Iranian Journal of Fuzzy Systems

Volume 20, Number 6, (2023), pp. 85-103

Original paper

University of Sistan and Baluchestan

Sharp elements in d_0-algebras

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Abstract

We introduce the notions of Sasaki mapping and of sharp elements on a d_0-algebra. We investigate the relationships of sharp elements with Sasaki mappings and with central elements, thus generalizing some results known for D-lattices. Namely, we give a characterization of sharp elements, by means of Sasaki mappings, which extends a result of Bennett and Foulis; we also prove that an element is central if and only if it is a sharp element and every element is compatible with it: This generalizes a result of Riečanová.

Keywords: d_0-algebra, BCK-algebra, D-lattice, effect algebra, orthomodular lattice, sharp element, Sasaki mapping, central element.

1 Introduction

In the 1990's, D-lattices (or, equivalently, lattice-ordered effect algebras) have been introduced as a simultaneous generalization of MV-algebras and orthomodular lattices (see [12, 14, 20]). Then the investigation of modular measures on D-lattices (see for instance [1, 2, 3, 4, 5, 6]) gave results which may be applied in both Fuzzy Measure Theory and Non-commutative Measure Theory.

Cancellative BCK-algebras (i.e., commutative BCK-algebras with the relative cancellation property, as defined in [19]) generalize MV-algebras, too. As shown in [18], they are related to other fuzzy structures such as clans [30, 31, 34], Vitali spaces [15] and Δ-ℓ-semigroups [33]. For a different generalization of MV-algebras, see [16, 17].

In 2011, d_0-algebras have been defined, which turned out to be a generalization of both D-lattices and cancellative BCK-algebras (see [26, 32]). It is also known that every d_0-algebra is at the same time a $\wedge$-semilattice and a generalized D-poset. In the recent paper [28], d_0-measures on d_0-algebras are introduced, of which both modular measures on D-lattices and measures on cancellative BCK-algebras are particular cases. Moreover, in the paper [27] suitable topological notions are introduced, which can be applied in measure theory. Then some measure-theoretic results have been established for d_0-algebras. See for instance [7, 8].

This paper is concerned with the algebraic structure of d_0-algebras. We introduce a concept of Sasaki mapping, of which the Sasaki mapping defined in [13] is a particular case. We also introduce a notion of sharp elements, and investigate its relationships with Sasaki mappings and with central elements, thus generalizing some results known for D-lattices, namely [13, Theorem 3.8] and [24, Theorem 2.5].

The paper is organized as follows. In Section 2 we give some preliminaries. In Section 3 we introduce Sasaki mappings, and in particular we show how to construct such mappings by restriction or by subdirect products. In Section 4 we consider sharp elements; we give a characterization of sharp elements, by means of Sasaki mappings, which

Corresponding Author: P. Vitolo
Received: November 2022; Revised: June 2023; Accepted: September 2023.
https://doi.org/10.22111/IJFS.2023.43899.7730

extends a result of Bennett and Foulis [13, Theorem 3.8]; we also prove that an element is central if and only if it is a sharp element and every element is compatible with it: this generalizes a result of Riečanová [24, Theorem 2.5].

2 Preliminaries

Definition 2.1. *Let A be a non-empty set, $\setminus$ a binary operation on A, which we call* difference*, and* 0 *a distinguished element of A, which we call* zero*. Define a relation $\leq$ and another operation on A as follows. For all $a, b \in A$*

$$a \leq b \iff a \setminus b = 0, \tag{2.1}$$

$$a - b = (a \setminus b) \setminus (b \setminus a). \tag{2.2}$$

The algebra $(A, \setminus, 0)$ is called d_0-algebra *if it satisfies the following axioms:*

(d0) $a \setminus 0 = a$;

(d1) $a \setminus (b \setminus a) \leq a$;

(d2) $a \setminus (a - b) = b \setminus (b - a)$;

(d3) $(a \setminus b) \setminus (c \setminus b) = (a \setminus c) \setminus (b \setminus c)$;

(d4) $a \setminus (a \setminus b) = b - (b \setminus a)$ $(a \setminus b) \setminus a = (b \setminus a) - b$.

We recall that d_0-algebras have been introduced in the paper [32]. Most of the times, instead of denoting a d_0-algebra by $(A, \setminus, 0)$, we will write simply A when the zero and the difference are understood.

In a d_0-algebra, the relation $\leq$, as defined in (2.1) above, is a partial order such that 0 is the smallest element of A [32, Theorem 3.1]. When we refer to an order in a d_0-algebra, we will always mean the relation above. We also use the usual notation for intervals. In particular, for every $b \in A$, we set $[0, b] = \{\, a \in A : a \leq b \,\}$.

As proved in [26], every d_0-algebra A is a semilattice with

$$a \wedge b = a \setminus (a - b) = b \setminus (b - a).$$

It is also shown in [26, Corollary 3.2] that, if we define

$$a \ominus b = a \setminus b \text{ whenever } b \leq a, \tag{2.3}$$

then $(A, \leq, \ominus)$ is a *generalized D-poset* according to [20, Definition 1.1.4]. As explained in [20, Section 1.1], A can be endowed with a partial sum $\oplus$, defined as follows:

$$\forall a, b, c \in A \qquad a \oplus b = c \iff \big[\, b \leq c \text{ and } c \setminus b = a \,\big]. \tag{2.4}$$

We say that a *is orthogonal to* b, and write $a \perp b$, if $a \oplus b$ exists.

The partial operation $\oplus$ defined in (2.4) is commutative and associative. In fact $(A, \oplus, 0)$ is a *generalized effect algebra*, according to [20, Definition 1.2.1].

We call a (necessarily unique) element 1 of A a *unit* if $a \setminus 1 = 0$ for every $a \in A$, i.e., 1 is the greatest element of A. More in general, we say that A is *directed*, if every pair of elements $a, b \in A$ has a common upper bound, i.e., A is directed by the relation $\geq$.

We say that a d_0-algebra A is *semicomplete* if every non-empty subset has infimum. As it is well known (since A is a partially ordered set with minimum), this is equivalent to being *Dedekind-complete*, i.e., every subset which is bounded above has a supremum.

The following theorem establishes a connection between d_0-algebras and generalized D-lattices. We recall here the relevant definitions, referring to [20] for more details.

Definition 2.2. *A* poset with difference *is a structure of the form $(P, \leq, \ominus)$, where P is nonempty set, $\leq$ is a partial order on P and $\ominus$ a partially defined operation on P such that the following conditions are satisfied for all $a, b, c \in P$:*

(DP1) *$a \ominus b$ is defined if and only if $b \leq a$;*

(DP2) *If $a \leq b$, then $b \ominus a \leq b$ and $b \ominus (b \ominus a) = a$;*

(DP3) *If $a \le b \le c$, then $c \ominus b \le c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.*

A poset with difference is called D-poset *if it has (smallest and) greatest element. A poset with difference which is also a lattice is called a* generalized D-lattice. *A D-poset which is also a lattice is called a* D-lattice.

We refer also to [20] for the well-known facts that D-lattices and generalized D-lattices are equivalent to lattice-ordered effect algebras and lattice generalized effect algebras, respectively.

Theorem 2.3.

(1) *Every directed d_0-algebra can be viewed as a generalized D-lattice by defining $\ominus$ as in* (2.3).
Conversely, if $(A, \le, \ominus)$ is a generalized D-lattice and we put

$$a \setminus b = (a \vee b) \ominus b \text{ for every } a, b \in A, \tag{2.5}$$

then $(A, \setminus, 0)$ is a (directed) d_0-algebra.

(2) *Every d_0-algebra with unit can be viewed as a D-lattice by defining $\ominus$ as* (2.3).
Conversely, if $(A, \le, \ominus)$ is a D-lattice and we define $\setminus$ as in (2.5) *above then $(A, \setminus, 0)$ is a d_0-algebra (with unit).*

Proof. See [32, Theorem 5.6 and Corollary 5.7]. □

Definition 2.4. *We say that the elements a and b of a d_0-algebra A are* compatible, *and write $a \leftrightarrow b$, if $a - b = a \setminus b$.*
If all elements of A are pairwise compatible, we say that A has the Riesz property *or that it is a* Riesz d_0-algebra (see [26]).

It follows from [20, Theorem 1.10.6] (with the help of Proposition 2.7(viii) below), that the foregoing notion of compatibility agrees with the one usually given in generalized D-lattices.

There is a connection also between d_0-algebras and BCK-algebras. For the sake of brevity, we will call *cancellative BCK-algebra* a commutative BCK-algebra with the relative cancellation property. For the definition of BCK algebra and commutative BCK-algebra we refer to [20, Section 5.1.1].

We also recall that, on a BCK-algebra $(X, *, 0)$, the relation $\le$ defined by

$$x \le y \iff x * y = 0,$$

is a partial order on X, with least element 0, called *BCK-order* [20, Proposition 5.1.1].

The relative cancellation property is defined as follows [20, Definition 5.2.1].

Definition 2.5. *A BCK-algebra $(X, *, 0)$ has the* relative cancellation property *if, for each $a, x, y \in X$,*

$$\big[\, a * x = a * y = 0 \ \text{ and } \ x * a = y * a \,\big] \implies x = y. \tag{2.6}$$

Cancellative BCK-algebras generalize MV-algebras, as shown in [19].

