+(\lambda)| \right\}.$$

It is worth remarking that

$$d_\infty(u,v) := \sup_{\lambda \in [0,1]} \left\{ d_H([u]_\lambda, [v]_\lambda) \right\}.$$

where d_H stands for the Hausdorff metric on the space of all closed intervals.

d_∞ is called the supremum metric on $\mathbb{E}^1$, and $(\mathbb{E}^1, d_\infty)$ is well-known to be a complete metric space. Notice that, by the definition of d_∞, $\mathbb{R}$ endowed with the euclidean topology can be topologically identified with the closed subspace $\tilde{R} = \{\, \tilde{x} \,:\, x \in \mathbb{R} \,\}$ of $(\mathbb{E}^1, d_\infty)$ where $\tilde{x}^+(\lambda) = \tilde{x}^-(\lambda) = x$ for all $\lambda \in [0,1]$. As a metric space, we shall always consider $\mathbb{E}^1$ equipped with the metric d_∞.

The following lemma is the key to find the coefficients in our approximation function.

Lemma 2.3. *Let n be positive integer number and set $a_j = \frac{j}{n}$ for $j = 0, 1, \ldots, n$. Given $0 < \epsilon < \frac{1}{2}$ and $0 < \delta < \frac{1}{2n}$, we can find $n+1$ continuous functions $\psi_j : [0,1] \to [0,1]$ such that*

(1) *$\psi_j(x) < \epsilon$, for $x \geq a_j + \delta$, $j = 0, \ldots, n-1$;*

(2) *$1 - \psi_j(x) < \epsilon$, for $x \leq a_j - \delta$, $j = 1, \ldots, n$.*

Proof. Apply [18, Lemma 2] to $a_j - \delta < a_j + \delta$ $j = 1, \ldots, n-1$ (or $a_0 < a_0 + \delta$ for $j = 0$ and $a_n - \delta < a_n$ for $j = n$). Therefore, we are able to find $n+1$ continuous functions $\psi_j : [0,1] \to [0,1]$ satisfying the requested conditions. □

Let us finally recall some related results concerning interval-valued functions.

Let $\mathcal{K}_\mathcal{C}$ denote the set of all compact intervals of the real line $\mathbb{R}$; that is

$$\mathcal{K}_\mathcal{C} = \{[a,b] \ : a, b \in \mathbb{R},\ a \leq b\},$$

where $[a,a]$ stands for the singleton $\{a\}$. We denote by $len(A)$ the diameter or length of the interval A; that is, $len(A) = a^+ - a^-$, when $A = [a^-, a^+]$.

Let us denote the family

$$\mathfrak{T}_n = \left\{ \sum_{i=0}^{n} \psi_i(x) A_i \ : \ \psi_i \in C([0,1],[0,1]);\ A_i \in \mathcal{K}_\mathcal{C},\ i = 0, 1, 2, \ldots n \right\}.$$

The authors proved in [7] that the set $\mathfrak{T} = \cup_{n \in \mathbb{N}} \mathfrak{T}_n$ is dense in $C([0,1], \mathcal{K}_\mathcal{C})$. Define the approximation error between $\mathfrak{T}_n$ and a continuous function $f \in C([0,1], \mathcal{K}_\mathcal{C})$ by

$$E_{n,f} := \inf_{g \in \mathfrak{T}_n} D(f,g),$$

where $D(f,g) := \sup\{d_H(f(x), g(x)) \ : \ x \in [0,1]\}$ is the supremum metric on $C([0,1], \mathcal{K}_\mathcal{C})$. Let us consider the modulus of continuity of f defined as

$$\omega^\mathcal{K}(f, \delta) := \sup\{d_H(f(x), f(y)) \ : \ x, y \in [0,1],\ |x - y| < \delta\}.$$

Relying on an idea, which we can trace back to Chen ([5]) and further developed by Hong and Hahm ([14], [15]), the authors provided in [7], using the generalized Hukuhara difference for intervals, a Jackson type approximation for interval-valued continuous functions defined on $[0,1]$ as follows.

Theorem 2.4. [7] *Let $f \in C([0,1], \mathcal{K}_\mathcal{C})$ satisfying*

$$len(f(x)) \geq len(f(y)), \quad \text{when } x \leq y. \tag{2}$$

or

$$len(f(x)) \leq len(f(y)), \quad \text{when } x \leq y. \tag{3}$$

Then, for every $n \in \mathbb{N}$,

$$E_{n,f} \leq 2\omega^\mathcal{K}\left(f, \frac{1}{n}\right).$$

In fact, from the proof of that theorem and using Lemma 2.3, we can get the following propositions.

