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Subcategories of the category of stratified (L, M)-semiuniform convergence tower spaces

L. Zhang[1], B. Pang[2] and W. Li[3]

[1] *School of Science, Qingdao University of Technology, Qingdao 266520, China*
[2] *Beijing Key Laboratory on MCAACI, School of Mathematics and Statistics, Beijing Institute of Technology, Beijing, China*
[3] *School of Mathematical Sciences, Mudanjiang Normal University, Mudanjiang 157011, China*

zhanglin116228@163.com, pangbin1205@163.com,1822811994@qq.com

Abstract

In this paper, we propose the concepts of stratified (L, M)-semiuniform convergence spaces and stratified (L, M)-semiuniform limit tower spaces. It is shown that (1) the category $\mathrm{S}(L, M)$-**SUC** of stratified (L, M)-semiuniform convergence spaces can be embedded in the category $\mathrm{S}(L, M)$-**SUCT** of stratified (L, M)-semiuniform convergence tower spaces as a bireflective subcategory; (2) the full subcategory of $\mathrm{S}(L, M)$-**SUCT**, consisting of stratified (L, M)-semiuniform limit tower spaces is strongly Cartesian closed; (3) the category $\mathrm{S}(L, M)$-**FT** of stratified (L, M)-filter tower spaces can be embedded in the category $\mathrm{S}(L, M)$-**SUCT** as a simultaneously bireflective and bicoreflective subcategory.

Keywords: (L, M)-semiuniform convergence space, (L, M)-filter tower space, strongly Cartesian closed, bireflective, bicoreflective.

1 Introduction

Topological spaces were introduced to overcome a "structural weakness" of metric spaces, namely that in the category (i.e., concrete category) of metric spaces, initial structures fails to exist. Unfortunately, in the category **Top** of topological spaces and continuous maps, "uniform concepts", such as Cauchy filters, completeness and uniform continuity cannot be defined in a satisfactory way. This leads to these generalizations such as uniform spaces introduced by Weil [32]. The category **Unif** of uniform spaces is also a topological category, i.e., initial structures exist. However, from a categorical point of view, **Unif** lacks other desirable properties, e.g., it is not Cartesian closed. For this reason, uniform convergence spaces, combining uniform spaces and convergence spaces, were introduced by Cook and Fischer [2]. Later, it was studied in a slightly modified form proposed by Wyler [33], since in this case natural function spaces exist, which had been demonstrated by Lee [23]. In [33], Wyler relaxed the defining axioms of uniform convergence spaces and introduced the concept of semiuniform convergence spaces. In [29], Preuss showed that the category of semiuniform convergence spaces remedies all the deficiencies of **Top** and **Unif**. The category **SUC** of semiuniform convergence spaces has nicer categorial properties: (1) it is Cartesian closed; (2) it is extensional; (3) it is closed under the product of quotient maps. Moreover, there are compatible categorical relations among semiuniform convergence spaces, uniform convergence spaces, generalized convergence spaces, filter spaces and Cauchy spaces.

In the lattice-valued case, different types of lattice-valued uniform spaces [10, 14], lattice-valued convergence spaces [15, 37, 38] and lattice-valued Cauchy spaces [18, 28, 36] were introduced. In order to unify these lattice-valued structures, Jäger and Burton [21] introduced stratified L-uniform convergence spaces and showed that the resulting category is a Cartesian closed superconstruct of the category of stratified L-uniform spaces [10]. For L a complete Heyting algebra, Fang [5] relaxed the axioms of defining a stratified L-uniform convergence space and introduced the concept of stratified L-semiuniform convergence spaces, which extend semiuniform convergence spaces to the lattice-valued case. Later, Fang and Fang [8] studied monoidal closedness of the category of stratified L-semiuniform convergence spaces.

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Fang [6, 7] further introduced stratified L-ordered quasiuniform limit spaces and stratified L-preuniform convergence spaces, and studied their relations with stratified L-generalized convergence spaces [15], stratified L-limit spaces [16] and stratified L-ordered convergence spaces [3, 4, 25], where L is also generalized from a complete Heyting algebra to the case of enriched lattices. Recently, Reid and Richardson [30] used $\top$-filters [13] to introduce the concept of $\top$-uniform convergence spaces and studied its completeness. Jäger [20, 22] researched diagonal conditions for $\top$-uniform convergence spaces and proved $\top$-uniform convergence spaces have nice categorical properties like topologicalness and Cartesian closedness. Zhang and Pang [35] proposed the concept of $\top$-semiuniform convergence spaces and studied its monoidal closedness. Fang and Yue [9] introduced the connectedness of disconnectedness of $\top$-semiuniform convergence spaces and established a one to one correspondence between connected varieties and disconnected varieties.

Jäger [17] proposed the notion of level spaces for stratified L-uniform convergence spaces, which contains probabilistic uniform convergence spaces and probabilistic uniform spaces as special cases. By relaxing some of the axioms defining a level space for a stratified L-uniform convergence space, Li [24] introduced the concept of stratified (L, M)-semiuniform convergence tower spaces and proved that the resulting category is a strong topological universe. In this paper, we will go on concentrating on stratified (L, M)-semiuniform convergence tower spaces. Precisely, we will focus on three subcategories of the category of stratified (L, M)-semiuniform convergence tower spaces and study their categorical relations as well as their categorical properties, which are different from the results in [24]. It is worth noting that the category of stratified (L, M)-semiuniform convergence tower spaces contains both the category of stratified L-semiuniform convergence spaces in the sense of Fang [5] and the category of stratified (L, M)-filter tower spaces in the sense of Yang and Li [34] as special subcategories.

The paper is organized as follows. In Section 2, we recall some necessary concepts and notations. In Section 3, we introduce the concept of stratified (L, M)-semiuniform convergence spaces and study its relations with stratified (L, M)-semiuniform convergence tower spaces. In Section 4, we introduce stratified (L, M)-semiuniform limit tower spaces and show that the resulting category as a full subcategory of $\mathrm{S}(L, M)$-**SUCT** is strongly Cartesian closed. In Section 5, we focus on the relations between stratified (L, M)-semiuniform convergence tower spaces and stratified (L, M)-filter tower spaces.

2 Preliminaries

Throughout this paper, L(resp.,M) denotes a frame, which means that L is a complete lattice, and for all $a,\ b_i\ (i \in I) \in L$, the following distributive law is valid

$$a \wedge \bigvee_{i\in I} b_i = \bigvee_{i\in I} (a \wedge b_i).$$

The smallest element and the biggest element of L are denoted by $\bot_L$ and $\top_L$, respectively.

For a nonempty set X, L^X denotes the set of all L-subsets on X. The set L^X with the pointwise order becomes a complete lattice. The smallest element and the biggest element in L^X are denoted by $\bot_L^X$ and $\top_L^X$, respectively. For each $a \in L$, $\underline{a}$ denotes the constant map $X \longrightarrow L,\ x \longmapsto a$.

Definition 2.1. [11] *A map $\mathcal{F} : L^X \longrightarrow L$ is called a stratified L-filter on X if it satisfies*
(F1) $\mathcal{F}(\bot_L^X) = \bot_L, \mathcal{F}(\top_L^X) = \top_L$;
(F2) $A \leqslant B \Rightarrow \mathcal{F}(A) \leqslant \mathcal{F}(B)$;
(F3) $\mathcal{F}(A \wedge B) \geqslant \mathcal{F}(A) \wedge \mathcal{F}(B)$;
(Fs) $a \wedge \mathcal{F}(A) \leqslant \mathcal{F}(\underline{a} \wedge A)$.

The family of all stratified L-filters on X will be denoted by $\mathcal{F}_L^s(X)$. For every $x \in X$, $[x] \in \mathcal{F}_L^s(X)$ is defined by $[x](A) = A(x)$ for all $A \in L^X$.

On the set $\mathcal{F}_L^s(X)$ of all stratified L-filters on X, an order $\leqslant$ defined by $\mathcal{F} \leqslant \mathcal{G}$ iff $\mathcal{F}(A) \leqslant \mathcal{G}(A)$ for all $A \in L^X$, was introduced in [13]. For a nonempty family $\{\mathcal{F}_\lambda\}_{\lambda\in\Upsilon}$ of stratified L-filters, the infimum $\bigwedge_{\lambda\in\Upsilon} \mathcal{F}_\lambda$ is given by $(\bigwedge_{\lambda\in\Upsilon} \mathcal{F}_\lambda)(A) = \bigwedge_{\lambda\in\Upsilon} \mathcal{F}_\lambda(A)$ for all $A \in L^X$. In order to guarantee the least upper bound for a family $\{\mathcal{F}_\lambda\}_{\lambda\in\Upsilon}$, Höhle and Šostak presented the following lemma.

Lemma 2.2. [13] *For a family $\{\mathcal{F}_\lambda\}_{\lambda\in\Upsilon}$ of stratified L-filters on X, there exists a stratified L-filter $\mathcal{F}$ such that $\mathcal{F}_\lambda \leqslant \mathcal{F}$ $(\forall \lambda \in \Upsilon)$, if and only if*

$$\mathcal{F}_{\lambda_1}(A_1) \wedge \cdots \wedge \mathcal{F}_{\lambda_n}(A_n) = \bot_L \text{ whenever } A_1 \wedge \cdots \wedge A_n = \bot_L^X,$$

for $n \in \mathbb{N}$, $A_1, \cdots, A_n \in L^X$, $\{\lambda_1, \cdots, \lambda_n\} \subseteq \Upsilon$. In the case of existence, the supremum $\bigvee_{\lambda\in\Upsilon} \mathcal{F}_\lambda$ of a nonempty family $\{\mathcal{F}_\lambda\}_{\lambda\in\Upsilon}$ of stratified L-filters is given by

$$\left(\bigvee_{\lambda\in\Upsilon} \mathcal{F}_\lambda\right)(A) = \bigvee_{n\in\mathbb{N}} \bigvee \{\mathcal{F}_{\lambda_1}(A_1) \wedge \cdots \wedge \mathcal{F}_{\lambda_n}(A_n) \mid A_1 \wedge \cdots \wedge A_n \leqslant A\},$$

for all $A \in L^X$.

