Iranian Journal of Fuzzy Systems

Volume 21, Number 4, (2024), pp. 51-65

Original paper

University of Sistan and Baluchestan

Trapezoidal fuzzy multi-number preference relations based on architecture multi-criteria decision-making application

D. Bakbak [1], V. Uluçay [2] and S. A. Edalatpanah [3]

[1] *TBMM Public relations building 2nd Floor, B206 room Ministries, Ankara06543-Turkey*
[2] *Department of Mathematics, Kilis 7 Aralık University, 79000 Kilis, Turkey*
[3] *Department of Applied Mathematics, Ayandegan Institute of Higher Education, Tonekabon, Iran*

derya.bakbak@tbmm.gov.tr, vulucay27@gmail.com, saedalatpanah@gmail.com

Abstract

There are many vague and uncertain issues in real-life. To address these problems, many models and theories have been developed. Therefore, the present study aims to describe the concept of trapezoidal fuzzy multi-number aggregation operators and also provide its application in architecture. Our main focus is on the trapezoidal fuzzy multi-number weighted arithmetic average (TFMNWAA) operator, the trapezoidal fuzzy multi-number weighted geometric average (TFMNWGA) operator and trapezoidal fuzzy multi-number hybrid aggregation (TFMNHA) operator. And then, we investigate the fundamental properties of the operators in detail such as idempotency, commutativity and boundedness. Then the notion of a new score function of ranking the trapezoidal fuzzy multi-numbers is defined. We apply the TFMNHA operator to multi-criteria decision-making trapezoidal fuzzy multi-number. Finally, a discussion is presented with a numerical example is provided to show its applicability and usefulness.

Keywords: Fuzzy set, trapezoidal fuzzy multi-number, aggregation operators, score function, multi-criteria decision-making.

1 Introduction

There are some situations where people don't make their own decisions. These situations are extraordinary situations and do not occur by the people's own preference. Often people cannot even understand the reason for what is happening. They can suddenly find themselves in a rapidly developing problem. This is the case in the internal turmoil, earthquakes and wars in their country [5]. It is not known when the earthquake will occur and when the war will end. In addition, even if there is an earthquake, it is uncertain when the investments such as infrastructure, education and health will end and when the normal life will be started. This complex situation forces people to do two things. Should they stay in the camps? Or should they go to normal life? As a matter of fact, a long-term life in emergency tents has brought many problems with it and container city arrangements have started from the camps. But nowadays, the process of decision making is a complex issue involving professionals of different style. Every organization has to take decision at one or another point as a part of managerial process. Therefore, every organization extensively needs a team of professional experts to make all sorts of complex decision. But remember, that an individual alone cannot come out with final decision because decision making problems consist of cumulative and consultative process. Decision making plays a significant role and is the key component to determine both organizational and managerial activities. Due to complexity in real life, all the information can not be represented by crips numbers. Therefore, to handle fuzziness and vagueness information, Zadeh [45] introduced the fuzzy sets by using only membership degree in [0,1]. The works on fuzzy set and fuzzy multiset sets and their hybrid structure in theories and application have been successfully applied to differently fields as; some aggregation operators for IVI-octahedron sets [4], T-spherical fuzzy maclaurin symmetric

Corresponding Author: S. A. Edalatpanah
Received: January 2023; Revised: October 2023; Accepted: December 2023.
https://doi.org/10.22111/IJFS.2023.44605.7853

mean operator [12], neutrosophic DEMATEL approach [39], regret theory-based three-way decision method [10], fuzzy multisets as a framework of information systems [21], decision-making with a fuzzy preference relation [24], interval-valued hesitant pythagorean fuzzy set of thermal energy storage technique [27], the novel augmented fermatean MCDM for optimal renewable energy power plant location [23], on generalized fuzzy multisets and their use in computation [34], decision-making and so on [46, 47, 48]. Recently, Karimi et al. [16] developed a fully fuzzy best worst MCDM method based on triangular fuzzy number. Parveen and Kamble [25] presented a solution methodology for fuzzy technique for hexagonal fuzzy numbers. Abdullah [1] gave the idea of sequence of trapezoidal cubic linguistic fuzzy aggregation operators. Akram and Shahzadi [2] presented some new operators. Aydin et al. [3] developed some operators and presenting its application in decision-making. Chakraborty et al. [7] defined new ranking for decision-making problems. Wei [40] presented the study of some new operators for Pythagorean fuzzy numbers. Fahmi et al. [11] introduced some Einstein operations on trapezoidal cubic fuzzy set. Jin et al.[15] studied new decision-method in interval-valued hesitant fuzzy sets. The theories have studied in various areas such as; some arithmetic averaging operators with their application in architecture [6], WA and OWA operators on interval-valued Atanassov's intuitionistic fuzzy [8], pythagorean fuzzy linguistic Muirhead mean operators [18], a new approach for ranking of intuitionistic fuzzy numbers [22], approaches to some induced Einstein geometric aggregation operators [26], induced Pythagorean trapezoidal fuzzy aggregation operators based on Einstein operations [33], Fermatean fuzzy weighted averaging/geometric operators [32], on neutrosophic sets [13, 28, 29, 36, 37, 38]. With the many methods that have been applied in decision-making literature as reviewed above, it is not hard to see that the fuzzy set concept methods have found more applications in the uncertainity evaluation process. This, however, can be tract to the unique ability of these fuzzy number concept methods to handle and address uncertainty, fuzziness, and vagueness in the decision-making process of evaluation stage. To further improve the decision-making, in this paper, however, an extended fuzzy multiset multi-criteria decision-making model which is based on the integration of the trapezoidal fuzzy multi-numbers weighted arithmetic average operator and the new score function is proposed. The trapezoidal fuzzy multi-number (TFM-number) has many grades of membership function, hence it considers the opinions of the experts from many dimensions before arriving at a preferred decision/alternative. The existing theories such as fuzzy number and triangular fuzzy number are the particular cases of the TFM-number. Consider the fact that the TFM-number has a useful ability to resolve the unclear and vague information which comes to our real-world problems. Aggregation operators and score function are an important tool to find the similarity among objects (sets). In the words, to find the degree of closeness among the objects (sets). But up to date, the Aggregation operators and score function are not introduced in the TFM-number. Thus, it is essential to interpret the Aggregation operators for TFM-number to cope with pattern recognition, medical diagnosis, information retrieval, multi-criteria decision-making issues, etc. to fulfill the needs of the decision analysts.

The aim of the current research is to propose a new operator and new score function using a the trapezoidal fuzzy multi numbers by in multi criteria decision-making problems. This study is aimed to a new operator and new score function using a the TFM-number by in MCDM problems. It is evident from the previous studies, in the literature, that the method in this study has not yet been proposed at this extent. In the fuzzy number, the element in a universe has a membership function but the a membership function is inadequate for providing complete information in some problems because there are situations where each element has different membership values.

Therefore, we studied on trapezoidal fuzzy multi-number. The remain of this paper is structured as follows: Section 2 consists the brief study of basic concepts. In Section 3, we introduce the trapezoidal fuzzy multi-number also their developable properties. In Section 4, we present the definition of completely consistent TFMNPRs and acceptably consistent TFMNPRs. In Section 5, we develop the technique and algorithm for MCDM, by using the operators and a new score function. In Section 6, we develop the algorithm for MCDM. Furthermore, an illustrative example is developed to describe the effectiveness of the developed operators. In Section 7, a discussion and comparative study of the investigated models with some existing literature are presented to show the applicability and superiority of the investigated models. Finally, some closing remarks are presented.

2 Preliminaries

Definition 2.1. [45] *A fuzzy set is of the following form*

$$\Upsilon = \{\langle x, \mu_\Upsilon(x)\rangle : x \in X\},$$

where $\mu_\Upsilon : X \to [0,1]$ *implies the membership function* $x \in X$.

Definition 2.2. [30] *Let* X *be a non-empty set. A multi-fuzzy set* G *on* X *is defined as:*

$$G = \{\langle x, \mu_G^1(x), \mu_G^2(x), ..., \mu_G^i(x), ...\rangle : x \in X\},$$

where $\mu_G^i : X \to [0,1]$ *for all* $i \in \{1,2,...,p\}$ *and* $x \in X$.

Definition 2.3. [49] *t-norms are associative, monotonic and commutative two valued functions t that map from* $[0,1] \times [0,1]$ *into* $[0,1]$. *These properties are formulated with the following conditions:*

1. $t(0,0) = 0$ *and* $t(\mu_{X_1}(x), 1) = t(1, \mu_{X_1}(x)) = \mu_{X_1}(x),\ x \in E$.
2. *If* $\mu_{X_1}(x) \leq \mu_{X_3}(x)$ *and* $\mu_{X_2}(x) \leq \mu_{X_4}(x)$, *then* $t(\mu_{X_1}(x), \mu_{X_2}(x)) \leq t(\mu_{X_3}x), \mu_{X_4}(x))$.
3. $t(\mu_{X_1}(x), \mu_{X_2}(x)) = t(\mu_{X_2}(x), \mu_{X_1}(x))$.
4. $t(\mu_{X_1}(x), t(\mu_{X_2}(x), \mu_{X_3}(x))) = t(t(\mu_{X_1}(x), \mu_{X_2})(x), \mu_{X_3}(x))$.