Proposition 2.5. [7, Theorem 4.3] *Let $f \in C([0,1], \mathcal{K}_\mathcal{C})$ satisfying equation (2) and let $M > 1$ be such that $d_H(f(x), f(y)) \leq 2M$, for all $x, y \in [0,1]$. Let $\varepsilon > 0$, $n \in \mathbb{N}$ and $0 < \delta < \frac{1}{2n}$. Let $a_j = \frac{j}{n}$ be given and take $\epsilon' < \frac{\varepsilon}{2(n+1)M}$. Let $\psi_j : [0,1] \to [0,1]$ be the $n+1$ continuous functions given by Lemma 2.3 for such ϵ' and δ.*

Then, the function

$$g(x) := A_n + \sum_{j=0}^{n-1} \psi_j(x) \Big(A_j \ominus A_{j+1} \Big),$$

where $A_j = f(a_j)$, $j = 0, 1, \ldots, n$, verifies that, for all $x \in [0,1]$,

$$d_H(g(x), f(x)) \leq 2\omega^\mathcal{K}\left(f, \frac{1}{n}\right) + \varepsilon.$$

Proposition 2.6. [7, Theorem 4.4] *Let $f \in C([0,1], \mathcal{K}_{\mathcal{C}})$ satisfying equation (3) and let $M > 1$ be such that $d_H(f(x), f(y)) \leq 2M$, for all $x, y \in [0,1]$. Let $\varepsilon > 0$, $n \in \mathbb{N}$ and $0 < \delta < \frac{1}{2n}$. Let $a_j = \frac{j}{n}$ be given and take $\epsilon' < \frac{\varepsilon}{2(n+1)M}$. Let $\psi_j : [0,1] \to [0,1]$ be the $n+1$ continuous functions given by Lemma 2.3 for such ϵ' and δ.*

Then, the function

$$g(x) := A_0 + \sum_{j=0}^{n-1} \phi_j(x)\Big(A_{j+1} \ominus A_j\Big),$$

where $A_j = f(a_j)$ and $\phi_j = 1 - \psi_j$, $j = 0, 1, \ldots, n$, verifies that, for all $x \in [0,1]$,

$$d_H(g(x), f(x)) \leq 2\omega^{\mathcal{K}}\left(f, \frac{1}{n}\right) + \varepsilon.$$

The generalized Hukuhara difference for fuzzy numbers was defined by Stefanini in [24] as follows:

Definition 2.7. *Given two fuzzy numbers $u, v \in \mathbb{E}^1$, the generalized Hukuhara difference (gH-difference for short) is the fuzzy number w, if it exists, such that*

$$u \ominus_{gH} v = w \Leftrightarrow \begin{cases} (i) & u = v + w \\ & or \\ (ii) & v = u + (-1)w. \end{cases}$$

It is easy to show that (i) and (ii) are both valid if and only if w is a crisp number. If $u \ominus_{gH} v$ exists, then its α-cuts are given by

$$[u \ominus_{gH} v]_\alpha = [u]_\alpha \ominus_{gH} [v]_\alpha.$$

Unfortunately, this substraction does not always define a fuzzy number. The conditions to guarantee its existence are given in [25, Proposition 21].

Let $f \in C([0,1], \mathbb{E}^1)$. Take now the interval-valued function $f_\alpha : [0,1] \to \mathcal{K}_{\mathcal{C}}$, defined as $f_\alpha(x) = [f(x)]_\alpha$, $\alpha \in [0,1]$. So, for such functions $f \in C([0,1], \mathbb{E}^1)$ we get a family of interval-valued functions f_α where our Theorem 2.4 in the previous section could be applied. Then we will need to retrieve the results back to our fuzzy-valued function f. Nevertheless, as it was pointed out above, a collection of closed intervals like

$$[u]_\alpha \ominus_{gH} [v]_\alpha,$$

can fail to define a proper fuzzy number. For instance, as shown in [4], given two triangular fuzzy numbers $u = \langle 12, 15, 19\rangle$ and $v = \langle 5, 9, 11\rangle$, $u \ominus_{gH} v$ does not exist as $u_\alpha^- - v_\alpha^- = -6 + \alpha$ and $u_\alpha^+ - v_\alpha^+ = -8 + 2\alpha$ are both increasing with respect to $\alpha \in [0,1]$, which fails to meet the conditions in [25, Proposition 21]. We will study that problem later. For now, we will add some restrictions to our continuous fuzzy-valued functions to prevent this from happening.