Theorem 2.6. *Every Riesz d_0-algebra $(A, \setminus, 0)$ can be viewed as a cancellative BCK-algebra by defining $*$ as $\setminus$. The BCK-order coincides with the order of A as a d_0-algebra.*
Conversely, if $(A, *, 0)$ is a cancellative BCK-algebra and we put $a \setminus b = a * b$ for every $a, b \in A$, then $(A, \setminus, 0)$ is a (Riesz) d_0-algebra.

Proof. See [26, Theorem 5.4 and Theorem 5.24]. □

The following result summarizes some properties of d_0-algebras.

Proposition 2.7. *In any d_0-algebra A the following hold, for every $a, b, c \in A$.*

(i) $0 \setminus a = a \setminus a = 0$;

(ii) $a \setminus b = b \setminus a \iff a = b$;

(iii) $a \le b \implies a \setminus c \le b \setminus c$;

(iv) $a \le b \implies \big[\, b \setminus a \le b \ \text{ and } \ b \setminus (b \setminus a) = a \,\big]$;

(v) $a - b \leq a$;

(vi) $a \leq b \iff a - b = 0$;

(vii) $a \setminus c = b \setminus c \implies (c \setminus b) \setminus (c \setminus a) = a - b$;

(viii) $a \setminus (a \wedge b) = a - b$;

(ix) $c \leq b \implies a - b \leq a - c$;

(x) $\big[b \perp c \text{ and } a \leq b\big] \implies \big[a \perp c \text{ and } a \oplus c \leq b \oplus c\big]$;

(xi) $\big[c \leq a \text{ and } c \leq b\big] \implies (a \setminus c) \wedge (b \setminus c) = (a \wedge b) \setminus c$;

(xii) $\big[c \leq a \text{ and } c \leq b\big] \implies (a \setminus c) - (b \setminus c) = a - b$;

(xiii) $a \perp b \implies \big[a \vee b \text{ exists and } (a \oplus b) \setminus (a \wedge b) = a \vee b\big]$;

(xiv) $a \vee b$ exists $\implies a \setminus b = (a \vee b) \setminus b$.

Proof. For Items (i)–(iv), see [32, Proposition 1.2]. For Items (v), (vi) and (vii), respectively, see [32, Proposition 1.4, Proposition 4.2(ii), and Proposition 1.7]. Item (viii) has been proved in [26, Proposition 4.3], Item (ix) in [26, Theorem 4.2] and Item (x) in [26, Proposition 3.5]. Item (xi) follows from [26, Proposition 4.4(a)] and Item (xii) has been proved in [26, Lemma 4.1]. Item (xiii) follows from [26, Proposition 4.6(a)]. For Item (xiv), see [26, Proposition 4.9]. □

Now we list some necessary and sufficent conditions for compatiblity.

Proposition 2.8. *Let A be a d_0-algebra. For every $a, b \in A$, the following are equivalent:*

(a) $a \leftrightarrow b$;

(b) $a \setminus b \leq a - b$;

(c) $a \setminus b \geq a - b$;

(d) $a \setminus b \leq a$;

(e) $b \leftrightarrow a$.

Proof. See [26, Proposition 6.3 and Corollary 6.4]. □

The following two propositions have been proved in [26], too.

Proposition 2.9. *Let a and b be elements of a d_0-algebra A. Suppose that at least one of the following conditions is satisfied:*

(1) *a is comparable with b;*

(2) *there exists $c \in A$ such that $a = c \setminus b$.*

Then $a \leftrightarrow b$.

Proof. See [26, Proposition 6.5]. □

Proposition 2.10. *Let A be a d_0-algebra. For every $a, b, e \in A$, with $a \leftrightarrow e$, $b \leftrightarrow e$ and $a \leq b$, we have $b \setminus a \leftrightarrow e$, too.*

Proof. See [26, Proposition 6.12]. □

Definition 2.11. *A* sub-d_0-algebra*, or simply a* subalgebra*, of a d_0-algebra A is a nonempty $B \subseteq A$ such that $a \setminus b \in B$ whenever $a, b \in B$* (see [26]).

If B is a subalgebra of A, it follows immediately from Proposition 2.7(i) that $0 \in B$.

Proposition 2.12. *Let A be a d_0-algebra and $b \in A$. Then the interval $[0, b]$ is a subalgebra of A and is a D-lattice.*

Proof. See [7, Proposition 5.4(1)]. □

Proposition 2.13. *If A is a D-lattice, then $B = A \smallsetminus \{1\}$ is a d_0-algebra and is a subalgebra of A. Moreover, B is a directed d_0-algebra if and only if A (equivalently B) is linearly ordered.*

Proof. See [9, Proposition 5]. □

Homomorphisms are defined in the usual way: if $(A_1, \setminus_1, 0_1)$ and $(A_2, \setminus_2, 0_2)$ are d_0-algebras, a mapping $f\colon A_1 \to A_2$ is a *homomorphism* if $f(a \setminus_1 b) = f(a) \setminus_2 f(b)$ for every $a, b \in A_1$; in this case, by Proposition 2.7(i), we also have $f(0_1) = 0_2$. An *isomorphism* is a bijective homomorphism (see [26, Definition 3.6]).

Definition 2.14. *An* ideal *of a d_0-algebra A* [32, Definition 2.2] *is a nonempty subset $I \subseteq A$ such that:*

(I1) $\forall a \in I, \quad \forall c \in A, \qquad a \setminus c \in I;$

(I2) $\forall a, b \in A, \qquad [a \in I \ \text{ and } \ b \setminus a \in I \implies b \in I].$

We say that an ideal I is central *if*

(I3) $\forall a \in A, \qquad \exists \max\{\, i \in I : i \leq a \,\}.$

Definition 2.15. *A mapping f of a d_0-algebra A into itself, satisfying the following conditions:*

(CP1) $f \circ f = f,$

(CP2) $\forall a, b \in A \quad f(a \setminus b) = f(a) \setminus b,$

is called a central projection [7, Definition 3.2].

Definition 2.16. *As shown in* [7]*, if I is a central ideal of a d_0-algebra A, then the mapping $f_I\colon x \mapsto \max\{\, i \in I : i \leq x \,\}$, which is well-defined by* (I3) *above, is a central projection. We call f_I the* central projection associated to I.

Actually, in the paper [7], a central ideal is defined as the range of a central projection (see [7, Definition 3.2 and Definition 4.1]). That definition is equivalent to the one given above, in view of [7, Theorem 4.12]. Moreover, the central projection associated with a central ideal I, as introduced in the foregoing definition, is in fact the unique central projection whose range is I: this follows from [7, Corollary 3.15(1) and Theorem 4.6(2)].

Proposition 2.17. *If f is a central projection on a d_0-algebra A, then:*

(1) *f is a homomorphism of A into itself;*

(2) $\forall a, b \in A \quad f(a \wedge b) = f(a) \wedge b.$

Proof.

(1) See [7, Proposition 3.11].

(2) See [10, Corollary 2.5]. □

Definition 2.18. *Let A be a d_0-algebra. A* pre-ideal *of A* [7, Definition 2.1] *is a nonempty subset $I \subseteq A$ satisfying* (I1) *above.*

We say that $B \subseteq A$ is lower*, or that it is a* lower set*, if for every $b \in B$ and every $a \in A$, with $a \leq b$, we have $a \in B$, too* [7, Definition 2.4]*. A lower set which is also a subalgebra will be called* lower subalgebra*.*

In view of [7, Proposition 2.5], every pre-ideal is a lower subalgebra.

Proposition 2.19. *A subset I of a d_0-algebra A is an ideal if and only if it satisfies* (I1) *and the following:*

(I2′) $\forall c, d \in I \qquad [c \perp d \implies c \oplus d \in I].$

Proof. See [7, Proposition 2.9]. □

Notation 2.20. *For a pre-ideal I of a d_0-algebra A, we set*

$$I^{\perp} = \{\, x \in A : \forall i \in I \quad x \wedge i = 0 \,\}.$$

Theorem 2.21. *Let A be a d_0-algebra, and I a pre-ideal of A. Then:*

(1) *$I^\perp$ is an ideal;*

(2) *$I^\perp$ is the largest lower set having intersection $\{0\}$ with I;*

(3) *$I^\perp$ is the largest ideal having intersection $\{0\}$ with I;*

(4) *if I is a central ideal, then $I^\perp$ is a central ideal, too, and every $a \in A$ may be written in a unique way as $f_I(a) \oplus f_{I^\perp}(a)$.*

Proof. See [7, Theorem 2.15, Proposition 3.10 and Theorem 4.6]. □

3 Sasaki mappings

In the paper [21], for a given element e of an orthomodular lattice L, a mapping $\varphi_e \colon L \to L$ is defined as $\varphi_e(f) = (f \vee e^\perp) \wedge e$ for every $f \in L$, where $e^\perp$ denotes the orthocomplement of e. It seems that such mappings were firstly introduced by U. Sasaki in [29], so that Nakamura [23] called them *Sasaki projections*. A full account on this topic can be found in the recent article [22].

Let L be a D-lattice (equivalently, a lattice-ordered effect algebra). In [13] a mapping $\phi \colon L \times L \to L$, called *Sasaki mapping*, is defined in such a way that, if L is an orthomodular lattice, one has $\phi(p, q) = \varphi_p(q)$ for every $p, q \in L$, where φ_p is the Sasaki mapping considered above (see also [20, Definition 1.8.17]).