Definition 2.4. [49] *s-norms are associative, monotonic and commutative two placed functions s that map from* $[0,1] \times [0,1]$ *into* $[0,1]$. *These properties are formulated with the following conditions:*

1. $s(1,1) = 1$ *and* $s(\mu_{X_1}(x), 0) = s(0, \mu_{X_1}(x)) = \mu_{X_1}(x), x \in E$.
2. *If* $\mu_{X_1}(x) \leq \mu_{X_3}(x)$ *and* $\mu_{X_2}(x) \leq \mu_{X_4}(x)$, *then* $s(\mu_{X_1}(x), \mu_{X_2}(x)) \leq s(\mu_{X_3}(x), \mu_{X_4}(x))$.
3. $s(\mu_{X_1}(x), \mu_{X_2}(x)) = s(\mu_{X_2}(x), \mu_{X_1}(x))$.
4. $s(\mu_{X_1}(x), s(\mu_{X_2}(x), \mu_{X_3}(x))) = s(s(\mu_{X_1}(x), \mu_{X_2})(x), \mu_{X_3}(x))$.

t-norm and t-conorm are related in a sense of logical duality. Typical dual pairs of non parametrized t-norm and t-conorm are complied below:

1. Drastic product:
$$t_w(\mu_{X_1}(x), \mu_{X_2}(x)) = \begin{cases} min\{\mu_{X_1}(x), \mu_{X_2}(x)\}, & \max\{\mu_{X_1}(x)\mu_{X_2}(x)\} = 1 \\ 0, & otherwise \end{cases}$$
2. Drastic sum:
$$s_w(\mu_{X_1}(x), \mu_{X_2}(x)) = \begin{cases} max\{\mu_{X_1}(x), \mu_{X_2}(x)\}, & \min\{\mu_{X_1}(x)\mu_{X_2}(x)\} = 0 \\ 1, & otherwise \end{cases}$$
3. Bounded product:
$$t_1(\mu_{X_1}(x), \mu_{X_2}(x)) = \max\{0, \mu_{X_1}(x) + \mu_{X_2}(x) - 1\}.$$
4. Bounded sum:
$$s_1(\mu_{X_1}(x), \mu_{X_2}(x)) = \min\{1, \mu_{X_1}(x) + \mu_{X_2}(x)\}.$$
5. Einstein product:
$$t_{1.5}(\mu_{X_1}(x), \mu_{X_2}(x)) = \frac{\mu_{X_1}(x).\mu_{X_2}(x)}{2 - [\mu_{X_1}(x) + \mu_{X_2}(x) - \mu_{X_1}(x).\mu_{X_2}(x)]}.$$
6. Einstein sum:
$$s_{1.5}(\mu_{X_1}(x), \mu_{X_2}(x)) = \frac{\mu_{X_1}(x) + \mu_{X_2}(x)}{1 + \mu_{X_1}(x).\mu_{X_2}(x)}.$$
7. Algebraic product:
$$t_2(\mu_{X_1}(x), \mu_{X_2}(x)) = \mu_{X_1}(x).\mu_{X_2}(x).$$
8. Algebraic sum:
$$s_2(\mu_{X_1}(x), \mu_{X_2}(x)) = \mu_{X_1}(x) + \mu_{X_2}(x) - \mu_{X_1}(x).\mu_{X_2}(x).$$
9. Hamacher product:
$$t_{2.5}(\mu_{X_1}(x), \mu_{X_2}(x)) = \frac{\mu_{X_1}(x).\mu_{X_2}(x)}{\mu_{X_1}(x) + \mu_{X_2}(x) - \mu_{X_1}(x).\mu_{X_2}(x)}.$$
10. Hamacher sum:
$$s_{2.5}(\mu_{X_1}(x), \mu_{X_2}(x)) = \frac{\mu_{X_1}(x) + \mu_{X_2}(x) - 2.\mu_{X_1}(x).\mu_{X_2}(x)}{1 - \mu_{X_1}(x).\mu_{X_2}(x)}.$$

11. Minumum:
$$t_3(\mu_{X_1}(x), \mu_{X_2}(x)) = min\{\mu_{X_1}(x), \mu_{X_2}(x)\}.$$

12. Maximum:
$$s_3(\mu_{X_1}(x), \mu_{X_2}(x)) = max\{\mu_{X_1}(x), \mu_{X_2}(x)\}.$$

Definition 2.5. [35] *Let $\eta_A^i \in [0,1]$ $(i \in \{1,2,...,p\})$ and $a,b,c,d \in R$ such that $a \leq b \leq c \leq d$. Then a TFM-number $\tilde{a} = \langle (a,b,c,d); \eta_A^1, \eta_A^2, ..., \eta_A^P \rangle$ is a special fuzzy multi-set on R, whose membership functions are defined as;*

$$\mu_A^i(x) = \begin{cases} (x-a_1)\eta_A^i/(b_1-a_1) & a_1 \leq x < b_1 \\ \eta_A^i & b_1 \leq x \leq c_1 \\ (d_1-x)\eta_A^i/(d_1-c_1) & c_1 < x \leq d_1 \\ 0 & otherwise, \end{cases}$$

Note that the set of all TFM-number on $\mathbb{R}$ will be denoted by Λ.

Definition 2.6. [35] *Let $A = \langle (a_1,b_1,c_1,d_1); \eta_A^1, \eta_A^2, ..., \eta_A^P \rangle$, $B = \langle (a_2,b_2,c_2,d_2); \eta_B^1, \eta_B^2, ..., \eta_B^P \rangle \in \Lambda$ and $\gamma \neq 0$ be any real number. Then,*

1. $A + B = \langle (a_1+a_2, b_1+b_2, c_1+c_2, d_1+d_2); s(\eta_A^1, \eta_B^1), s(\eta_A^2, \eta_B^2), ..., s(\eta_A^p, \eta_B^p) \rangle$.
2. $A - B = \langle (a_1-d_2, b_1-c_2, c_1-b_2, d_1-a_2); s(\eta_A^1, \eta_B^1), s(\eta_A^2, \eta_B^2), ..., s(\eta_A^p, \eta_B^p) \rangle$.
3. $A.B = \begin{cases} \langle (a_1a_2, b_1b_2, c_1c_2, d_1d_2); t(\eta_A^1, \eta_B^1), t(\eta_A^2, \eta_B^2), ..., t(\eta_A^p, \eta_B^p) \rangle & (d_1 > 0, d_2 > 0) \\ \langle (a_1d_2, b_1c_2, c_1b_2, d_1a_2); t(\eta_A^1, \eta_B^1), t(\eta_A^2, \eta_B^2), ..., t(\eta_A^p, \eta_B^p) \rangle & (d_1 < 0, d_2 > 0) \\ \langle (d_1d_2, c_1c_2, b_1b_2, a_1a_2); t(\eta_A^1, \eta_B^1), t(\eta_A^2, \eta_B^2), ..., t(\eta_A^p, \eta_B^p) \rangle & (d_1 < 0, d_2 < 0) \end{cases}$
4. $A/B = \begin{cases} \langle (a_1/d_2, b_1/c_2, c_1/b_2, d_1/a_2); t(\eta_A^1, \eta_B^1), t(\eta_A^2, \eta_B^2), ..., t(\eta_A^p, \eta_B^p) \rangle & (d_1 > 0, d_2 > 0) \\ \langle ((d_1/d_2, c_1/c_2, b_1/b_2, a_1/a_2); t(\eta_A^1, \eta_B^1), t(\eta_A^2, \eta_B^2), ..., t(\eta_A^p, \eta_B^p) \rangle & (d_1 < 0, d_2 > 0) \\ \langle (d_1/a_2, c_1/b_2, b_1/c_2, a_1/d_2); t(\eta_A^1, \eta_B^1), t(\eta_A^2, \eta_B^2), ..., t(\eta_A^p, \eta_B^p) \rangle & (d_1 < 0, d_2 < 0) \end{cases}$
5. $\gamma A = \langle (\gamma a_1, \gamma b_1, \gamma c_1, \gamma d_1); 1-(1-\eta_A^1)^\gamma, 1-(1-\eta_A^2)^\gamma, ..., 1-(1-\eta_A^p)^\gamma \rangle (\gamma \geq 0)$.
6. $A^\gamma = \langle (a_1^\gamma, b_1^\gamma, c_1^\gamma, d_1^\gamma); (\eta_A^1)^\gamma, (\eta_A^2)^\gamma, ..., (\eta_A^P)^\gamma \rangle (\gamma \geq 0)$.

Definition 2.7. [41] *A fuzzy preference relation (FPR) B on the set $A = \{A_1, A_2, ..., A_m\}$ is represented by a matrix $B = (b_{ik})_{n \times m}$, where (b_{ik}) is the intensity of preference of A_i over A_k, and satisfies:*

$$b_{ik} + b_{ki} = 1, b_{ij} \in [0,1], \forall A_i, A_k \in A.$$

3 Score function and operators on trapezoidal fuzzy multi-number

Definition 3.1. *Let $A = \langle (a_1,b_1,c_1,d_1); \eta_A^1, \eta_A^2, ..., \eta_A^P \rangle$ and $B = \langle (a_2,b_2,c_2,d_2); \eta_B^1, \eta_B^2, ..., \eta_B^P \rangle$ be two TFMNs $\gamma \geq 0$, then new operation for TFMNs is defined as follows:(In this work we will only define the addition operator. The other operations are as in Definition 2.6.)*