3 Approximation of continuous fuzzy-valued functions

We will present similar results as those mentioned in the section above but for functions $f \in C([0,1], \mathbb{E}^1)$, endowed with the distance

$$D(f, g) := \sup_{x \in [0,1]} d_\infty(f(x), g(x)).$$

Firstly, we study how the fuzzy modulus of continuity of $f \in C([0,1], \mathbb{E}^1)$ is related to the modulus of continuity of each level function f_α. We define the fuzzy modulus of continuity of f as

$$\omega^{\mathcal{F}}(f, \delta) := \sup\{d_\infty(f(x), f(y)) \ : \ x, y \in [0,1], \ |x - y| < \delta\}.$$

Lemma 3.1. *Let $f \in C([0,1], \mathbb{E}^1)$. Then, for each $\alpha \in [0,1]$, we have*

$$\omega^{\mathcal{K}}(f_\alpha, \delta) \leq \omega^{\mathcal{F}}(f, \delta).$$

Proof. Given any $\alpha \in [0,1]$, we have

$$\begin{aligned}\omega^{\mathcal{K}}(f_\alpha,\delta) &= \sup\{d_H(f_\alpha(x),f_\alpha(y)) \ : \ x,y\in[0,1],\ |x-y|<\delta\}\\ &\le \sup\Big\{\sup_{\alpha\in[0,1]}\{d_H(f_\alpha(x),f_\alpha(y))\} \ : \ x,y\in[0,1],\ |x-y|<\delta\Big\}\\ &= \sup\{d_\infty(f(x),f(y)) \ : \ x,y\in[0,1],\ |x-y|<\delta\}\\ &= \omega^{\mathcal{F}}(f,\delta).\end{aligned}$$

□

In order to apply Theorem 2.4 we need to work with functions $f \in C([0,1],\mathbb{E}^1)$ with the property that $f(x)\ominus_{gH} f(y)$ exists. Some examples of such functions are provided after Theorem 3.2.

Let us consider the family

$$\mathfrak{F}_n = \left\{\sum_{i=0}^{n}\psi_i(x)u_i \ : \ \psi_i\in C([0,1],[0,1]);\ u_i\in\mathbb{E}^1,\ i=0,1,2,\dots n\right\}.$$

Define the approximation error between a continuous function $f\in C([0,1],\mathbb{E}^1)$ and $\mathfrak{F}_n$ by

$$E_{n,f} := \inf_{g\in\mathfrak{F}_n} D(f,g).$$

Let us recall that, given $u,v\in\mathbb{E}^1$, the fuzzy inclusion $u\subseteq v$, means $[u]_\alpha\subseteq[v]_\alpha$, for all $\alpha\in[0,1]$. It is worth remarking that $u\subseteq v$ implies $len([u]_\alpha)\le len([v]_\alpha)$, for all α.

The following theorem provides a Jackson-type approximation for such class of continuous functions.

Theorem 3.2. *Let $f\in C([0,1],\mathbb{E}^1)$ satisfying the following conditions:*

(1) *There always exists $f(x)\ominus_{gH} f(y)$, for all $0\le x<y\le 1$.*

(2) *$f(y)\subseteq f(x)$, for all $x\le y$.*

Then, for all $n\in\mathbb{N}$,

$$E_{n,f}\le 2\omega^{\mathcal{F}}\left(f,\frac{1}{n}\right).$$

Proof. Fix $\varepsilon>0$ and $n\in\mathbb{N}$. Set $a_j=j/n$ and $u_j=f(a_j)$, $j=0,1,2,\dots,n$. Take now $0<\delta<\frac{1}{2n}$. Condition (2) for the function f implies that every function f_α satisfies equation (2). Then, we can apply Proposition 2.5 to find $n+1$ continuous functions $\psi_j:[0,1]\to[0,1]$, such that, for each $\alpha\in[0,1]$, the interval-valued function

$$g_\alpha = f_\alpha(a_n)+\sum_{j=0}^{n-1}\psi_j(x)\Big(f_\alpha(a_j)\ominus f_\alpha(a_{j+1})\Big),$$

satisfies

$$d_H(f_\alpha(x),g_\alpha(x))\le 2\omega^{\mathcal{K}}(f_\alpha,1/n)+\varepsilon,\quad \text{for every } x\in[0,1],$$

and, by Lemma 3.1, we get, for every $x\in[0,1]$,

$$d_H(f_\alpha(x),g_\alpha(x))\le 2\omega^{\mathcal{F}}(f,1/n)+\varepsilon.$$

On the other hand, the condition (1) on the function f yields that the gH-difference $u_j\ominus_{gH} u_{j+1}$ is well defined for all j. So, we can define the fuzzy function $g\in\mathfrak{F}_n$ by

$$g(x) := u_n+\sum_{j=0}^{n-1}\psi_j(x)(u_j\ominus_{gH} u_{j+1}),$$

and then, for every $x \in [0,1]$, since $[u_j]_\alpha = f_\alpha(a_j)$, we have $[g(x)]_\alpha = g_\alpha(x)$ and

$$\begin{aligned} d_\infty(f(x),g(x)) &= \sup_{\alpha\in[0,1]} d_H([f(x)]_\alpha,[g(x)]_\alpha) \\ &= \sup_{\alpha\in[0,1]} d_H(f_\alpha(x),g_\alpha(x)) \le 2\omega^{\mathcal{F}}(f,1/n)+\varepsilon, \end{aligned}$$

which means $D(f,g) \le 2\omega^{\mathcal{F}}(f,1/n)+\varepsilon$. Finally,

$$E_{n,f} := \inf_{h\in\mathfrak{F}_n} D(f,h) \le D_\infty(f,g) \le 2\omega^{\mathcal{F}}(f,1/n)+\varepsilon,$$

and, being true for all $\varepsilon > 0$, we get the conclusion.