Our aim is to further generalize this concept to d_0-algebras. In fact what we deal with a binary operation, so that instead of $\phi(p, q)$ one could write something simpler such as $p \cdot q$; nevertheless we prefer to use the former notation, in order to be consistent with the particular case of D-lattices.

In what follows, we will always denote by A a d_0-algebra.

Definition 3.1. *A* Sasaki mapping *on A is a $\phi \colon A \times A \to A$ such that, for every $p, q, r \in A$, the following hold:*

(S0) $\phi(p, q) \leq p$;

(S1) $\phi(p, q) = p \iff \forall t \in A \quad p \wedge (t \setminus q) = 0$;

(S2) $q \leq r \implies \phi(p, q) \leq \phi(p, r)$;

(S3) $r \leq p,\ r \perp q \implies \phi(p \setminus r, q \oplus r) = \phi(p, q)$;

(S4) $\phi(p, q) \leq \phi(q, p) \iff p \leftrightarrow q$.

Proposition 3.2. *Let ϕ be a Sasaki mapping on A. For every $p, q \in A$, we have:*

(1) $\phi(p, q) = \phi(q, p) \iff p \leftrightarrow q$;

(2) $p \leftrightarrow q \implies \phi(p, q) \leq p \wedge q$;

(3) $\phi(p, 0) = \phi(0, q) = 0$;

(4) $p \perp q \implies \phi(p, q) = 0$;

(5) $\phi(p, q - p) = \phi(p - q, q)$.

Proof.

(1) This is a straightforward consequence of (S4), as compatibility is symmetric.

(2) It follows immediately from (S4) and (S0).

(3) Since comparable elements are compatible, the previous item applies.

(4) By (S3) one has $\phi(p, q) = \phi(p \setminus p, q \oplus p) = \phi(0, q \oplus p) = 0$.

(5) Let $r = p \wedge q$. As $r \leq p$ and $r \perp p - q$, we get from (S3):

$$\phi(p, q - p) = \phi\big((p \setminus r, (q - p) \oplus r\big) = \phi\big((p \setminus (p \wedge q)), (q \setminus (p \wedge q)) \oplus (p \wedge q)\big) = \phi(p - q, q). \qquad \square$$

Corollary 3.3. *Let A be a d_0-algebra, and ϕ a Sasaki mapping on A. Then A has the Riesz property if and only if ϕ is symmetric, i.e.,*

$$\forall p, q \in A \quad \phi(p, q) = \phi(q, p).$$

Proof. Immediate from Proposition 3.2(1). $\square$

We are going to prove that, in case A has unit 1 (i.e., A is a D-lattice), there exists a unique Sasaki mapping on A, namely $(p, q) \mapsto p - (1 \setminus q)$, which is just the Sasaki mapping on a D-lattice as defined in [13] (see also [20, Definition 1.8.7]).

Lemma 3.4. *Let $a, b, c \in A$, where $a \leq c$, $b \leq c$ and $a \perp b$. Then $c \setminus (a \oplus b) = (c \setminus a) \setminus b$.*

Proof. Set $s = a \oplus b$. We have $a \leq s$ and $s \setminus a = b$, so that, by (d3),

$$(c \setminus a) \setminus b = (c \setminus a) \setminus (s \setminus a) = (c \setminus s) \setminus (a \setminus s) = (c \setminus s) \setminus 0 = c \setminus s = c \setminus (a \oplus b). \qquad \square$$

Theorem 3.5. *If A is a D-lattice, then $\phi\colon A \times A \to A$ is a Sasaki mapping on A if and only if*

$$\forall p, q \in A \qquad \phi(p, q) = p - (1 \setminus q). \tag{3.1}$$

Proof. We begin assuming that ϕ is defined by (3.1), and proving that it satisfies the conditions in Definition 3.1. Condition (S0) follows immediately from Proposition 2.7(v). To prove (S1), observe that

$$p \setminus \phi(p, q) = p \setminus \big(p - (1 \setminus q)\big) = p \wedge (1 \setminus q).$$

Hence $\phi(p, q) = p$ is equivalent to $p \wedge (1 \setminus q) = 0$; in view of Proposition 2.7(iii), this is in turn equivalent to

$$\forall t \in A \quad p \wedge (t \setminus q) = 0.$$

If $r \geq q$, then $1 \setminus r \leq 1 \setminus q$, so that by Proposition 2.7(ix),

$$\phi(p, q) = p - (1 \setminus q) \leq p - (1 \setminus r) = \phi(p, r),$$

and (S2) is proved. Now suppose that $r \leq p$ is orthogonal to q, so that $r \leq 1 \setminus q$. We have, applying Lemma 3.4 and Proposition 2.7(xii),

$$\begin{aligned} \phi(p \setminus r, q \oplus r) &= (p \setminus r) - \big(1 \setminus (q \oplus r))\big) \\ &= (p \setminus r) - \big((1 \setminus q) \setminus r)\big) \\ &= p - (1 \setminus q) = \phi(p, q), \end{aligned}$$

which proves (S3). Finally, from Proposition 2.10, taking Proposition 2.7(iv) into account, we get that $p \leftrightarrow q$ is equivalent to $p \leftrightarrow 1 \setminus q$, and this, in view of Proposition 2.8, is in turn equivalent to

$$p - (1 \setminus q) \leq p \setminus (1 \setminus q).$$

It follows, using Items (iv) and (vii) of Proposition 2.7, that $p \leftrightarrow q$ if and only if

$$\phi(p, q) \leq p \setminus (1 \setminus q) = 1 \setminus (1 \setminus q) - (1 \setminus p) = q - (1 \setminus p) = \phi(q, p).$$

This gives (S4).

Conversely, assume that ϕ is a Sasaki mapping. Given $p, q \in A$, let $r = p \wedge (1 \setminus q)$, so that $r \leq p$ and $r \perp q$. For every $t \in A$, in view of Proposition 2.7(iii), taking into account Lemma 3.4 and Proposition 2.7(xi), we have

$$\begin{aligned} (p \setminus r) \wedge \big(t \setminus (q \oplus r)\big) &\leq (p \setminus r) \wedge \big(1 \setminus (q \oplus r)\big) \\ &= (p \setminus r) \wedge \big((1 \setminus q) \setminus r\big) = \big(p \wedge (1 \setminus q)\big) \setminus r = r \setminus r = 0, \end{aligned}$$

hence, applying (S3) and (S1),

$$\phi(p, q) = \phi(p \setminus r, q \oplus r) = p \setminus r = p \setminus \big(p \wedge (1 \setminus q)\big).$$

$\square$

As a consequence we get the following (see [20, Theorem 1.8.23]).

Corollary 3.6. *Let A be a D-lattice, with Sasaki mapping ϕ. Then A is an MV-algebra if and only if ϕ is symmetric, i.e.,*

$$\forall p, q \in A \quad \phi(p, q) = \phi(q, p).$$

Proof. The previous theorem and Corollary 3.3 imply that ϕ is symmetric if and only if A is a D-lattice with the Riesz property. Equivalently (as it is well-known—see for instance [20, Theorem 1.8.12]) A is an MV-algebra. □

In view of Theorem 3.5, being a D-lattice is a sufficient condition in order to have a uniquely defined Sasaki mapping. But this condition is not necessary, as we are going to see.

Example 3.7.

(1) *Let V_1 be the three-element d_0-algebra $\{0, a, b\}$, where the only nontrivial differences are $a \setminus b = b$ and $b \setminus a = a$* [32, Example 3.5]. *Clearly A is not a D-lattice. On the other hand, there exists a unique Sasaki mapping ϕ on A, whose only nonzero values are $\phi(a, b) = a$ and $\phi(b, a) = b$.*

(2) *Similary, let V_2 be the three-element d_0-algebra $\{0, a, b\}$, where the only nontrivial differences are $a \setminus b = a$ and $b \setminus a = b$* [32, Example 4.5]. *Of course A is not a D-lattice, too. Again, there exists a unique Sasaki mapping ϕ on A. In this case, the only nonzero values of ϕ are $\phi(a, a) = a$ and $\phi(b, b) = b$.*

Now we investigate the restriction of a Sasaki mapping to a subalgebra.

Lemma 3.8. *Suppose that ϕ is a Sasaki mapping on A. Let B be a lower subalgebra of A, and suppose that*

$$\forall p, q \in B \quad \big[\, [\exists t \in A \quad p \wedge (t \setminus q) \neq 0] \implies [\exists s \in B \quad p \wedge (s \setminus q) \neq 0] \,\big]. \tag{3.2}$$

Then the restriction ϕ_B of ϕ to $B \times B$ is a Sasaki mapping on B.

Proof. Fix $p, q \in B$. We have $\phi_B(p, q) = \phi(p, q) \leq p$, by (S0); since B is lower, we get that ϕ_B maps $B \times B$ into B and satisfies (S0).

Now assume that $\phi_B(p, q) < p$, i.e., $\phi(p, q) < p$. As ϕ satisfies (S1), there exists $t \in A$ such that $p \wedge (t \setminus q) \neq 0$; applying (3.2) gives $s \in B$ with $p \wedge (s \setminus q) \neq 0$. Conversely, if there exists $t \in B$ such that $p \wedge (t \setminus q) \neq 0$, then $t \in A$, and (S1) yields $\phi(p, q) < p$, i.e., $\phi_B(p, q) < p$. This proves that ϕ_B satisfies (S1).

