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Some novel approaches on the $(\overline{N}, p, q)$ summability of double sequences of fuzzy numbers and its applications

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Abstract

In this paper, our aim is to make a novel interpretation of relation between $(\overline{N}, p, q)$ method and P-convergence for double sequences of fuzzy numbers. In accordance with this aim, we derive some Tauberian conditions, controlling O-oscillatory behavior of a double sequence of fuzzy numbers in certain senses, from $(\overline{N}, p, q)$ summability to P-convergence with some restrictions on the weight sequences. As special cases, we indicate that O-condition of Hardy type with respect to (P_m) and (Q_n) are Tauberian conditions for $(\overline{N}, p, q)$ summability under some additional conditions. In the sequel, we prove a fuzzy Korovkin-type approximation theorem by using the $(\overline{N}, p, q)$ summability method for fuzzy positive linear operators.

Keywords: Double sequences of fuzzy numbers, convergence in Pringsheim's sense, $(\overline{N}, p, q)$ summability, regularly varying sequences, slowly decreasing sequences, slowly oscillating sequences, weighted mean summability method, fuzzy Korovkin theory.

1 Introduction

Among the several variances in science and especially mathematics in the 20th century, one of the prominent ones was about the concept of fuzziness. In science, all those variances have been manifested by a gradual transition from the conventional viewpoint, which persists that fuzziness is undesired and should be avoided in all its parts, toward an alternating viewpoint, which is tolerant of fuzziness and defends that science cannot disclaim it. Science according to the conventional viewpoint should tend to respect what is quantitative, certain, and consistent, and keep away from what is qualitative and fuzzy. On the other hand, science according to the alternative/contemporary viewpoint should be aware of the necessity of uncertainty which has utility in the sense that comprehending a mathematical model of situations encountered in the real physical world.

The first stage of the transition from the traditional view to the contemporary one on fuzziness began in the mid-20th century when researchers became concerned with the classes of objects which do not constitute in the usual mathematical sense and withal play a considerable role in human thinking, especially in the domains of pattern classification, communication of information, artificial intelligence. Improving a concept more general than the crisp set which is defined in such a way as to divide the objects in a given universe into two groups as members and non-members, Zadeh [51] introduced the concept of the fuzzy set which is characterized by a membership function assigning to each object a grade of membership ranking between 0 and 1. Following this study, Dubois and Prade [20] offered the concept of fuzzy numbers and extended fundamental algebraic operations to fuzzy sets. In [23], Goetschel and Voxman presented a definition, including a less condition, of fuzzy numbers. Utilizing this definition, Matloka [28] introduced the concepts of convergent and bounded sequences of fuzzy numbers, examined some properties of them, and indicated that if a

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sequence of fuzzy numbers is convergent, then it is also bounded. In [31], Nanda established the spaces of convergent and bounded sequences of fuzzy numbers and proved that both of them are complete metric spaces.

Although there is extensive literature on the fuzzy set theory and its approximations, it can be difficult for researchers who wish to know the development process in summability methods, especially the weighted mean method being the point of this paper, of the theory to find a study that both presents an up-to-date exposition of fuzzy Tauber theory and includes Korovkin-type approximations of the theory. To help researchers working on these subjects and to better understand the position of this manuscript in the literature, the studies carried out within the framework of fuzzy Tauber theory and its approaches are evaluated here according to the authors' treatment of this subject matter. At this point, it should be firstly mentioned Subrahmanyam [43] which defined the Cesàro or briefly $(C,1)$ method for sequences of fuzzy numbers. In [43], the author stated fuzzy analogs of certain well-known Tauberian theorems. Talo and Çakan [44] obtained some Tauberian theorems for sequences of fuzzy numbers that convergence follows from $(C,1)$ summability under Hardy-type condition $D(u_n,u_{n-1})=O(1/n)$, Schmidt-type slow oscillation condition and more general condition involving the notion of the deferred mean. There are also some interesting outcomes about Tauberian theorems in which $(C,1)$ and its iterative means, named as Hölder, methods of sequences of fuzzy numbers are used (see, [11, 12, 13, 15]). When it comes to the logarithmic or $(\ell,1)$ summability method for sequences of fuzzy numbers, it is encountered with Yavuz and Çoşkun [50]. In [50] the authors indicated regularity of the $(\ell,1)$ method and proved that necessary conditions for convergence of a sequence of fuzzy numbers being $(\ell,1)$ summable are Landau-type condition $(n\ln n)u_n \succeq (n\ln n)u_{n-1}-\overline{H}$, slow decrease condition concerning $(\ell,1)$ summability and more general condition involving the notion of deferred mean concerning $(\ell,1)$ summability, independently from each other. Carrying the mentioned results in a step further, Sezer [40] pointed out the validity of each implication in the following diagram

$$\text{convergence} \Longrightarrow (\ell,1)\text{ summability} \Longrightarrow (\ell,2)\text{ summability},$$

and established some Tauberian theorems to retrieve the convergence of a sequence of fuzzy numbers from its $(\ell,2)$ summability. As concerns the weighted mean or briefly $(\overline{N},p)$ method for sequences of fuzzy numbers including $(C,1)$ and $(\ell,1)$ methods, the following statement

$$(\overline{N},p)-\lim u_n=\mu \ \text{ implies } \ \lim u_n=\mu,$$

was indicated to be substantiated under the slow oscillation condition in [47], the condition based upon the relationship between a sequence and its de la Vallée Poussin means in [14], the slow decrease condition in [35] and the slow decrease condition concerning $(\overline{N},p)$ summability in [33] with some additional conditions imposed on the weight (P_n).

Thanks to the concept of double sequence of fuzzy numbers defined by Savaş [39], Tripathy and Dutta [48] investigated certain spaces of double sequences of fuzzy numbers and pointed out in [49] that every P-convergent double sequence of fuzzy numbers is $(C,1,1)$ summable to its P-limit under the boundedness of the sequence. Following this, Çanak et al. [16] determined Tauberian theorems for double sequences of fuzzy numbers dealing with the implication from $(C,1,1)$ summability to P-convergence under Hardy-type conditions, slow oscillation conditions in certain senses and more general conditions involving the concept of deferred means. Taking inspiration from [25] on the space of real-valued double sequences, Jena et al. [26] stated that conditions needed for $(C,1,1)$ and (C,k,r) summable double sequences of fuzzy numbers to be P-convergent are slow oscillation condition of sequence and its double generator sequence in sense $(1,1)$. There are also current studies including fuzzy analogues of some Tauberian results, given by Jena et al. [24], Parida et al. [36] and Edely and Mursaleen [22] for Cesàro or statistical Cesàro summable sequences. In the sequel, Totur and Çanak [46] defined the $(\overline{N},p,q)$ method for double sequences of fuzzy numbers and present necessary and sufficient conditions under which P-convergence follows from the $(\overline{N},p,q)$ summability.

When it comes to applying Korovkin-type approximation theorems to the above-said summability methods within the scope of fuzzy spaces, it is possible to say that Anastassiou and Duman [5] are (to our knowledge) pioneers in this field. After a Korovkin-type approximation theorem was proved by them for fuzzy positive linear operators by using the concept of A-statistical convergence, where A is a non-negative regular summability matrix, this kind of approximation theorems was constructed by Anastassiou et al. [4] via A-summability method; by Nayak et al. [32] via the deferred statistical A-summability method; by Demirci and Karakuş [18] for n-multiple sequences of fuzzy positive linear operators via statistical convergence; by Demirci and Karakuş [19] via A-statistical convergence for four-dimensional summability matrices; by Acar and Mohiuddine [1] via statistical $(C,1)(E,1)$ summability method; by Braha and Loku [10] via $(C,2)$ summability method, as well.

In essence, one of the matters we focused on is the idea of improving the result, which was formulated by Önder and Çanak [34] for single sequences of fuzzy numbers as follows, by extending to $(\overline{N},p,q)$ summable double ones.

Theorem 1.1. *Let (p_n) be a sequence which has the property*

$$\frac{P_n}{P_{n+1}}\to 1 \quad as \quad n\to\infty.$$

If a sequence (u_n) of fuzzy numbers is $(\overline{N},p)$ summable to $\mu\in E^1$ and slowly oscillating relative to (P_n), then (u_n) is convergent to μ.

The other is to indicate that our results obtained in this paper include all of the fuzzy analogues of classical Tauberian theorems for double sequences which P-convergence follows from $(C,1,1)$ and $(\ell,1,1)$ under slow oscillation conditions relative to Schmidt and slow oscillation conditions relative to logarithmic summability for double sequences of fuzzy numbers in certain senses, respectively. Herein, the main issue to be discussed is in what ways the conditions imposed on the weights $(p_m),(q_n)$ or its partial sums $(P_m),(Q_n)$ should change while the ones imposed on the sequence (u_{mn}) of fuzzy numbers become more inclusive. To reach an answer about this, we need the class $SVA_{reg(\alpha)}$ and its characterization.

In the present paper, we are interested in the relation between the $(\overline{N},p,q)$ method and P-convergence for double sequences of fuzzy numbers. In line with this purpose, we derive some Tauberian conditions, controlling O-oscillatory behavior of a double sequence of fuzzy numbers in certain senses, from $(\overline{N},p,q)$ summability to P- convergence with some restrictions on the weight sequences. In the sequel, we prove a fuzzy Korovkin-type approximation theorem by using the $(\overline{N},p,q)$ summability method for fuzzy positive linear operators.

2 Preliminaries

We refer basic definitions and notations relating to fuzzy numbers, their linear structure and algebraic properties. In [23], Goetschel and Voxman defined the concept of fuzzy numbers.

Consider a fuzzy subset of the real line $u:\mathbb{R}\to[0,1]$. Then, the mapping u is a fuzzy number if it satisfies the following $(i)-(iv)$ properties:

(i) u is normal, i.e., there is a $t_0\in\mathbb{R}$ such that $u(t_0)=1$.

(ii) u is fuzzy convex, i.e., for any $t_0,t_1\in\mathbb{R}$ and $\alpha\in[0,1]$, $u(\alpha t_0+(1-\alpha)t_1)\geq\min\{u(t_0),u(t_1)\}$.

(iii) u is upper semicontinuous on $\mathbb{R}$.

(iv) The support of u, $[u]_0:=\overline{\{t\in\mathbb{R}:u(t)>0\}}$ is compact, where $\overline{K}$ denotes closure of the set K in the usual topology of $\mathbb{R}$.

The set of all fuzzy numbers on $\mathbb{R}$ is denoted by E^1.