1. $A \oplus B$
   (i) If $c_1 + d_1 \neq a_1 + b_1$ *and* $c_2 + d_2 \neq a_2 + b_2$, *then*
$$A \oplus B = \langle [a_1+a_2, b_1+b_2, c_1+c_2, d_1+d_2], (\frac{\phi(A)\eta_A^1 + \phi(B)\eta_B^1}{\phi(A)+\phi(B)}, \frac{\phi(A)\eta_A^2 + \phi(B)\eta_B^2}{\phi(A)+\phi(B)}, ..., \frac{\phi(A)\eta_A^p + \phi(B)\eta_B^p}{\phi(A)+\phi(B)}) \rangle,$$
   where $\phi(A) = \frac{c_1-b_1+d_1-a_1}{2}$ *and* $\phi(B) = \frac{c_2-b_2+d_2-a_2}{2}$;
   (ii) If $a_1 = b_1 = c_1 = d_1 = a$ *and* $c_2 + d_2 \neq a_2 + b_2$, *then*
$$A \oplus B = \langle [a+a_2, a+b_2, a+c_2, a+d_2], (\frac{a\eta_A^1 + \phi(B)\eta_B^1}{a+\phi(B)}, \frac{a\eta_A^2 + \phi(B)\eta_B^2}{a+\phi(B)}, ..., \frac{a\eta_A^p + \phi(B)\eta_B^p}{a+\phi(B)}) \rangle,$$
   (iii) If $c_1 + d_1 \neq a_1 + b_1$ *and* $a_2 = b_2 = c_2 = d_2 = b$, *then*
$$A \oplus B = \langle [a_1+b, b_1+b, c_1+b, d_1+b], (\frac{\phi(A)\eta_A^1 + b\eta_B^1}{\phi(A)+b}, \frac{\phi(A)\eta_A^2 + b\eta_B^2}{\phi(A)+b}, ..., \frac{\phi(A)\eta_A^p + b\eta_B^p}{\phi(A)+b}) \rangle,$$

(iv) If $a_1 = b_1 = c_1 = d_1 = a$ *and* $a_2 = b_2 = c_2 = d_2 = b$, *then*

$$A \oplus B = \langle [a+b], (\frac{a\eta_A^1 + b\eta_B^1}{a+b}, \frac{a\eta_A^2 + b\eta_B^2}{a+b}, ..., \frac{a\eta_A^p + b\eta_B^p}{a+b}) \rangle.$$

Example 3.2. *Let* $A = \langle (1,3,5,8); 0.3, 0.4, 0.9 \rangle$, $B = \langle (2,4,6,7); 0.2, 0.4, 0.7 \rangle \in \Lambda$,

$$\phi(A) = \frac{5-3+8-1}{2} = 4.5 \quad \textit{and} \quad \phi(B) = \frac{6-4+7-2}{2} = 3.5;$$

Since $5+8 = 13 \neq 1+3 = 4$ *and* $6+7 = 13 \neq 2+4 = 6$, *then*

$$A \oplus B = \langle [1+2, 3+4, 5+6, 8+7], (\frac{(4.5)(0.3)+(3.5)(0.2)}{(4.5)+(3.5)}, \frac{(4.5)(0.4)+(3.5)(0.4)}{(4.5)+(3.5)}, \frac{(4.5)(0.9)+(3.5)(0.7)}{(4.5)+(3.5)}) \rangle,$$

$$A \oplus B = \langle (3,7,11,15); 0.2562, 0.4, 0.8125 \rangle.$$

Example 3.3. *Let* $A = \langle (1,1,1,1); 0.2, 0.3, 0.5 \rangle$, $B = \langle (2,4,6,7); 0.2, 0.4, 0.7 \rangle \in \Lambda$.
Since $1 = 1 = 1 = 1$ *and* $6+7 = 13 \neq 2+4 = 6$, *then*

$$A \oplus B = \langle [1+2, 1+4, 1+6, 1+7], (\frac{(1)(0.3)+(3.5)(0.2)}{(1)+(3.5)}, \frac{(1)(0.4)+(3.5)(0.4)}{(1)+(3.5)}, \frac{(1)(0.9)+(3.5)(0.7)}{(1)+(3.5)}) \rangle,$$

$$A \oplus B = \langle (3,5,7,8); 0.2222, 0.4, 0.7444 \rangle.$$

Example 3.4. *Let* $A = \langle (1,3,5,8); 0.2, 0.3, 0.5 \rangle$, $B = \langle (2,2,2,2); 0.2, 0.4, 0.7 \rangle \in \Lambda$.
Since $5+8 = 13 \neq 1+3 = 4$ *and* $2 = 2 = 2 = 2$, *then*

$$A \oplus B = \langle [1+2, 3+2, 5+2, 8+2], (\frac{(1)(0.2)+(2)(0.2)}{(4.5)+(2)}, \frac{(3)(0.3)+(2)(0.4)}{(4.5)+(2)}, \frac{(5)(0.5)+(2)(0.7)}{(4.5)+(2)}) \rangle,$$

$$A \oplus B = \langle (3,5,7,10); 0.0923, 0.2615, 0.6 \rangle.$$

Example 3.5. *Let* $A = \langle (1,1,1,1); 0.2, 0.3, 0.5 \rangle$, $B = \langle (2,2,2,2); 0.2, 0.4, 0.7 \rangle \in \Lambda$.
Since $1 = 1 = 1 = 1$ *and* $2 = 2 = 2 = 2$, *then*

$$A \oplus B = \langle [1+2, 1+2, 1+2, 1+2], (\frac{(1)(0.2)+(2)(0.2)}{(1)+(2)}, \frac{(1)(0.3)+(2)(0.4)}{(1)+(2)}, \frac{(1)(0.5)+(2)(0.7)}{(1)+(2)}) \rangle,$$

$$A \oplus B = \langle (3); 0.1333, 0.2333, 0.4 \rangle.$$

Theorem 3.6. *Let* $A = \langle (a_1, b_1, c_1, d_1); \eta_A^1, \eta_A^2, ..., \eta_A^P \rangle$, $B = \langle (a_2, b_2, c_2, d_2); \eta_B^1, \eta_B^2, ..., \eta_B^P \rangle$,

$$C = \langle (a_3, b_3, c_3, d_3); \eta_C^1, \eta_C^2, ..., \eta_C^P \rangle \in \Lambda.$$

Then, we have

1. $B \oplus A = A \oplus B$
2. $C \oplus (A \oplus B) = A \oplus (B \oplus C)$
3. $B \otimes A = A \otimes B$
4. $C \otimes (A \otimes B) = (B \otimes C) \otimes A$
5. $\lambda_1.A \oplus \lambda_2.A = (\lambda_1 + \lambda_2).A, (\lambda_1, \lambda_2 \geq 0)$
6. $\lambda.(A \oplus B) = \lambda.A \oplus \lambda.B, (\lambda \geq 0)$.
7. $(A \otimes B)^\lambda = A^\lambda \otimes B^\lambda, (\lambda \geq 0)$.

Proof. 1. Based on Definition 3.1, it can be seen that; seen that;

$$\begin{aligned} A \otimes B &= \langle (a_1a_2, b_1b_2, c_1c_2, d_1d_2); t(\eta_A^1, \eta_B^1), t(\eta_A^2, \eta_B^2), ..., t(\eta_A^p, \eta_B^p) \rangle \\ &= \langle (a_2a_1, b_2b_1, c_2c_1, d_2d_1); t(\eta_B^1, \eta_A^1), t(\eta_B^2, \eta_A^2), ..., t(\eta_B^p, \eta_A^p) \rangle \\ &= B \otimes A. \end{aligned}$$

The proof of (1), (2), (4), (5),(6) and (7) can be made similarly.

□

3.1 New comparison methods for trapezoidal fuzzy multi-number

Definition 3.7. *Let $A = \langle (a_1, b_1, c_1, d_1); \eta_A^1, \eta_A^2, ..., \eta_A^P \rangle$ be a TFMN, then the score function and the accuracy function of A are respectively defined as:*

$$E(A) = \frac{a_1 + 2b_1 + 2c_1 + d_1}{6} \times \frac{(\eta_A^1 + \eta_A^2 + ... + \eta_A^P)}{p},$$

$$\mathcal{A}(A) = \frac{b_1 + c_1}{3} \times \frac{(\eta_A^1 + \eta_A^2 + ... + \eta_A^P)}{p}.$$

Definition 3.8. *Let $A = \langle (a_1, b_1, c_1, d_1); \eta_A^1, \eta_A^2, ..., \eta_A^P \rangle$ and $B = \langle (a_2, b_2, c_2, d_2); \eta_B^1, \eta_B^2, ..., \eta_B^P \rangle$ be two TFMNs. The comparison method for A and B can be defined as:*

1. *If $E(A) > E(B)$, then $A \succ B$, meaning that A is superior to B.*
2. *If $E(A) = E(B)$, and $\mathcal{A}(A) < \mathcal{A}(B)$, then $A \prec B$ meaning that A is inferior to B.*
3. *If $E(A) = E(B)$, and $\mathcal{A}(A) > \mathcal{A}(B)$, then $A \succ B$ meaning that A is superior to B.*
4. *If $E(A) = E(B)$, and $\mathcal{A}(A) = \mathcal{A}(B)$, then $A \sim B$ meaning that A is indifferent to B.*

Example 3.9. *Let $A = \langle (1, 2, 3, 10); 0.1, 0.3, 0.5, 0.9 \rangle$, $B = \langle (1, 2, 3, 9); 0.1, 0.3, 0.5, 0.9 \rangle \in \Lambda$, we can obtain that*

$$E(A) = \frac{(1 + 2 \times 2 + 2 \times 3 + 10)}{6} \times \frac{(0.1 + 0.3 + 0.5 + 0.9)}{4} = 1.575,$$

$$E(B) = \frac{(1 + 2 \times 2 + 2 \times 3 + 9)}{6} \times \frac{(0.1 + 0.3 + 0.5 + 0.9)}{4} = 1.5.$$

Then $A \succ B$, that is A is superior to B which makes up the deficiency and is consistent with our intuition.