For explicitness, note that elements in $\mathcal{F}_L^s(X)$ are denoted by $\mathcal{F}, \mathcal{G}, \mathcal{H}, \cdots$, and elements in $\mathcal{F}_L^s(X \times X)$ are denoted by $\mathscr{F}, \mathscr{G}, \mathscr{H}, \cdots$.

Let $\varphi : X \longrightarrow Y$ be a map and $\mathcal{F}$ be a stratified L-filter on X. Define $\varphi^{\rightarrow} : L^X \longrightarrow L^Y$ and $\varphi^{\leftarrow} : L^Y \longrightarrow L^X$ (see [31]) by $\varphi^{\rightarrow}(A)(y) = \bigvee_{\varphi(x)=y} A(x)$ for $A \in L^X$ and $y \in Y$, and $\varphi^{\leftarrow}(B) = B \circ \varphi$, respectively. Then the map $\varphi^{\Rightarrow}(\mathcal{F}) : L^Y \longrightarrow L$ defined by $\varphi^{\Rightarrow}(\mathcal{F})(B) = \mathcal{F}(\varphi^{\leftarrow}(B))$ for $B \in L^Y$, is a stratified L-filter on Y, which is called the image of $\mathcal{F}$ under φ (see [13]). In [15], Jäger defined the map $\varphi^{\Leftarrow}(\mathcal{G}) : L^X \longrightarrow L$ by:

$$\forall A \in L^X, \quad \varphi^{\Leftarrow}(\mathcal{G})(A) = \bigvee_{\varphi^{\leftarrow}(B)\leqslant A} \mathcal{G}(B),$$

where $\mathcal{G} \in \mathcal{F}_L^s(Y)$. Then the following proposition holds.

Lemma 2.3. [13, 15] *Let $\mathcal{G}$ be a stratified L-filter on Y and $\varphi : X \longrightarrow Y$ be a map. The following statements are equivalent:*

(1) *$\varphi^{\Leftarrow}(\mathcal{G})$ is a stratified L-filter.*

(2) *$\mathcal{G}(B) = \perp_L$ whenever $\varphi^{\leftarrow}(B) = \perp_L^X$ for all $B \in L^Y$.*

(3) *$\mathcal{G}(\top_{Y-\varphi(X)}) = \perp_L$, where $\top_{Y-\varphi(X)}(y) = \top_L$ whenever $y \in Y - \varphi(X)$, and $\perp_L$, otherwise.*

In [21], Jäger defined the composition of two stratified L-filters on $X \times X$. Let $\mathscr{F}, \mathscr{G} \in \mathcal{F}_L^s(X \times X)$, $A \in L^{X\times X}$ and $(x, y) \in X \times X$. We define $(\mathscr{F} \circ \mathscr{G})(A) = \bigvee\{\mathscr{F}(B) \wedge \mathscr{G}(C) \mid B \circ C \leqslant A\}$, where $(B \circ C)(x, y) = \bigvee\{B(x, z) \wedge C(z, y) \mid z \in X\}$. Then $\mathscr{F} \circ \mathscr{G} \in \mathcal{F}_L^s(X \times X)$ if and only if $B \circ C = \perp_L^{X\times X}$ implies $\mathscr{F}(B) \wedge \mathscr{G}(C) = \perp_L$. Let $\mathscr{H} \in \mathcal{F}_L^s(X \times X)$. We define $\mathscr{H}^{-1}(A) = \mathscr{H}(A^{-1})$ for each $A \in L^{X\times X}$, where $A^{-1}(x, y) = A(y, x)$ for each $(x, y) \in X \times X$. Further, $(\varphi \times \varphi)^{\Rightarrow}(\mathscr{H}^{-1}) = ((\varphi \times \varphi)^{\Rightarrow}(\mathscr{H}))^{-1}$.

In [15], Jäger proposed that the product $\prod_{\lambda\in\Upsilon} \mathcal{F}_\lambda$ of a family of stratified L-filters $\{\mathcal{F}_\lambda\}_{\lambda\in\Upsilon}$, where for each $\lambda \in \Upsilon$, X_λ is a nonempty set and $\mathcal{F}_\lambda \in \mathcal{F}_L^s(X_\lambda)$, is defined as follows:

$$\prod_{\lambda\in\Upsilon} \mathcal{F}_\lambda := \bigvee_{\lambda\in\Upsilon} p_\lambda^{\Leftarrow}(\mathcal{F}_\lambda) \in \mathcal{F}_L^s\left(\prod_{\lambda\in\Upsilon} X_\lambda\right),$$

where for each $\lambda \in \Upsilon$, $p_\lambda : \prod_{\mu\in\Upsilon} X_\mu \longrightarrow X_\lambda$ is the projection map.

Lemma 2.4. [15] *Let $\{X_\lambda\}_{\lambda\in\Upsilon}$ be a family of nonempty sets, $p_\lambda : \prod_{\mu\in\Upsilon} X_\mu \to X_\lambda$ the projection map, $\mathcal{F}_\lambda \in \mathcal{F}_L^s(X_\lambda)$ $(\forall \lambda \in \Upsilon)$ and $\mathscr{F} \in \mathcal{F}_L^s\left(\prod_{\lambda\in\Upsilon} X_\lambda\right)$. Then the following statements hold:*

(1) *$\prod_{\lambda\in\Upsilon} p_\lambda^{\Rightarrow}(\mathscr{F}) \leqslant \mathscr{F}$.*

(2) *$p_\lambda^{\Rightarrow}\left(\prod_{\mu\in\Upsilon} \mathcal{F}_\mu\right) \geqslant \mathcal{F}_\lambda$, $\forall \lambda \in \Upsilon$.*

(3) *$p_\lambda^{\Rightarrow}\left(\prod_{\mu\in\Upsilon} p_\mu^{\Rightarrow}(\mathscr{F})\right) = p_\lambda^{\Rightarrow}(\mathscr{F})$, $\forall \lambda \in \Upsilon$.*

Definition 2.5. *A category $\mathbf{C}$ is called Cartesian closed provided that the following conditions are satisfied:*

(1) *For each pair (X, Y) of $\mathbf{C}$-objects, there exists a product $X \times Y$ in $\mathbf{C}$.*

(2) *For each pair of $\mathbf{C}$-objects X and Y, there exists a $\mathbf{C}$-object Y^X (called power object) and a $\mathbf{C}$-morphism $ev_{[X,Y]} : Y^X \times X \longrightarrow Y$ (called evaluation morphism) such that for each $\mathbf{C}$-object Z and each $\mathbf{C}$-morphism $\varphi : Z \times X \longrightarrow Y$, there exists a unique $\mathbf{C}$-morphism $\varphi_* : Z \longrightarrow Y^X$ such that $ev_{[X,Y]} \circ (\varphi_* \times id_X) = \varphi$.*

Several convenient properties for a topological category are proposed by Preuss in the book [29], namely

(CP1) $\mathbf{C}$ is Cartesian closed.

(CP2) $\mathbf{C}$ is extensional.

(CP3) In $\mathbf{C}$ product of quotient maps is a quotient map.

According to the terminology of [29], a topological category $\mathbf{C}$ is called

(1) strongly Cartesian closed provided that $\mathbf{C}$ fulfills (CP1) and (CP3),

(2) a topological universe provided that **C** fulfills (CP1) and (CP2),

(3) a strong topological universe provided that **C** fulfills (CP1)–(CP3).

The class of objects of a category **C** is denoted by $|\mathbf{C}|$. For more notions related to category theory, we refer to [1] and [29].

Definition 2.6. [24] *A stratified (L,M)-semiuniform convergence tower structure on X is $\Lambda = \{\Lambda_\alpha\}_{\alpha\in M}$, where $\Lambda_\alpha \subseteq \mathcal{F}_L^s(X\times X)$ satisfies:*

(LMUCT1) *For all $\alpha \in M$ and $x \in X$, $[x]\times[x] \in \Lambda_\alpha$.*

(LMUCT2) *If $\mathscr{F} \in \Lambda_\alpha$ and $\mathscr{F} \leqslant \mathscr{G}$, then $\mathscr{G} \in \Lambda_\alpha$.*

(LMUCT3) *If $\mathscr{F} \in \Lambda_\alpha$, then $\mathscr{F}^{-1} \in \Lambda_\alpha$.*

(LMP1) *If $\beta \leqslant \alpha$, then $\Lambda_\alpha \subseteq \Lambda_\beta$.*

(LMP2) $\Lambda_{\perp_M} = \mathcal{F}_L^s(X\times X)$.

For a stratified (L,M)-semiuniform convergence tower structure Λ on X, the pair (X,Λ) is called a stratified (L,M)-semiuniform convergence tower space. A stratified (L,M)-semiuniform convergence tower structure Λ on X is called left continuous if it satisfies:

(LMPC) *For all $N \subseteq M$, we have $\Lambda_{\bigvee N} = \bigcap_{\alpha\in N}\Lambda_\alpha$.*

For a left continuous stratified (L,M)-semiuniform convergence tower structure Λ on X, the pair (X,Λ) is called a left continuous stratified (L,M)-semiuniform convergence tower space.