For $u\in E^1$, the α-level set of u is defined by

$$[u]_\alpha:=\begin{cases}\{x\in\mathbb{R}:u(x)\geq\alpha\}, & 0<\alpha\leq 1,\\ \overline{\{x\in\mathbb{R}:u(x)>\alpha\}}, & \alpha=0.\end{cases}$$

Then, $u\in E^1$ if and only if $[u]_\alpha$ is a closed, bounded and nonempty interval for each $\alpha\in[0,1]$ with $[u]_\beta\subseteq[u]_\alpha$ if $0\leq\alpha\leq\beta\leq 1$. According to this characterization, $u\in E^1$ is determined by the end points of interval $[u]_\alpha=[u^-(\alpha),u^+(\alpha)]$ where $u^-(\alpha)\leq u^+(\alpha)$ and $u^-(\alpha),u^+(\alpha)\in\mathbb{R}$ for each $\alpha\in[0,1]$ (see [21]).

Theorem 2.1. (Representation Theorem) [23] *Let $u\in E^1$ and let $[u]_\alpha=[u^-(\alpha),u^+(\alpha)]$. Then, the functions $u^-,u^+:[0,1]\to\mathbb{R}$ satisfy (i)-(iii) conditions:*

(i) *$u^-(\alpha)$ is a bounded, non-decreasing, left continuous on $(0,1]$ and right continuous at $\alpha=0$.*

(ii) *$u^+(\alpha)$ is a bounded, non-increasing, left continuous on $(0,1]$ and right continuous at $\alpha=0$.*

(iii) *$u^-(1)\leq u^+(1)$.*

Conversely, if the functions f and g satisfy (i)-(iii), then there is a unique $u\in E^1$ such that $[u]_\alpha:=[f(\alpha),g(\alpha)]$ for each $\alpha\in[0,1]$ and $u(x):=\sup_{\alpha\in[0,1]}\{\alpha:f(\alpha)\leq x\leq g(\alpha)\}$.

Let $u,\omega\in E^1$ be represented by $[u^-(\alpha),u^+(\alpha)]$ and $[\omega^-(\alpha),\omega^+(\alpha)]$ for each $\alpha\in[0,1]$, respectively. Then, the addition operation on E^1 is defined by $[u+\omega]_\alpha=[u]_\alpha+[\omega]_\alpha=[u^-(\alpha)+\omega^-(\alpha),u^+(\alpha)+\omega^+(\alpha)]$, and the scalar multiplication operation on E^1 is defined by $[c\omega]_\alpha=c[\omega]_\alpha=[c\omega^-(\alpha),c\omega^+(\alpha)]$ if $c\geq 0$; and $[c\omega]_\alpha=[c\omega^+(\alpha),c\omega^-(\alpha)]$ if $c<0$.

The set of all real numbers can be embedded in E^1 since for each $r\in\mathbb{R}$, $\bar{r}\in E^1$ is defined by $\bar{r}(x):=1$ if $x=r$, and $\bar{r}(x):=0$ if $x\neq r$.

Lemma 2.2. [6] *The following is verified for any $u,v,w\in E^1$ and $a,b\in\mathbb{R}$:*

(i) *The addition of fuzzy numbers is associative and commutative.*

(ii) *$\bar{0}\in E^1$ is neutral element with respect to $+$, i.e., $u+\bar{0}=\bar{0}+u=u$.*

(iii) *With respect to $+$, none of $u\in E^1\setminus\mathbb{R}$ has opposite in E^1.*

(iv) *$(a+b)u=au+bu$ holds provided that $ab\geq 0$. For general $a,b\in\mathbb{R}$, it does not hold.*

(v) *$a(u+v)=au+av$ and $(ab)u=a(bu)$ hold.*

As a result of Lemma 2.2, it can be observed that E^1 is not a linear space.

Let W be the set of all bounded and closed intervals. For the intervals $\varphi = [\varphi^-, \varphi^+]$, $\omega = [\omega^-, \omega^+]$ of real numbers, the classical Hausdorff distance on W is defined by

$$d(\varphi, \omega) := \max\left\{\left|\varphi^- - \omega^-\right|, \left|\varphi^+ - \omega^+\right|\right\}.$$

According to the Haussdorff distance, W is a complete separable metric space (cf. Nanda [31]). By means of the Hausdorff metric d, let $D : E^1 \times E^1 \to \mathbb{R}_+$ be defined as

$$D(u, v) := \sup_{\alpha\in[0,1]} d([u]_\alpha, [v]_\alpha) = \sup_{\alpha\in[0,1]} \max\left\{\left|u^-(\alpha) - v^-(\alpha)\right|, \left|u^+(\alpha) - v^+(\alpha)\right|\right\}.$$

Then, the metric D is called the Hausdorff distance between fuzzy numbers u and v (cf. [6]).

Proposition 2.3 indicates some useful properties of the Hausdorff distance between fuzzy numbers.

Proposition 2.3. [6] *Let $u, v, \omega, \varphi \in E^1$ and $c \in \mathbb{R}$. Then, the following statements hold true.*
(i) *(E^1, D) is a complete metric space.*
(ii) *D is translation invariant, i.e., $D(u + \omega, v + \omega) = D(u, v)$.*
(iii) *$D(cu, cv) = |c| D(u, v)$.*
(iv) *$D(u + v, \omega + \varphi) \le D(u, \omega) + D(v, \varphi)$.*
(v) *$|D(u, \bar{0}) - D(v, \bar{0})| \le D(u, v) \le D(u, \bar{0}) + D(v, \bar{0})$.*

Herein, we recall some definitions relating to double sequences of fuzzy numbers. In [39], Savaş introduced the following definitions for double sequences of fuzzy numbers.

A double sequence $u = (u_{mn})$ of fuzzy numbers is a function u from $\mathbb{N}\times\mathbb{N}$ into the set E^1. The value of the function at a point $(m, n) \in \mathbb{N} \times \mathbb{N}$ is denoted by $u_{mn} \in E^1$ and is called the (m, n)-term of the double sequence.

The set of all double sequences of fuzzy numbers is denoted by $w^2(F)$.

A double sequence $u = (u_{mn})$ of fuzzy numbers is said to be P-convergent to $\mu \in E^1$, if for every $\epsilon > 0$ there exists a $n_0(\epsilon) \in \mathbb{N}$ such that $D(u_{mn}, \mu) < \epsilon$ whenever $m, n \ge n_0$, written as $P - \lim u_{mn} = \mu$. The number μ is called the P-limit of u.

A double sequence $u = (u_{mn})$ of fuzzy numbers is said to be bounded if there exists a number $M > 0$ such that $D(u_{mn}, \bar{0}) < M$ for all m, n, i.e, if $||u||_{\infty,2} = \sup_{m,n} D(u_{mn}, \bar{0}) < \infty$.

The set of all P-convergent and bounded double sequences of fuzzy numbers is denoted by $c^2(F)$ and $\ell^2_\infty(F)$, respectively.

Note that unlike $(u_m) \in w(F)$, $(u_{mn}) \in c(F)$ need not be bounded.

Example 2.4. *Consider the double sequence $u = (u_{mn})$ of fuzzy numbers defined by*

$$u_{mn} = \begin{cases} \omega_n & \text{if } m = 0, n \in \mathbb{N}, \\ \bar{0} & \text{otherwise,} \end{cases} \quad \text{where } \omega_n = \sum_{k=0}^{n} \nu_k \quad \text{and} \quad \nu_k(t) = \begin{cases} 1 - t\left(4 + \frac{1}{k}\right) & \text{if } t \in \left[0, \frac{k}{4k+1}\right], \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that (u_{mn}) is P-convergent to $\bar{0}$. The endpoints of the α-level set of the double sequence (u_{mn}) are

$$u^-_{mn}(\alpha) = 0 \quad \text{and} \quad u^+_{mn}(\alpha) = \begin{cases} \omega^+_n(\alpha) & \text{if } m = 0, n \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

for all $m, n \in \mathbb{N}$. Hence, we have

$$\begin{aligned} ||u||_{\infty,2} = \sup_{m,n\in\mathbb{N}} D(u_{mn}, \bar{0}) &= \sup_{m,n\in\mathbb{N}} \sup_{\alpha\in[0,1]} \max\left\{|u^-_{mn}(\alpha) - 0|, |u^+_{mn}(\alpha) - 0|\right\} \\ &= \sup_{m,n\in\mathbb{N}} \sup_{\alpha\in[0,1]} |u^+_{mn}(\alpha)| = \sup_{m,n\in\mathbb{N}} |u^+_{mn}(0)| = \sup_{m\in\mathbb{N}} |\omega^+_n(0)|. \end{aligned}$$

Since the sequence $(\omega^+_n(0)) = \left(\sum_{k=0}^{n} \frac{k}{4k+1}\right)$ is divergent as $n \to \infty$, the double sequence (u_{mn}) is not bounded.

Some notations that will be used in places throughout this paper are given below.
Let (u_{mn}) be a double sequence of fuzzy numbers.

(i) The symbol $u_{mn} = O(1)$ means that $D(u_{mn}, \overline{0}) < M$ for some constant $M > 0$ and each m, n.
(ii) The symbol $u_{mn} = o(1)$ means that $u_{mn} \to 0$ as $m, n \to \infty$.

Let $p = (p_m)$ and $q = (q_n)$ be two sequences of positive real numbers such that

$$P_m := \sum_{i=0}^{m} p_i \to \infty \quad \text{and} \quad Q_n := \sum_{j=0}^{n} q_j \to \infty \ \text{ as } \ m, n \to \infty. \tag{1}$$

The weighted means of a double sequence $u = (u_{mn})$ of fuzzy numbers are defined by

$$\sigma_{mn} := \frac{1}{P_m Q_n} \sum_{i=0}^{m} \sum_{j=0}^{n} p_i q_j u_{ij},$$

for all $(m, n) \in \mathbb{N} \times \mathbb{N}$ and $P_m Q_n > 0$.
A double sequence (u_{mn}) of fuzzy numbers is said to be summable by the weighted mean method determined by the sequences (p_m) and (q_n) or shortly $(\overline{N}, p, q)$ summable to a fuzzy number μ if (σ_{mn}) is P-convergent to μ. In [46], Totur and Çanak stated that the $(\overline{N}, p, q)$ summability method is a regular method for bounded double sequences of fuzzy numbers and presented the following theorem.