If $r \in B$ and $r \geq q$, then, by (S2),

$$\phi_B(p, q) = \phi(p, q) \leq \phi(p, r) = \phi_B(p, r).$$

Which proves (S2) for ϕ_B.

Since B is a subalgebra of A, compatibility of p and q in B is the same as compatibility in A, which, by (S4), is equivalent to $\phi(p, q) \leq \phi(q, p)$, i.e., $\phi_B(p, q) \leq \phi_B(q, p)$. Hence (S4) also holds for ϕ_B.

Finally consider $r \in B$ with $r \leq p$, and suppose that r is orthogonal in B to q: this means that there exists $s \in B$ such that $s \geq q$ and $s \setminus q = r$, hence $r \perp q$ in A, too. Applying (S3), we obtain

$$\phi_B(p \setminus r, q \oplus r) = \phi(p \setminus r, q \oplus r) = \phi(p, q) = \phi_B(p, q),$$

and we have verified (S3) for ϕ_B. This completes the proof. □

Lemma 3.9. *If A is a D-lattice, then $A \setminus \{1\}$ is a lower subalgebra of A and is directed if and only if A is linearly ordered.*

Proof. It is clear that $A \setminus \{1\}$ is lower. The remaining facts have been established in Proposition 2.13. □

Theorem 3.10. *Let A be a D-lattice, with Sasaki mapping ϕ. If $B \subseteq A$, then the restriction ϕ_B of ϕ to $B \times B$ is a Sasaki mapping on B, provided that one of the following two conditions is satisfied:*

(1) *$B = A \setminus \{1\}$ and A is not linearly ordered;*

(2) *B is an ideal of A.*

Proof. Since in both cases B is a lower subalgebra of A, it suffices to prove condition (3.2) of Lemma 3.8 above. To this end, fix $p, q \in B$ such that there exists $t \in A$ with $p \wedge (t \setminus q) \neq 0$. Denoting $1 \setminus q$ by $q^\perp$, as usual, Proposition 2.7(iii) gives

$$p \wedge q^\perp = p \wedge (1 \setminus q) \geq p \wedge (t \setminus q) > 0. \tag{3.3}$$

Now, we prove that there exists $s \in B$ such that $p \wedge (s \setminus q) > 0$.

In case (1), suppose first that $p \geq q^\perp$. Observe that there is some $s_* \in B$ such that $s_* \not\leq q$, i.e., $s_* \setminus q \neq 0$ (otherwise q should be the greatest element of B, and hence Lemma 3.9 would imply that A is linearly ordered). By Proposition 2.7(iii) we have $s_* \setminus q \leq 1 \setminus q = q^\perp \leq p$, hence

$$p \wedge (s_* \setminus q) = s_* \setminus q > 0.$$

Next, suppose that $p \not\geq q^\perp$, i.e., $p \wedge q^\perp < q^\perp$. Let

$$s_0 = q \oplus (p \wedge q^\perp). \tag{3.4}$$

Clearly $s_0 \setminus q = p \wedge q^\perp$; thus, from (3.3), it follows that

$$p \wedge (s_0 \setminus q) = p \wedge (p \wedge q^\perp) = p \wedge q^\perp > 0. \tag{3.5}$$

As $p \wedge q^\perp < q^\perp$, we have $s_0 < q \oplus q^\perp = 1$, that is $s_0 \in B$.

Finally, in case (2), we have $p \wedge q^\perp \in B$, because B is lower. Therefore, defining s_0 as in (3.4), we obtain (3.5) as above, and $s_0 \in B$ by (I2′). □

One cannot drop the condition that A is not linearly ordered in Theorem 3.10(1) above, as the following example shows.

Example 3.11. *Let $A = \{0, h, 1\}$ be a three-element d_0-algebra where $1 \setminus h = h$ (see Figure 1 below), and set $B = A \setminus \{1\}$, so that A and B are linearly ordered D-lattices. Theorem 3.5 says that the only Sasaki mappings on A and B are respectively $\phi\colon (p, q) \mapsto p - (1 \setminus q)$ and $\phi_*\colon (p, q) \mapsto p - (h \setminus q)$. But ϕ_* is not a restriction of ϕ, because $\phi_*(h, h) = h$, while $\phi(h, h) = 0$.*

$\setminus$	0	h	1
0	0	0	0
h	h	0	0
1	1	h	0

Figure 1: The Cayley table of the d_0-algebra A of Example 3.11.

Now we give another application of Lemma 3.8.

Proposition 3.12. *Let I be a central ideal of A. If ϕ is a Sasaki mapping on A then the restriction ϕ_I of ϕ to $I \times I$ is a Sasaki mapping on I.*

Proof. First note that I, being a central ideal, is a lower subalgebra, so we can use Lemma 3.8. Fix $p, q \in I$ such that there exist $t \in A$ with $p \wedge (t \setminus q) \neq 0$. Set $s = f(t)$, where f is the central projection associated to I, in particular $s \in I$; since I is lower, $f(x) = x$ for $x \in I$, and f is a homomorphism, we have

$$p \wedge (s \setminus q) = f(p) \wedge \big(f(t) \setminus f(q)\big) = f\big(p \wedge (t \setminus q)\big) = p \wedge (t \setminus q) \neq 0.$$ □

The remaining part of this section is concerned with the relationships between Sasaki mappings and subdirect products.

Notation 3.13. *Given a family $(A_\lambda)_{\lambda \in \Lambda}$ of d_0-algebras, let $A^{(\kappa)}$ be the subset of the direct product $A = \prod_{\lambda \in \Lambda} A_\lambda$, consisting of those $a = (a_\lambda)_{\lambda \in \Lambda}$ such that $a_\lambda = 0$ for $\lambda \neq \kappa$.*

Definition 3.14. *A* d_0-subdirect product *of a family $(A_\lambda)_{\lambda \in \Lambda}$ of d_0-algebras is a subalgebra S of the direct product $\prod_{\lambda \in \Lambda} A_\lambda$ such that $A^{(\kappa)} \subseteq S$, for each $\kappa \in \Lambda$.*

As shown in [9, Proposition 7], a d_0-subdirect product is a subdirect product. Moreover, among all possible d_0-subdirect products, there are always $\bigcup_{\lambda\in\Lambda} A^{(\lambda)}$ (which is the smallest one) and the whole direct product (which is the biggest one).

Lemma 3.15. *Let S be a d_0-subdirect product of a family $(A_\lambda)_{\lambda\in\Lambda}$ of d_0-algebras. For every $\kappa\in\Lambda$, the subset $A^{(\kappa)}$ (see Notation 3.13 above) is a central ideal of S, isomorphic to A_κ.*

Proof. See [9, Theorem 5]. □

Theorem 3.16. *Let S be a lower d_0-subdirect product of a family $(A_\lambda)_{\lambda\in\Lambda}$ of d_0-algebras. Given, for each $\lambda\in\Lambda$, a mapping $\phi_\lambda\colon A_\lambda\times A_\lambda\to A_\lambda$, denote by ϕ the mapping of $S\times S$ into S, defined as*

$$\big((p_\lambda)_{\lambda\in\Lambda},(q_\lambda)_{\lambda\in\Lambda}\big)\mapsto\big(\phi_\lambda(p_\lambda,q_\lambda)\big)_{\lambda\in\Lambda}. \tag{3.6}$$

Then ϕ is a Sasaki mapping on S if and only if ϕ_λ is a Sasaki mapping on A_λ for every $\lambda\in\Lambda$.

Proof. Suppose that ϕ is a Sasaki mapping on S, and fix $\kappa\in\Lambda$: we prove that ϕ_κ is a Sasaki mapping.

Let $A^{(\kappa)}$ be as in Notation 3.13, and consider the restriction $\phi^{(\kappa)}$ of ϕ to $A^{(\kappa)}\times A^{(\kappa)}$. Denoting by π_κ the projection of A onto A_κ, we have

$$\forall p,q\in A^{(\kappa)}\quad \pi_\kappa\big(\phi^{(\kappa)}(p,q)\big)=\phi_\kappa\big(\pi_\kappa(p),\pi_\kappa(q)\big). \tag{3.7}$$

As seen in [26, Proposition 3.15], the restriction of π_κ to $A^{(\kappa)}$ is an isomorphism, and therefore it remains to prove that $\phi^{(\kappa)}$ is a Sasaki mapping on $A^{(\kappa)}$. But this follows from Proposition 3.12, because $A^{(\kappa)}$ is a central ideal of S by Lemma 3.15.

Conversely suppose that ϕ_λ is a Sasaki mapping on A_λ for every $\lambda\in\Lambda$. We first consider the case in which S is the whole direct product.

Fix $p=(p_\lambda)_{\lambda\in\Lambda}$ and $q=(q_\lambda)_{\lambda\in\Lambda}$ in S. For each $\lambda\in\Lambda$ we have $\phi_\lambda(p_\lambda,q_\lambda)\le p_\lambda$, hence $\phi(p,q)\le p$.