3.2 Aggregation operators for trapezoidal fuzzy multi-number

Definition 3.10. *Let $A_j = \langle (a_{j1}, b_{j1}, c_{j1}, d_{j1}); \eta_A^{j1}, \eta_A^{j2}, ..., \eta_A^{jP} \rangle$ $(j = 1, 2, ..., n)$ be a set of TFMNs and $\overline{w} = (\overline{w}_1, \overline{w}_2, ..., \overline{w}_n)^T$ be the weight vector of A_j $(j = 1, 2, ..., n)$ with $\overline{w}_j \in [0, 1]$ and $\sum_{j=1}^n \overline{w}_j = 1$ then the TFMNWAA operator can be defined as follows:*

$$TFMNWAA_w(A_1, A_2, ..., A_n) = \bigoplus_{j=1}^n \overline{w}_j A_j.$$

Theorem 3.11. *Let $A_j = \langle (a_{j1}, b_{j1}, c_{j1}, d_{j1}); \eta_{A_j}^1, \eta_{A_j}^2, ..., \eta_{A_j}^P \rangle$ $(j = 1, 2, ..., n)$ be a set of TFMNs then, the aggregated value utilizing the TFMNWAA operator is still a TFMN, which is shown as follows:*

$$\begin{aligned}
&TFMNWAA_w(A_1, A_2, ..., A_n) \\
&= \bigoplus_{j=1}^n \overline{w}_j A_j \\
&= \langle [\sum_{j=1}^n \overline{w}_j a_{j1}, \sum_{j=1}^n \overline{w}_j b_{j1}, \sum_{j=1}^n \overline{w}_j c_{j1}, \sum_{j=1}^n \overline{w}_j d_{j1}], (\frac{\sum_{j=1}^n \overline{w}_j \phi(A_j \eta_A^1)}{\sum_{j=1}^n \overline{w}_j \phi(A_j)}, \frac{\sum_{j=1}^n \overline{w}_j \phi(A_j \eta_A^2)}{\sum_{j=1}^n \overline{w}_j \phi(A_j)}, ..., \frac{\sum_{j=1}^n \overline{w}_j \phi(A_j \eta_A^p)}{\sum_{j=1}^n \overline{w}_j \phi(A_j)}) \rangle.
\end{aligned}$$

where $\phi(A_j) = \frac{c_{j1} - b_{j1} + d_{j1} - a_{j1}}{2}$.

In the following, Theorem 3.11 will be proved using the mathematical induction on n.

Proof. (1) When $n = 2$, the following equation can be obtained.

$$\begin{aligned}
TFMNWAA_w(A_1, A_2) = &\ \langle [\overline{w}_1 a_{11} + \overline{w}_1 b_{11} + \overline{w}_1 c_{11} + \overline{w}_1 d_{11}], (\tfrac{\overline{w}_1 \phi(A_1 \eta_A^1)}{\overline{w}_1 \phi(A_1)}, \tfrac{\overline{w}_1 \phi(A_1 \eta_A^2)}{\overline{w}_1 \phi(A_1)}, ..., \tfrac{\overline{w}_1 \phi(A_1 \eta_A^p)}{\overline{w}_1 \phi(A_1)}) \rangle \\
+ &\ \langle [\overline{w}_2 a_{21} + \overline{w}_2 b_{21} + \overline{w}_2 c_{21} + \overline{w}_2 d_{21}], (\tfrac{\overline{w}_2 \phi(A_2 \eta_A^1)}{\overline{w}_2 \phi(A_2)}, \tfrac{\overline{w}_2 \phi(A_2 \eta_A^2)}{\overline{w}_2 \phi(A_2)}, ..., \tfrac{\overline{w}_2 \phi(A_2 \eta_A^p)}{\overline{w}_2 \phi(A_2)}) \rangle \\
= &\ \langle [\overline{w}_1 a_{11} + \overline{w}_2 a_{21} + \overline{w}_1 b_{11} + \overline{w}_2 b_{21} + \overline{w}_1 c_{11} + \overline{w}_2 c_{21} + \overline{w}_1 d_{11} + \overline{w}_2 d_{21}], \\
&\ (\tfrac{\overline{w}_1 \phi(A_1 \eta_A^1) + \overline{w}_2 \phi(A_2 \eta_A^1)}{\overline{w}_1 \phi(A_1) + \overline{w}_2 \phi(A_2)}, \tfrac{\overline{w}_1 \phi(A_1 \eta_A^2) + \overline{w}_2 \phi(A_2 \eta_A^2)}{\overline{w}_1 \phi(A_1) + \overline{w}_2 \phi(A_2)}, ..., \tfrac{\overline{w}_1 \phi(A_1 \eta_A^p) + \overline{w}_2 \phi(A_2 \eta_A^p)}{\overline{w}_1 \phi(A_1) + \overline{w}_2 \phi(A_2)}) \rangle \\
= &\ \langle [\sum_{j=1}^2 \overline{w}_j a_{j1}, \sum_{j=1}^2 \overline{w}_j b_{j1}, \sum_{j=1}^2 \overline{w}_j c_{j1}, \sum_{j=1}^2 \overline{w}_j d_{j1}], \\
&\ (\tfrac{\sum_{j=1}^2 \overline{w}_j \phi(A_j \eta_A^1)}{\sum_{j=1}^2 \overline{w}_j \phi(A_j)}, \tfrac{\sum_{j=1}^2 \overline{w}_j \phi(A_j \eta_A^2)}{\sum_{j=1}^2 \overline{w}_j \phi(A_j)}, ..., \tfrac{\sum_{j=1}^2 \overline{w}_j \phi(A_j \eta_A^p)}{\sum_{j=1}^2 \overline{w}_j \phi(A_j)}) \rangle,
\end{aligned}$$

where $\phi(A_1) = \frac{c_{11}-b_{11}+d_{11}-a_{11}}{2}$ and $\phi(A_2) = \frac{c_{21}-b_{21}+d_{21}-a_{21}}{2}$. Clearly, when, $n = 2$, Theorem 3.11 is true.
(2) Suppose that when $n = k$, Theorem 3.11 is true. That is,

$$\begin{aligned}
&TFMNWAA_w(A_1, A_2, ..., A_k)\\
&= \bigoplus_{j=1}^{k} \overline{w}_j A_j\\
&= \langle [\sum_{j=1}^{k} \overline{w}_j a_{j1}, \sum_{j=1}^{k} \overline{w}_j b_{j1}, \sum_{j=1}^{k} \overline{w}_j c_{j1}, \sum_{j=1}^{k} \overline{w}_j d_{j1}], (\frac{\sum_{j=1}^{k} \overline{w}_j \phi(A_j \eta_A^1)}{\sum_{j=1}^{k} \overline{w}_j \phi(A_j)}, \frac{\sum_{j=1}^{k} \overline{w}_j \phi(A_j \eta_A^2)}{\sum_{j=1}^{k} \overline{w}_j \phi(A_j)}, ..., \frac{\sum_{j=1}^{k} \overline{w}_j \phi(A_j \eta_A^p)}{\sum_{j=1}^{k} \overline{w}_j \phi(A_j)}) \rangle.
\end{aligned}$$

Then when $n = k + 1$, the following equation can be calculated.

$$\begin{aligned}
\overline{w}_1 A_1 \oplus \overline{w}_2 A_2 \oplus \overline{w}_3 A_3 \oplus ... \oplus \overline{w}_k A_k \oplus \overline{w}_{k+1} A_{k+1} &= TFMNWAA_w(A_1, A_2, ..., A_k) \oplus \overline{w}_{k+1} A_{k+1}\\
&= \textstyle\bigoplus_{j=1}^{n} \overline{w}_j A_j \oplus \overline{w}_{k+1} A_{k+1}\\
&= \langle [\textstyle\sum_{j=1}^{k} \overline{w}_j a_{j1}, \sum_{j=1}^{k} \overline{w}_j b_{j1}, \sum_{j=1}^{k} \overline{w}_j c_{j1}, \sum_{j=1}^{k} \overline{w}_j d_{j1}],\\
&\quad (\tfrac{\sum_{j=1}^{k} \overline{w}_j \phi(A_j \eta_A^1)}{\sum_{j=1}^{k} \overline{w}_j \phi(A_j)}, \tfrac{\sum_{j=1}^{k} \overline{w}_j \phi(A_j \eta_A^2)}{\sum_{j=1}^{k} \overline{w}_j \phi(A_j)}, ..., \tfrac{\sum_{j=1}^{k} \overline{w}_j \phi(A_j \eta_A^p)}{\sum_{j=1}^{k} \overline{w}_j \phi(A_j)}) \rangle\\
&= \langle [\overline{w}_{k+1} a_{k+1,1} + \overline{w}_{k+1} b_{k+1,1} + \overline{w}_{k+1} c_{k+1,1} + \overline{w}_{k+1} d_{k+1,1}],\\
&\quad (\tfrac{\overline{w}_{k+1} \phi(A_{k+1} \eta_A^1)}{\overline{w}_{k+1} \phi(A_{k+1})}, \tfrac{\overline{w}_{k+1} \phi(A_{k+1} \eta_A^2)}{\overline{w}_{k+1} \phi(A_{k+1})}, ..., \tfrac{\overline{w}_{k+1} \phi(A_{k+1} \eta_A^p)}{\overline{w}_{k+1} \phi(A_{k+1})}) \rangle\\
&= \langle [\textstyle\sum_{j=1}^{k+1} \overline{w}_j a_{j1}, \sum_{j=1}^{k+1} \overline{w}_j b_{j1}, \sum_{j=1}^{k+1} \overline{w}_j c_{j1}, \sum_{j=1}^{k+1} \overline{w}_j d_{j1}],\\
&\quad (\tfrac{\sum_{j=1}^{k+1} \overline{w}_j \phi(A_j \eta_A^1)}{\sum_{j=1}^{k+1} \overline{w}_j \phi(A_j)}, \tfrac{\sum_{j=1}^{k+1} \overline{w}_j \phi(A_j \eta_A^2)}{\sum_{j=1}^{k+1} \overline{w}_j \phi(A_j)}, ..., \tfrac{\sum_{j=1}^{k+1} \overline{w}_j \phi(A_j \eta_A^p)}{\sum_{j=1}^{k+1} \overline{w}_j \phi(A_j)}) \rangle
\end{aligned}$$