Theorem 2.5. *Let a double sequence $u = (u_{mn})$ of fuzzy numbers be bounded. If (u_{mn}) is P-convergent to $\mu \in E^1$, then (u_{mn}) is also $(\overline{N}, p, q)$ summable to μ.*

In the meantime, Totur and Çanak [46] constructed an example indicating that the opposite of this proposition is not true in general. The question of whether some (nontrivial) condition on the terms u_{mn} under which its $(\overline{N}, p, q)$ summability implies its P-convergence exist comes to mind at this point. The condition $T\{u_{mn}\}$ making such a situation possible is called a *Tauberian condition.* The resulting theorem stating that P-convergence follows from its $(\overline{N}, p, q)$ summability and $T\{u_{mn}\}$ is called a *Tauberian Theorem* for the $(\overline{N}, p, q)$ method.

In the remainder of this section, we mention the classes including all positive sequences (p_m) whose partial sum sequence (P_m) is
(i) regularly varying sequence of positive index,
(ii) rapidly varying sequence of index ∞ (see [8] for more details).

Let $p = (p_m)$ be a sequence that satisfies $(p_m) = (P_m - P_{m-1})$ where $P_{-1} = 0$.

(i) A sequence (P_m) of positive numbers is said to be *regularly varying* if for all $\lambda > 0$

$$\lim_{m \to \infty} \frac{P_{\lambda_m}}{P_m} = \varphi(\lambda) \quad \text{exists}, \tag{2}$$

where $0 < \varphi(\lambda) < \infty$ (cf. [9]).
In spite of the fact that this definition has been used by many authors as a starting point for studies including regularly varying sequences, these sequences possess quite useful properties, the most important of which is probably the following characterization theorem.

Characterization Theorem 2.6. [27] *The following statements are equivalent:*

(a) *A sequence (P_m) of positive numbers is a regularly varying sequence.*
(b) *There exists a real number $\alpha > 0$ such that $\varphi(\lambda) = \lambda^\alpha$, for all $\lambda > 0$.*
(c) *The sequence (P_m) has the form $P_m = (m+1)^\alpha L(m)$, for $m \geq 0$ with constant $\alpha \geq 0$ and slowly varying function $L(.)$ on $(0, \infty)$, i.e. the function $L(.)$ is positive, measurable, and satisfies*

$$\lim_{t \to \infty} \frac{L(\lambda t)}{L(t)} = 1, \quad \textit{for all} \ \ \lambda > 0.$$

To emphasize such α, a sequence (P_m) is called a *regularly varying sequence of positive index* α, as well. Note that a regularly varying sequence of index $\alpha = 0$ corresponds to a slowly varying sequence.
By $SVA_{reg(\alpha)}$, we denote the set of all sequences of positive numbers (p_m) satisfying (c).
Here, it is useful to give the following implication proved by Bojanic and Seneta [9].

Lemma 2.7. [9] *If a sequence $P = (P_m)$ of positive numbers is regularly varying, then $P_m / P_{m-1} \to 1$ as $m \to \infty$.*

(ii) A sequence (P_m) of positive numbers is said to be *rapidly varying of index* ∞ if

$$\frac{P_{\lambda_m}}{P_m} \to \begin{cases} 0 & \text{if } 0 < \lambda < 1, \\ 1 & \text{if } \lambda = 1, \\ \infty & \text{if } \lambda > 1 \end{cases} \qquad \text{as} \quad m \to \infty. \tag{3}$$

By SVA_{rap}, we denote the set of all sequences of positive numbers (p_m) satisfying (3).
In addition, it may be written conventionally as λ^{∞} because the right hand side of (3) is the limit of λ^{α} as $\alpha \to \infty$.

3 Auxiliary results

In this section, we state an auxiliary result to be benefitted in the proofs of main results. The following lemma indicates two representations for the distance between general terms of double sequences (u_{mn}) and (σ_{mn}) of fuzzy numbers.

Lemma 3.1. *Let $u = (u_{mn})$ be a double sequence of fuzzy numbers.*

(i) *For sufficiently large $\mu > m$ and $\eta > n$, we have*

$$D(u_{mn}, \sigma_{mn}) \le \frac{P_\mu Q_\eta}{(P_\mu - P_m)(Q_\eta - Q_n)} \left(D(\sigma_{\mu\eta}, \sigma_{\mu n}) + D(\sigma_{mn}, \sigma_{m\eta})\right) + \frac{P_\mu}{P_\mu - P_m} D(\sigma_{\mu n}, \sigma_{mn}) + \frac{Q_\eta}{Q_\eta - Q_n} D(\sigma_{m\eta}, \sigma_{mn}) + \frac{1}{(P_\mu - P_m)(Q_\eta - Q_n)} \sum_{i=m+1}^{\mu} \sum_{j=n+1}^{\eta} p_i q_j D(u_{ij}, u_{mn}).$$

(ii) *For sufficiently large $\mu < m$ and $\eta < n$, we have*

$$D(u_{mn}, \sigma_{mn}) \le \frac{P_\mu Q_\eta}{(P_m - P_\mu)(Q_n - Q_\eta)} \left(D(\sigma_{\mu\eta}, \sigma_{\mu n}) + D(\sigma_{mn}, \sigma_{m\eta})\right) + \frac{P_\mu}{P_m - P_\mu} D(\sigma_{mn}, \sigma_{\mu n}) + \frac{Q_\eta}{Q_n - Q_\eta} D(\sigma_{mn}, \sigma_{m\eta}) + \frac{1}{(P_m - P_\mu)(Q_n - Q_\eta)} \sum_{i=\mu+1}^{m} \sum_{j=\eta+1}^{n} p_i q_j D(u_{mn}, u_{ij}).$$

Proof. (i) For sufficiently large $\mu > m$ and $\eta > n$, we have from definition of the weighted means of (u_{mn})

$$\begin{aligned} D(u_{mn}, \sigma_{mn}) &= D\Bigg(\frac{1}{(P_\mu - P_m)(Q_\eta - Q_n)} \sum_{i=m+1}^{\mu} \sum_{j=n+1}^{\eta} p_i q_j u_{mn} \\ &+ \frac{1}{(P_\mu - P_m)(Q_\eta - Q_n)} \left(\sum_{i=0}^{m} \sum_{j=n+1}^{\eta} + \sum_{i=m+1}^{\mu} \sum_{j=0}^{n} + \sum_{i=0}^{m} \sum_{j=0}^{n} + \sum_{i=m+1}^{\mu} \sum_{j=n+1}^{\eta} \right) p_i q_j u_{ij}, \\ &\sigma_{mn} + \frac{1}{(P_\mu - P_m)(Q_\eta - Q_n)} \left(\sum_{i=0}^{m} \sum_{j=n+1}^{\eta} + \sum_{i=m+1}^{\mu} \sum_{j=0}^{n} + \sum_{i=0}^{m} \sum_{j=0}^{n} + \sum_{i=m+1}^{\mu} \sum_{j=n+1}^{\eta} \right) p_i q_j u_{ij} \Bigg) \\ &= D\Bigg(\frac{1}{(P_\mu - P_m)(Q_\eta - Q_n)} \sum_{i=m+1}^{\mu} \sum_{j=n+1}^{\eta} p_i q_j u_{mn} + \frac{1}{(P_\mu - P_m)(Q_\eta - Q_n)} \sum_{i=0}^{m} \sum_{j=0}^{n} p_i q_j u_{ij} \\ &+ \frac{1}{(P_\mu - P_m)(Q_\eta - Q_n)} \sum_{i=m+1}^{\mu} \sum_{j=0}^{\eta} p_i q_j u_{ij} + \frac{1}{(P_\mu - P_m)(Q_\eta - Q_n)} \sum_{i=0}^{m} \sum_{j=0}^{n} p_i q_j u_{ij}, \\ &\sigma_{mn} + \frac{1}{(P_\mu - P_m)(Q_\eta - Q_n)} \sum_{i=0}^{m} \sum_{j=0}^{\eta} p_i q_j u_{ij} + \frac{1}{(P_\mu - P_m)(Q_\eta - Q_n)} \sum_{i=0}^{\mu} \sum_{j=0}^{n} p_i q_j u_{ij} \\ &+ \frac{1}{(P_\mu - P_m)(Q_\eta - Q_n)} \sum_{i=m+1}^{\mu} \sum_{j=n+1}^{\eta} p_i q_j u_{ij} \Bigg) \end{aligned}$$

$$
\begin{aligned}
&= D\left(\frac{1}{(P_\mu - P_m)(Q_\eta - Q_n)}\sum_{i=m+1}^{\mu}\sum_{j=n+1}^{\eta} p_i q_j u_{mn} + \frac{1}{(P_\mu - P_m)(Q_\eta - Q_n)}\sum_{i=0}^{\mu}\sum_{j=0}^{\eta} p_i q_j u_{ij}\right.\\
&+ \frac{1}{(P_\mu - P_m)(Q_\eta - Q_n)}\sum_{i=0}^{m}\sum_{j=0}^{n} p_i q_j u_{ij}, \sigma_{mn} + \frac{1}{(P_\mu - P_m)(Q_\eta - Q_n)}\sum_{i=0}^{m}\sum_{j=0}^{\eta} p_i q_j u_{ij}\\
&+ \left.\frac{1}{(P_\mu - P_m)(Q_\eta - Q_n)}\sum_{i=0}^{\mu}\sum_{j=0}^{n} p_i q_j u_{ij} + \frac{1}{(P_\mu - P_m)(Q_\eta - Q_n)}\sum_{i=m+1}^{\mu}\sum_{j=n+1}^{\eta} p_i q_j u_{ij}\right)\\
&= D\left(\frac{1}{(P_\mu - P_m)(Q_\eta - Q_n)}\sum_{i=m+1}^{\mu}\sum_{j=n+1}^{\eta} p_i q_j u_{mn} + \frac{P_\mu Q_\eta}{(P_\mu - P_m)(Q_\eta - Q_n)}\sigma_{\mu\eta}\right.\\
&+ \frac{P_m Q_n}{(P_\mu - P_m)(Q_\eta - Q_n)}\sigma_{mn} + \frac{Q_\eta}{Q_\eta - Q_n}\sigma_{m\eta} + \frac{P_\mu}{P_\mu - P_m}\sigma_{\mu n}, \sigma_{mn}\\
&+ \left.\frac{P_\mu Q_\eta}{(P_\mu - P_m)(Q_\eta - Q_n)}\left(\sigma_{m\eta} + \sigma_{\mu n}\right) + \frac{1}{(P_\mu - P_m)(Q_\eta - Q_n)}\sum_{i=m+1}^{\mu}\sum_{j=n+1}^{\eta} p_i q_j u_{ij}\right)\\
&\leq \frac{P_\mu Q_\eta}{(P_\mu - P_m)(Q_\eta - Q_n)} D\left(\sigma_{\mu\eta}, \sigma_{\mu n}\right)\\
&+ D\left(\frac{P_\mu Q_\eta}{(P_\mu - P_m)(Q_\eta - Q_n)}\sigma_{mn} + \frac{Q_\eta}{Q_\eta - Q_n}\sigma_{m\eta} + \frac{P_\mu}{P_\mu - P_m}\sigma_{\mu n}, \frac{P_\mu Q_\eta}{(P_\mu - P_m)(Q_\eta - Q_n)}\sigma_{m\eta}\right.\\
&+ \left.\frac{Q_\eta}{Q_\eta - Q_n}\sigma_{mn} + \frac{P_\mu}{P_\mu - P_m}\sigma_{mn}\right) + \frac{1}{(P_\mu - P_m)(Q_\eta - Q_n)}\sum_{i=m+1}^{\mu}\sum_{j=n+1}^{\eta} p_i q_j D\left(u_{mn}, u_{ij}\right),
\end{aligned}
$$