Definition 2.7. [24] *A map $\varphi:(X,\Lambda)\longrightarrow(Y,\Sigma)$ between stratified (L,M)-semiuniform convergence tower spaces is called uniformly continuous provided that $\mathscr{F}\in\Lambda_\alpha$ implies $(\varphi\times\varphi)^{\rightarrow}(\mathscr{F})\in\Sigma_\alpha$ for all $\alpha\in M$.*

The category of stratified (L,M)-semiuniform convergence tower spaces and uniformly continuous maps is denoted by S(L,M)-**SUCT**. The full subcategory of S(L,M)-**SUCT**, consisting of left continuous stratified (L,M)-semiuniform convergence tower spaces, is denoted by LC-S(L,M)-**SUCT**.

Remark 2.8. [24] (1) *For $L=\{0,1\}$ and $M=[0,1]$, we can identify the notion of (L,M)-semiuniform convergence tower spaces with the notion of probabilistic semiuniform convergence spaces* [27].

(2) *For $L=\{0,1\}$ and $M=[0,\infty]$ with the opposite order, a left continuous stratified (L,M)-semiuniform convergence tower space is a semi-approach uniform convergence space in the definition of Nauwelaerts* [26].

(3) *For $L=M$, we can identify the notion of left continuous (L,M)-semiuniform convergence tower spaces with the notion of stratified L-semiuniform convergence spaces* [5].

Theorem 2.9. [24] *S(L,M)-**SUCT** is a strong topological universe.*

3 Stratified (L,M)-semiuniform convergence spaces

In this section, we will introduce the concept of stratified (L,M)-semiuniform convergence spaces, which is a generalization of stratified L-semiuniform convergence spaces in the sense of Fang [5]. Then we will investigate its relations with stratified (L,M)-semiuniform convergence tower spaces.

Definition 3.1. *A stratified (L,M)-semiuniform convergence space is a pair $(X,\mathcal{J})$, where $\mathcal{J}:\mathcal{F}_L^s(X\times X)\longrightarrow M$ satisfies:*

(LMUC1) $\mathcal{J}([x]\times[x]) = \top_M$.

(LMUC2) *If $\mathscr{F}\leqslant\mathscr{G}$, then $\mathcal{J}(\mathscr{F})\leqslant\mathcal{J}(\mathscr{G})$.*

(LMUC3) $\mathcal{J}(\mathscr{F}) = \mathcal{J}(\mathscr{F}^{-1})$.

A stratified (L,M)-semiuniform limit space is a stratified (L,M)-semiuniform convergence space which satisfies:

(LMUC4) $\mathcal{J}(\mathscr{F}\wedge\mathscr{G}) = \mathcal{J}(\mathscr{F})\wedge\mathcal{J}(\mathscr{G})$.

A stratified (L,M)-uniform limit space is a stratified (L,M)-semiuniform limit space which satisfies:

(LMUC5) $\mathcal{J}(\mathscr{F}\circ\mathscr{G}) = \mathcal{J}(\mathscr{F})\wedge\mathcal{J}(\mathscr{G})$ *whenever $\mathscr{F}\circ\mathscr{G}$ exists.*

Definition 3.2. *A map $\varphi:(X,\mathcal{J}_X)\longrightarrow(Y,\mathcal{J}_Y)$ between two stratified (L,M)-semiuniform convergence spaces is called uniformly continuous provided that $\mathcal{J}_X(\mathscr{F})\leqslant\mathcal{J}_Y((\varphi\times\varphi)^{\rightarrow}(\mathscr{F}))$ for all $\mathscr{F}\in\mathcal{F}_L^s(X\times X)$.*

The category of stratified (L,M)-semiuniform convergence spaces (resp. stratified (L,M)-semiuniform limit spaces, stratified (L,M)-uniform limit spaces) is denoted by $\mathrm{S}(L,M)$-**SUC** (resp. $\mathrm{S}(L,M)$-**SUL**, $\mathrm{S}(L,M)$-**UL**).

Remark 3.3. (1) *When $L=M$, we can identify stratified (L,M)-semiuniform convergence spaces and stratified (L,M)-uniform limit spaces with stratified L-semiuniform convergence spaces in the sense of* [5] *and stratified L-uniform convergence spaces in the sense of Jäger and Burton* [21]*, respectively.*

(2) *If $L=M=\mathbf{2}=\{0,1\}$, then $S(\mathbf{2},\mathbf{2})$-**SUC**$\simeq$ **SUC** (resp., $S(\mathbf{2},\mathbf{2})$-**SUL**$\simeq$ **SUL**, $S(\mathbf{2},\mathbf{2})$-**UL**$\simeq$ **UC**), the category of semiuniform convergence spaces (resp., the category of semiuniform limit spaces, the category of uniform limit spaces) in* [29].

Next we will study the relations between stratified (L,M)-semiuniform convergence spaces and stratified (L,M)-semiuniform convergence tower spaces. It is worth mentioning that there is some overlap with some conclusions in [19], but the authors therein only provided one transformation formula and gave a claim without detailed proof. So we give some necessary proofs in our paper for self-completeness.

Proposition 3.4. [19] *Let (X,Λ) be a stratified (L,M)-semiuniform convergence tower space and define $\mathcal{J}:\mathcal{F}_L^s(X\times X)\longrightarrow M$ as follows:*

$$\forall\mathscr{F}\in\mathcal{F}_L^s(X\times X),\quad \mathcal{J}^{\Lambda}(\mathscr{F})=\bigvee\{\alpha\in M\mid\mathscr{F}\in\Lambda_\alpha\}.$$

Then $\mathcal{J}^{\Lambda}$ is a stratified (L,M)-semiuniform convergence structure on X.

Proposition 3.5. *Let $(X,\mathcal{J})$ be a stratified (L,M)-semiuniform convergence space and define $\Lambda^{\mathcal{J}}=\{\Lambda_\alpha^{\mathcal{J}}\mid\alpha\in M\}$ by*

$$\forall\alpha\in M,\Lambda_\alpha^{\mathcal{J}}=\{\mathscr{F}\in\mathcal{F}_L^s(X\times X)\mid\mathcal{J}(\mathscr{F})\geqslant\alpha\}.$$

Then $(X,\Lambda^{\mathcal{J}})$ is a left continuous stratified (L,M)-semiuniform convergence tower space.

Proof. (LMUCT1)–(LMUCT3) and (LMP1)–(LMP2) are obvious. It suffices to show that $(X,\mathcal{J})$ satisfies (LMPC). For each $U\subseteq M$, take any $\mathscr{F}\in\bigcap_{\alpha\in U}\Lambda_\alpha^{\mathcal{J}}$. Then for all $\alpha\in U$, it follows that $\mathscr{F}\in\Lambda_\alpha^{\mathcal{J}}$, i.e., $\mathcal{J}(\mathscr{F})\geqslant\alpha$. By the arbitrariness of α, we have $\mathcal{J}(\mathscr{F})\geqslant\bigvee_{\alpha\in U}\alpha=\bigvee U$. This means that $\mathscr{F}\in\Lambda_{\bigvee U}^{\mathcal{J}}$. Thus, we have $\bigcap_{\alpha\in U}\Lambda_\alpha^{\mathcal{J}}\subseteq\Lambda_{\bigvee U}^{\mathcal{J}}$. Moreover, $\Lambda_{\bigvee U}^{\mathcal{J}}\subseteq\bigcap_{\alpha\in U}\Lambda_\alpha^{\mathcal{J}}$ is obvious. Therefore, we obtain $\Lambda_{\bigvee U}^{\mathcal{J}}=\bigcap_{\alpha\in U}\Lambda_\alpha^{\mathcal{J}}$, as desired. □

Proposition 3.6. (1) *If $\varphi:(X,\Lambda^X)\longrightarrow(Y,\Lambda^Y)$ is uniformly continuous, then so is $\varphi:(X,\mathcal{J}^{\Lambda^X})\longrightarrow(Y,\mathcal{J}^{\Lambda^Y})$.*

(2) *If $\varphi:(X,\mathcal{J}_X)\longrightarrow(Y,\mathcal{J}_Y)$ is uniformly continuous, then so is $\varphi:(X,\Lambda^{\mathcal{J}_X})\longrightarrow(Y,\Lambda^{\mathcal{J}_Y})$.*

Proof. (1) Take any $\mathscr{F}\in\mathcal{F}_L^s(X\times X)$. Then by the uniform continuity of $\varphi:(X,\Lambda^X)\longrightarrow(Y,\Lambda^Y)$, we have

$$\begin{aligned}\mathcal{J}^{\Lambda^X}(\mathscr{F}) &= \bigvee\{\alpha\in M\mid\mathscr{F}\in\Lambda_\alpha^X\}\\ &\leqslant \bigvee\{\alpha\in M\mid(\varphi\times\varphi)^{\rightarrow}(\mathscr{F})\in\Lambda_\alpha^Y\}\\ &= \mathcal{J}^{\Lambda^Y}((\varphi\times\varphi)^{\rightarrow}(\mathscr{F})).\end{aligned}$$

Thus, $\varphi:(X,\mathcal{J}^{\Lambda^X})\longrightarrow(Y,\mathcal{J}^{\Lambda^Y})$ is uniformly continuous.