□

Example 3.3. Let g be a positive real-valued decreasing continuous function defined on $[0,1]$. Let $f \in C([0,1],\mathbb{E}^1)$ be defined as $f(x) := g(x)\cdot u$ being u a fixed fuzzy number. Then, for every $0 \le x < y \le 1$, there always exists $f(x) \ominus_{gH} f(y)$. Since $g(x)$ is decreasing, $len([f(x)]_\alpha) \ge len([f(y)]_\alpha)$, for every $\alpha \in [0,1]$. On the other hand, if $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$, then

$$\begin{aligned} w_\alpha^- &= g(x)u_\alpha^- - g(y)u_\alpha^- = (g(x)-g(y))u_\alpha^-, \\ w_\alpha^+ &= g(x)u_\alpha^+ - g(y)u_\alpha^+ = (g(x)-g(y))u_\alpha^+, \end{aligned}$$

and w_α^- is increasing and w_α^+ is decreasing with respect to α, which means that the condition (1) in [25, Proposition 21] is fulfilled and then, it always defines a fuzzy number $f(x) \ominus_{gH} f(y) \in \mathbb{E}^1$ with α-cuts

$$[f(x) \ominus_{gH} f(y)]_\alpha = [w_\alpha^-, w_\alpha^+] = [f(x)]_\alpha \ominus_{gH} [f(y)]_\alpha.$$

It is worth mentioning that if the function g is increasing, then the function $f(x) := g(x)\cdot u$ also satisfies that $f(x) \ominus_{gH} f(y) \in \mathbb{E}^1$, verifying now condition (2) in [25, Proposition 21].

Using Proposition 2.6 we can show, similarly,

Theorem 3.4. *Let $f \in C([0,1],\mathbb{E}^1)$ satisfying the following conditions:*

(1) *There always exists $f(y) \ominus_{gH} f(x)$, for all $0 \le x < y \le 1$.*

(2) *$f(y) \supseteq f(x)$, for all $x \le y$.*

Then, for all $n \in \mathbb{N}$,

$$E_{n,f} \le 2\omega^{\mathcal{F}}\left(f, \frac{1}{n}\right).$$

As mentioned above, the gH-difference of fuzzy numbers does not always exist. To overcome this problem, Stefanini proposed a new difference between fuzzy numbers that always exists:

Definition 3.5. [4, 13, 25] *Given two fuzzy numbers u, v, the generalized difference (g-difference, for short) of u and v is given by their level sets as*

$$[u \ominus_g v]_\alpha = \mathrm{cl}\left(conv \bigcup_{\beta\ge\alpha} [u]_\beta \ominus [v]_\beta\right),$$

where the gH-difference $\ominus$ is used with interval operands $[u]_\beta$ and $[v]_\beta$.

It turns out that the α-cuts can be calculated as

$$[u \ominus_g v]_\alpha = [\inf_{\beta\ge\alpha} \min\{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+\}, \sup_{\beta\ge\alpha} \max\{u_\beta^- - v_\beta^-, u_\beta^+ - v_\beta^+\}].$$

Although, the gH-difference of intervals $[u]_\alpha \ominus [v]_\alpha$ could not coincide with $[u \ominus_g v]_\alpha$, except when $u \ominus_{gH} v$ exists, there is still some connection for the Hausdorff metric.

Lemma 3.6. *Let $u, v \in \mathbb{E}^1$ be two fuzzy numbers with their α-cuts satisfying that $[v]_\alpha \subseteq [u]_\alpha$, for all $\alpha \in [0,1]$. Then,*

$$d_H\Big([u]_\alpha \ominus [v]_\alpha,\ [u \ominus_g v]_\alpha\Big) \leq \sup_{\beta \geq \alpha} d_H\Big([u]_\alpha \ominus [v]_\alpha,\ [u]_\beta \ominus [v]_\beta\Big).$$