which require to Lemma 3.1(i).
(ii) For sufficiently large $m > \mu$ and $n > \eta$, we have from definition of the weighted means of (u_{mn})

$$
\begin{aligned}
D\left(u_{mn}, \sigma_{mn}\right) &= D\left(\frac{1}{(P_m - P_\mu)(Q_n - Q_\eta)}\sum_{i=\mu+1}^{m}\sum_{j=\eta+1}^{n} p_i q_j u_{mn}\right.\\
&+ \frac{1}{(P_m - P_\mu)(Q_n - Q_\eta)}\left(\sum_{i=0}^{\mu}\sum_{j=\eta+1}^{n} + \sum_{i=\mu+1}^{m}\sum_{j=0}^{\eta} + \sum_{i=0}^{\mu}\sum_{j=0}^{\eta} + \sum_{i=\mu+1}^{m}\sum_{j=\eta+1}^{n}\right) p_i q_j u_{ij},\\
&\left.\sigma_{mn} + \frac{1}{(P_m - P_\mu)(Q_n - Q_\eta)}\left(\sum_{i=0}^{\mu}\sum_{j=\eta+1}^{n} + \sum_{i=\mu+1}^{m}\sum_{j=0}^{\eta} + \sum_{i=0}^{\mu}\sum_{j=0}^{\eta} + \sum_{i=\mu+1}^{m}\sum_{j=\eta+1}^{n}\right) p_i q_j u_{ij}\right)\\
&= D\left(\frac{1}{(P_m - P_\mu)(Q_n - Q_\eta)}\sum_{i=\mu+1}^{m}\sum_{j=\eta+1}^{n} p_i q_j u_{mn} + \frac{1}{(P_m - P_\mu)(Q_n - Q_\eta)}\sum_{i=0}^{\mu}\sum_{j=0}^{n} p_i q_j u_{ij}\right.\\
&+ \frac{1}{(P_m - P_\mu)(Q_n - Q_\eta)}\sum_{i=\mu+1}^{m}\sum_{j=0}^{n} p_i q_j u_{ij} + \frac{1}{(P_m - P_\mu)(Q_n - Q_\eta)}\sum_{i=0}^{\mu}\sum_{j=0}^{\eta} p_i q_j u_{ij},\\
&\sigma_{mn} + \frac{1}{(P_m - P_\mu)(Q_n - Q_\eta)}\sum_{i=0}^{\mu}\sum_{j=0}^{n} p_i q_j u_{ij} + \frac{1}{(P_m - P_\mu)(Q_n - Q_\eta)}\sum_{i=0}^{m}\sum_{j=0}^{\eta} p_i q_j u_{ij}\\
&+ \left.\frac{1}{(P_m - P_\mu)(Q_n - Q_\eta)}\sum_{i=\mu+1}^{m}\sum_{j=\eta+1}^{n} p_i q_j u_{ij}\right)\\
&= D\left(\frac{1}{(P_m - P_\mu)(Q_n - Q_\eta)}\sum_{i=\mu+1}^{m}\sum_{j=\eta+1}^{n} p_i q_j u_{mn} + \frac{1}{(P_m - P_\mu)(Q_n - Q_\eta)}\sum_{i=0}^{m}\sum_{j=0}^{n} p_i q_j u_{ij}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{1}{(P_m-P_\mu)(Q_n-Q_\eta)}\sum_{i=0}^{\mu}\sum_{j=0}^{\eta} p_iq_ju_{ij},\sigma_{mn} + \frac{1}{(P_m-P_\mu)(Q_n-Q_\eta)}\sum_{i=0}^{\mu}\sum_{j=0}^{n} p_iq_ju_{ij}\\
&+ \quad \frac{1}{(P_m-P_\mu)(Q_n-Q_\eta)}\sum_{i=0}^{m}\sum_{j=0}^{\eta} p_iq_ju_{ij} + \frac{1}{(P_m-P_\mu)(Q_n-Q_\eta)}\sum_{i=\mu+1}^{m}\sum_{j=\eta+1}^{n} p_iq_ju_{ij}\Bigg)\\
&= \quad D\Bigg(\frac{1}{(P_m-P_\mu)(Q_n-Q_\eta)}\sum_{i=\mu+1}^{m}\sum_{j=\eta+1}^{n} p_iq_ju_{mn} + \frac{P_mQ_n}{(P_m-P_\mu)(Q_n-Q_\eta)}\sigma_{mn}\\
&+ \quad \frac{P_\mu Q_\eta}{(P_m-P_\mu)(Q_n-Q_\eta)}\sigma_{\mu\eta},\sigma_{mn} + \frac{P_\mu Q_\eta}{(P_m-P_\mu)(Q_n-Q_\eta)}\left(\sigma_{\mu n}+\sigma_{m\eta}\right)\\
&+ \quad \frac{Q_\eta}{Q_n-Q_\eta}\sigma_{m\eta} + \frac{P_\mu}{P_m-P_\mu}\sigma_{\mu n} + \frac{1}{(P_m-P_\mu)(Q_n-Q_\eta)}\sum_{i=\mu+1}^{m}\sum_{j=\eta+1}^{n} p_iq_ju_{ij}\Bigg)\\
&\leq \quad \frac{P_\mu Q_\eta}{(P_m-P_\mu)(Q_n-Q_\eta)}D\left(\sigma_{\mu\eta},\sigma_{\mu n}\right) + \frac{1}{(P_m-P_\mu)(Q_n-Q_\eta)}\sum_{i=\mu+1}^{m}\sum_{j=\eta+1}^{n} p_iq_jD\left(u_{mn},u_{ij}\right)\\
&+ \quad D\Bigg(\left(\frac{P_mQ_n}{(P_m-P_\mu)(Q_n-Q_\eta)} + \frac{P_\mu Q_\eta}{(P_m-P_\mu)(Q_n-Q_\eta)} + \frac{Q_\eta}{Q_n-Q_\eta} + \frac{P_\mu}{P_m-P_\mu}\right)\sigma_{mn},\\
&\qquad \sigma_{mn} + \frac{P_\mu Q_\eta}{(P_m-P_\mu)(Q_n-Q_\eta)}\left(\sigma_{m\eta}+\sigma_{mn}\right) + \frac{Q_\eta}{Q_n-Q_\eta}\left(\sigma_{m\eta}+\sigma_{mn}\right) + \frac{P_\mu}{P_m-P_\mu}\left(\sigma_{\mu n}+\sigma_{mn}\right)\Bigg),
\end{aligned}
$$

which require to Lemma 3.1(ii). □

4 Main results

The rest of this paper is constructed by considering the following headings for $(\overline{N},p,q)$ summable double sequences of fuzzy numbers:

(a) Determining certain subsets $\mathcal{T}\{u_{mn}\}$ of double sequence space of fuzzy numbers and certain subclasses $\mathcal{C}\{p_m,q_n\}$ of single sequence space of real numbers having the property: "Under conditions $\mathcal{C}\{p_m,q_n\}$, a double sequence $(u_{mn})\in\mathcal{T}\{u_{mn}\}$ of fuzzy numbers being $(\overline{N},p,q)$ summable to $l\in E^1$ is also P-convergent to same value."

(b) Demonstrating with examples how the subsets $\mathcal{T}\{u_{mn}\}$ and the subclasses $\mathcal{C}\{p_m,q_n\}$ change for special means occurring depends on choosing of weight sequences (p_m) and (q_n).

(c) Proving the property given in (a) for the related subsets and subclasses.

(d) Applying the $(\overline{N},p,q)$ method to prove fuzzy Korovkin-type approximation theorem in the space of fuzzy positive linear operators.

Herein, we firstly introduce the concepts of slow oscillation relative to (P_m) and (Q_n) for double sequences of fuzzy numbers, and we make a few observations about the relation between them.
A double sequence (u_{mn}) of fuzzy numbers is said to be slowly oscillating relative to both (P_m) and (Q_n) provided that

$$\lim_{\substack{\lambda\to1^+\\ \kappa\to1^+}}\limsup_{m,n\to\infty}\max_{\substack{P_m\leq P_i\leq\lambda P_m\\ Q_n\leq Q_j\leq\kappa Q_n}} D\left(u_{ij},u_{mn}\right)=0; \tag{4}$$

that is, for each $\epsilon>0$ there exist $n_0=n_0(\epsilon)\in\mathbb{N}$, $\lambda=\lambda(\epsilon)>1$ and $\kappa=\kappa(\epsilon)>1$ such that

$$D(u_{ij},u_{mn})\leq\epsilon \text{ whenever } n_0\leq m\leq i,\ \ n_0\leq n\leq j \text{ and } 1\leq\frac{P_i}{P_m}\leq\lambda,\ \ 1\leq\frac{Q_j}{Q_n}\leq\kappa.$$

Condition (4) is equivalent to

$$\lim_{\substack{\lambda\to1^-\\ \kappa\to1^-}}\limsup_{m,n\to\infty}\max_{\substack{\lambda P_m<P_i\leq P_m\\ \kappa Q_n<Q_j\leq Q_n}} D\left(u_{mn},u_{ij}\right)=0. \tag{4$'$}$$

A double sequence (u_{mn}) of fuzzy numbers is said to be slowly oscillating relative to (P_m) provided that

$$\lim_{\lambda\to 1^+}\limsup_{m,n\to\infty}\max_{P_m\le P_i\le \lambda P_m} D\left(u_{in},u_{mn}\right)=0, \tag{5}$$

or equivalently,

$$\lim_{\lambda\to 1^-}\limsup_{m,n\to\infty}\max_{\lambda P_m< P_i\le P_m} D\left(u_{mn},u_{in}\right)=0, \tag{5$'$}$$

besides it is said to be slowly oscillating relative to (P_m) in the strong sense if (5) is satisfied with

$$\max_{\substack{P_m\le P_i\le \lambda P_m\\ Q_n\le Q_j\le \kappa Q_n}} D\left(u_{ij},u_{mj}\right) \quad \text{instead of} \quad \max_{P_m\le P_i\le \lambda P_m} D\left(u_{in},u_{mn}\right). \tag{6}$$

In a similar way, the concept of slow oscillation relative to (Q_n) *(in the strong sense)* for a double sequence of fuzzy numbers can be analogously defined.