(2) Since $\varphi:(X,\mathcal{J}_X)\longrightarrow(Y,\mathcal{J}_Y)$ is uniformly continuous, it follows that for each $\mathscr{F}\in\mathcal{F}_L^s(X\times X)$,

$$\mathcal{J}_X(\mathscr{F})\leqslant\mathcal{J}_Y((\varphi\times\varphi)^{\rightarrow}(\mathscr{F})).$$

Take any $\mathscr{F}\in\Lambda_\alpha^{\mathcal{J}_X}$, i.e., $\mathcal{J}_X(\mathscr{F})\geqslant\alpha$. Then it follows that $\mathcal{J}_Y((\varphi\times\varphi)^{\rightarrow}(\mathscr{F}))\geqslant\alpha$. Hence, $(\varphi\times\varphi)^{\rightarrow}(\mathscr{F})\in\Lambda_\alpha^{\mathcal{J}_Y}$. This shows the uniform continuity of $\varphi:(X,\Lambda^{\mathcal{J}_X})\longrightarrow(Y,\Lambda^{\mathcal{J}_Y})$. □

Proposition 3.7. *If (X,Λ) is a left continuous stratified (L,M)-semiuniform convergence tower space and $(X,\mathcal{J})$ is a stratified (L,M)-semiuniform convergence space, then $\mathcal{J}^{\Lambda^{\mathcal{J}}}=\mathcal{J}$ and $\Lambda^{\mathcal{J}^{\Lambda}}=\Lambda$.*

Proof. (1) Take any $\mathscr{F}\in\mathcal{F}_L^s(X\times X)$. Then

$$\mathcal{J}^{\Lambda^{\mathcal{J}}}(\mathscr{F})=\bigvee\{\alpha\in M\mid\mathscr{F}\in\Lambda_\alpha^{\mathcal{J}}\}=\bigvee\{\alpha\in M\mid\mathcal{J}(\mathscr{F})\geqslant\alpha\}=\mathcal{J}(\mathscr{F}).$$

This shows $\mathcal{J}^{\Lambda^{\mathcal{J}}}=\mathcal{J}$.

(2) Take any $\mathscr{F} \in \mathcal{F}_L^s(X \times X)$. On one hand, if $\mathscr{F} \in \Lambda_\alpha$, then $\mathcal{J}^\Lambda(\mathscr{F}) = \bigvee\{\beta \in M \mid \mathscr{F} \in \Lambda_\beta\} \geqslant \alpha$, i.e., $\mathscr{F} \in \Lambda_\alpha^{\mathcal{J}^\Lambda}$. On the other hand, we have

$$\begin{aligned}
\mathscr{F} \in \Lambda_\alpha^{\mathcal{J}^\Lambda} &\iff \mathcal{J}^\Lambda(\mathscr{F}) \geqslant \alpha \\
&\implies \Lambda_{\mathcal{J}^\Lambda(\mathscr{F})} \subseteq \Lambda_\alpha \quad \text{(by (LMP1))} \\
&\iff \Lambda_{\bigvee\{\beta \in M \mid \mathscr{F} \in \Lambda_\beta\}} \subseteq \Lambda_\alpha \\
&\iff \bigcap_{\mathscr{F} \in \Lambda_\beta} \Lambda_\beta \subseteq \Lambda_\alpha \quad \text{(by (LMPC))} \\
&\implies \mathscr{F} \in \Lambda_\alpha.
\end{aligned}$$

This shows $\Lambda_\alpha^{\mathcal{J}^\Lambda} \subseteq \Lambda_\alpha$, as desired. □

By Propositions 3.6 and 3.7, we obtain

Theorem 3.8. [19] *The category* LC-S(L, M)-**SUCT** *is isomorphic to the category* S(L, M)-**SUC**.

The following results show the relations between S(L, M)-**SUCT** and S(L, M)-**SUC**.

Theorem 3.9. *The category* LC-S(L, M)-**SUCT** *is bireflective in the category* S(L, M)-**SUCT**.

Proof. Let (X, Λ) be a stratified (L, M)-semiuniform convergence tower space. By Propositions 3.4 and 3.5, we know $(X, \Lambda^{\mathcal{J}^\Lambda})$ is a left continuous stratified (L, M)-semiuniform convergence tower space. Further, we claim that $id_X : (X, \Lambda) \longrightarrow (X, \Lambda^{\mathcal{J}^\Lambda})$ is the LC-S(L, M)-**SUCT**-bireflection. For this, we need only show the following statements.

(1) $id_X : (X, \Lambda) \longrightarrow (X, \Lambda^{\mathcal{J}^\Lambda})$ is uniformly continuous.

(2) For a left continuous stratified (L, M)-semiuniform convergence tower space (Y, Σ), the uniform continuity of $\varphi : (X, \Lambda) \longrightarrow (Y, \Sigma)$ implies the uniform continuity of $\varphi : (X, \Lambda^{\mathcal{J}^\Lambda}) \longrightarrow (Y, \Sigma)$.

For (1), by Proposition 3.7, it is straightforward.

For (2), if $\varphi : (X, \Lambda) \longrightarrow (Y, \Sigma)$ is uniformly continuous, then by Proposition 3.6, we have $\varphi : (X, \Lambda^{\mathcal{J}^\Lambda}) \longrightarrow (Y, \Sigma^{\mathcal{J}^\Sigma})$ is uniformly continuous. Since (Y, Σ) is left continuous, by Proposition 3.7, it follows that $\Sigma^{\mathcal{J}^\Sigma} = \Sigma$. Thus, $\varphi : (X, \Lambda^{\mathcal{J}^\Lambda}) \longrightarrow (Y, \Sigma)$ is uniformly continuous, as desired. □

Corollary 3.10. *The category* S(L, M)-**SUC** *can be embedded in the category* S(L, M)-**SUCT** *as a bireflective subcategory.*

4 Stratified (L, M)-semiuniform limit tower spaces

In this section, we will introduce the concept of stratified (L, M)-semiuniform limit tower spaces and study its relations with stratified (L, M)-semiuniform convergence tower spaces. Then we will investigate its categorical properties.

Definition 4.1. *A stratified (L, M)-semiuniform convergence tower structure Λ on X is called a stratified (L, M)-semiuniform limit tower structure on X if it satisfies*

(LMUCT4) *If $\mathscr{F} \in \Lambda_\alpha$ and $\mathscr{G} \in \Lambda_\alpha$, then $\mathscr{F} \wedge \mathscr{G} \in \Lambda_\alpha$.*

For a stratified (L, M)-semiuniform limit tower structure Λ on X, the pair (X, Λ) is called a stratified (L, M)-semiuniform limit tower space.

The full subcategory of S(L, M)-**SUCT**, consisting of stratified (L, M)-semiuniform limit tower spaces, is denoted by S(L, M)-**SULT**.

In the classical case, the category of semiuniform limit spaces is bireflective in the category of semiuniform convergence spaces. Next we generalize this conclusion to the lattice-valued case.

Proposition 4.2. *Let (X, Λ) be a stratified (L, M)-semiuniform convergence tower space and define $\Lambda^L = \{\Lambda_\alpha^L \mid \alpha \in M\}$ as follows:*

$$\Lambda_\alpha^L = \{\mathscr{F} \in \mathcal{F}_L^s(X \times X) \mid \exists \mathscr{F}_1, \cdots, \mathscr{F}_n \in \Lambda_\alpha \text{ s.t. } \mathscr{F} \geqslant \wedge_{i=1}^{i=n} \mathscr{F}_i, n \in \mathbb{N}\}.$$

Then (X, Λ^L) is a stratified (L, M)-semiuniform limit tower space.

Proof. It suffices to show that (X, Λ^L) satisfies (LMUCT1)–(LMUCT4) and (LMP1)–(LMP2).

(LMUCT1), (LMUCT2), (LMP1) and (LMP2) are straightforward.

(LMUCT3) Take any $\mathscr{F} \in \Lambda^L_\alpha$. Then there exist $\mathscr{F}_1, \cdots, \mathscr{F}_n \in \Lambda_\alpha$ such that $\mathscr{F} \geqslant \wedge_{i=1}^{i=n} \mathscr{F}_i$. Further, we have $\mathscr{F}^{-1} \geqslant (\wedge_{i=1}^{i=n} \mathscr{F}_i)^{-1} = \wedge_{i=1}^{i=n} \mathscr{F}_i^{-1}$. Since $\mathscr{F}_i^{-1} \in \Lambda_\alpha$ for $i = 1, 2, \cdots, n$, it follows that $\mathscr{F}^{-1} \in \Lambda^L_\alpha$.

(LMUCT4) Take any $\mathscr{F}, \mathscr{G} \in \Lambda^L_\alpha$. Then there exist $\mathscr{F}_1, \cdots, \mathscr{F}_n, \mathscr{G}_1, \cdots, \mathscr{G}_m \in \Lambda_\alpha$ such that $\mathscr{F} \geqslant \wedge_{i=1}^{i=n} \mathscr{F}_i$ and $\mathscr{G} \geqslant \wedge_{j=1}^{j=m} \mathscr{G}_j$. This implies that $\mathscr{F} \wedge \mathscr{G} \geqslant \wedge_{i=1}^{i=n} \mathscr{F}_i \wedge \wedge_{j=1}^{j=m} \mathscr{G}_j$. Thus, $\mathscr{F} \wedge \mathscr{G} \in \Lambda^L_\alpha$. □

Theorem 4.3. *The category* $\mathrm{S}(L, M)$-**SULT** *is bireflective in the category* $S(L, M)$-**SUCT**.