That is, Theorem 3.11 is true for $n = k + 1$.
Thus, Theorem 3.11 holds for all n. □

Theorem 3.12. *(Idempotency)Let* $A_j = \langle (a_{j1}, b_{j1}, c_{j1}, d_{j1}); \eta_A^{j1}, \eta_A^{j2}, ..., \eta_A^{jP} \rangle$ $(j = 1, 2, ..., n)$ *be a collection of TFMNs and* $\overline{w} = (\overline{w}_1, \overline{w}_2, ..., \overline{w}_n)^T$ *be the weight vector of* A_j $(j = 1, 2, ..., n)$ *with* $\overline{w}_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \overline{w}_j = 1$ *and* A_j $(j = 1, 2, ..., n)$ *are equal, i.e.,* $A_j = A$ *for all* $(j = 1, 2, ..., n)$*, then*

$$TFMNWAA_w(A_1, A_2, ..., A_n) = A.$$

Theorem 3.13. *(Commutativity) Let* $A_j = \langle (a_{j1}, b_{j1}, c_{j1}, d_{j1}); \eta_A^{j1}, \eta_A^{j2}, ..., \eta_A^{jP} \rangle$ $(j = 1, 2, ..., n)$ *be a collection of TFMNs and* $\acute{A}_1, \acute{A}_2, ..., \acute{A}_n$ *be any permutation of* $A_1, A_2, ..., A_n$*, their weight vector* $(\acute{\overline{w}}_1, \acute{\overline{w}}_2, ..., \acute{\overline{w}}_n)$ *be permutation of* $(\overline{w}_1, \overline{w}_2, ..., \overline{w}_n)$ *then*

$$TFMNWAA_w(A_1, A_2, ..., A_n) = TFMNWAA_w(\acute{A}_1, \acute{A}_2, ..., \acute{A}_n).$$

Theorem 3.14. *(Boundedness)Let* $A_j = \langle (a_{j1}, b_{j1}, c_{j1}, d_{j1}); \eta_A^{j1}, \eta_A^{j2}, ..., \eta_A^{jP} \rangle$ $(j = 1, 2, ..., n)$ *be a collection of TFMNs, and let*

$$A_{\min} = \langle (\min_j a_{j1}, \min_j b_{j1}, \min_j c_{j1}, \min_j d_{j1}); \min_j \eta_A^{j1}, \min_j \eta_A^{j2}, ..., \min_j \eta_A^{jP} \rangle,$$

$$A_{\max} = \langle (\max_j a_{j1}, \max_j b_{j1}, \max_j c_{j1}, \max_j d_{j1}); \max_j \eta_A^{j1}, \max_j \eta_A^{j2}, ..., \max_j \eta_A^{jP} \rangle,$$

then

$$A_{\min} \le TFMNWAA_w(A_1, A_2, ..., A_n) \le A_{\max}.$$

Theorem 3.15. *Let* $A_j = \langle (a_{j1}, b_{j1}, c_{j1}, d_{j1}); \eta_A^{j1}, \eta_A^{j2}, ..., \eta_A^{jP} \rangle$ $(j = 1, 2, ..., n)$ *be a collection of TFMNs and* $\overline{w} = (\overline{w}_1, \overline{w}_2, ..., \overline{w}_n)^T$ *be the weighting vector of the TFMNWAA operator with* $\overline{w}_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \overline{w}_j = 1$*. Then*

1) *If* $\overline{w} = (1, 0, ..., 0)^T$*, then*

$$TFMNWAA_w(A_1, A_2, ..., A_n) = \max_j(A_j).$$

2) *If* $\overline{w} = (0, 0, ..., 1)^T$*, then*

$$TFMNWAA_w(A_1, A_2, ..., A_n) = \min_j(A_j).$$

3) *If* $\overline{w}_j = 1, \overline{w}_i = 0$*, and* $i \ne j$*, then*

$$TFMNWAA_w(A_1, A_2, ..., A_n) = \dot{A}_{\sigma(j)}.$$

where $\dot{A}_{\sigma(j)}$ *is the* j*th largest of* A_j $(j = 1, 2, ..., n)$.

Definition 3.16. *Let* $A_j = \langle (a_{j1}, b_{j1}, c_{j1}, d_{j1}); \eta_A^{j1}, \eta_A^{j2}, ..., \eta_A^{jP} \rangle$ $(j = 1, 2, ..., n)$ *be a set of TFMNs and* $\overline{w} = (\overline{w}_1, \overline{w}_2, ..., \overline{w}_n)^T$ *is the weight vector of* A_j $(j = 1, 2, ..., n)$ *with* $\overline{w}_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \overline{w}_j = 1$*, then the TFMNWGA operator can be defined as follows:*

$$TFMNWGA_w(A_1, A_2, ..., A_n) = \prod_{j=1}^{n} A_j^{\overline{w}_j}.$$

Theorem 3.17. *Let* $A_j = \langle (a_{j1}, b_{j1}, c_{j1}, d_{j1}); \eta_{A_j}^1, \eta_{A_j}^2, ..., \eta_{A_j}^P \rangle$ $(j = 1, 2, ..., n)$ *be a set of TFMNs Then, the aggregated value using the TFMNWGA operator and Definition 3.16 is still a TFMN, the aggregated results is shown as follows:*

$$\begin{aligned} TFMNWGA_w(A_1, A_2, ..., A_n) &= \prod_{j=1}^{n} A_j^{\overline{w}_j} \\ &= \langle [\prod_{j=1}^{n} a_{j1}^{\overline{w}_j}, \prod_{j=1}^{n} b_{j1}^{\overline{w}_j}, \prod_{j=1}^{n} c_{j1}^{\overline{w}_j}, \prod_{j=1}^{n} d_{j1}^{\overline{w}_j}], (\prod_{j=1}^{n} (\eta_A^1)^{\overline{w}_j}, \prod_{j=1}^{n} (\eta_A^2)^{\overline{w}_j}, ..., \prod_{j=1}^{n} (\eta_A^p)^{\overline{w}_j}) \rangle. \end{aligned}$$

As the proof is similar to Theorem 3.11, so it is omitted here.

3.3 Hybrid aggregation operators for trapezoidal fuzzy multi-number

At this point, we argue the condition after taking into account cases of repeat judgment where the a condition is no longer valid. What will happen if the value of membership is greater than one? To overcome this limitation, motivated by the idea of combining the weighted average and the OWA operator [40, 43], in what follows, we improved an trapezoidal fuzzy multi-number hybrid aggregation (TFMNHA) operator, which weights both the given trapezoidal fuzzy multi value and its ordered position.

Definition 3.18. *A TFMNHA operator of dimension n is a mapping* $TFMNHA : \Delta^n \to \Delta$*, which has an associated vector* $\overline{w} = (\overline{w}_1, \overline{w}_2, ..., \overline{w}_n)^T$ *is the weight vector of* A_j $(j = 1, 2, ..., n)$ *with* $\overline{w}_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \overline{w}_j = 1$ *such that*

$$TFMNHA_w(A_1, A_2, ..., A_n) = \overline{w}_1 \dot{A}_{\sigma(1)} + \overline{w}_2 \dot{A}_{\sigma(2)} + ... + \overline{w}_n \dot{A}_{\sigma(n)}.$$

where $\dot{A}_{\sigma(j)}$ *is the j th largest of the weighted trapezoidal fuzzy multi values* $\dot{A}_j$ $(\dot{A}_j = n\overline{w}_j A_j, j = 1, 2, ..., n)$ $\overline{w} = (\overline{w}_1, \overline{w}_2, ..., \overline{w}_n)^T$ *is the weight vector of* $A_j, (j = 1, 2, ..., n)$ *with* $\overline{w}_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \overline{w}_j = 1$ *and is the balancing coefficient, which plays a role of balance (in such a case, if the vector* $(\overline{w}_1, \overline{w}_2, ..., \overline{w}_n)^T$ *approaches* $(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$*, then the vector* $(n\overline{w}_1 A_1, n\overline{w}_2 A_2, ..., n\overline{w}_n A_n)^T$ *approaches* $(A_1, A_2, ..., A_n)^T$*). Let* $A_{\sigma(j)} = \eta_{A_{\sigma(j)}}, (j = 1, 2, ..., n)$ *then, similar to Theorem 3.11, we have*

$$TFMNHA_{w,w}(A_1, A_2, ..., A_n) = \langle [\prod_{j=1}^{n} a_{j1}^{\overline{w}_j}, \prod_{j=1}^{n} b_{j1}^{\overline{w}_j}, \prod_{j=1}^{n} c_{j1}^{\overline{w}_j}, \prod_{j=1}^{n} d_{j1}^{\overline{w}_j}], (1 - \prod_{j=1}^{n} (1 - \eta_A^j)^{\overline{w}_j}) \rangle, \quad (1)$$

and the aggregated value derived by using the TFMNHA operator is also a TFMN.
From Definition 3.18, we know that the TFMNHA operator is composed of the following three phases.