Observation 4.1. *If a double sequence (u_{mn}) of fuzzy numbers is slowly oscillating relative to (P_m) in the strong sense and slowly oscillating relative to (Q_n), then (u_{mn}) is slowly oscillating relative to both (P_m) and (Q_n).*

Proof. For all large enough m and n, that is, $m,n\ge n_0$, $\lambda>1$, and $\kappa>1$, we find by means of Proposition 2.3 (ii) and (iv)

$$\begin{aligned}\max_{\substack{P_m\le P_i\le \lambda P_m\\ Q_n\le Q_j\le \kappa Q_n}} D\left(u_{ij},u_{mn}\right) &= \max_{\substack{P_m\le P_i\le \lambda P_m\\ Q_n\le Q_j\le \kappa Q_n}} D\left(u_{ij}+u_{mj},u_{mj}+u_{mn}\right)\\ &\le \max_{\substack{P_m\le P_i\le \lambda P_m\\ Q_n\le Q_j\le \kappa Q_n}} D\left(u_{ij},u_{mj}\right)+\max_{Q_n\le j\le \kappa Q_n} D\left(u_{mj},u_{mn}\right).\end{aligned} \tag{7}$$

Taking lim sup and limit of both sides of (7) as $m,n\to\infty$ and $\lambda,\kappa\to 1^+$ respectively, we attain that the terms on the right-hand side of (7) are equal to 0. Therefore, we reach that (u_{mn}) is slowly oscillating relative to both (P_m) and (Q_n). □

In harmony with that, we can give the following lemma without proof, as well.

Observation 4.2. *If a double sequence (u_{mn}) of fuzzy numbers is slowly oscillating relative to (P_m) and slowly oscillating relative to (Q_n) in the strong sense, then (u_{mn}) is slowly oscillating relative to both (P_m) and (Q_n).*

Here, we formulate our main results for double sequences of fuzzy numbers as follows:

Theorem 4.3. *Let $(p_m),(q_n)\in SVA_{reg(\alpha)}$ and a double sequence (u_{mn}) of fuzzy numbers be $(\overline{N},p,q)$ summable to a number $\ell\in E^1$. If (u_{mn}) is slowly oscillating relative to (P_m), slowly oscillating relative to (Q_n), and slowly oscillating relative to (P_m) or (Q_n) in the strong sense, then (u_{mn}) is P-convergent to ℓ.*

Theorem 4.4. *Let $(p_m),(q_n)\in SVA_{reg(\alpha)}$ and a double sequence (u_{mn}) of fuzzy numbers be $(\overline{N},p,q)$ summable to a number $\ell\in E^1$. If (u_{mn}) satisfies conditions*

$$\frac{P_m}{p_m}D\left(u_{mn},u_{m-1,n}\right)=O(1) \quad \text{and} \quad \frac{Q_n}{q_n}D\left(u_{mn},u_{m,n-1}\right)=O(1), \tag{8}$$

then (u_{mn}) is P-convergent to ℓ.

Example 4.5. *Some means occurring depends on choosing of the sequences (p_m) and (q_m) are listed with their corresponding subclasses $\mathcal{C}\{p_m,q_n\}$ and subsets $\mathcal{T}\{u_{mn}\}$ for double sequences of fuzzy numbers as follows.*

(i) *In case $p_m=q_n=1$, it leads to the $(C,1,1)$ means of a double sequence of fuzzy numbers where $P_mQ_n=(m+1)(n+1)$ for all $m,n\in\mathbb{N}$. Under such circumstances, conditions in Theorem 4.3 correspond to slow oscillation conditions in senses $(1,0),(0,1)$ and in the strong sense $(1,0)$ or $(0,1)$, besides conditions (8) correspond to conditions*

$$mD\left(u_{mn},u_{m-1,n}\right)=O(1) \quad \text{and} \quad nD\left(u_{mn},u_{m,n-1}\right)=O(1).$$

As a result of Theorem 4.3 and Theorem 4.4, the above conditions are Tauberian conditions for the $(C,1,1)$ summability of double sequences of fuzzy numbers.

(ii) *In case* $p_m q_n = 1/((m+1)(n+1))$, *it leads to the logarithmic means of a double sequence of fuzzy numbers where* $P_m Q_n \sim \log m \log n$ *for all* $m, n \in \mathbb{N}$. *Under such circumstances, conditions in Theorem 4.3 correspond to conditions of slow oscillation with respect to summability* $(L,1)$ *in sense* $(1,0),(0,1)$ *and the strong sense* $(1,0)$ *or* $(0,1)$, *i.e.,*

$$\lim_{\lambda\to 1^+} \limsup_{m,n\to\infty} \max_{m\le i\le m^\lambda} D(u_{in}, u_{mn}) = 0, \tag{9}$$

$$\lim_{\kappa\to 1^+} \limsup_{m,n\to\infty} \max_{n\le j\le n^\kappa} D(u_{mj}, u_{mn}) = 0, \tag{10}$$

and shall we say

$$\lim_{\lambda,\kappa\to 1^+} \limsup_{m,n\to\infty} \max_{\substack{m\le i\le m^\lambda \\ n\le j\le n^\kappa}} D(u_{ij}, u_{in}) = 0, \tag{11}$$

which was discussed by Yavuz and Çoşkun [50] and Sezer [40] for the logarithmic summability of single sequences of fuzzy numbers in different aspects. Additionally, conditions (8) correspond to conditions

$$m\log(m+1)D\left(u_{mn}, u_{m-1,n}\right) = O(1) \quad \textit{and} \quad n\log(n+1)D\left(u_{mn}, u_{m,n-1}\right) = O(1). \tag{12}$$

As a result of Theorem 4.3 and Theorem 4.4, the above conditions are Tauberian conditions for the logarithmic summability of double sequences of fuzzy numbers.

Now, we give the proofs of our main results for double sequences of fuzzy numbers.

***Proof of Theorem* 4.3**. Assume that $(u_{mn}) \in w^2(F)$ being $(\overline{N}, p, q)$ summable to $\ell \in E^1$ is slowly oscillating relative to (P_m) and (Q_n) and slowly oscillating relative to (Q_n) in the strong sense. In order to prove that (u_{mn}) is P-convergent to same number, we firstly consider the distance $D\left(u_{mn}, \sigma_{mn}\right)$. By means of Lemma 3.1, we have for $\mu > m$ and $\eta > n$

$$\begin{aligned}
D\left(u_{mn}, \sigma_{mn}\right) &\le \frac{P_\mu Q_\eta}{(P_\mu - P_m)(Q_\eta - Q_n)}\left(D\left(\sigma_{\mu\eta}, \sigma_{\mu n}\right) + D\left(\sigma_{mn}, \sigma_{m\eta}\right)\right) + \frac{P_\mu}{P_\mu - P_m} D\left(\sigma_{\mu n}, \sigma_{mn}\right) \\
&+ \frac{Q_\eta}{Q_\eta - Q_n} D\left(\sigma_{m\eta}, \sigma_{mn}\right) + \frac{1}{(P_\mu - P_m)(Q_\eta - Q_n)} \sum_{i=m+1}^{\mu} \sum_{j=n+1}^{\eta} p_i q_j D\left(u_{ij}, u_{mn}\right) \\
&= \frac{P_\mu Q_\eta}{(P_\mu - P_m)(Q_\eta - Q_n)}\left(D\left(\sigma_{\mu\eta}, \sigma_{\mu n}\right) + D\left(\sigma_{mn}, \sigma_{m\eta}\right)\right) + \frac{P_\mu}{P_\mu - P_m} D\left(\sigma_{\mu n}, \sigma_{mn}\right) \\
&+ \frac{Q_\eta}{Q_\eta - Q_n} D\left(\sigma_{m\eta}, \sigma_{mn}\right) + \frac{1}{(P_\mu - P_m)(Q_\eta - Q_n)} \sum_{i=m+1}^{\mu} \sum_{j=n+1}^{\eta} p_i q_j D\left(u_{ij} + u_{in}, u_{mn} + u_{in}\right) \\
&\le \frac{P_\mu Q_\eta}{(P_\mu - P_m)(Q_\eta - Q_n)}\left(D\left(\sigma_{\mu\eta}, \sigma_{\mu n}\right) + D\left(\sigma_{mn}, \sigma_{m\eta}\right)\right) + \frac{P_\mu}{P_\mu - P_m} D\left(\sigma_{\mu n}, \sigma_{mn}\right) \\
&+ \frac{Q_\eta}{Q_\eta - Q_n} D\left(\sigma_{m\eta}, \sigma_{mn}\right) + \max_{\substack{m\le i\le \mu \\ n\le j\le \eta}} D\left(u_{ij}, u_{in}\right) + \max_{m\le i\le \mu} D\left(u_{in}, u_{mn}\right).
\end{aligned} \tag{13}$$