Proof. Let (X, Λ) be a stratified (L, M)-semiuniform convergence tower space and define $\Lambda^L = \{\Lambda^L_\alpha \mid \alpha \in M\}$ as follows:

$$\Lambda^L_\alpha = \{\mathscr{F} \in \mathcal{F}^s_L(X \times X) \mid \exists \mathscr{F}_1, \cdots, \mathscr{F}_n \in \Lambda_\alpha \ s.t. \ \mathscr{F} \geqslant \wedge_{i=1}^{i=n} \mathscr{F}_i, n \in \mathbb{N}\}.$$

By Proposition 4.2, we know (X, Λ^L) is stratified (L, M)-semiuniform limit tower space. Further, we claim that $id_X : (X, \Lambda) \longrightarrow (X, \Lambda^L)$ is the $\mathrm{S}(L, M)$-**SULT**-bireflector. For this, it suffices to prove that:

(1) $id_X : (X, \Lambda) \longrightarrow (X, \Lambda^L)$ is uniformly continuous.

(2) For a stratified (L, M)-semiuniform limit tower space (Y, Λ^Y), the uniform continuity of $\varphi : (X, \Lambda) \longrightarrow (Y, \Lambda^Y)$ implies the uniform continuity of $\varphi : (X, \Lambda^L) \longrightarrow (Y, \Lambda^Y)$.

For (1), it follows from $\Lambda_\alpha \subseteq \Lambda^L_\alpha$.

For (2), take any $\mathscr{F} \in \Lambda^L_\alpha$. Then there exist $\mathscr{F}_1, \cdots, \mathscr{F}_n \in \Lambda_\alpha$ such that $\mathscr{F} \geqslant \wedge_{i=1}^{i=n} \mathscr{F}_i$. By the uniform continuity of $\varphi : (X, \Lambda) \longrightarrow (Y, \Lambda^Y)$, we have $(\varphi \times \varphi)^{\rightrightarrows}(\mathscr{F}_i) \in \Lambda^Y_\alpha$ for $i = 1, 2, \cdots, n$. Since (Y, Λ^Y) is a stratified (L, M)-semiuniform limit tower space, it follows that $\wedge_{i=1}^{i=n} (\varphi \times \varphi)^{\rightrightarrows}(\mathscr{F}_i) \in \Lambda^Y_\alpha$. Further,

$$(\varphi \times \varphi)^{\rightrightarrows}(\mathscr{F}) \geqslant (\varphi \times \varphi)^{\rightrightarrows}\left(\wedge_{i=1}^{i=n} \mathscr{F}_i\right) = \wedge_{i=1}^{i=n} (\varphi \times \varphi)^{\rightrightarrows}(\mathscr{F}_i).$$

This implies that $(\varphi \times \varphi)^{\rightrightarrows}(\mathscr{F}) \in \Lambda^Y_\alpha$. The uniform continuity of $\varphi : (X, \Lambda^L) \longrightarrow (Y, \Lambda^Y)$ is proved. □

Corollary 4.4. *The category* $\mathrm{S}(L, M)$-**SULT** *is a topological category.*

In order to show the categorical properties of stratified (L, M)-semiuniform limit tower spaces, the following lemma is necessary.

Lemma 4.5. [29] *Let* **C** *be a topological category. Then*

(1) *If* **D** *is a bireflective (full and isomorphism-closed) subcategory of* **C** *which is closed under the formation of power objects in* **C**, *then* **D** *fulfills* (CP1) *whenever* **C** *fulfills* (CP1).

(2) *If* **D** *is a bireflective (full and isomorphism-closed) subcategory of* **C** *which is closed under the formation of quotient objects in* **C**, *then* **D** *fulfills* (CP3) *whenever* **C** *fulfills* (CP3).

Definition 4.6. *A uniformly continuous map* $\varphi : (X, \Lambda^X) \longrightarrow (Y, \Lambda^Y)$ *is called an isomorphism provided that* φ *is bijective and its inverse map* $\psi : (Y, \Lambda^Y) \longrightarrow (X, \Lambda^X)$ *is uniformly continuous.*

Remark 4.7. *Since morphisms in* $S(L, M)$-**SUCT** *are uniformly continuous maps, the isomorphisms in Definition 4.6 are exactly isomorphisms in* $S(L, M)$-**SUCT**.

Proposition 4.8. *The category* $\mathrm{S}(L, M)$-**SULT** *is an isomorphism-closed subcategory of the category* $\mathrm{S}(L, M)$-**SUCT**.

Proof. Let $\varphi : (X, \Lambda^X) \longrightarrow (Y, \Lambda^Y)$ be an isomorphism in $\mathrm{S}(L, M)$-**SUCT** and (X, Λ^X) be a stratified (L, M)-semiuniform limit tower space. It suffices to show that (Y, Λ^Y) fulfils (LMUCT4). Now let ψ denote the inverse map of φ. Take any $\mathscr{G}, \mathscr{H} \in \mathcal{F}^s_L(Y \times Y)$. Then by the uniform continuity of φ and ψ, we have

$$\begin{aligned} \mathscr{G}, \mathscr{H} \in \Lambda^Y_\alpha &\implies \psi^{\rightrightarrows}(\mathscr{G}), \psi^{\rightrightarrows}(\mathscr{H}) \in \Lambda^X_\alpha \\ &\implies \psi^{\rightrightarrows}(\mathscr{G}) \wedge \psi^{\rightrightarrows}(\mathscr{H}) \in \Lambda^X_\alpha \\ &\implies \varphi^{\rightrightarrows}(\psi^{\rightrightarrows}(\mathscr{G}) \wedge \psi^{\rightrightarrows}(\mathscr{H})) \in \Lambda^Y_\alpha \\ &\implies \mathscr{G} \wedge \mathscr{H} \in \Lambda^Y_\alpha. \end{aligned}$$

This means that $\mathscr{G}, \mathscr{H} \in \Lambda^Y_\alpha$ implies $\mathscr{G} \wedge \mathscr{H} \in \Lambda^Y_\alpha$, as desired. □

In order to show that the category $\mathrm{S}(L,M)$-**SULT** is Cartesian closed, we need the following preparations, which can be found in [5, 21, 24].

Let (X,Λ_X) and (Y,Λ_Y) be two stratified (L,M)-semiuniform convergence tower spaces. Define the set

$$[X,Y]_{\mathbf{S}(L,M)-\mathbf{SUTC}} = \{\varphi \mid \varphi : (X,\Lambda_X) \longrightarrow (Y,\Lambda_Y) \text{ is uniformly continuous}\}.$$

For convenience, denote $[X,Y]_{\mathbf{S}(L,M)-\mathbf{SUTC}}$ by $[X,Y]$.

Let $ev_{[X,Y]} : [X,Y]\times X \longrightarrow Y$ be the evaluation map such that $(\varphi,x)\longmapsto \varphi(x)$ and

$$j : ([X,Y]\times[X,Y])\times(X\times X) \longrightarrow ([X,Y]\times X)\times([X,Y]\times X),$$

be the canonical map such that $((\varphi,\psi),(x,x'))\longmapsto((\varphi,x),(\psi,x'))$.

For each $\Phi\in\mathcal{F}_L^s([X,Y]\times[X,Y])$, we define a map $\Gamma_\Phi : \mathcal{F}_L^s(X\times X)\longrightarrow\mathcal{F}_L^s(Y\times Y)$ by

$$\forall\mathscr{F}\in\mathcal{F}_L^s(X\times X),\quad \Gamma_\Phi(\mathscr{F}) = ((ev_{[X,Y]}\times ev_{[X,Y]})\circ j)^{\rightarrow}(\Phi\times\mathscr{F}),$$

which will be called the transformation map of stratified L-filters induced by Φ. To obtain a stratified (L,M)-semiuniform convergence tower structure on $[X,Y]$, we define $\Lambda_\alpha^{[X,Y]}\subseteq\mathcal{F}_L^s([X,Y]\times[X,Y])$ for all $\alpha\in M$ as follows:

$$\Lambda_\alpha^{[X,Y]} = \{\Phi\in\mathcal{F}_L^s([X,Y]\times[X,Y]) \mid \mathscr{F}\in\Lambda_\beta^X \textit{ implies } \Gamma_\Phi(\mathscr{F})\in\Lambda_\beta^Y \textit{ for all } \beta\leqslant\alpha\}.$$

Denote $\Lambda^{[X,Y]} = \{\Lambda_\alpha^{[X,Y]} \mid \alpha\in M\}$.

Proposition 4.9. [24] $([X,Y],\Lambda^{[X,Y]})$ *is a stratified* (L,M)*-semiuniform convergence tower space.*

Theorem 4.10. *The category* $\mathrm{S}(L,M)$-**SULT** *is Cartesian closed.*

Proof. By Theorem 2.9, Lemma 4.5 and Propositions 4.8 and 4.9, it suffices to prove that $\mathrm{S}(L,M)$-**SULT** is closed under the formation of power objects in $\mathrm{S}(L,M)$-**SUCT**. Let (X,Λ^X) and (Y,Λ^Y) be two stratified (L,M)-semiuniform limit tower spaces and $([X,Y],\Lambda^{[X,Y]})$ be the power object in $\mathrm{S}(L,M)$-**SUCT**. We need only show that $([X,Y],\Lambda^{[X,Y]})$ satisfies (LMUCT4).