1. *It weights the TFM number* $A_j, (j = 1, 2, ..., n)$ *by the associated weights* $\overline{w}_j, (j = 1, 2, ..., n)$ *and multiplies these values by a balancing coefficient n, and then gets the weighted TFM number* $n\overline{w}_j A_j, (j = 1, 2, ..., n)$*.*
2. *It reorders the weighted TFMN* $n\overline{w}_j A_j, (j = 1, 2, ..., n)$*. in descending order* $(A_{\sigma(1)}, A_{\sigma(2)}, ..., A_{\sigma(n)})$ *where* $A_{\sigma(j)}$ *is the jth largest of* $n\overline{w}_i A_i, (i = 1, 2, ..., n)$*.*
3. *It weights these ordered weighted TFM number* $A_{\sigma(j)}, (j = 1, 2, ..., n)$ *by the TFMNHA weights* $\overline{w}_j (j = 1, 2, ..., n)$ *and then aggregates all the weighted TFMN* $\overline{w}_j A_{\sigma(j)} (j = 1, 2, ..., n)$ *into a collective one.*

4 Trapezoidal fuzzy multi-number preference relations

In this section, the traditional FPRs is extended to TFMNPRs and some of its prominent characteristic is explored.

Definition 4.1. *Let a matrix* $\ddot{R} = [\ddot{A}_{ij}]_{m\times m}$ *on the set* $A = \{A_1, A_2, ..., A_m\}$ *be composed of TFMNs, where* $\ddot{A}_{ij} = \langle (a_{ij}, b_{ij}, c_{ij}, d_{ij}); \eta^1_{\ddot{A}_{ij}}, \eta^2_{\ddot{A}_{ij}}, ..., \eta^p_{\ddot{A}_{ij}} \rangle$ $(i, j = 1, 2, ..., m)$ *and* $0 \le a_{ij} \le b_{ij} \le c_{ij} \le d_{ij} \le 1$, $0 \le \eta^1_{\ddot{A}_{ij}}, \eta^2_{\ddot{A}_{ij}}, ..., \eta^p_{\ddot{A}_{ij}} \le 1$. *When comparing the alternative* A_i *over* A_j, *the* $\eta^1_{\ddot{A}_{ij}}, \eta^2_{\ddot{A}_{ij}}, ..., \eta^p_{\ddot{A}_{ij}}$ *indicates the certainty degree to which the alternative* A_i *is prior to* A_j, *with a degree of* $[a_{ij}, b_{ij}, c_{ij}, d_{ij}]$ *which is represented as a TFMNs.*

Then the matrix $\ddot{R} = [\ddot{A}_{ij}]_{m\times m}$ *is denoted as a TFMNPR on A, if the following rules can be acceptable:*

1. $a_{ij} + d_{ji} = 1, b_{ij} + c_{ji} = 1, c_{ij} + b_{ji} = 1, d_{ij} + a_{ji} = 1$;
2. $\eta^1_{\ddot{A}_{ij}} \eta^2_{\ddot{A}_{ij}} ... \eta^p_{\ddot{A}_{ij}} = (1 - \eta^1_{\ddot{A}_{ij}})(1 - \eta^2_{\ddot{A}_{ij}})...(1 - \eta^p_{\ddot{A}_{ij}})$;
3. $\ddot{A}_{ij} = \langle [1, 1, 1, 1], (1, 1, ..., 1) \rangle$;
4. $0 \le \eta^1_{\ddot{A}_{ij}}, \eta^2_{\ddot{A}_{ij}}, ..., \eta^p_{\ddot{A}_{ij}} \le 1$.

Theorem 4.2. *A TFMNPR* $\ddot{R}$ *on the set of* $A = \{A_1, A_2, ..., A_m\}$ *is represented by a matrix* $\ddot{R} = [\ddot{A}_{ij}]_{m\times m}$, *where* $\ddot{A}_{ij} = \langle (a_{ij}, b_{ij}, c_{ij}, d_{ij}); \eta^1_{\ddot{A}_{ij}}, \eta^2_{\ddot{A}_{ij}}, ..., \eta^p_{\ddot{A}_{ij}} \rangle$ *for all* $(i, j = 1, 2, ..., m)$, *if the elements in i-th row and i-th column are removed from* $\ddot{R} = [\ddot{A}_{ij}]_{m\times m}$, *then the preference relation composed by the remainder elements of* $\ddot{R}$ *is still a TFMNPR.*

Proof. In terms of Definition 4.1, the proof can be completed easily and is omitted here.

□

Definition 4.3. *A TFMNPR* $R = [A_{ij}]_{m\times n}$ *with* $A_{ij} = \langle (a_{ij}, b_{ij}, c_{ij}, d_{ij}); \eta^1_{A_{ij}}, \eta^2_{A_{ij}}, ..., \eta^p_{A_{ij}} \rangle$ *for all* $(i, j = 1, 2, ..., m)$ *is called completely consistent TFMNPR if the following statements are equivalent for any* i, j.

1. $a_{ij} + d_{ij} = b_{ij} + c_{ij} = C(A)$ *where, C(A) is number of alternatives;*
2. $\eta^1_{\ddot{A}_{ij}} \eta^2_{\ddot{A}_{ij}} ... \eta^p_{\ddot{A}_{ij}} = (1 - \eta^1_{\ddot{A}_{ij}})(1 - \eta^2_{\ddot{A}_{ij}})...(1 - \eta^p_{\ddot{A}_{ij}})$.

Definition 4.4. *A TFMNPR* $R = [A_{ij}]_{m\times m}$ *with* $A_{ij} = \langle (a_{ij}, b_{ij}, c_{ij}, d_{ij}); \eta^1_{A_{ij}}, \eta^2_{A_{ij}}, ..., \eta^p_{A_{ij}} \rangle$ *for all* $(i, j, k = 1, 2, ..., m)$ *is called acceptably consistent TFMNPR if the following statements are equivalent for any* i, j, k.

1. $a_{ij} \sim a_{ii}$ *and* $a_{jk} \sim a_{ii}$, *then* $a_{ik} \sim a_{ii}$;
2. $a_{ij} \prec a_{ii}$ *and* $a_{jk} \prec a_{ii}$, *then* $a_{ik} \prec a_{ii}$;
3. $a_{ij} \succ a_{ii}$ *and* $a_{jk} \succ a_{ii}$, *then* $a_{ik} \succ a_{ii}$.

5 An approach to MCDM problems with TFMNPR and complete weight information

The following application is adopted from [9]. In this section, we define a MCDM approach is proposed based on TFMNPRs, and the main procedures of the proposed method are provided. Thus for a matrix $\ddot{R} = [\ddot{A}_{ij}]_{m\times n}$, consider a set of m alternatives $A = (A_1, A_2, ..., A_m)$ and n criteria $U = (U_1, U_2, ..., U_n)$.The weight vector of criteria is denoted by $\overline{w} = (\overline{w}_1, \overline{w}_2, ..., \overline{w}_n)^T$ with subject to $\overline{w}_j \in [0, 1]$ such that $\sum_{j=1}^n \overline{w}_j = 1$.

Now, we can give an algorithm of the TFM-numbers multi-criteria decision-making method as following steps are demonstrated by the technique:,

Algorithm:

Step 1. Normalize the TFMNPRs Normalize the decision-making information $[A_{ij}]$ in the matrices $\ddot{R} = [\ddot{A}_{ij}]_{m\times n}$. The criteria in the decision matrices are classified as benefit and cost types according to Definition 3.1.

The normalized evaluation information matrices are denoted as

$$\bar{\ddot{R}} = [\bar{\ddot{A}}_{ij}]_{m\times n}.$$

Step 2. Check the complete consistency or acceptable consistency of TFMNPR $\bar{\ddot{R}}$.

If $\bar{\ddot{R}}$ meets complete consistency or acceptable consistency, then proceed to Step 6. Otherwise, change according to Definition 4.3 and Definition 4.4 should continue until complete consistent.

Step 3. Improve the consistency degree, find the completely consistent of TFMNPR, and then the overall preference information according to Equation (1) in Definition 3.18,

Step 4. Calculate the ordering vector of each alternative.

Step 5. Calculated the score values of every alternative.

Step 6. Pick the best alternative depending on the maximum score or the value of accuracy.