Proof. The condition $[v]_\alpha \subseteq [u]_\alpha$ gives $u_\beta^- - v_\beta^- \leq u_\beta^+ - v_\beta^+$, for all β, which implies that

$$\begin{aligned} [u \ominus_g v]_\alpha &= [\inf_{\beta \geq \alpha}\{u_\beta^- - v_\beta^-\}, \sup_{\beta \geq \alpha}\{u_\beta^+ - v_\beta^+\}]; \\ [u]_\alpha \ominus [v]_\alpha &= [u_\alpha^- - v_\alpha^-,\ u_\alpha^+ - v_\alpha^+], \end{aligned}$$

so

$$d_H\Big([u]_\alpha \ominus [v]_\alpha,\ [u \ominus_g v]_\alpha\Big) = \max\{\Big|u_\alpha^- - v_\alpha^- - \inf_{\beta \geq \alpha}\{u_\beta^- - v_\beta^-\}\Big|,\ \Big|u_\alpha^+ - v_\alpha^+ - \sup_{\beta \geq \alpha}\{u_\beta^+ - v_\beta^+\}\Big|\}.$$

Assume that

$$d_H\Big([u]_\alpha \ominus [v]_\alpha,\ [u \ominus_g v]_\alpha\Big) = \Big|u_\alpha^- - v_\alpha^- - \inf_{\beta \geq \alpha}\{u_\beta^- - v_\beta^-\}\Big|.$$

Then, removing the absolute value, we have

$$\begin{aligned} d_H\Big([u]_\alpha \ominus [v]_\alpha,\ [u \ominus_g v]_\alpha\Big) &= u_\alpha^- - v_\alpha^- - \inf_{\beta \geq \alpha}\{u_\beta^- - v_\beta^-\} \\ &= \sup_{\beta \geq \alpha}\{u_\alpha^- - v_\alpha^- - (u_\beta^- - v_\beta^-)\} \\ &\leq \sup_{\beta \geq \alpha} d_H\Big([u]_\alpha \ominus [v]_\alpha,\ [u]_\beta \ominus [v]_\beta\Big). \end{aligned}$$

On the other case, if

$$d_H\Big([u]_\alpha \ominus [v]_\alpha,\ [u \ominus_g v]_\alpha\Big) = \Big|u_\alpha^+ - v_\alpha^+ - \sup_{\beta \geq \alpha}\{u_\beta^+ - v_\beta^+\}\Big|,$$

then,

$$\begin{aligned} d_H\Big([u]_\alpha \ominus [v]_\alpha,\ [u \ominus_g v]_\alpha\Big) &= \sup_{\beta \geq \alpha}\{u_\beta^+ - v_\beta^+\} - (u_\alpha^+ - v_\alpha^+) \\ &= \sup_{\beta \geq \alpha}\{u_\beta^+ - v_\beta^+ - (u_\alpha^+ - v_\alpha^+)\} \\ &\leq \sup_{\beta \geq \alpha} d_H\Big([u]_\alpha \ominus [v]_\alpha,\ [u]_\beta \ominus [v]_\beta\Big), \end{aligned}$$

and we are done. □

Unfortunately, the generalized difference of two fuzzy numbers has no cancellation law and the technique used in Theorem 3.2 is not applicable. However, using Lemma 3.6 we are still able to provide a Jackson-type approximation for continuous fuzzy-valued functions, thus avoiding one of the restrictions in Theorem 3.2, although with a nonconstant bound.

Theorem 3.7. *Let $f \in C([0,1], \mathbb{E}^1)$ satisfying*

$$f(y) \subseteq f(x),\ \textit{for all } x \leq y.$$

Then, for all $n \in \mathbb{N}$,

$$E_{n,f} \leq (2n+2)\omega^{\mathcal{F}}\left(f, \frac{1}{n}\right).$$

Proof. Proceed as in Theorem 3.2. Fix $\varepsilon > 0$ and $n \in \mathbb{N}$. Set $a_j = j/n$ and $u_j = f(a_j)$, $j = 0, 1, 2, \ldots, n$ and find $n+1$ continuous functions $\psi_j : [0,1] \to [0,1]$ satisfying conditions in Proposition 2.5 for $0 < \delta < \frac{1}{2n}$.

We already know that, for each $\alpha \in [0,1]$, the interval-valued function

$$g_\alpha = f_\alpha(a_n) + \sum_{j=0}^{n-1} \psi_j(x)\Big(f_\alpha(a_j) \ominus f_\alpha(a_{j+1})\Big),$$

satisfies, for every $x \in [0,1]$,

$$d_H(f_\alpha(x), g_\alpha(x)) \leq 2\omega^{\mathcal{F}}(f, 1/n) + \varepsilon,$$

where $f_\alpha(x) = [f(x)]_\alpha$.