Take any $\Phi,\Psi\in\Lambda_\alpha^{[X,Y]}$. Then for each $\mathscr{F}\in\Lambda_\beta^X$ for $\beta\leqslant\alpha$, it follows that $\Gamma_\Phi(\mathscr{F}),\Gamma_\Psi(\mathscr{F})\in\Lambda_\beta^Y$. By the definition of $\Gamma_{\Phi\wedge\Psi}$, we have

$$\begin{aligned}
\Gamma_{\Phi\wedge\Psi}(\mathscr{F}) &= ((ev_{[X,Y]}\times ev_{[X,Y]})\circ j)^{\rightarrow}((\Phi\wedge\Psi)\times\mathscr{F})\\
&= ((ev_{[X,Y]}\times ev_{[X,Y]})\circ j)^{\rightarrow}((\Phi\times\mathscr{F})\wedge(\Psi\times\mathscr{F}))\\
&= ((ev_{[X,Y]}\times ev_{[X,Y]})\circ j)^{\rightarrow}(\Phi\times\mathscr{F})\wedge((ev_{[X,Y]}\times ev_{[X,Y]})\circ j)^{\rightarrow}(\Psi\times\mathscr{F})\\
&= \Gamma_\Phi(\mathscr{F})\wedge\Gamma_\Psi(\mathscr{F})\\
&\in \Lambda_\beta^Y. \quad (\text{since } \Lambda^Y \text{ satisfies (LMUCT4)})
\end{aligned}$$

This completes the proof. □

Proposition 4.11. [24] *Let* (X,Λ) *be a stratified* (L,M)*-semiuniform convergence tower space and* $\varphi:(X,\Lambda)\longrightarrow Y$ *be a surjective map. For each* $\alpha\in M$*, define* $\Sigma_\alpha\subseteq\mathcal{F}_L^s(Y\times Y)$ *as follows:*

$$\mathscr{G}\in\Sigma_\alpha \Longleftrightarrow \exists\ \mathscr{F}\in\Lambda_\alpha \textit{ s.t. } (\varphi\times\varphi)^{\rightarrow}(\mathscr{F})\leqslant\mathscr{G}.$$

Let $\Sigma=\{\Sigma_\alpha\mid\alpha\in M\}$*. Then* (Y,Σ) *is the quotient space of* (X,Λ) *and* φ *is the quotient map.*

Theorem 4.12. *The category* $\mathrm{S}(L,M)$-**SULT** *satisfies* (CP3).

Proof. By Theorem 2.9, Lemma 4.5 and Propositions 4.8 and 4.11, it suffices to show that $\mathrm{S}(L,M)$-**SULT** is closed under the formation of quotient objects in $\mathrm{S}(L,M)$-**SUCT**. Let (X,Λ^X) be a stratified (L,M)-semiuniform limit tower space and (Y,Λ^Y) be the quotient space of (X,Λ^X) in $\mathrm{S}(L,M)$-**SUCT** w.r.t. the quotient map $\varphi:X\longrightarrow Y$. We need only show that (Y,Λ^Y) satisfies (LMUCT4).

Take any $\mathscr{G}_1,\mathscr{G}_2\in\Lambda_\alpha^Y$. Then by the definition of quotient space, there exist $\mathscr{F}_1,\mathscr{F}_2\in\Lambda_\alpha^X$ such that $(\varphi\times\varphi)^{\rightarrow}(\mathscr{F}_1)\leqslant\mathscr{G}_1$ and $(\varphi\times\varphi)^{\rightarrow}(\mathscr{F}_2)\leqslant\mathscr{G}_2$. Let $\mathscr{F}=\mathscr{F}_1\wedge\mathscr{F}_2$. Then $\mathscr{F}\in\Lambda_\alpha^X$ and

$$(\varphi\times\varphi)^{\rightarrow}(\mathscr{F}) = (\varphi\times\varphi)^{\rightarrow}(\mathscr{F}_1\wedge\mathscr{F}_2) = (\varphi\times\varphi)^{\rightarrow}(\mathscr{F}_1)\wedge(\varphi\times\varphi)^{\rightarrow}(\mathscr{F}_2)\leqslant\mathscr{G}_1\wedge\mathscr{G}_2.$$

This implies that $\mathscr{G}_1\wedge\mathscr{G}_2\in\Lambda_\alpha^Y$, as desired. □

By Theorems 4.10, 4.12 and Corollary 4.4, we obtain

Theorem 4.13. $\mathrm{S}(L,M)$-**SULT** *is a strongly Cartesian closed topological category.*

5 Stratified (L, M)-filter tower spaces

In the classical case, there are close relations between semiuniform convergence spaces and filter spaces. In [34], Yang and Li introduced the concept of stratified (L, M)-filter tower spaces. In [24], Li proved that the category of stratified (L, M)-filter tower spaces can be embedded in $\mathrm{S}(L, M)$-**SUCT** as a bicoreflective subcategory. In this section, we will further discuss the relations between stratified (L, M)-semiuniform convergence tower spaces and stratified (L, M)-filter tower spaces.

Definition 5.1. [34] *A stratified (L, M)-filter tower structure $\mathbb{F}$ on X is a set $\{\mathbb{F}_\alpha \mid \alpha \in M\}$, where $\mathbb{F}_\alpha \subseteq \mathcal{F}_L^s(X)$ satisfies:*

(LMF1) *For all $\alpha \in M$ and $x \in X$, $[x] \in \mathbb{F}_\alpha$.*

(LMF2) *If $\mathcal{F} \in \mathbb{F}_\alpha$ and $\mathcal{F} \leqslant \mathcal{G}$, then $\mathcal{G} \in \mathbb{F}_\alpha$.*

(LMT1) $\mathbb{F}_{\perp_M} = \mathcal{F}_L^s(X)$.

(LMT2) *If $\beta \leqslant \alpha$, then $\mathbb{F}_\alpha \subseteq \mathbb{F}_\beta$.*

For a stratified (L, M)-filter tower structure $\mathbb{F}$ on X, the pair $(X, \mathbb{F})$ is called a stratified (L, M)-filter tower space.

Definition 5.2. [34] *A map $\varphi : (X, \mathbb{F}) \longrightarrow (Y, \mathbb{F}')$ between stratified (L, M)-filter tower spaces is called Cauchy continuous provided that $\mathcal{F} \in \mathbb{F}_\alpha$ implies $\varphi^{\Rightarrow}(\mathcal{F}) \in \mathbb{F}'_\alpha$ for all $\alpha \in M$.*

The category of stratified (L, M)-filter tower spaces and Cauchy continuous maps is denoted by $\mathrm{S}(L, M)$-**FT**.

Next, we establish the relations between stratified (L, M)-semiuniform convergence tower spaces and stratified (L, M)-filter tower spaces.

Proposition 5.3. [24] (1) *Let (X, Λ) be a stratified (L, M)-semiuniform convergence tower space and define $\mathbb{F}^\Lambda = \{\mathbb{F}^\Lambda_\alpha \mid \alpha \in M\}$ as follows:*

$$\mathbb{F}^\Lambda_\alpha = \{\mathcal{F} \in \mathcal{F}_L^s(X) \mid \mathcal{F} \times \mathcal{F} \in \Lambda_\alpha\}.$$

Then $\mathbb{F}^\Lambda$ is a stratified (L, M)-filter tower structure on X.

(2) *Let $(X, \mathbb{F})$ be a stratified (L, M)-filter tower space and define $\Lambda^\mathbb{F} = \{\Lambda^\mathbb{F}_\alpha \mid \alpha \in M\}$ as follows:*

$$\Lambda^\mathbb{F}_\alpha = \{\mathscr{F} \in \mathcal{F}_L^s(X \times X) \mid \exists \mathcal{F} \in \mathbb{F}_\alpha \text{ s.t. } \mathcal{F} \times \mathcal{F} \leqslant \mathscr{F}\}.$$

Then $\Lambda^\mathbb{F}$ is a stratified (L, M)-semiuniform convergence tower structure on X.

Proposition 5.4. [24] (1) *If $\varphi : (X, \Lambda^X) \longrightarrow (Y, \Lambda^Y)$ is uniformly continuous, then $\varphi : (X, \mathbb{F}^{\Lambda^X}) \longrightarrow (Y, \mathbb{F}^{\Lambda^Y})$ is Cauchy continuous.*

(2) *If $(X, \mathbb{F}^X) \longrightarrow (Y, \mathbb{F}^Y)$ is Cauchy continuous, then $\varphi : (X, \Lambda^{\mathbb{F}^X}) \longrightarrow (Y, \Lambda^{\mathbb{F}^Y})$ is uniformly continuous.*

Definition 5.5. *A stratified (L, M)-semiuniform convergence tower space (X, Λ) is called Fil-determined provided that $\Lambda = \Lambda^{\mathbb{F}^\Lambda}$ holds.*

The full subcategory of $\mathrm{S}(L, M)$-**SUCT**, consisting of Fil-determined stratified (L, M)-semiuniform convergence tower spaces, is denoted by Fil-D-$\mathrm{S}(L, M)$-**SUCT**.

Proposition 5.6. *If $\mathbb{F}$ is a stratified (L, M)-filter tower structure on X and Λ is a Fil-determined stratified (L, M)-semiuniform convergence tower structure on X, then $\mathbb{F}^{\Lambda^\mathbb{F}} = \mathbb{F}$ and $\Lambda^{\mathbb{F}^\Lambda} = \Lambda$.*

Proof. It suffices to show $\mathbb{F}^{\Lambda^\mathbb{F}} = \mathbb{F}$. Take any $\mathcal{F} \in \mathcal{F}_L^s(X)$. Then

$$\begin{aligned}
\mathcal{F} \in \mathbb{F}^{\Lambda^\mathbb{F}}_\alpha &\iff \mathcal{F} \times \mathcal{F} \in \Lambda^\mathbb{F}_\alpha \\
&\iff \exists \mathcal{G} \in \mathbb{F}_\alpha \text{ s.t. } \mathcal{G} \times \mathcal{G} \leqslant \mathcal{F} \times \mathcal{F} \\
&\iff \exists \mathcal{G} \in \mathbb{F}_\alpha \text{ s.t. } \mathcal{G} \leqslant \mathcal{F} \\
&\iff \mathcal{F} \in \mathbb{F}_\alpha.
\end{aligned}$$

This shows $\mathbb{F}^{\Lambda^\mathbb{F}}_\alpha = \mathbb{F}_\alpha$ for all $\alpha \in M$. □

By Propositions 5.3, 5.4 and 5.6, we obtain

Theorem 5.7. *The category* Fil-D-S(L,M)-**SUCT** *is isomorphic to the category* S(L,M)-**FT**.