Putting

$$\mu = \operatorname{argmin}\left\{P_i \ge \left(1+\frac{\delta}{2}\right) P_m\right\} = \min\left\{i > m : P_i \ge \left(1+\frac{\delta}{2}\right) P_m\right\},$$

and

$$\eta = \operatorname{argmin}\left\{Q_j \ge \left(1+\frac{\gamma}{2}\right) Q_n\right\} = \min\left\{j > n : Q_j \ge \left(1+\frac{\gamma}{2}\right) Q_n\right\},$$

with $\delta, \gamma > 0$, we get $\mu \ge m$, $\eta \ge n$ and $P_{\mu-1} < \left(1+\frac{\delta}{2}\right) P_m$ and $Q_{\eta-1} < \left(1+\frac{\gamma}{2}\right) Q_n$. Moreover, on account of $(p_m), (q_n) \in SVA_{reg(\alpha)}$, we have $\frac{P_m}{P_{m-1}} \le 1 + \frac{\delta}{4}$ and $\frac{Q_n}{Q_{n-1}} \le 1 + \frac{\gamma}{4}$ for sufficiently large m, n. By means of Lemma 2.7, it can be easily seen that $p_m/P_m \to 0$ and $q_n/Q_n \to 0$ as $m, n \to \infty$, as well. As a result of these findings, we obtain by Lemma 2.7 that

$$P_\mu = \frac{P_\mu}{P_{\mu-1}} P_{\mu-1} \le \left(1+\frac{\delta}{4}\right) P_m \left(1+\frac{\delta}{2}\right) \le P_m(1+\delta) = \lambda P_m, \tag{14}$$

or in another saying,

$$\frac{P_\mu}{P_m} = \frac{P_{\mu-1}}{P_m} + \frac{p_\mu}{P_m} \leq \lambda + \frac{p_\mu}{P_\mu}\frac{P_\mu}{P_m} = \lambda(1+o(1)) \quad \text{and} \quad \frac{P_\mu}{P_m} \geq \left(1+\frac{\delta}{2}\right) = \frac{\lambda+1}{2}, \tag{15}$$

and

$$Q_\eta = \frac{Q_\eta}{Q_{\eta-1}}Q_{\eta-1} \leq \left(1+\frac{\gamma}{4}\right) Q_n \left(1+\frac{\gamma}{2}\right) \leq Q_n(1+\gamma) = \kappa Q_n, \tag{16}$$

or in another saying,

$$\frac{Q_\eta}{Q_n} = \frac{Q_{\eta-1}}{Q_n} + \frac{q_\eta}{Q_n} \leq \kappa + \frac{q_\eta}{Q_\eta}\frac{Q_\eta}{Q_n} = \kappa(1+o(1)) \quad \text{and} \quad \frac{Q_\eta}{Q_n} \geq \left(1+\frac{\gamma}{2}\right) = \frac{\kappa+1}{2}, \tag{17}$$

for λ, $\kappa > 1$. From the assumption, we have $D(\sigma_{mn}, \ell) = o(1)$ and so, $D(\sigma_{ij}, \sigma_{mn}) = D(\sigma_{ij}+\ell, \sigma_{mn}+\ell) \leq D(\sigma_{ij}, \ell) + D(\sigma_{mn}, \ell) = o(1)$ for $i = m$ or μ and $j = n$ or η as $m, n \to \infty$. Since the sequences (P_m) and (Q_n) are strictly increasing sequences, we attain from (13) that

$$\begin{aligned} D(u_{mn}, \sigma_{mn}) &\leq \frac{P_\mu Q_\eta}{(P_\mu - P_m)(Q_\eta - Q_n)} \left(D(\sigma_{\mu\eta}, \sigma_{\mu n}) + D(\sigma_{mn}, \sigma_{m\eta})\right) + \frac{P_\mu}{P_\mu - P_m} D(\sigma_{\mu n}, \sigma_{mn}) \\ &+ \frac{Q_\eta}{Q_\eta - Q_n} D(\sigma_{m\eta}, \sigma_{mn}) + \max_{\substack{P_m \leq P_i \leq \lambda P_m \\ Q_n \leq Q_j \leq \kappa Q_n}} D(u_{ij}, u_{in}) + \max_{P_m \leq P_i \leq \lambda P_m} D(u_{in}, u_{mn}) \\ &\leq \frac{4\lambda\kappa + \lambda(\kappa-1) + (\lambda-1)\kappa}{(\lambda-1)(\kappa-1)}(1+o(1))o(1) \\ &+ \max_{\substack{P_m \leq P_i \leq \lambda P_m \\ Q_n \leq Q_j \leq \kappa Q_n}} D(u_{ij}, u_{in}) + \max_{P_m \leq P_i \leq \lambda P_m} D(u_{in}, u_{mn}), \end{aligned} \tag{18}$$

where

$$\frac{P_\mu}{P_\mu - P_m} = \frac{P_\mu/P_m}{P_\mu/P_m - 1} \leq \frac{2\lambda}{(\lambda-1)}(1+o(1)) \quad \text{and} \quad \frac{Q_\eta}{Q_\eta - Q_n} = \frac{Q_\eta/Q_n}{Q_\eta/Q_n - 1} \leq \frac{2\kappa}{(\kappa-1)}(1+o(1)).$$

If we take lim sup of both sides of inequality (18) as $m, n \to \infty$, then we reach for any $\lambda, \kappa > 1$

$$\limsup_{m,n\to\infty} D(u_{mn}, \sigma_{mn}) \leq \limsup_{m,n\to\infty} \max_{\substack{P_m \leq P_i \leq \lambda P_m \\ Q_n \leq Q_j \leq \kappa Q_n}} D(u_{ij}, u_{in}) + \limsup_{m,n\to\infty} \max_{P_m \leq P_i \leq \lambda P_m} D(u_{in}, u_{mn}).$$

If we get limit of both sides of last inequality as $\lambda, \kappa \to 1^+$, then we find

$$\limsup_{m,n\to\infty} D(u_{mn}, \sigma_{mn}) \leq 0, \tag{19}$$

due to the fact that (u_{mn}) is slowly oscillating relative to (P_m) and (Q_n) and slowly oscillating relative to (Q_n) in the strong sense. Therefore, we conclude that (u_{mn}) is P-convergent to ℓ.
Following a similar procedure to above for $\tilde{\mu} < m$ and $\tilde{\eta} < n$, we have by means of Lemma 3.1

$$\begin{aligned} D(u_{mn}, \sigma_{mn}) &\leq \frac{P_{\tilde{\mu}} Q_{\tilde{\eta}}}{(P_m - P_{\tilde{\mu}})(Q_n - Q_{\tilde{\eta}})} \left(D(\sigma_{\tilde{\mu}\tilde{\eta}}, \sigma_{\tilde{\mu}n}) + D(\sigma_{mn}, \sigma_{m\tilde{\eta}})\right) + \frac{P_{\tilde{\mu}}}{P_m - P_{\tilde{\mu}}} D(\sigma_{mn}, \sigma_{\tilde{\mu}n}) \\ &+ \frac{Q_{\tilde{\eta}}}{Q_n - Q_{\tilde{\eta}}} D(\sigma_{mn}, \sigma_{m\tilde{\eta}}) + \frac{1}{(P_m - P_{\tilde{\mu}})(Q_n - Q_{\tilde{\eta}})} \sum_{i=\tilde{\mu}+1}^{m} \sum_{j=\tilde{\eta}+1}^{n} p_i q_j D(u_{mn}, u_{ij}) \\ &\leq \frac{P_{\tilde{\mu}} Q_{\tilde{\eta}}}{(P_m - P_{\tilde{\mu}})(Q_n - Q_{\tilde{\eta}})} \left(D(\sigma_{\tilde{\mu}\tilde{\eta}}, \sigma_{\tilde{\mu}n}) + D(\sigma_{mn}, \sigma_{m\tilde{\eta}})\right) + \frac{P_{\tilde{\mu}}}{P_m - P_{\tilde{\mu}}} D(\sigma_{mn}, \sigma_{\tilde{\mu}n}) \\ &+ \frac{Q_{\tilde{\eta}}}{Q_n - Q_{\tilde{\eta}}} D(\sigma_{mn}, \sigma_{m\tilde{\eta}}) + \max_{\tilde{\mu} \leq i \leq m} D(u_{mn}, u_{in}) + \max_{\substack{\tilde{\mu} \leq i \leq m \\ \tilde{\eta} \leq j \leq n}} D(u_{in}, u_{ij}). \end{aligned} \tag{20}$$

Putting

$$\tilde{\mu} = \operatorname{argmax}\{P_m \geq \left(1+\frac{\delta}{2}\right) P_i\} = \max\{m > i : P_m \geq \left(1+\frac{\delta}{2}\right) P_i\},$$

and

$$\tilde{\eta} = \operatorname{argmax}\{Q_n \geq \left(1+\frac{\gamma}{2}\right) Q_j\} = \max\{n > j : Q_n \geq \left(1+\frac{\delta}{2}\right) Q_j\},$$

with $\delta, \gamma > 0$, we get $\tilde{\mu} \leq m$, $\tilde{\eta} \leq n$ and $\left(1+\frac{\delta}{2}\right) P_{\tilde{\mu}+1} > P_m$ and $\left(1+\frac{\gamma}{2}\right) Q_{\tilde{\eta}+1} > Q_n$. Moreover, on account of $(p_m), (q_n) \in SVA_{reg(\alpha)}$, we have $\frac{P_{\tilde{\mu}+1}}{P_{\tilde{\mu}}} \leq 1+\frac{\delta}{4}$ and $\frac{Q_{\tilde{\eta}+1}}{Q_{\tilde{\eta}}} \leq 1+\frac{\gamma}{4}$ for sufficiently large $\tilde{\mu}, \tilde{\eta}$. As a result of these findings, we obtain by Lemma 2.7 that

$$P_{\tilde{\mu}} = \frac{P_{\tilde{\mu}}}{P_{\tilde{\mu}+1}} P_{\tilde{\mu}+1} \geq \frac{1}{\left(1+\frac{\delta}{4}\right)} \frac{1}{\left(1+\frac{\delta}{2}\right)} P_m \geq \frac{1}{\left(1+\frac{\delta}{2}\right)^2} P_m = \tilde{\lambda} P_m \quad \text{and} \quad \frac{P_{\tilde{\mu}}}{P_m} \leq \frac{1}{\left(1+\frac{\delta}{2}\right)} = \sqrt{\tilde{\lambda}},$$

and

$$Q_{\tilde{\eta}} = \frac{Q_{\tilde{\eta}}}{Q_{\tilde{\eta}+1}} Q_{\tilde{\eta}+1} \geq \frac{1}{\left(1+\frac{\gamma}{4}\right)} \frac{1}{\left(1+\frac{\gamma}{2}\right)} Q_n \geq \frac{1}{\left(1+\frac{\gamma}{2}\right)^2} Q_n = \tilde{\kappa} Q_n \quad \text{and} \quad \frac{Q_{\tilde{\eta}}}{Q_n} \leq \frac{1}{\left(1+\frac{\gamma}{2}\right)} = \sqrt{\tilde{\kappa}},$$