6 Application

People sought a solution to the need for protection, a natural motive, to sustain their lives and assets. In this sense, architecture started with the existence of human beings. Architecture that meets the need for protection and has been developing since the creation of a restricted space is different from other building actions with the feature of sustaining life in the space. Architecture designs not only the structures themselves, but also the life that will last in it.Turkey geological and topographic structure and natural disasters that cause life and property losses due to climatic conditions is one of the countries often encounter. Therefore, Containers are needed for an urgent and fast solution. Containers, which are examples of mobile structures, emerged at the end of the need for temporary use and entered the architectural literature. In addition to the uses such as shelter, construction site unit, wc, warehouse, it is also used as an emergency shelter unit resulting from natural disasters that are frequently encountered in our country. This paper deals with the development of modular architectural design process of container city for the post-disaster shelter response. After thorough investigation, five containers (alternatives) are taken into consideration, i.e.,$\{A_1, A_2, A_3, A_4, A_5\}$. There are many factors that affect the architecture of container city and three factors are considered based on the experience of the department personnel, including u_1; structure and population density u_2; location u_3; route selection ($\omega = (0.35, 0.20, 0.45)^T$). Our solution is to examine the needy people at different time intervals, which in turn give arise to different memberships function for each needy people, by the form of TFMNs, the results are listed as follows:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{pmatrix} \langle(0.5,0.5,0.5,0.5);0.5,0.5,0.5,0.5\rangle & \langle(0.3,0.4,0.6,0.7);0,0.3,0.6,1\rangle & \langle(0.4,0.5,0.3,0.9);0,0.5,0.2,1\rangle \\ \langle(0.4,0.4,0.6,0.6);0,0.6,0.8,1\rangle & \langle(0.5,0.5,0.5,0.5);0.5,0.5,0.5,0.5\rangle & \langle(0.4,0.5,0.7,0.8);0,0.6,0.8,1\rangle \\ \langle(0.25,0.30,0.70,0.75);0,0.3,0.5,1\rangle & \langle(0.1,0.2,0.8,0.9);0,0.2,0.5,1\rangle & \langle(0.5,0.5,0.5,0.5);0.5,0.5,0.5,0.5\rangle \\ \langle(0.35,0.45,0.55,0.65);0,0.2,0.4,1\rangle & \langle(0.2,0.4,0.6,0.8);0,0.1,0.3,1\rangle & \langle(0.1,0.2,0.4,0.6);0,0.3,0.7,1\rangle \\ \langle(0.20,0.45,0.55,0.80);0,0.3,0.2,1\rangle & \langle(0.1,0.5,0.5,0.9);0,0.2,0.4,1\rangle & \langle(0.3,0.5,0.5,0.7);0,0.2,0.5,1\rangle \end{pmatrix} \end{array}$$

Step 1. Since all the criteria are benefit ones, the decision information need not to be normalized. In that case,

$$\ddot{R} = [\ddot{A}_{ij}]_{m\times n} = \bar{\ddot{R}} = [\bar{\ddot{A}}_{ij}]_{m\times n}.$$

Step 2. Based on Definition 4.3, the result by calculating the elements in TFMNPR $\ddot{R}$ are shown as follows:

$$a_{13}+d_{13}+a_{23}+d_{23}+a_{33}+d_{33}+a_{43}+d_{43}+a_{53}+d_{53} = 0.4+0.9+0.4+0.8+0.5+0.5+0.1+0.6+0.3+0.7 = 5.2 \neq 5$$

$$b_{13}+c_{13}+b_{23}+c_{23}+b_{33}+c_{33}+b_{43}+c_{43}+b_{53}+c_{53} = 0.5+0.3+0.5+0.7+0.5+0.5+0.2+0.4+0.5+0.5 = 4.6 \neq 5$$

then $\ddot{R}$ is not completely consistent. Also, using the score function defined in Definition 3.7, we can obtain $E(\ddot{A}_{23}) = 0.2, E(\ddot{A}_{31}) = 0.1125, E(\ddot{A}_{21}) = 0.18$ and $E(\ddot{A}_{11}) = 0.167$, thus $\ddot{A}_{23} \prec \ddot{A}_{11}, \ddot{A}_{31} \prec \ddot{A}_{11}$ and $\ddot{A}_{21} \prec \ddot{A}_{11}$ according to Definition 4.4, $\ddot{R}$ is not acceptably consistent either.

Step 3.

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{pmatrix} \langle(0.5,0.5,0.5,0.5);0.5,0.5,0.5,0.5\rangle & \langle(0.3,0.4,0.6,0.7);0,0.3,0.6,1\rangle & \langle(0.1,0.2,0.8,0.9);0,0.5,0.2,1\rangle \\ \langle(0.4,0.4,0.6,0.6);0,0.6,0.8,1\rangle & \langle(0.5,0.5,0.5,0.5);0.5,0.5,0.5,0.5\rangle & \langle(0.2,0.5,0.5,0.8);0,0.6,0.8,1\rangle \\ \langle(0.25,0.30,0.70,0.75);0,0.3,0.5,1\rangle & \langle(0.1,0.2,0.8,0.9);0,0.2,0.5,1\rangle & \langle(0.5,0.5,0.5,0.5);0.5,0.5,0.5,0.5\rangle \\ \langle(0.35,0.45,0.55,0.65);0,0.2,0.4,1\rangle & \langle(0.2,0.4,0.6,0.8);0,0.1,0.3,1\rangle & \langle(0.4,0.4,0.6,0.6);0,0.3,0.7,1\rangle \\ \langle(0.20,0.45,0.55,0.80);0,0.3,0.2,1\rangle & \langle(0.1,0.5,0.5,0.9);0,0.2,0.4,1\rangle & \langle(0.3,0.3,0.7,0.7);0,0.2,0.5,1\rangle \end{pmatrix} \end{array}$$

Step 4.

$$\begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{pmatrix} \langle(0.219, 0.316, 0.641, 0.0697); 0, 0.004, 0.003, 0.215\rangle \\ \langle(0.306, 0.462, 0.533, 0.659); 0, 0.120, 0.029, 0.129\rangle \\ \langle(0.284, 0.348, 0.618, 0.648); 0, 0.001, 0.007, 0.268\rangle \\ \langle(0.332, 0.417, 0.582, 0.654); 0, 0.0002, 0.005, 1\rangle \\ \langle(0.209, 0.383, 0.601, 0.771); 0, 0.0005, 0.002, 1\rangle \end{pmatrix}$$

By Definition 3.7, we calculate $E(A_i)$ of $A_i(i = 1, 2, 3, 4, 5)$;

$$E(A_1) = 0.0263, E(A_2) = 0.0210, E(A_3) = 0.0330, E(A_4) = 0.1249 \text{ and } E(A_5) = 0.1232$$

since

$$E(A_4) > E(A_5) > E(A_3) > E(A_1) > E(A_2)$$

then

$$\begin{array}{c} \ddot{A}_4 = A_{\sigma(1)} \\ \ddot{A}_5 = A_{\sigma(2)} \\ \ddot{A}_3 = A_{\sigma(3)} \\ \ddot{A}_1 = A_{\sigma(4)} \\ \ddot{A}_2 = A_{\sigma(5)} \end{array} \begin{pmatrix} \langle(0.35, 0.45, 0.55, 0.65); 0, 0.2, 0.4, 1\rangle & \langle(0.2, 0.4, 0.6, 0.8); 0, 0.1, 0.3, 1\rangle & \langle(0.4, 0.4, 0.6, 0.6); 0, 0.3, 0.7, 1\rangle \\ \langle(0.20, 0.45, 0.55, 0.80); 0, 0.3, 0.2, 1\rangle & \langle(0.1, 0.5, 0.5, 0.9); 0, 0.2, 0.4, 1\rangle & \langle(0.3, 0.3, 0.7, 0.7); 0, 0.2, 0.5, 1\rangle \\ \langle(0.25, 0.30, 0.70, 0.75); 0, 0.3, 0.5, 1\rangle & \langle(0.1, 0.2, 0.8, 0.9); 0, 0.2, 0.5, 1\rangle & \langle(0.5, 0.5, 0.5, 0.5); 0.5, 0.5, 0.5, 0.5\rangle \\ \langle(0.5, 0.5, 0.5, 0.5); 0.5, 0.5, 0.5, 0.5\rangle & \langle(0.3, 0.4, 0.6, 0.7); 0, 0.3, 0.6, 1\rangle & \langle(0.1, 0.2, 0.8, 0.9); 0, 0.5, 0.2, 1\rangle \\ \langle(0.4, 0.4, 0.6, 0.6); 0, 0.6, 0.8, 1\rangle & \langle(0.5, 0.5, 0.5, 0.5); 0.5, 0.5, 0.5, 0.5\rangle & \langle(0.2, 0.5, 0.5, 0.8); 0, 0.6, 0.8, 1\rangle \end{pmatrix}$$

Step 5. Assume that $\omega = (0.3, 0.4, 0.3)^T$ and the $E(A_{\sigma(i)})$ of alternatives $A_i(i = 1, 2, 3, 4, 5)$ according to last matrix in Step 4 calculated as;

$$E(\ddot{A}_1) = 0.0266, E(\ddot{A}_2) = 0.0291, E(\ddot{A}_3) = 0.0269, E(\ddot{A}_4) = 0.1249 \text{ and } E(\ddot{A}_5) = 0.1232$$

Step 6. The alternatives $A_i(i = 1, 2, 3, 4, 5)$ by using $E(A_i)$ ranked as;

$$E(A_4) > E(A_5) > E(A_2) > E(A_3) > E(A_1).$$

7 Comparative analysis and discussion

In the following, we provides some suitable examples to show the feasibility as well as effectiveness of the proposed novel MCDM based on TFMNRs method and make a comparison with the existing studies. Their common advantage is that they all have a score function that makes the process of decision making more easily. Therefore, we can differentiate effective and useful of proposed score function for MCDM from [9, 14, 44] inconsistent in this study. We would like to show this difference with examples in the following subsection;

Some drawbacks, which will be using in the following examples, still exist in the operations described in Theorem 3 in Wang [14], Definition 8 in Ye [44] and Definition 4.2 in Deli and Subas [9].
1-The comparison method proposed in Wang [14]

Example 7.1. *Let* $A = \langle[0.1, 0.2, 0.6, 0.9]; 0.2, 0.9\rangle$,$B = \langle[0.2, 0.3, 0.6, 0.7]; 0.1, 0.8\rangle$ *be two intuitionistic trapezoidal fuzzy numbers, it is obviously* $A \neq B$. *According to Theorem 3 in* [14], *we can obtain that* $s(A) = s(B) = 0.0675, H(a) = H(B) = 0.03375$ *according to Theorem 3 in* [14], $A = B$ *which is illogical.*

2- The comparison method proposed in Ye [44]