Next we discuss the relations between Fil-D-S(L,M)-**SUCT** and S(L,M)-**SUCT**. Firstly, by Theorem 5.7 and Corollary 7.6 in [24], we have the following corollary.

Corollary 5.8. *The category* Fil-D-S(L,M)-**SUCT** *is bicoreflective in the category* S(L,M)-**SUCT**.

In order to show that Fil-D-S(L,M)-**SUCT** is bireflective in S(L,M)-**SUCT**, the following lemmas are necessary.

Lemma 5.9. *Let* $\{\Lambda_i \mid i\in I\}$ *be a family of stratified* (L,M)*-semiuniform convergence tower spaces. Then for each* $\alpha\in M$*, the following statements hold:*

(1) $\mathbb{F}_\alpha^{\bigcap_{i\in I}\Lambda_i}=\bigcap_{i\in I}\mathbb{F}_\alpha^{\Lambda_i}$.

(2) $\Lambda_\alpha^{\bigcap_{i\in I}\mathbb{F}^{\Lambda_i}}\subseteq\bigcap_{i\in I}(\Lambda_i)_\alpha$*, where "=" holds whenever* $\{\Lambda_i \mid i\in I\}$ *are Fil-determined.*

Proof. (1) Straightforward.

(2) Take any $\mathscr{F}\in\Lambda_\alpha^{\bigcap_{i\in I}\mathbb{F}^{\Lambda_i}}$. Then there exists $\mathcal{F}\in\bigcap_{i\in I}\mathbb{F}_\alpha^{\Lambda_i}=\mathbb{F}_\alpha^{\bigcap_{i\in I}\Lambda_i}$ such that $\mathcal{F}\times\mathcal{F}\leqslant\mathscr{F}$. This implies that $\mathcal{F}\times\mathcal{F}\in(\Lambda_i)_\alpha$ for all $i\in I$. Then it follows that $\mathscr{F}\in\bigcap_{i\in I}(\Lambda_i)_\alpha$. Further, whenever $\{\Lambda_i \mid i\in I\}$ are Fil-determined, take any $\mathscr{F}\in\bigcap_{i\in I}(\Lambda_i)_\alpha$. Then $\mathscr{F}\in(\Lambda_i)_\alpha=\Lambda_\alpha^{\mathbb{F}^{\Lambda_i}}$ for all $i\in I$. This implies that for each $i\in I$, there exists $\mathcal{F}_i\in\mathbb{F}_\alpha^{\Lambda_i}$ such that $\mathcal{F}_i\times\mathcal{F}_i\leqslant\mathscr{F}$. Let $p_1,p_2:X\times X\longrightarrow X$ be projection maps, respectively. Then $p_1^{\rightrightarrows}(\mathscr{F})\geqslant\mathcal{F}_i$ and $p_2^{\rightrightarrows}(\mathscr{F})\geqslant\mathcal{F}_i$. Let $\mathcal{F}=p_1^{\rightrightarrows}(\mathscr{F})\wedge p_2^{\rightrightarrows}(\mathscr{F})$. Then $\mathcal{F}\geqslant\mathcal{F}_i$ for all $i\in I$. Thus, it follows from $\mathcal{F}_i\in\mathbb{F}_\alpha^{\Lambda_i}$ that $\mathcal{F}\in\mathbb{F}_\alpha^{\Lambda_i}$ for all $i\in I$, i.e., $\mathcal{F}\in\bigcap_{i\in I}\mathbb{F}_\alpha^{\Lambda_i}$. By Lemma 2.4, we have $\mathscr{F}\geqslant p_1^{\rightrightarrows}(\mathscr{F})\times p_2^{\rightrightarrows}(\mathscr{F})\geqslant\mathcal{F}\times\mathcal{F}$. This means $\mathscr{F}\in\Lambda_\alpha^{\bigcap_{i\in I}\mathbb{F}^{\Lambda_i}}$. By the arbitrariness of $\mathscr{F}$, we obtain $\bigcap_{i\in I}(\Lambda_i)_\alpha\subseteq\Lambda_\alpha^{\bigcap_{i\in I}\mathbb{F}^{\Lambda_i}}$, as desired. □

Lemma 5.10. *Let* $\varphi:X\longrightarrow Y$ *be a map and* $\mathcal{G}\in\mathcal{F}_L^s(Y)$*. If* $(\varphi\times\varphi)^{\Leftarrow}(\mathcal{G}\times\mathcal{G})\in\mathcal{F}_L^s(X\times X)$*, then* $\varphi^{\Leftarrow}(\mathcal{G})\in\mathcal{F}_L^s(X)$ *and* $\varphi^{\Leftarrow}(\mathcal{G})\times\varphi^{\Leftarrow}(\mathcal{G})=(\varphi\times\varphi)^{\Leftarrow}(\mathcal{G}\times\mathcal{G})$.

Proof. Take each $B\in L^Y$ such that $\varphi^{\leftarrow}(B)=\perp_L^X$. Then $(\varphi\times\varphi)^{\leftarrow}(B\times B)=\varphi^{\leftarrow}(B)\times\varphi^{\leftarrow}(B)=\perp_L^{X\times X}$. Since $(\varphi\times\varphi)^{\Leftarrow}(\mathcal{G}\times\mathcal{G})\in\mathcal{F}_L^s(X\times X)$, we have

$$(\mathcal{G}\times\mathcal{G})(B\times B)=\bigvee_{C\times D\leqslant B\times B}(\mathcal{G}(C)\wedge\mathcal{G}(D))=\perp_L.$$

This implies $\mathcal{G}(B)=\mathcal{G}(B)\wedge\mathcal{G}(B)\leqslant(\mathcal{G}\times\mathcal{G})(B\times B)=\perp_L$. By Lemma 2.3, we obtain $\varphi^{\Leftarrow}(\mathcal{G})\in\mathcal{F}_L^s(X)$. Further, take each $A\in L^{X\times X}$. On one hand,

$$\begin{aligned}
(\varphi^{\Leftarrow}(\mathcal{G})\times\varphi^{\Leftarrow}(\mathcal{G}))(A) &= \bigvee_{C\times D\leqslant A}(\varphi^{\Leftarrow}(\mathcal{G})(C)\wedge\varphi^{\Leftarrow}(\mathcal{G})(D))\\
&\geqslant \bigvee_{\varphi^{\leftarrow}(E)\times\varphi^{\leftarrow}(F)\leqslant A}(\varphi^{\Leftarrow}(\mathcal{G})(\varphi^{\leftarrow}(E))\wedge\varphi^{\Leftarrow}(\mathcal{G})(\varphi^{\leftarrow}(F)))\\
&\geqslant \bigvee_{\varphi^{\leftarrow}(E)\times\varphi^{\leftarrow}(F)\leqslant A}(\mathcal{G}(E)\wedge\mathcal{G}(F))\\
&= \bigvee_{(\varphi\times\varphi)^{\leftarrow}(E\times F)\leqslant A}(\mathcal{G}(E)\wedge\mathcal{G}(F))\\
&\geqslant \bigvee_{(\varphi\times\varphi)^{\leftarrow}(G)\leqslant A}\ \bigvee_{(\varphi\times\varphi)^{\leftarrow}(E\times F)\leqslant(\varphi\times\varphi)^{\leftarrow}(G)}(\mathcal{G}(E)\wedge\mathcal{G}(F))\\
&\geqslant \bigvee_{(\varphi\times\varphi)^{\leftarrow}(G)\leqslant A}\ \bigvee_{E\times F\leqslant G}(\mathcal{G}(E)\wedge\mathcal{G}(F))\\
&= \bigvee_{(\varphi\times\varphi)^{\leftarrow}(G)\leqslant A}(\mathcal{G}\times\mathcal{G})(G)\\
&= (\varphi\times\varphi)^{\Leftarrow}(\mathcal{G}\times\mathcal{G})(A).
\end{aligned}$$

This means that $\varphi^{\Leftarrow}(\mathcal{G})\times\varphi^{\Leftarrow}(\mathcal{G})\geqslant(\varphi\times\varphi)^{\Leftarrow}(\mathcal{G}\times\mathcal{G})$. On the other hand,

$$(\varphi\times\varphi)^{\Leftarrow}(\mathcal{G}\times\mathcal{G})(A) = \bigvee_{(\varphi\times\varphi)^{\leftarrow}(G)\leqslant A}(\mathcal{G}\times\mathcal{G})(G)$$

$$\begin{aligned}
&\geqslant \bigvee_{(\varphi\times\varphi)^{\leftarrow}(E\times F)\leqslant A} (\mathcal{G}\times\mathcal{G})(E\times F)\\
&= \bigvee_{(\varphi\times\varphi)^{\leftarrow}(E\times F)\leqslant A} \bigvee_{C\times D\leqslant E\times F} (\mathcal{G}(C)\wedge\mathcal{G}(D))\\
&\geqslant \bigvee_{(\varphi\times\varphi)^{\leftarrow}(E\times F)\leqslant A} (\mathcal{G}(E)\wedge\mathcal{G}(F))\\
&= \bigvee_{\varphi^{\leftarrow}(E)\times\varphi^{\leftarrow}(F)\leqslant A} (\mathcal{G}(E)\wedge\mathcal{G}(F))\\
&\geqslant \bigvee_{C\times D\leqslant A} \bigvee_{\varphi^{\leftarrow}(E)\leqslant C} \bigvee_{\varphi^{\leftarrow}(F)\leqslant D} (\mathcal{G}(E)\wedge\mathcal{G}(F))\\
&= \bigvee_{C\times D\leqslant A} \left(\bigvee_{\varphi^{\leftarrow}(E)\leqslant C} \mathcal{G}(E) \wedge \bigvee_{\varphi^{\leftarrow}(F)\leqslant D} \mathcal{G}(F) \right)\\
&= \bigvee_{C\times D\leqslant A} (\varphi^{\Leftarrow}(\mathcal{G})(C)\wedge\varphi^{\Leftarrow}(\mathcal{G})(D))\\
&= (\varphi^{\Leftarrow}(\mathcal{G})\times\varphi^{\Leftarrow}(\mathcal{G}))(A).
\end{aligned}$$