Let us consider the continuous fuzzy-valued function in the family $\mathfrak{F}_n$ defined by

$$h(x) := u_n + \sum_{j=0}^{n-1} \psi_j(x)(u_j \ominus_g u_{j+1}).$$

Fix $x \in [0,1]$ and $\alpha \in [0,1]$. Since

$$\begin{aligned} d_H\left([f(x)]_\alpha, [h(x)]_\alpha\right) &\leq d_H([f(x)]_\alpha, g_\alpha(x)) + d_H(g_\alpha(x), [h(x)]_\alpha) \\ &\leq 2\omega^{\mathcal{F}}(f, 1/n) + \varepsilon + d_H(g_\alpha(x), [h(x)]_\alpha), \end{aligned}$$

we need to compute $d_H(g_\alpha(x), [h(x)]_\alpha)$. Since, by Lemma 3.6

$$\begin{aligned} d_H\Big([u_j]_\alpha \ominus [u_{j+1}]_\alpha), [u_j \ominus_g u_{j+1}]_\alpha\Big) &\leq \sup_{\beta \geq \alpha} d_H\Big([u_j]_\alpha \ominus [u_{j+1}]_\alpha,\ [u_j]_\beta \ominus [u_{j+1}]_\beta\Big) \\ &\leq \sup_{\beta \geq \alpha} \Big[d_H\Big([u_j]_\alpha \ominus [u_{j+1}]_\alpha, \mathbf{0}\Big) + d_H\Big([u_j]_\beta \ominus [u_{j+1}]_\beta, \mathbf{0}\Big)\Big] \\ &\leq \omega^{\mathcal{F}}(f, 1/n) + \omega^{\mathcal{F}}(f, 1/n), \end{aligned}$$

we get

$$\begin{aligned} d_H\left(g_\alpha(x), [h(x)]_\alpha\right) &\leq \sum_{j=0}^{n-1} d_H\Big(\psi_j(x)([u_j]_\alpha \ominus [u_{j+1}]_\alpha), \psi_j(x)[u_j \ominus_g u_{j+1}]_\alpha\Big) \\ &\leq \sum_{j=0}^{n-1} \psi_j(x) d_H\Big([u_j]_\alpha \ominus [u_{j+1}]_\alpha, [u_j \ominus_g u_{j+1}]_\alpha\Big) \\ &\leq 2\omega^{\mathcal{F}}(f, 1/n) \sum_{j=0}^{n-1} \psi_j(x) \\ &\leq 2n\omega^{\mathcal{F}}(f, 1/n). \end{aligned}$$

So, we have just shown that

$$d_H\left([f(x)]_\alpha, [h(x)]_\alpha\right) \leq 2\omega^{\mathcal{F}}(f, 1/n) + \varepsilon + 2n\omega^{\mathcal{F}}(f, 1/n),$$

for every x and α, which means

$$D(f, g) \leq (2n+2)\omega^{\mathcal{F}}(f, 1/n) + \varepsilon.$$

Then,

$$\inf_{h \in \mathfrak{F}_n} D(f, h) \leq D(f, g) \leq (2n+2)\omega^{\mathcal{F}}(f, 1/n) + \varepsilon,$$

and, being true for all $\varepsilon > 0$, we finally get the conclusion

$$E_{n,f} \leq (2n+2)\omega^{\mathcal{F}}(f, 1/n).$$

□

The problem with the nonconstant bound seems to be caused by the fact that the sum of the coefficients $\sum \psi_k$ is not bounded. So, in order to improve this result, we need to find a family of functions ϕ_k well behaved for the modulus of continuity and with the property that $\sum_{k=0}^{n} \phi_k = 1$. Thus, using the technique of Prolla [23], we are able to provide a Jackson type approximation, with a slight worse bound than the one in Theorem 3.2, but with the advantage that it can be applied to any continuous fuzzy-valued function defined on $[0,1]$.

Proposition 3.8. *There exist $n+1$ continuous functions $\phi_j : [0,1] \to [0,1]$ such that, for every $x \in [0,1]$,*

$$\sum_{j=0}^{n} \phi_j(x) = 1,$$

and with the property that, at each $x \in [0,1]$, there exist, at most, two nonzero terms in the sum.

Proof. Take $a_j = \frac{j}{n}$, $j = 0, 1, \ldots, n$ and let $0 < \delta < \frac{1}{2n}$. Define the continuous functions $f_j : [0,1] \to [0,1]$, $j = 0, 1, \ldots, n$, as follows

$$f_0(x) := \begin{cases} \frac{1}{\delta}(\delta - x), & \text{if } 0 \le x < \delta \\ 0, & \text{if } \delta \le x \le 1 \end{cases}$$

$$f_j(x) := \begin{cases} 1, & \text{if } 0 \le x \le a_j - \delta \\ \frac{1}{2\delta}(a_j + \delta - x), & \text{if } a_j - \delta < x < a_j + \delta \\ 0, & \text{if } a_j + \delta \le x \le 1 \end{cases}$$

$$f_n(x) := \begin{cases} 1, & \text{if } 0 \le x \le a_n - \delta \\ \frac{1}{\delta}(a_n - x), & \text{if } a_n - \delta < x \le 1 \end{cases}$$

that is, f_j takes value 1 before $a_j - \delta$, takes value 0 after $a_j + \delta$ and it is linear in the gap between. Now, let us define,

$$\begin{aligned} \phi_0 &= f_0, \\ \phi_1 &= (1 - f_0) f_1, \\ \phi_2 &= (1 - f_0)(1 - f_1) f_2, \\ &\vdots \\ \phi_{n-1} &= (1 - f_0)(1 - f_1) \cdots (1 - f_{n-2}) f_{n-1}. \end{aligned}$$