for $0 < \tilde{\lambda}, \tilde{\kappa} < 1$. From the assumption, we have $D(\sigma_{mn}, \ell) = o(1)$ and so, $D(\sigma_{ij}, \sigma_{mn}) = D(\sigma_{ij} + \ell, \sigma_{mn} + \ell) \leq D(\sigma_{ij}, \ell) + D(\sigma_{mn}, \ell) = o(1)$ for $i = m$ or $\tilde{\mu}$ and $j = n$ or $\tilde{\eta}$ as $\tilde{\mu}, \tilde{\eta} \to \infty$. Since the sequences (P_m) and (Q_n) are strictly increasing sequences, we attain from (20) that

$$\begin{aligned} D(u_{mn}, \sigma_{mn}) &\leq \frac{P_{\tilde{\mu}} Q_{\tilde{\eta}}}{(P_m - P_{\tilde{\mu}})(Q_n - Q_{\tilde{\eta}})} \left(D(\sigma_{\tilde{\mu}\tilde{\eta}}, \sigma_{\tilde{\mu}n}) + D(\sigma_{mn}, \sigma_{m\tilde{\eta}})\right) + \frac{P_{\tilde{\mu}}}{P_m - P_{\tilde{\mu}}} D(\sigma_{mn}, \sigma_{\tilde{\mu}n}) \\ &+ \frac{Q_{\tilde{\eta}}}{Q_n - Q_{\tilde{\eta}}} D(\sigma_{mn}, \sigma_{m\tilde{\eta}}) + \max_{\tilde{\lambda} P_m < P_i \leq P_m} D(u_{mn}, u_{in}) + \max_{\substack{\tilde{\lambda} P_m \leq P_i \leq P_m \\ \tilde{\kappa} Q_n \leq Q_j \leq Q_n}} D(u_{in}, u_{ij}) \\ &\leq \frac{\sqrt{\tilde{\lambda}} + \sqrt{\tilde{\kappa}}}{(1-\sqrt{\tilde{\lambda}})(-1\sqrt{\tilde{\lambda}})} o(1) + \max_{\tilde{\lambda} P_m < P_i \leq P_m} D(u_{mn}, u_{in}) + \max_{\substack{\tilde{\lambda} P_m \leq P_i \leq P_m \\ \tilde{\kappa} Q_n \leq Q_j \leq Q_n}} D(u_{in}, u_{ij}), \end{aligned} \tag{21}$$

for $0 < \tilde{\lambda}, \tilde{\kappa} < 1$. If we take lim sup of both sides of inequality (21) as $\tilde{\mu}, \tilde{\eta} \to \infty$, then we reach for any $0 < \tilde{\lambda}, \tilde{\kappa} < 1$

$$\limsup_{m,n\to\infty} D(u_{mn}, \sigma_{mn}) \leq \limsup_{m,n\to\infty} \max_{\tilde{\lambda} P_m < P_i \leq P_m} D(u_{mn}, u_{in}) + \max_{\substack{\tilde{\lambda} P_m \leq P_i \leq P_m \\ \tilde{\kappa} Q_n \leq Q_j \leq Q_n}} D(u_{in}, u_{ij}).$$

If we get limit of both sides of last inequality as $\tilde{\lambda}, \tilde{\kappa} \to 1^-$, then we find

$$\limsup_{m,n\to\infty} D(u_{mn}, \sigma_{mn}) \leq 0, \tag{22}$$

due to the fact that (u_{mn}) is slowly decreasing relative to (P_m) and (Q_n) and slowly decreasing relative to (Q_n) in the strong sense.If we consider (19) and (22), then we conclude for both of $\lambda, \kappa > 1$ and $0 < \tilde{\lambda}, \tilde{\kappa} < 1$

$$\lim_{m,n\to\infty} D(u_{mn}, \sigma_{mn}) = 0,$$

which means that (u_{mn}) is P-convergent to ℓ. □

Proof of Theorem 4.4. Assume that $(u_{mn}) \in w^2(F)$ being $(\overline{N}, p, q)$ summable to $\ell \in E^1$ satisfies conditions (8). If we indicate that conditions (8) imply slow oscillation condition relative to (P_m) and (Q_n) and slow oscillation condition relative to (Q_n) in the strong sense, then we could this proof with the help of Theorem 4.3.
Putting $\mu = \operatorname{argmin}\left\{P_i \geq \left(1+\frac{\delta}{2}\right) P_m\right\}$ and $\eta = \operatorname{argmin}\left\{Q_j \geq \left(1+\frac{\gamma}{2}\right) Q_n\right\}$ = with $\delta, \gamma > 0$, we can observe inequalities (15) and (17). Then, we have for $n_0 \leq m \leq i \leq \mu$ and $n_0 \leq n$

$$D(u_{in}, u_{mn}) \leq \sum_{k=m+1}^{i} D(u_{kn}, u_{k-1,n}) \leq M_1 \sum_{k=m+1}^{i} \frac{p_k}{P_k} \leq M_1 \left(\frac{P_\mu}{P_m} - 1\right) \leq M_1(\lambda - 1 + \lambda o(1)),$$

for any constant $M_1 > 0$ and $1 + \delta = \lambda > 1$. If we take lim sup and limit of both sides of last inequality as $m, n \to \infty$ and $\lambda \to 1^+$ respectively, then we reach

$$\lim_{\lambda \to 1^+} \limsup_{m,n \to \infty} \max_{P_m \leq P_i \leq \lambda P_m} D\left(u_{in}, u_{mn}\right) = 0,$$

which means that (u_{mn}) is slowly oscillating relative to (P_m). Similarly, we obtain $n_0 \leq n \leq j \leq \eta$ and $n_0 \leq m$

$$D\left(u_{mj}, u_{mn}\right) \leq \sum_{r=n+1}^{j} D\left(u_{mr}, u_{m,r-1}\right) \leq M_2 \sum_{r=n+1}^{j} \frac{q_r}{Q_r} \leq M_2 \left(\frac{Q_\eta}{Q_n} - 1\right) \leq M_2(\kappa - 1 + \kappa o(1)),$$

for any constant $M_2 > 0$ and $1 + \gamma = \kappa > 1$. If we get lim sup and limit of both sides of last inequality as $m, n \to \infty$ and $\kappa \to 1^+$ respectively, then we reach

$$\lim_{\kappa \to 1^+} \limsup_{m,n \to \infty} \max_{Q_n \leq Q_j \leq \kappa Q_n} D\left(u_{mj}, u_{mn}\right) = 0,$$

which means that (u_{mn}) is slowly oscillating relative to (Q_n). Therefore, we conclude with the help of Theorem 4.3 that (u_{mn}) is P-convergent to ℓ. □

5 Applications to fuzzy Korovkin-type approximation theorem

Korovkin-type theorems have an important place in approximation theory that gives conditions for uniform approximation of continuous functions defined on a compact metric space using sequences of positive linear operators. The most notable instances of such approximation theorems were formulated for single sequences of positive linear operators by M. Mursaleen et al. [30] via weighted statistical convergence; by Srivastava et al. [41] via statistical deferred Nörlund summability method; by Srivastava et al. [42] via statistical probability convergence for the deferred Nörlund summability method; for double sequences of positive linear operators by Belen et al. [7] and Mursaleen and Alotaibi [29] via statistical A-summability method. During the ongoing process, the results achieved in the field of fuzzy Korovkin-type approximation theory were extended to encompass their non-fuzzy counterpart, as well. Laid the foundations by Anastassiou [2, 3], Anastassiou and Duman [5] and Anastassiou et al. [4], fuzzy Korovkin-type approximation theory was improved by Demirci and Karakuş [19] for fuzzy-valued functions of two variables. In this context, Demirci and Karakuş [19] presented a fuzzy Korovkin-type approximation theorem for fuzzy positive linear operators by using A-statistical convergence for four-dimensional summability matrices. Besides, Talo et al. [45] applied the statistical version of the $(\overline{N}, p, q)$ summability method to prove a fuzzy Korovkin-type approximation theorem for a double sequence of fuzzy positive linear operators.

Let $C(K)$, the set of all continuous real-valued functions on K where K is a compact subset of $\mathbb{R}^2$, be equipped with the supremum norm

$$\|f\|_\infty = \sup_{(x,y) \in K} |f(x,y)|.$$

Consider a fuzzy-valued function of two variable $f : K \to E^1$. Then, $f \in C(K)$ can be represented by $[f(x,y)]_\alpha = [f_\alpha^-(x,y), f_\alpha^+(x,y)]$, for each $(x,y) \in K$ and $\alpha \in [0,1]$. Let $C_{\mathcal{F}}(K)$ be the set of all continuous fuzzy-valued functions on K. For the intervals $[f_\alpha^-(x,y), f_\alpha^+(x,y)], [g_\alpha^-(x,y), g_\alpha^+(x,y)]$, the distance based on the above supremum norm is defined by

$$D^*(f,g) = \sup_{(x,y) \in K} D(f(x,y), g(x,y)) \quad = \sup_{(x,y) \in K} \sup_{\alpha \in [0,1]} \max\left\{|f_\alpha^-(x,y) - g_\alpha^-(x,y)|, |f_\alpha^+(x,y) - f_\alpha^+(x,y)|\right\}.$$

Let $\mathcal{L} : C_{\mathcal{F}}(K) \to C_{\mathcal{F}}(K)$ be an operator.Then, $\mathcal{L}$ is said to be fuzzy positive linear operator if it satisfies the following $(i) - (ii)$ properties for any $f, g \in C_{\mathcal{F}}(K)$ and $(x,y) \in K$:

(i) $\mathcal{L}$ is fuzzy linear, i.e., for any $a, b \in \mathbb{R}$, $\mathcal{L}\left(af + bg; x, y\right) = a\mathcal{L}\left(f; x, y\right) + b\mathcal{L}\left(g; x, y\right)$.

(ii) The condition $f(x,y) \preceq g(x,y)$ implies $L(f; x, y) \preceq L(g; x, y)$.