Now, we have $\phi(p,q)=p$ if and only if $\phi_\lambda(p_\lambda,q_\lambda)=p_\lambda$ for each $\lambda\in\Lambda$, and this, as ϕ_λ is a Sasaki mapping, is in turn equivalent to

$$\forall\lambda\in\Lambda\quad\forall t_\lambda\in A_\lambda\quad p_\lambda\wedge(t_\lambda\setminus q_\lambda)=0,$$

i.e.,

$$\forall t\in A\quad p\wedge(t\setminus q)=0.$$

Therefore ϕ satisfies (S1).

Let $r=(r_\lambda)_{\lambda\in\Lambda}$, and suppose that $r\ge q$. For each $\lambda\in\Lambda$ we have $r_\lambda\ge q_\lambda$, hence $\phi_\lambda(p_\lambda,q_\lambda)\le\phi_\lambda(p_\lambda,r_\lambda)$: thus $\phi(p,q)\le\phi(p,r)$.

Now suppose $r\le p$ and $r\perp q$. For each $\lambda\in\Lambda$ we have $r_\lambda\le p_\lambda$ and $r_\lambda\perp q_\lambda$, hence $\phi_\lambda(p_\lambda\setminus r_\lambda,q_\lambda\oplus r_\lambda)=\phi_\lambda(p_\lambda,q_\lambda)$: thus $\phi(p\setminus r,q\oplus r)=\phi(p,q)$.

Finally we have $p\leftrightarrow q$ if and only if $p_\lambda\leftrightarrow q_\lambda$ for each $\lambda\in\Lambda$ and, since ϕ_λ is a Sasaki mapping, this is equivalent to

$$\forall\lambda\in\Lambda\quad\phi_\lambda(p_\lambda,q_\lambda)\le\phi_\lambda(q_\lambda,p_\lambda),$$

i.e., $\phi(p,q)\le\phi(q,p)$.

In the general case, denote by A the direct product, by ϕ_A the mapping defined on A according to (3.6), and by ϕ its restriction to $S\times S$. We have just seen that ϕ_A is a Sasaki mapping on A. Since S is lower by assumption, it remains to prove that the condition of Lemma 3.8 is satisfied.

To this end, fix $p=(p_\lambda)_{\lambda\in\Lambda}$ and $q=(q_\lambda)_{\lambda\in\Lambda}$ in S. Suppose that there exists $t=(t_\lambda)_{\lambda\in\Lambda}\in A$ such that $p\wedge(t\setminus q)=\big(p_\lambda\wedge(t_\lambda\setminus q_\lambda)\big)_{\lambda\in\Lambda}\ne 0$, and hence $p_\kappa\wedge(t_\kappa\setminus q_\kappa)\ne 0$, for some $\kappa\in\Lambda$. Let $s=(s_\lambda)_{\lambda\in\Lambda}$, where for each $\lambda\in\Lambda$

$$s_\lambda=\begin{cases}0, & \text{if } \lambda\ne\kappa;\\ t_\kappa, & \text{if } \lambda=\kappa.\end{cases}$$

We have $p\wedge(s\setminus q)\ne 0$, because $p_\kappa\wedge(s_\kappa\setminus q_\kappa)=p_\kappa\wedge(t_\kappa\setminus q_\kappa)\ne 0$. Moreover, it is clear that $s\in A^{(\kappa)}$ and therefore $s\in S$, which completes the proof. □

Definition 3.17. *A* decomposition *of A (as introduced in [11, Definition 6.8]) is a collection $\mathcal{I}$ of central ideals of A such that:*

(D1) *for each $I\in\mathcal{I}$ we have $I\ne\{0\}$;*

(D2) *for every $I', I'' \in \mathcal{I}$, if $I' \neq I''$, then $I' \cap I'' = \{0\}$;*

(D3) *$\mathcal{I}$ is maximal with respect to inclusion.*

In [11], the following result has been established.

Lemma 3.18. *Let $\mathcal{I} = \{ I_\lambda : \lambda \in \Lambda \}$, where $I_{\lambda'} \neq I_{\lambda''}$ for $\lambda' \neq \lambda''$, be a decomposition of A. If either $\mathcal{I}$ is finite or A is semicomplete, then there is an isomorphism η of A onto the lower d_0-subdirect product S of $(I_\lambda)_{\lambda\in\Lambda}$ consisting of those $a \in \prod_{\lambda\in\Lambda} I_\lambda$ whose coordinates are bounded above by some $m_a \in A$. The isomorphism η is given by $\eta\colon a \mapsto \big(f_\lambda(a)\big)_{\lambda\in\Lambda}$, where f_λ is the central projection associated to I_λ, for each $\lambda \in \Lambda$.*

Proof. See [11, Theorem 6.10(2)]. □

By Proposition 3.12, if ϕ is a Sasaki mapping on A, then, for every decompostion $\mathcal{I}$ of A and every $I \in \mathcal{I}$, the restriction ϕ_I of ϕ to $I \times I$ is a Sasaki mapping on I.

The following result, exploiting Theorem 3.16, provides a sort of converse.

Proposition 3.19. *Let $\mathcal{I} = \{ I_\lambda : \lambda \in \Lambda \}$, where $I_{\lambda'} \neq I_{\lambda''}$ for $\lambda' \neq \lambda''$, be a decomposition of A. Suppose given, for each $\lambda \in \Lambda$, a Sasaki mapping ϕ_λ on I_λ. If either $\mathcal{I}$ is finite or A is semicomplete, then there exists a Sasaki mapping ϕ on A whose restriction on $I_\lambda \times I_\lambda$ is ϕ_λ, for every $\lambda \in \Lambda$.*

Proof. By Lemma 3.18, there is a lower d_0-subdirect product S of $(I_\lambda)_{\lambda\in\Lambda}$ and an isomorphism η of A onto S, defined by

$$x \mapsto \big(f_\lambda(x)\big)_{\lambda\in\Lambda}, \tag{3.8}$$

where f_λ is the central projection associated to I_λ, for each $\lambda \in \lambda$.

Define $\bar\phi\colon S \times S \to S$ as in (3.6), i.e.

$$\bar\phi\colon \big((p_\lambda)_{\lambda\in\Lambda}, (q_\lambda)_{\lambda\in\Lambda}\big) \mapsto \big(\phi_\lambda(p_\lambda, q_\lambda)\big)_{\lambda\in\Lambda}.$$

By Theorem 3.16, $\bar\phi$ is a Sasaki mapping on S. Hence

$$\phi\colon (p, q) \mapsto \eta^{-1}\big(\bar\phi\big(\eta(p), \eta(q)\big)\big),$$

is a Sasaki mapping on A, because η^{-1} is an isomorphism, too.

It remains to prove that, for every $\lambda \in \Lambda$, the restriction of ϕ to $I_\lambda \times I_\lambda$ coincides with ϕ_λ. To this end, fix a $\kappa \in \Lambda$ and a pair $(a, b) \in I_\kappa \times I_\kappa$. We claim that $\phi(a, b) = \phi_\kappa(a, b)$.

Let $p = \eta(a)$ and $q = \eta(b)$. Write $p = (p_\lambda)_{\lambda\in\Lambda}$ and $q = (q_\lambda)_{\lambda\in\Lambda}$; in view of (3.8), for each $\lambda \in \Lambda$ we have

$$p_\lambda = \begin{cases} f_\lambda(a) = 0, & \text{if } \lambda \neq \kappa, \\ f_\kappa(a) = a, & \text{if } \lambda = \kappa; \end{cases} \qquad q_\lambda = \begin{cases} f_\lambda(b) = 0, & \text{if } \lambda \neq \kappa, \\ f_\kappa(b) = b, & \text{if } \lambda = \kappa. \end{cases}$$

Therefore $\bar\phi\big(\eta(a), \eta(b)\big) = \bar\phi(p, q) = r = (r_\lambda)_{\lambda\in\Lambda}$, where, for each $\lambda \in \Lambda$,

$$r_\lambda = \begin{cases} \phi_\lambda(p_\lambda, q_\lambda) = \phi_\lambda(0, 0) = 0 = f_\lambda\big(\phi_\kappa(a, b)\big), & \text{if } \lambda \neq \kappa, \\ \phi_\kappa(p_\kappa, q_\kappa) = \phi_\kappa(a, b) = f_\lambda\big(\phi_\kappa(a, b)\big), & \text{if } \lambda = \kappa. \end{cases}$$

We conlcude that $\phi(a, b) = \eta^{-1}(r) = \phi_\kappa(a, b)$, as claimed. □

4 Sharp elements

Definition 4.1. *We say that an element $a \in A$ is* sharp *if, for every $b \in A$, we have $a \setminus (b \setminus a) = a$. We say that A is a* sharp d_0-algebra *if each element of A is sharp.*

Proposition 4.2. *For an element $a \in A$, the following are equivalent:*

(a) *a is sharp;*

(b) *for every $b \in A$, we have $a \leq a \setminus (b \setminus a)$;*

(c) *for every* $b \in A$, *we have* $a \wedge (b \setminus a) = 0$;

(d) *for every* $b \in A$, *we have* $(b \setminus a) \setminus a = b \setminus a$;

(e) *for every* $b \in A$, *we have* $b \setminus a \leq (b \setminus a) \setminus a$.