We claim that

$$\phi_0 + \phi_1 + \phi_2 + \ldots + \phi_{n-1} = 1 - (1 - f_0)(1 - f_1) \cdots (1 - f_{n-1}).$$

Then, taking $\phi_n = (1 - f_0)(1 - f_1) \cdots (1 - f_{n-1})$ we obtain

$$\sum_{j=0}^{n} \phi_j(x) = 1, \quad \text{for all } x \in [0,1].$$

We have partitioned $[0,1]$ into two disjoint sets, say,

$$\begin{aligned} V &:= [0, \delta) \cup \Big(\bigcup_{j=1}^{n-1} (a_j - \delta, a_j + \delta) \Big) \cup (a_n - \delta, 1]; \\ W &:= [0,1] \setminus V. \end{aligned}$$

We will study what happens to each $\phi_j(x)$ depending on where x is located.

Assume that $x \in V$. Then $x \in (a_k - \delta, a_k + \delta)$ for some $k \in \{1, \ldots, n-1\}$ or $x \in [0, \delta)$ or $x \in (a_n - \delta, 1]$.

If $x \in [0, \delta)$, then $f_1(x) = 1$ and $\phi_j(x) = 0$, for all $j > 1$. We have only two nonzero summands, $\phi_0(x)$ and $\phi_1(x)$. If $x \in (a_n - \delta, 1]$ then $f_j(x) = 0$ and therefore $\phi_j(x) = 0$, for all $0 \le j \le n-1$. So we have only one nonzero term $\phi_n(x) = 1$.

If $x \in (a_k - \delta, a_k + \delta)$, $1 \le k \le n-1$, then

$$x > a_{k-1} - \delta > a_{k-2} - \delta > \ldots > \delta,$$

implying that $f_j(x) = 0$ and $\phi_j(x) = 0$ as well, for $0 \leq j \leq k-1$. On the other hand,

$$x < a_{k+1} - \delta < a_{k+2} - \delta < \ldots < a_n - \delta,$$

which yields $f_j(x) = 1$ for $j \geq k+1$ and then $\phi_j(x) = 0$, for $j > k+1$. Again, that gives us only two nonzero terms, $\phi_k(x)$ and $\phi_{k+1}(x)$.

We study now the case $x \in W$. This implies that $x \in [a_{k-1} + \delta, a_k - \delta]$, for some $1 \leq k \leq n$. Then

$$x \leq a_k - \delta < a_{k+1} - \delta < \ldots < a_n - \delta,$$

which means $f_j(x) = 1$ for $j \geq k$ and then $\phi_j(x) = 0$, for $j \geq k+1$. On the other hand,

$$x \geq a_{k-1} + \delta > a_{k-2} + \delta > \ldots > \delta,$$

implying that $f_j(x) = 0$ and also $\phi_j(x) = 0$, for $0 \leq j \leq k-1$. Again we have only one nonzero term, $\phi_k(x) = 1$. □

The fuzzy modulus of continuity has the following properties [11].

Proposition 3.9. *Let f be in $C([0,1], \mathbb{E}^1)$. Then,*

(1) $\omega^{\mathcal{F}}(f, \delta_1 + \delta_2) \leq \omega^{\mathcal{F}}(f, \delta_1) + \omega^{\mathcal{F}}(f, \delta_2)$, $\delta_1, \delta_2 > 0$.

(2) $\omega^{\mathcal{F}}(f, n\delta) \leq n\, \omega^{\mathcal{F}}(f, \delta)$, $\delta > 0$, $n \in \mathbb{N}$.

(3) $\omega^{\mathcal{F}}(f, \lambda\delta) \leq (\lambda + 1)\, \omega^{\mathcal{F}}(f, \delta)$, $\delta > 0$, $\lambda > 0$.

The functions $\phi_k(x)$ in Proposition 3.8 are piecewise linear and, in fact, they have a trapezoidal graphic. For each $0 < \delta < \frac{1}{2n}$, we define:

$$T^f_{n,\delta}(x) := \sum_{k=0}^{n} \phi_k(x) \cdot f\left(\frac{j}{n}\right).$$

Now we can prove the following theorem.