Theorem 5.1. *Let $(\mathcal{L}_{mn})$ be a double sequence of fuzzy positive linear operators from $C_{\mathcal{F}}(K)$ into itself. Assume that there exists a corresponding double sequence $\tilde{\mathcal{L}}_{mn}$ of positive linear operators from $C(K)$ into itself with the property $\{\mathcal{L}_{mn}(f; x, y)\}_\alpha^\pm = \tilde{\mathcal{L}}_{mn}(f_\alpha^\pm; x, y)$ for all $(x,y) \in K, \alpha \in [0,1]$ and $f \in C_{\mathcal{F}}(K)$. Assume also that*

$$\lim_{m,n \to \infty} \left\| \frac{1}{P_m Q_n} \sum_{i=0}^{m} \sum_{j=0}^{n} p_i q_j \tilde{\mathcal{L}}_{ij}(e_k) - e_k \right\|_\infty = 0, \quad i = 0, 1, 2, 3, \tag{23}$$

where $e_0(x,y)=1, e_1(x,y)=x, e_2(x,y)=y, e_3(x,y)=x^2+y^2$. *Then, for any function* $f\in C_{\mathcal{F}}(K)$, *we have*

$$\lim_{m,n\to\infty} D^*\left(\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\mathcal{L}_{ij}(f),f\right)=0. \tag{24}$$

Proof. Let $f\in C_{\mathcal{F}}(K)$, $(x,y)\in K$ and $\alpha\in[0,1]$. Since $f_\alpha^{\pm}\in C(K)$, it follows from the hypothesis that for a given $\epsilon>0$ there exists a number $\delta>0$ such that $|f_\alpha^{\pm}(s,t)-f_\alpha^{\pm}(x,y)|<\epsilon$ for every $(s,t)\in K$ satisfying $\sqrt{(s-x)^2+(t-y)^2}<\delta$. By means of the continuity of $f_\alpha^{\pm}$ on K, we have $|f_\alpha^{\pm}(x,y)|\le M$ for all $(x,y)\in K$, where $0<M:=\|f_\alpha^{\pm}\|_\infty$.Then, we obtain for all $(s,t)\in K$ that

$$|f_\alpha^{\pm}(s,t)-f_\alpha^{\pm}(x,y)|\le\epsilon+\frac{2M}{\delta^2}\left((s-x)^2+(t-y)^2\right). \tag{25}$$

Using the linearity and positivity of the operators $\tilde{\mathcal{L}}_{mn}$ and taking into (25) account, we attain

$$\begin{aligned}
&\left|\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}\left(f_\alpha^{\pm};x,y\right)-f_\alpha^{\pm}(x,y)\right|\\
\le\;&\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}\left(\left|f_\alpha^{\pm}(s,t)-f_\alpha^{\pm}(x,y)\right|;x,y\right)+\left|f_\alpha^{\pm}(x,y)\right|\left|\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}(1;x,y)-1\right|\\
\le\;&\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}\left(\epsilon+\frac{2M}{\delta^2}\left((s-x)^2+(t-y)^2\right);x,y\right)+\left\|f_\alpha^{\pm}\right\|_\infty\left|\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}(1;x,y)-1\right|\\
\le\;&\epsilon+\epsilon\left|\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}(1;x,y)-1\right|+\frac{2M}{\delta^2}\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}\left((s-x)^2+(t-y)^2;x,y\right)\\
+\;&M\left|\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}(1;x,y)-1\right|\\
\le\;&\epsilon+(\epsilon+M)\left|\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}(1;x,y)-1\right|+\frac{2M}{\delta^2}\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}\left((s-x)^2+(t-y)^2;x,y\right)\\
=\;&\epsilon+(\epsilon+M)\left|\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}(1;x,y)-1\right|\\
+\;&\frac{2M}{\delta^2}\left(\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}(s^2+t^2;x,y)-2x\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}(s;x,y)\right.\\
-\;&\left.2y\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}(t;x,y)+(x^2+y^2)\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}(1;x,y)\right)\\
\le\;&\epsilon+(\epsilon+M)\left|\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}(1;x,y)-1\right|\\
+\;&\frac{2M}{\delta^2}\left(\left|\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}(s^2+t^2;x,y)-(x^2+y^2)\right|+2|x|\left|\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}(s;x,y)-x\right|\right.\\
+\;&\left.2|y|\left|\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}(t;x,y)-y\right|+(x^2+y^2)\left|\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}(1;x,y)-1\right|\right)\\
\le\;&\epsilon+\left(\epsilon+M+\frac{4MC^2}{\delta^2}\right)\left|\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}(e_0;x,y)-e_0(x,y)\right|
\end{aligned}$$

$$+ \quad \frac{2M}{\delta^2}\left|\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}(e_3;x,y)-e_3(x,y)\right|+\frac{4MC}{\delta^2}\left|\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}(e_1;x,y)-e_1(x,y)\right|$$

$$+ \quad \frac{4MC}{\delta^2}\left|\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}(e_2;x,y)-e_2(x,y)\right|,$$

where $C := \max\{|x|, |y|\}$. If we take

$$K(\epsilon) := \max\left\{\epsilon + M + \frac{4MC^2}{\delta^2}, \frac{2M}{\delta^2}, \frac{4MC}{\delta^2}\right\},$$

then we reach by taking supremum over $(x, y) \in K$

$$\left\|\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}\left(f_\alpha^{\pm};x,y\right)-f_\alpha^{\pm}(x,y)\right\|_\infty \leq \epsilon + K(\epsilon)\sum_{k=0}^{3}\left\|\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}(e_k;x,y)-e_k(x,y)\right\|_\infty, \quad (26)$$

for $\alpha \in [0, 1]$. From definition of the metric D^*, we find by means of (26)

$$\begin{aligned}
& D^*\left(\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\mathcal{L}_{ij}(f), f\right) = \sup_{(x,y)\in K} D\left(\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\mathcal{L}_{ij}(f), f\right)\\
= & \sup_{\substack{(x,y)\in K\\ \alpha\in[0,1]}}\max\left\{\left|\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}\left(f_\alpha^{-}\right)-f_\alpha^{-}\right|, \left|\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}\left(f_\alpha^{+}\right)-f_\alpha^{+}\right|\right\}\\
= & \sup_{\alpha\in[0,1]}\max\left\{\left\|\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}\left(f_\alpha^{-}\right)-f_\alpha^{-}\right\|_\infty, \left\|\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}\left(f_\alpha^{+}\right)-f_\alpha^{+}\right\|_\infty\right\}\\
\leq & \epsilon + K(\epsilon)\sum_{k=0}^{3}\left\|\frac{1}{P_mQ_n}\sum_{i=0}^{m}\sum_{j=0}^{n}p_iq_j\tilde{\mathcal{L}}_{ij}(e_k;x,y)-e_k(x,y)\right\|_\infty,
\end{aligned}$$

If we get limit of both sides of last inequality as $m, n \to \infty$, then we reach (24) due to the hypothesis (23). □

Example 5.2. *Let $(x, y) \in K = [0, 1] \times [0, 1]$ and $f \in C_{\mathcal{F}}(K)$. Consider the double sequence $(\mathcal{L}_{mn})$ of positive linear operators defined by $\mathcal{L}_{mn}(f; x, y) = (3/4 + \omega_{mn})\, B_{mn}(f; x, y)$ where (B_{mn}) are the Bernstein polynomials of two variables defined on $C(K)$ by*

$$B_{mn}(f;x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} f\left(\frac{i}{m}, \frac{j}{n}\right)\binom{m}{i}x^i(1-x)^{m-i}\binom{n}{j}y^j(1-y)^{n-j},$$

and (ω_{mn}) is a double sequence defined by $(\omega_{mn}) = \left(\sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\right)$. Then, it can be observe

$$\mathcal{L}_{mn}(e_0;x,y) = (3/4+\omega_{mn})\, e_0(x,y), \qquad \mathcal{L}_{mn}(e_1;x,y) = (3/4+\omega_{mn})\, e_1(x,y),$$

$$\mathcal{L}_{mn}(e_2;x,y) = (3/4+\omega_{mn})\, e_2(x,y), \qquad \mathcal{L}_{mn}(e_3;x,y) = (3/4+\omega_{mn})\left(e_3(x,y) + \frac{x-x^2}{m} + \frac{y-y^2}{n}\right).$$

If we take $p_m = q_n = 1$, then we can say that the double sequence (ω_{mn}) is $(C, 1, 1)$ summable to $1/4$ but not P-convergent. Hence, we conclude

$$\lim_{m,n\to\infty}\left\|\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{\mathcal{L}}_{ij}(e_k) - e_k\right\|_\infty = 0, \quad i = 0, 1, 2, 3,$$

which means by Theorem 5.1 that

$$\lim_{m,n\to\infty} D^*\left(\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}\mathcal{L}_{ij}(f), f\right) = 0.$$

6 Conclusion

In this paper, we have discussed relation between the $(\overline{N}, p, q)$ method and P-convergence for double sequences of fuzzy numbers. As a beginning, we have presented two representations for the distance between general terms of double sequences (u_{mn}) and (σ_{mn}) of fuzzy numbers. We have introduced the concepts of slow oscillation relative to (P_m) and (Q_n) for double sequences of fuzzy numbers, and we have made a few observations about the relation between them. In the sequel, we have determined that conditions needed for $P-\lim \sigma_{mn}(u) = \ell$ to be $P-\lim u_{mn} = \ell$ are slow oscillation relative to (P_m) and (Q_n) and slow oscillation relative to (P_m) or (Q_n) in the strong sense under additional conditions $(p_m), (q_n) \in SVA_{reg(\alpha)}$. As special cases, we have deduced that if slow oscillation conditions in certain senses are replaced by

$$\frac{P_m}{p_m} D\left(u_{mn}, u_{m-1,n}\right) = O(1) \quad \text{and} \quad \frac{Q_n}{q_n} D\left(u_{mn}, u_{m,n-1}\right) = O(1),$$

the result remains the same under conditions $(p_m), (q_n) \in SVA_{reg(\alpha)}$. We have indicated with examples how Tauberian conditions imposed on (u_{mn}) and additional conditions imposed on $(p_m), (q_n)$ change for special means occurring based on choosing of weight sequences (p_m) and (q_n). In the last instance,we have applied the $(\overline{N}, p, q)$ method to prove a fuzzy Korovkin-type approximation theorem in the space of fuzzy positive linear operators.

In the recent times, Parida et al. [37, 38] have established some Tauberian theorems for statistical Cesàro integrability of continuous function, incorporating the concepts of oscillation behavior and De la Vallée Poussin mean of integral based on post-quantum calculus and locally convex spaces, respectively. Besides, Das et al. [17] have given a fuzzy Korovkin-type approximation theorem and another result for the fuzzy rate of convergence by using statistical $(C,1)(E,\mu)$ summability method in space of fuzzy-valued sequences. In the light of these significant contributions, we seek to advance the understanding of the $(\overline{N}, p, q)$ summability by introducing innovative concepts, techniques, and results that can enhance the existing knowledge and deepen this understanding in a statistical framework in Tauber theory. Our aim in future works is to contribute to the development and enrichment of Tauberian theory by considering various spaces and extending the applicability of statistical $(\overline{N}, p, q)$ summability in different mathematical contexts.

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