This proves that $(\varphi\times\varphi)^{\Leftarrow}(\mathcal{G}\times\mathcal{G}) \geqslant \varphi^{\Leftarrow}(\mathcal{G})\times\varphi^{\Leftarrow}(\mathcal{G})$, as desired. □

Proposition 5.11. *Let (Y, Σ) be a Fil-determined stratified (L, M)-semiuniform convergence tower space and $\varphi : X \longrightarrow Y$ be a map. Define $\overline{\Lambda} = \{\overline{\Lambda}_\alpha \mid \alpha \in M\}$ as follows:*

$$\overline{\Lambda}_\alpha = \{\mathscr{F} \in \mathcal{F}^s_L(X\times X) \mid (\varphi\times\varphi)^{\Rightarrow}(\mathscr{F}) \in \Sigma_\alpha\}.$$

Then $(X, \overline{\Lambda})$ is a Fil-determined stratified (L, M)-semiuniform convergence tower space.

Proof. (LMUCT1)–(LMUCT3) and (LMP1)–(LMP2) are straightforward. It suffices to show that $(X, \overline{\Lambda})$ is Fil-determined. Take any $\mathscr{F} \in \overline{\Lambda}_\alpha$. Then $(\varphi\times\varphi)^{\Rightarrow}(\mathscr{F}) \in \Sigma_\alpha = \Sigma_\alpha^{\mathbb{F}^\Sigma}$. This implies that there exists $\mathcal{G} \in \mathbb{F}^\Sigma_\alpha$ such that $\mathcal{G}\times\mathcal{G} \leqslant (\varphi\times\varphi)^{\Rightarrow}(\mathscr{F})$. Further, it follows that

$$\mathscr{F} \geqslant (\varphi\times\varphi)^{\Leftarrow}((\varphi\times\varphi)^{\Rightarrow}(\mathscr{F})) \geqslant (\varphi\times\varphi)^{\Leftarrow}(\mathcal{G}\times\mathcal{G}).$$

Since $\mathcal{G} \in \mathbb{F}^\Sigma_\alpha$, i.e., $\mathcal{G}\times\mathcal{G} \in \Sigma_\alpha$ and $(\varphi\times\varphi)^{\Rightarrow}((\varphi\times\varphi)^{\Leftarrow}(\mathcal{G}\times\mathcal{G})) \geqslant \mathcal{G}\times\mathcal{G}$, we have $(\varphi\times\varphi)^{\Rightarrow}((\varphi\times\varphi)^{\Leftarrow}(\mathcal{G}\times\mathcal{G})) \in \Sigma_\alpha$. Thus, $(\varphi\times\varphi)^{\Leftarrow}(\mathcal{G}\times\mathcal{G}) \in \overline{\Lambda}_\alpha$. By Lemma 5.10, we obtain $\varphi^{\Leftarrow}(\mathcal{G})\times\varphi^{\Leftarrow}(\mathcal{G}) = (\varphi\times\varphi)^{\Leftarrow}(\mathcal{G}\times\mathcal{G}) \in \overline{\Lambda}_\alpha$. This implies that $\varphi^{\Leftarrow}(\mathcal{G}) \in \mathbb{F}^{\overline{\Lambda}}_\alpha$. Then it follows from $\mathscr{F} \geqslant (\varphi\times\varphi)^{\Leftarrow}(\mathcal{G}\times\mathcal{G}) = \varphi^{\Leftarrow}(\mathcal{G})\times\varphi^{\Leftarrow}(\mathcal{G})$ that $\mathscr{F} \in \Lambda^{\mathbb{F}^{\overline{\Lambda}}}_\alpha$. By the arbitrariness of $\mathscr{F}$, we have $\overline{\Lambda}_\alpha \subseteq \Lambda^{\mathbb{F}^{\overline{\Lambda}}}_\alpha$. The inverse inequality $\Lambda^{\mathbb{F}^{\overline{\Lambda}}}_\alpha \subseteq \overline{\Lambda}_\alpha$ holds obviously. Therefore, $\overline{\Lambda}_\alpha = \Lambda^{\mathbb{F}^{\overline{\Lambda}}}_\alpha$ for all $\alpha \in M$, i.e., $\overline{\Lambda} = \Lambda^{\mathbb{F}^{\overline{\Lambda}}}$, as desired. □

Theorem 5.12. *The category* Fil-D-S(L, M)-**SUCT** *is bireflective in the category* S(L, M)-**SUCT**.

Proof. Let (X, Λ^X) be stratified (L, M)-semiuniform convergence tower space and let

$$\Lambda_F = \{\Lambda \mid (X, \Lambda) \in| \text{ Fil-D-S(L,M)-}\mathbf{SUCT} \mid \text{ and } (X, \Lambda^X) \leqslant_\Lambda (X, \Lambda)\}.$$

Denote $\Lambda_F = \{\Lambda^i \mid i \in I\}$ and put $\widetilde{\Lambda} = \bigcap \Lambda_F = \bigcap_{i\in I} \Lambda^i$. By Lemma 5.9, it follows that

$$\Lambda^{\mathbb{F}^{\widetilde{\Lambda}}} = \Lambda^{\mathbb{F}^{\bigcap_{i\in I}\Lambda^i}} = \Lambda^{\bigcap_{i\in I}\mathbb{F}^{\Lambda^i}} = \bigcap_{i\in I} \Lambda^{\mathbb{F}^{\Lambda^i}} = \bigcap_{i\in I} \Lambda^i = \widetilde{\Lambda}.$$

This shows $(X, \widetilde{\Lambda})$ is a Fil-determined stratified (L, M)-semiuniform convergence tower space. Further, we claim that $id_X : (X, \Lambda^X) \longrightarrow (X, \widetilde{\Lambda})$ is the bireflection w.r.t. Fil-D-S(L, M)-**SUCT**. For this, it suffices to show that

(1) $id_X : (X, \Lambda^X) \longrightarrow (X, \widetilde{\Lambda})$ is uniformly continuous.

(2) For a Fil-determined stratified (L, M)-semiuniform convergence tower space (Y, Σ), the uniform continuity of $\varphi : (X, \Lambda^X) \longrightarrow (Y, \Sigma)$ implies the uniform continuity of $\varphi : (X, \widetilde{\Lambda}) \longrightarrow (Y, \Sigma)$.

For (1), by the definition of $\widetilde{\Lambda}$, it is obvious.

For (2), for a Fil-determined stratified (L, M)-semiuniform convergence tower space (Y, Σ), define $(X, \overline{\Lambda})$ as in Proposition 5.11. Then $(X, \overline{\Lambda})$ is a Fil-determined stratified (L, M)-semiuniform convergence tower space. Since $\varphi : (X, \Lambda^X) \longrightarrow (Y, \Sigma)$ is uniformly continuous, we know $\mathscr{F} \in \Lambda_\alpha^X$ implies $(\varphi \times \varphi)^{\rightarrow}(\mathscr{F}) \in \Sigma_\alpha$ for all $\alpha \in M$. This means $\mathscr{F} \in \overline{\Lambda}_\alpha$. Thus, $\Lambda_\alpha^X \subseteq \overline{\Lambda}_\alpha$ for all $\alpha \in M$, i.e., $\Lambda^X \leqslant_\Lambda \overline{\Lambda}$. Hence, $\overline{\Lambda} \in \Lambda_F$. This means $\widetilde{\Lambda} \leqslant_\Lambda \overline{\Lambda}$. Therefore, take any $\mathscr{F} \in \widetilde{\Lambda}_\alpha$. Then $\mathscr{F} \in \overline{\Lambda}_\alpha$, i.e., $(\varphi \times \varphi)^{\rightarrow}(\mathscr{F}) \in \Sigma_\alpha$. This shows the uniform continuity of $\varphi : (X, \widetilde{\Lambda}) \longrightarrow (Y, \Sigma)$, as desired. □

By Theorems 5.7, 5.12 and Corollary 5.8, we obtain

Corollary 5.13. *$S(L, M)$-**FT** can be embedded in $S(L, M)$-**SUCT** both as a simultaneously bireflective and a bicoreflective subcategory.*

6 Conclusions

In this paper, we focused on some subcategories of the category $\mathrm{S}(L, M)$-**SUCT**. Concretely, we proposed the concepts of stratified (L, M)-semiuniform convergence spaces and stratified (L, M)-semiuniform limit tower spaces, and discussed their categorical properties. Also, we investigated the categorical relations between stratified (L, M)-semiuniform convergence tower spaces and stratified (L, M)-filter tower spaces. These contents can be considered as further explorations on fuzzy semiuniform convergence structures compared with the results in [24]. Following this paper, we will consider the following problems in the future.

- Generalizing the lattice background from a frame to a complete residuated lattice.
- Considering subcategories of the category of $\top$-semiuniform convergence spaces and study their categorical properties.

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