Proof. $(b) \Rightarrow (a)$ By (d1).
$(a) \Rightarrow (c)$ Suppose a is sharp, and fix $b \in A$. From Proposition 2.9(2) we get $b \setminus a \leftrightarrow a$, hence

$$a \wedge (b \setminus a) = a \setminus \big(a - (b \setminus a)\big) = a \setminus \big(a \setminus (b \setminus a)\big) = a \setminus a = 0.$$

$(c) \Rightarrow (d)$ Suppose that for every $b \in A$ we have $a \wedge (b \setminus a) = 0$. As $a \leftrightarrow b \setminus a$, we have, using Proposition 2.7(viii),

$$(b \setminus a) \setminus a = (b \setminus a) - a = (b \setminus a) \setminus \big(a \wedge (b \setminus a)\big) = (b \setminus a) \setminus 0 = b \setminus a.$$

$(d) \Rightarrow (e)$ This is obvious.
$(e) \Rightarrow (b)$ Fix any $b \in A$. Applying the first equality in (d4) and Proposition 2.7(vi), we obtain

$$a \setminus \big(a \setminus (b \setminus a)\big) = (b \setminus a) - \big((b \setminus a) \setminus a\big) = 0,$$

that is $a \leq a \setminus (b \setminus a)$. □

Example 4.3.

(1) *Let* S_0 *be any set, of cardinality at least* 3, *with a distinguished element* 0; *define an operation* $\setminus$ *on* S_0 *as follows:*

$$\forall x \in S_0 \quad \big[\, 0 \setminus x = x \setminus x = 0,\ x \setminus 0 = x \,\big]; \qquad \forall x, y \in S_0 \smallsetminus \{0\} \quad \big[\, x \neq y \implies x \setminus y = y \,\big].$$

Then $(S_0, \setminus, 0)$ *is a* d_0*-algebra in which no element, except* 0, *is sharp. Indeed, if* x, y *are distinct elements in* $S_0 \smallsetminus \{0\}$, *we have*

$$x \setminus (y \setminus x) = x \setminus x = 0 \neq x.$$

When $S_0 = \{0, a, b\}$, *we get the* d_0*-algebra* V_1 *of Example 3.7*(1).

(2) *Let* S_1 *be the* d_0*-algebra on the four-element set* $\{0, a, b, c\}$ *whose operation is described in Figure 2(1) below. It is easy to see that* a *and* c *are sharp elements, while* b *is not sharp.*

(3) *The* d_0*-algebra* S_2 *on the five-element set* $\{0, a, b, c, d\}$, *whose operation is described in Figure 2(2) below, is a sharp* d_0*-algebra.*

(1)

$\setminus$	0	a	b	c
0	0	0	0	0
a	a	0	b	a
b	b	c	0	a
c	c	c	b	0

(2)

$\setminus$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	d	a	b
b	b	c	0	a	b
c	c	c	d	0	b
d	d	c	d	a	0

Figure 2: The Cayley tables of the d_0-algebras S_1 and S_2 of Example 4.3.

Lemma 4.4. *Given* $a \in A$, *the following are equivalent:*

(a) *for every* $b \perp a$, *we have* $a \wedge b = 0$;

(b) *for every* $c \leq a$, *if* $c \perp a$, *then* $c = 0$.

Proof. Assume (a), and consider $c \leq a$ with $c \perp a$. Then $c = c \wedge a = 0$.

Conversely, suppose that (b) hold. If $b \perp a$ then, by Proposition 2.7(xiii), we have $a \oplus b = (a \vee b) \oplus (a \wedge b)$ so that the element $c = a \wedge b$ is orthogonal to $a \vee b$ and hence to a. Since $c \leq a$, we have $c = 0$, i.e., $a \wedge b = 0$. □

Definition 4.5. *We say that an element $a \in A$ is* weakly sharp *if it satisfies the two equivalent conditions of Lemma 4.4 above.*

Proposition 4.6. *Every sharp element is weakly sharp.*

Proof. Suppose that a is sharp. Fix an element b which is orthogonal to a, and denote by s the sum $a \oplus b$. Applying Proposition 4.2 we get

$$a \wedge b = a \wedge (s \setminus a) = 0,$$

which gives condition (a) of Lemma 4.4, so that a is weakly sharp. □

The converse of the foregoing proposition is not true, in general.

Example 4.7. *Let $V_1 = \{0, a, b\}$ be the d_0-algebra of Example 3.7(1). For every $x, y \in V_1$, if $x \perp y$ then at least one of them is 0. Thus every element of V_1 is weakly sharp. On the other hand, we have seen in Example 4.3(1) above that no element of V_1, except 0, is sharp.*

The notions of sharp and weakly sharp elements coincide in case of a generalized D-lattice.

Proposition 4.8.

(1) *Suppose A is a generalized D-lattice. Then an element a is sharp if and only if it is weakly sharp.*

(2) *Suppose A is a D-lattice. Then an element a is sharp if and only if $a \wedge (1 \setminus a) = 0$.*

Proof.

(1) Let a be weakly sharp. Theorem 2.3 implies that $b \setminus a$ is orthogonal to a, so that $a \wedge (b \setminus a) = 0$. Hence a is sharp by virtue of Proposition 4.2.

(2) Necessity is immediate (applying Proposition 4.2). To prove sufficiency, assume that $a \wedge (1 \setminus a) = 0$. By Proposition 2.7(iii), for every $b \in A$ we have

$$a \wedge (b \setminus a) \leq a \wedge (1 \setminus a) = 0,$$

and Proposition 4.2 yields that a is sharp. □

In view of Proposition 4.8(2), our definition of sharp elements agrees with the usual one for D-lattices.

Corollary 4.9. *A sharp d_0-algebra with unit is an orthomodular lattice.*

Proof. Indeed every sharp d_0-algebra with unit is a D-lattice where each element is sharp, which, in turn, is an orthomodular lattice, as it is well-known (see, for instance [20, Proposition 1.5.3 and Proposition 1.5.8]). □

Now we investigate the relationships between sharp elements and Sasaki mappings.

Proposition 4.10. *Let ϕ be a Sasaki mapping. An element p of A is sharp if and only if $\phi(p, p) = p$.*

Proof. Indeed, by (S1), one has $\phi(p, p) = p$ if and only if the equality $p \wedge (t \setminus p) = 0$ holds for every $t \in A$. In view of Proposition 4.2, this is equivalent to saying that p is sharp. □

Lemma 4.11. *If ϕ is a Sasaki mapping and c is sharp then, for every $p \leq c$ and every $q \geq c$, we have $\phi(p, q) = p$.*

Proof. Fix $p \leq c$ and $q \geq c$. By Proposition 4.2, for every $t \in A$ we have $c \wedge (t \setminus c) = 0$, hence $p \wedge (t \setminus c) = 0$, too. Now (S2) and (S1) yield

$$p = \phi(p, c) \leq \phi(p, q) \leq p.$$ □

Corollary 4.12. *Let ϕ be a Sasaki mapping, and let c be a sharp element. For every $b \in A$ such that $b \vee c$ exists, one has $\phi(b \wedge c, b \vee c) = b \wedge c$.*

Proof. Apply the previous lemma with $p = b \wedge c$ and $q = b \vee c$. □

Lemma 4.13. *Let ϕ be a Sasaki mapping, and let c be a sharp element. For every $b \in A$ such that $b \vee c$ exists and $b \leftrightarrow c$, one has $\phi(b, c) = \phi(c, b) = b \wedge c$.*

Proof. Since $b \leftrightarrow c$, from Proposition 2.8 we get $c \setminus b \leq c$, and $c \setminus (c \setminus b) = c \setminus (c - b) = b \wedge c$. Moreover, by Proposition 2.7(xiv), we have $c \setminus b = (b \vee c) \setminus b$. Hence b is orthogonal to $c \setminus b$, and $b \oplus (c \setminus b) = b \vee c$. Therefore applying Proposition 3.2(1), as well as (S3) and the previous corollary, we obtain

$$\phi(b,c) = \phi(c,b) = \phi\big(c \setminus (c \setminus b), b \oplus (c \setminus b)\big) = \phi(b \wedge c, b \vee c) = b \wedge c. \qquad \square$$

Theorem 4.14. *Let ϕ be a Sasaki mapping, and let c be an element of A. The following are equivalent:*

(a) *c is sharp;*

(b) *$\phi(c,a) \geq a \wedge c$, for every $a \in A$;*

(c) *$\phi(c,a) = \phi(a,c) = a \wedge c$, whenever $a \leftrightarrow c$;*

(d) *$\phi(c,a) = \phi(a,c) = a \wedge c$, whenever a is comparable with c.*

Proof. Implication (b)⇒(c) is an immediate consequence of Proposition 3.2. Implication (c)⇒(d) is also straightforward, because comparable elements are compatible. Implication (d)⇒(a) follows at once by putting $a = c$ and applying Proposition 4.10. It remains to prove that (a)⇒(b).

Let $a \in A$, and put $b = a \wedge c$. Proposition 2.9 says that $b \leftrightarrow c$, and clearly $b \vee c$ exists, hence we may apply Lemma 4.13, obtaining, with the help of (S2),

$$\phi(c,a) \geq \phi(c,b) = b \wedge c = a \wedge c. \qquad \square$$

Definition 4.15. *Given a Sasaki mapping ϕ, we say that A is* ϕ-ranged *if ϕ maps $A \times A$ onto A.*

Proposition 4.16. *If ϕ is a Sasaki mapping on a D-lattice L, then L is ϕ-ranged.*

Proof. Indeed, from Theorem 3.5 we infer that $\phi(p,1) = p$ for each $p \in L$. $\square$

Example 4.17. *Let A be the set of nonnegative integers, with the difference defined as follows:*

$$\forall x, y \in A \qquad x \setminus y = \max\{x - y, 0\},$$

where $x - y$ denotes the usual difference of the numbers x and y. Then A is a d_0-algebra and, in fact, a cancellative BCK-algebra [20, Example 5.2.4], whose order coincides with the usual one. (Hence the elements are pairwise compatible.) Observe further that, for every $x, y \in A$, we have $x \perp y$, and $x \oplus y$ coincides with the usual sum $x + y$ of the numbers x and y. With this in mind, one easily sees that $\phi\colon (p,q) \mapsto 0$ is a Sasaki mapping on A, which shows that A is not ϕ-ranged.