Theorem 3.10. *Let $f \in C([0,1], \mathbb{E}^1)$. Then, for all $n \in \mathbb{N}$ and $0 < \delta < \frac{1}{2n}$,*

$$D\left(f, T^f_{n,\delta}\right) \leq 3\omega^{\mathcal{F}}\left(f, \frac{1}{n}\right).$$

Proof. Set $a_j = \frac{j}{n}$, $j = 0, 1, \ldots, n$. Set $u_j = f(a_j)$ and take $0 < \delta < \frac{1}{2n}$. Let us take $g(x) := T^f_{n,\delta}(x)$.

Then, for each $x \in [0,1]$ and $\alpha \in [0,1]$,

$$\begin{aligned} d_H([f(x)]_\alpha, [g(x)]_\alpha) &= d_H\Big(\sum_{j=0}^{n} \phi_j(x)[f(x)]_\alpha, \sum_{j=0}^{n} \phi_j(x)[u_j]_\alpha\Big) \\ &\leq \sum_{j=0}^{n} \phi_j(x) d_H\Big([f(x)]_\alpha, [u_j]_\alpha\Big). \end{aligned}$$

Let us consider

$$\begin{aligned} V &:= [0, \delta) \cup \Big(\bigcup_{j=1}^{n-1} (a_j - \delta, a_j + \delta)\Big) \cup (a_n - \delta, 1], \\ W &:= [0,1] \setminus V. \end{aligned}$$

Now, if $x \in V$, then we have two nonzero terms, $\phi_k(x)$ and $\phi_{k+1}(x)$ for $k = 0, 1, \ldots, n-1$ when $x \in (a_k - \delta, a_k + \delta)$ or $x \in [0, \delta)$ and we have only one $\phi_n(x) = 1$, when $x \in (a_n - \delta, 1]$. Then, in the worst case and bearing in mind that the position of x means $|x - a_k| < \frac{1}{n}$ and $|x - a_{k+1}| < \frac{2}{n}$, we have

$$\begin{aligned} d_H\Big([f(x)]_\alpha, [u_k]_\alpha\Big) &\leq \omega^{\mathcal{F}}\left(f, \frac{1}{n}\right), \quad \text{and} \\ d_H\Big([f(x)]_\alpha, [u_{k+1}]_\alpha\Big) &\leq \omega^{\mathcal{F}}\left(f, \frac{2}{n}\right). \end{aligned}$$

Then, we obtain

$$\begin{aligned} d_H([f(x)]_\alpha, [g(x)]_\alpha) &\leq \phi_k(x) d_H\Big([f(x)]_\alpha, [u_k]_\alpha\Big) + \phi_{k+1}(x) d_H\Big([f(x)]_\alpha, [u_{k+1}]_\alpha\Big) \\ &\leq \omega^{\mathcal{F}}\left(f, \frac{1}{n}\right) + \omega^{\mathcal{F}}\left(f, \frac{2}{n}\right) \leq \omega^{\mathcal{F}}\left(f, \frac{1}{n}\right) + 2\omega^{\mathcal{F}}\left(f, \frac{1}{n}\right) \\ &= 3\omega^{\mathcal{F}}\left(f, \frac{1}{n}\right). \end{aligned}$$

Now, when $x \in W$, there exists only one nonzero term $\phi_k(x)$. So,

$$\begin{aligned} d_H([f(x)]_\alpha, [g(x)]_\alpha) &\leq \phi_k(x) d_H\Big([f(x)]_\alpha, [u_k]_\alpha\Big) \\ &\leq \omega^{\mathcal{F}}\left(f, \frac{1}{n}\right) \leq 3\omega^{\mathcal{F}}\left(f, \frac{1}{n}\right). \end{aligned}$$

So, we have just shown that,

$$d_H([f(x)]_\alpha, [g(x)]_\alpha) \leq 3\omega^{\mathcal{F}}\left(f, \frac{1}{n}\right),$$

for every $x \in [0,1]$ and for every α-cut, which means that

$$D(f, T^f_{n,\delta}) \leq 3\omega^{\mathcal{F}}\left(f, \frac{1}{n}\right).$$

□

Conclusions

In this paper we provide approximation results for fuzzy-valued continuous functions and give also an estimate of the approximation error using the modulus of continuity. Specifically, given $f \in C([0,1], \mathbb{E}^1)$, we find a family of real-valued functions $\phi_j(x)$ and fuzzy numbers u_j in such a way that the combination $g(x) = \sum_{j=0}^{n} \phi_j(x) u_j$ is close enough to $f(x)$. Moreover, we are able to measure the approximation based on the fuzzy modulus of continuity $\omega^{\mathcal{F}}\left(f, \frac{1}{n}\right)$.

We have also shown that if f satisfies some additional conditions, then we get a bound of two times the modulus of the continuity. In order to avoid such restrictions, we construct a family of piecewise linear functions $\phi_j(x)$ which provides a bound of three times the modulus of continuity which, however, is valid for any continuous fuzzy-valued function, thus improving some of the earlier results in the literature.

It remains to show whether these bounds are sharp.

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