On the other hand, the following holds.

Proposition 4.18. *If ϕ is a Sasaki mapping on a sharp d_0-algebra A, then A is ϕ-ranged.*

Proof. It follows immediately from Proposition 4.10. $\square$

Definition 4.19. *Given a Sasaki mapping ϕ on A, we say that A is* ϕ-idempotent *if, for each $p, q \in A$, one has $\phi\big(p, \phi(p,q)\big) = \phi(p,q)$.*

Theorem 4.20. *Let ϕ be a Sasaki mapping on A. The following are equivalent:*

(a) *A is sharp;*

(b) *A is ϕ-ranged and ϕ-idempotent.*

Proof. $(a) \Rightarrow (b)$ Suppose that A is sharp. We have seen in Proposition 4.18 that A is ϕ-ranged. To prove that A is ϕ-idempotent, fix $p, q \in A$ and let $r = \phi(p,q)$. Since r is comparable with p by (S0), Theorem 4.14 yields

$$\phi(p,r) = p \wedge r = r.$$

$(b) \Rightarrow (a)$ Assume that A is ϕ-ranged and ϕ-idempotent. We take any $r \in A$ and show that it is sharp. As A is ϕ-ranged we may find $p, q \in A$ such that $r = \phi(p,q)$. Hence $p \geq r$ by (S0), and Proposition 2.9 gives $p \leftrightarrow r$. Now $\phi(p,r) = r$ because A is ϕ-idempotent, and $\phi(r,p) = r$ by Proposition 3.2(1). Applying ϕ-idempotency again we obtain

$$\phi(r,r) = \phi\big(r, \phi(r,p)\big) = \phi(r,p) = r,$$

and therefore r is sharp by Proposition 4.10. $\square$

As a particular case, we obtain a well-known result of Bennett and Foulis [13, Theorem 3.8].

Corollary 4.21. *A D-lattice is an orthomodular lattice if and only if it is ϕ-idempotent, where ϕ is given by* (3.1) *above.*

Proof. Let A be a D-lattice. Then the ϕ given by (3.1) is a Sasaki mapping by Theorem 3.5, and A is ϕ-ranged by Proposition 4.16. Moreover, Corollary 4.9 says that A is an orthomodular lattice if and only if it is sharp. By the previous theorem this is in turn equivalent to A being ϕ-idempotent. □

Finally we consider central elements.

Definition 4.22. *A $c \in A$ is a* central element *if $[0, c]$ is a central ideal.*

Example 4.23. *Let S be the d_0-algebra on the six-element set $\{0, a, b, c, d, e\}$ whose operation is described in Figure 3 below. One readily recognizes that c is a central element, while the other elements are not central (except 0, of course).*

$\setminus$	0	a	b	c	d	e
0	0	0	0	0	0	0
a	a	0	b	a	0	b
b	b	a	0	b	a	0
c	c	c	c	0	0	0
d	d	c	e	a	0	b
e	e	d	c	b	a	0

Figure 3: The Cayley table of the d_0-algebra S of Example 4.23.

As already observed in [9], in case A is a D-lattice the above definition of central element coincides with the usual one. Actually, the notion of central element for a generalized D-lattice agrees with Definition 4.22, too. For this fact, we refer to [25].

Proposition 4.24. *Let $c \in A$. Then:*

(1) *c is central if and only if $[0, c]$ is an ideal;*

(2) *in case the equivalent conditions above hold, the central projection associated to I is $x \mapsto x \wedge c$.*

Proof. Suppose that $I = [0, c]$ is an ideal. For every $a \in A$, it is clear that $a \wedge c$ is the greatest element $i \in I$ such that $i \leq a$. Hence I is a central ideal, whose associated central projection is $x \mapsto x \wedge c$. □

We are going to characterize central elements as particular sharp elements, thus generalizing a result by Z. Riečanová [24, Theorem 2.5].

Lemma 4.25. *If $c \in A$ is sharp, then, given orthogonal elements $a, b \in A$ with $a \leq c$ and $b \leq c$, we have $a \oplus b \leq c$*

Proof. Let $s = a \oplus b$. Observe that $a \leftrightarrow c$ (because $a \leq c$), and $a \leftrightarrow b$ (because $b = s \setminus a$). Since $b \leq c$, by Proposition 2.10 we have $c \setminus b \leftrightarrow a$, too. Hence $a \setminus (c \setminus b) \leq a$. On the other hand, using (d3) and the assumption $b \setminus c = 0$, we get

$$a \setminus (c \setminus b) = (s \setminus b) \setminus (c \setminus b) = (s \setminus c) \setminus (b \setminus c) = (s \setminus c) \setminus 0 = s \setminus c,$$

So that $s \setminus c \leq a \leq c$. It follows, applying Proposition 4.2, that $s \setminus c = c \wedge (s \setminus c) = 0$, which means that $s \leq c$. □

Lemma 4.26. *Let $c \in A$. Then $[0, c]$ is a pre-ideal if and only if $a \leftrightarrow c$ for every $a \in A$.*

Proof. Suppose that $a \leftrightarrow c$ for every $a \in A$. If $b \in [0, c]$, i.e., $b \leq c$, then, for every $a \in A$, by Proposition 2.7(iii) and Proposition 2.8, we have

$$b \setminus a \leq c \setminus a \leq c.$$

Hence $b \setminus a \in [0, c]$.

Conversely, assume $[0, c]$ is a pre-ideal. For every $a \in A$, we must have $c \setminus a \in [0, c]$, i.e., $c \setminus a \leq c$, so that Proposition 2.8 gives $a \leftrightarrow c$. □

Theorem 4.27. *For an element $c \in A$, the following are equivalent:*

(1) *c is central;*

(2) $\forall a \in A, \quad a \leftrightarrow c$ *and* $c \wedge (a - c) = 0$*;*

(3) $\forall a \in A, \quad a \leftrightarrow c$ *and* $c \wedge (a \setminus c) = 0$*;*

(4) $\forall a \in A, \quad a \leftrightarrow c$ *and* $a = (a \wedge c) \vee (a - c)$*;*

(5) $\forall a \in A, \quad a = (a \wedge c) \vee (a \setminus c)$*;*

(6) $\forall a \in A, \quad a \leftrightarrow c$ *and c is sharp.*

Moreover, if ϕ is a Sasaki mapping on A, then the above conditions are also equivalent to:

(7) $\forall a \in A, \quad \phi(a, c) = a \wedge c$*;*

(8) $\forall a \in A, \quad a - c = a \setminus \phi(a, c)$.

Proof. Equivalence (2)⇔(3) and implication (4)⇒(5) are immediate consequences of the definition of compatibility, while (3)⇔(6) follows from Proposition 4.2.
(1) ⇒ (4) By assumption $I = [0, c]$ is a central ideal.

Given any $a \in A$, from Lemma 4.26 we get that $a \leftrightarrow c$. Applying Theorem 2.21(4), and taking Proposition 4.24(2) into account, we may write

$$a = f_I(a) \oplus f_{I^\perp}(a) = (a \wedge c) \oplus f_{I^\perp}(a),$$

whence, using Proposition 2.7(viii), we get

$$f_{I^\perp}(a) = a \setminus (a \wedge c) = a - c,$$

so that $a - c \in I^\perp$, which yields $c \wedge (a - c) = 0$. Therefore with the use of Proposition 2.7(xiii), we have

$$\begin{aligned}(a \wedge c) \vee (a - c) &= \big((a \wedge c) \oplus (a - c)\big) \setminus \big((a \wedge c) \wedge (a - c)\big) \\ &= \big((a \wedge c) \oplus (a - c)\big) \setminus 0 = (a \wedge c) \oplus (a - c) = a.\end{aligned}$$

(4) ⇒ (2) Using items (iv) and (viii) of Proposition 2.7, and applying [26, Proposition 4.6(a)], we obtain

$$\begin{aligned}c \wedge (a - c) = a \wedge c \wedge (a - c) &= a \setminus \Big(a \setminus \big((a \wedge c) \wedge (a - c)\big)\Big) \\ &= a \setminus \Big(\big(a \setminus (a \wedge c)\big) \vee \big(a \setminus (a - c)\big)\Big) = a \setminus \big((a - c) \vee (a \wedge c)\big) = a \setminus a = 0.\end{aligned}$$

(5) ⇒ (4) Let $a \in A$. It clearly suffices to show that $a \leftrightarrow c$. But this follows at once from Proposition 2.8, as $a \geq a \setminus c$ by assumption.
(6) ⇒ (1) In view of Lemma 4.26, the set $[0, c]$ is a pre-ideal, and Lemma 4.25 says that it also satisfies (I2′). Applying Proposition 2.19 we obtain that $[0, c]$ is an ideal, and therefore c is central by Proposition 4.24(1).
Finally, suppose we are given a Sasaki mapping ϕ on A.
(7) ⇒ (8) Just apply Proposition 2.7(viii).
(8) ⇒ (7) From (S0) and Proposition 2.7(iv), we get $\phi(a, c) = a \setminus \big(a \setminus \phi(a, c)\big) = a \setminus (a - c) = a \wedge c$.
(6) ⇔ (7) By Proposition 3.2 and Theorem 4.14. □

As a consequence, we get a result of Riečanová for generalized D-lattices [24, Theorem 2.5].

Corollary 4.28.

(1) *For an element z of a generalized D-lattice A, the following are equivalent:*

(i) *z is central;*

(ii) *$z \leftrightarrow x$ and $\big(x \ominus (x \wedge z)\big) \wedge z = 0$ for every $x \in A$;*

(iii′) *$z \leftrightarrow x$ and $x = (x \wedge z) \vee \big(x \ominus (x \wedge z)\big)$ for every $x \in A$;*

(iii″) *$x = (x \wedge z) \vee \big((x \vee z) \ominus z\big)$ for every $x \in A$.*

(2) *If A is a D-lattice, the above conditions are equivalent to any of the following:*

(iv) $z \leftrightarrow x$ *for every* $x \in A$ *and* z *is sharp;*

(v) $x \ominus (x \wedge z) = x \wedge z^{\perp}$ *for every* $x \in A$.

Proof. Immediate from the previous theorem. To obtain (v) from condition (7) above, just observe that $\phi(x, z) = x \ominus (x \wedge z^{\perp})$ and hence

$$x \ominus \phi(x, z) = x \ominus \big(x \ominus (x \wedge z^{\perp})\big) = x \wedge z^{\perp}. \qquad \square$$

Remark 4.29. *The reader should note that the foregoing result is not exactly the same as stated in the cited paper* [24], *which was not quite correct. Indeed, in that paper, instead of our conditions* (iii′) *and* (iii″), *there is a condition* (iii), *which reads as follows:*

(iii) $x = (x \wedge z) \vee \big(x \ominus (x \wedge z)\big)$ *for every* $x \in A$.

It is obvious that (iii) *is a weaker condition than* (iii′). *But they are not equivalent, even in case of a D-lattice, as shown in the example below.*

Example 4.30. *The five-element set* $\{0, a, b, c, 1\}$ *can be endowed with a structure of d_0-algebra by defining the operation* $\backslash$ *as in Figure 4. The resulting d_0-algebra D is in fact a D-lattice. Now, for every $x \in D$, one easily verifies that* $x = (x \wedge c) \vee \big(x \ominus (x \wedge c)\big)$. *On the other hand, a is not compatible with c. Indeed* $a - c = (a \backslash c) \backslash (c \backslash a) = b \backslash a = a \neq b = a \backslash c$.

$\backslash$	0	a	b	c	1
0	0	0	0	0	0
a	a	0	c	b	0
b	b	a	0	b	0
c	c	a	c	0	0
1	1	a	c	b	0

Figure 4: The Cayley table of the D-lattice D of Example 4.30.

Conclusion

Sharp (and weakly sharp) elements in d_0-algebras are defined in order to generalize the corresponding concepts in D-lattices, in particular a d_0-algebra in which every element is sharp is a generalization of an orthomodular lattice. A number of properties of sharp element are investigated; also, having defined Sasaki mapping, the relationships between them and sharp elements are explored. Finally, central elements are characterized as sharp elements which are compatible with every other element, and a result of Riečanová for D-lattices [24, Theorem 2.5] is extended to d_0-algebras. A flaw in that result is fixed, too.

References

[1] A. Avallone, G. Barbieri, P. Vitolo, H. Weber, *Decomposition of effect algebras and the Hammer-Sobczyk theorem*, Algebra Universalis, **60**(1) (2009), 1-18.

[2] A. Avallone, A. de Simone, P. Vitolo, *Effect algebras and extensions of measures*, Bollettino dell'Unione Matematica Italiana, **9**(2) (2006), 423-444.

[3] A. Avallone, P. Vitolo, *Hahn decomposition of modular measures and applications*, Annales Soc. Math. Polon. Series I: Commentationes Mathematicae, **43** (2003), 149-168.

[4] A. Avallone, P. Vitolo, *Decomposition and control theorems on effect algebras*, Scientiae Mathematicae Japonicae, **58**(1) (2003), 1-14.

[5] A. Avallone, P. Vitolo, *Lattice uniformities on effect algebras*, International Journal of Theoretical Physics, **44**(7) (2005), 793-806.

[6] A. Avallone, P. Vitolo, *Lyapunov decomposition of measures on effect algebras*, Scientiae Mathematicae Japonicae, **69**(1) (2009), 79-87.

[7] A. Avallone, P. Vitolo, *Hahn decomposition in d_0-algebras*, Soft Computing, **23**(22) (2019), 11373-11388.

[8] A. Avallone, P. Vitolo, *Lyapunov decomposition in d_0-algebras*, Rendiconti del Circolo Matematico di Palermo Series 2, **69** (2020), 837-859.

[9] A. Avallone, P. Vitolo, *The center of a d_0-algebra*, Reports on Mathematical Physics, **86**(1) (2020), 63-78.

[10] A. Avallone, P. Vitolo, *Kalmbach Measurability in d_0-algebras*, Mathematica Slovaca, **72**(6) (2022), 1387-1402.

[11] A. Avallone, P. Vitolo, *Decomposition of d_0-algebras*, Submitted for publication.

[12] M. K. Bennett, D. J. Foulis, *Effect algebras and unsharp quantum logics*, Foundations of Physics, **24**(10) (1994), 1331-1352.

[13] M. K. Bennett, D. J. Foulis, *Phi-symmetric effect algebras*, Foundations of Physics, **25**(12) (1994), 1699-1722.

[14] F. Chovanec, F. Kôpka, *D-lattices*, International Journal of Theoretical Physics, **34** (1995), 1297-1302.

[15] C. Constantinescu, *Some properties of spaces of measures*, Atti del Seminario Matematico e Fisico dell' Universita di Modena, **35**(suppl.), Modena, Italy, 1989.

[16] A. Dvurečenskij, *States on weak pseudo EMV-algebras. I. States and states morphisms*, Iranian Journal of Fuzzy Systems, **19**(4) (2022), 1-15.

[17] A. Dvurečenskij, *States on weak pseudo EMV-algebras. II. Representations of states*, Iranian Journal of Fuzzy Systems, **19**(4) (2022), 17-26.

[18] A. Dvurečenskij, M. G. Graziano, *Remarks on representations of minimal clans*, Tatra Mountains Mathematical Publications, **15** (1998), 31-53.

[19] A. Dvurečenskij, M. G. Graziano, *On representations of commutative BCK-algebras*, Demonstratio Mathematica, **32**(2) (1999), 227–246.

[20] A. Dvurečenskij, S. Pulmannová, *New trends in quantum structures*, Kluwer Academic Publishers, Dordrecht, 2000.

[21] D. J. Foulis, *A note on orthomodular lattices*, Portugaliae Mathematica, **21**(1) (1962), 65-72.

[22] J. J. M. Gabriëls, S. M. Gagola, M. Navara, *Sasaki projections*, Algebra Universalis, **77**(3) (2017), 305-320.

[23] M. Nakamura, *The permutability in a certain orthocomplemented lattice*, Kodai Mathematical Seminar Reports, **9**(4) (1957), 158-160.

[24] Z. Riečanová, *Compatibility and central elements in effect algebras*, Tatra Mountains Mathematical Publications, **16** (1999), 151-158.

[25] Z. Riečanová, *Subalgebras, intervals and central elements of generalized effect algebras*, International Journal of Theoretical Physics, **38** (1999), 3209-3220.

[26] M. Rosa, P. Vitolo, *Blocks and compatibility in d_0-algebras*, Algebra Universalis, **78**(4) (2017), 489-513.

[27] M. Rosa, P. Vitolo, *Topologies and uniformities on d_0-algebras*, Mathematica Slovaca, **67**(6) (2017), 1301-1322.

[28] M. Rosa, P. Vitolo, *Measures and submeasures on d_0-algebras*, Ricerche di Matematica, **67** (2018), 373-386.

[29] U. Sasaki, *Orthocomplemented lattices satisfying the exchange axiom*, Journal of Science of the Hiroshima University, Series A, **17** (1954), 293-302.

[30] K. D. Schmidt, *Minimal clans: A class of ordered partial semigroups including Boolean rings and lattice-ordered groups*, in *Semigroups, theory and applications (Oberwolfach, 1986)*, volume **1320** of Lecture Notes in Math., Springer, Berlin, (1988), 300-341.

[31] K. D. Schmidt, *Jordan decompositions of generalized vector measures*, volume **214** of Pitman Research Notes in Mathematics Series, Longman Scientific and Technical, Harlow, 1989.

[32] P. Vitolo, *A generalization of set-difference*, Mathematica Slovaca, **61**(6) (2011), 835-850.

[33] H. Weber, *An abstraction of clans of fuzzy sets*, Ricerche di Matematica, **46**(2) (1997), 457-472.

[34] O. Wyler, *Clans*, Compositio Mathematica, **17** (1965), 172-189.