Example 7.2. *Let* $A = \langle[0.1, 0.2, 0.6, 0.9]; 0.2, 0.9, 0\rangle$,$B = \langle[0.2, 0.3, 0.6, 0.7]; 0.1, 0.8, 0\rangle$ *be two single valued trapezoidal neutrosophic numbers, it is obviously* $A \neq B$. *According to Definition 4.2 in* [44], *we can obtain that* $s(A) = s(B) = 0.195, H(a) = H(B) = 0.195$ *according to Definition 8 in* [44], $A = B$ *which is illogical.*

3- The comparison method proposed in Deli and Subas [9]

Example 7.3. *Let* $A = \langle[0.1, 0.2, 0.6, 0.9]; 0.2, 0.9, 0\rangle$,$B = \langle[0.2, 0.3, 0.6, 0.7]; 0.1, 0.8, 0\rangle$ *be two single valued trapezoidal neutrosophic numbers, it is obviously* $A \neq B$. *According to Definition 4.2 in* [9], *we can obtain that* $s(A) = s(B) = 0.14625, H(a) = H(B) = 0.14625$ *according to Definition 4.3 in* [9], $A = B$ *which is illogical.*

8 Conclusion

TFMNPRs play a very important role in the decision-making process, and they are more realistic to express the TFMNPRs among alternatives flexibly without scoring all alternatives under the corresponding attributes. Therefore, TFMNPRs have been applied to many real-life decision-making problems. The research significance and value of this paper are highlighted below:

1. In this article presented a new MCDM method for trapezoidal fuzzy multi-numbers based on TFMNPR in architecture.
2. New operations based on TFMNs were defined and studied their basic properties.
3. Based these operation laws, different operators, namely, the TFMNWAA, the TFMNWGA and the TFMNHA were presented.
4. Then the notion of the score function of ranking the trapezoidal fuzzy multi numbers were defined.
5. And then, a comparative study is also conducted to test its efficiency with several existing works.

Further a numerical example is developed to illustrate the flexibility and applicability of the proposed operators. In the future, we will employ the initiated work on fuzzy set extension and extended of fuzzy numbers. Such that hesitant fuzzy set, intuitionistic fuzzy set fuzzy soft set etc.

Acknowledgement:

The authors wish to express their appreciation for several excellent suggestions for improvements in this paper made by the referees.

Authors contributions: All the authors have equal contribution for the preparation of this article.

References

[1] S. Abdullah, A. Fahmi, M. Aslam, *Generalized trapezoidal cubic linguistic fuzzy ordered weighted average operator and group decision-making*, Soft Computing, **24**(5) (2020), 3155-3171. https://doi.org/10.1007/s00500-020-04672-7

[2] M. Akram, G. Shahzadi, *A hybrid decision-making model under q-rung orthopair fuzzy Yager aggregation operators*, Granular Computing, **6** (2021), 763-777. https://doi.org/10.1007/s41066-020-00229-z

[3] S. Aydin, C. Kahraman, M. Kabak, *Development of harmonic aggregation operator with trapezoidal Pythagorean fuzzy numbers*, Soft Computing, **24**(15) (2020), 11791-11803. https://doi.org/10.1007/s00500-019-04638-4

[4] J. I. Baek, A. Borumand Saeid, S. H. Han, K. Hur, *Some aggregation operators for IVI-octahedron sets and their application to MCDGM*, Iranian Journal of Fuzzy Systems, **20**(2) (2023), 135-149. https://doi.org/10.22111/IJFS.2023.7561

[5] D. Bakbak, *Suriyeli siginmacilarin konteyner kamplariýnailiskin bir arastirma*, Gazi Akademik Bakis, **11**(23) (2018), 249-287. https://dergipark.org.tr/tr/download/article-file/610381

[6] D. Bakbak, V. Uluçay, M. Sahin, *Intuitionistic trapezoidal fuzzy multi-numbers and some arithmetic averaging operators with their application in architecture*, 6th International Multidisciplinary Studies Congress (Multicongress 19) Gaziantep, Turkiye, (2019).

[7] A. Chakraborty, S. Maity, S. Jain, S. P. Mondal, S. Alam, *Hexagonal fuzzy number and its distinctive representation, ranking, defuzzification technique and application in production inventory management problem*, Granular Computing, **6** (2021), 507-521. https://doi.org/10.1007/s41066-020-00212-8

[8] I. A. Da Silva, B. Bedregal, B. Bedregal, R. H. N. Santiago, *An interval-valued Atanassov's intuitionistic fuzzy multi-attribute group decision making method based on the best representation of the WA and OWA operators*, Journal of Fuzzy Extension and Applications, **2**(3) (2021), 239-261. https://doi.org/10.22105/jfea.2021.306164.1162

[9] I. Deli, Y. Subas, *Single valued neutrosophic numbers and their applications to multicriteria decision making problem*, Neutrosophic Sets and Systems, **2**(1) (2014), 1-13. https://fs.unm.edu/SN/Neutro-SingleValuedNeutroNumbers.pdf

[10] J. Deng, J. Zhan, E. Herrera-Viedma, F. Herrera, *Regret theory-based three-way decision method on incomplete multi-scale decision information systems with interval fuzzy numbers*, IEEE Transactions on Fuzzy Systems, **31** (2023), 982-996. https://doi.org/10.1109/TFUZZ.2022.3193453

[11] A. Fahmi, S. Abdullah, F. Amin, M. S. A. Khan, *Trapezoidal cubic fuzzy number Einstein hybrid weighted averaging operators and its application to decision making*, Soft Computing, **23**(14) (2019), 5753-5783. https://doi.org/10.1007/s00500-018-3242-6

[12] H. Garg, A. Ahmad, K. Ullah, T. Mahmood, Z. Ali, *Algorithm for multiple attribute decision-making using T-spherical fuzzy Maclaurin symmetric mean operator*, Iranian Journal of Fuzzy Systems, **19**(6) (2022), 111-124. https://doi.org/10.22111/IJFS.2022.7215

[13] N. Hassan, V. Ulucay, M. Sahin, *Q-neutrosophic soft expert set and its application in decision making*, International Journal of Fuzzy System Applications (IJFSA), **7**(4) (2018), 37-61.

[14] W. Jianqiang, Z. Zhong, *Aggregation operators on intuitionistic trapezoidal fuzzy number and its application to multi-criteria decision making problems*, Journal of Systems Engineering and Electronics, **20**(2) (2009), 321-326. https://ieeexplore.ieee.org/document/6074660

[15] F. Jin, Z. Ni, H. Chen, *Interval-valued hesitant fuzzy Einstein prioritized aggregation operators and their applications to multi-attribute group decision making*, Soft Computing, **20**(5) (2016), 1863-1878. https://doi.org/10.1007/s00500-015-1887-y

[16] H. Karimi, M. Sadeghi-Dastaki, M. Javan, *A fully fuzzy best-worst multi attribute decision making method with triangular fuzzy number: A case study of maintenance assessment in the hospitals*, Applied Soft Computing, **86** (2020). https://doi.org/10.1016/j.asoc.2019.105882

[17] A. Kaufmann, M. M. Gupta, *Fuzzy mathematical models in engineering and management science*, Elsevier Science Publishers, Amsterdam, Netherland, 1988.

[18] Y. Liu, J. Liu, Y. Qin, *Pythagorean fuzzy linguistic Muirhead mean operators and their applications to multiattribute decision-making*, International Journal of Intelligent Systems, **35**(2) (2020), 300-332. https://doi.org/10.1002/int.22212

[19] S. Miyamoto, *Fuzzy multisets and their generalizations, multiset processing, lecture notes in computer science*, Springer-Verlag, Berlin, **2235** (2001), 225-235. https://doi.org/10.1007/3-540-45523-X_11

[20] S. Miyamoto, *Data structure and operations for fuzzy multisets, transactions on rough sets II*, Lecture Notes in Computer Science, Springer-Verlag, Berlin, **3135** (2004), 189-200. https://doi.org/10.1007/978-3-540-27778-1_10

[21] S. Miyamoto, *Multisets and fuzzy multisets as a framework of information systems*, In International Conference on Modeling Decisions for Artificial Intelligence, Springer, Berlin, Heidelberg, (2004, August), 27-40. https://doi.org/10.1007/978-3-540-27774-3_4

[22] S. Mohan, A. P. Kannusamy, V. Samiappan, *A new approach for ranking of intuitionistic fuzzy numbers*, Journal of fuzzy extension and applications, **1**(1) (2020), 15-26. https://www.journal-fea.com/article_114137_f4d9bae346ee774968748b4e0c43960c.pdf

[23] S. Narayanamoorthy, T. N. Parthasarathy, S. Pragathi, P. Shanmugam, D. Baleanu, A. Ahmadian, D. Kang, *The novel augmented Fermatean MCDM perspectives for identifying the optimal renewable energy power plant location*, Sustainable Energy Technologies and Assessments, **53** (2022), 102488. https://doi.org/10.1016/j.seta.2022.102488

[24] S. Orlovsky, *Decision-making with a fuzzy preference relation*, Fuzzy Sets and Systems, **1**(3) (1978), 155-167. https://doi.org/10.1016/0165-0114(78)90001-5

[25] N. Parveen, P. N. Kamble, *Decision-making problem using fuzzy TOPSIS method with hexagonal fuzzy number*, In Computing in Engineering and Technology, Springer, Singapore, (2020), 421-430. https://doi.org/10.1007/978-981-32-9515-5_40

[26] K. Rahman, *Approaches to some induced Einstein geometric aggregation operators based on interval-valued Pythagorean fuzzy numbers and their application*, New Mathematics and Natural Computation, **16**(02) (2020), 211-230. https://doi.org/10.1142/S1793005720500131

[27] L. Ramya, S. Narayanamoorthy, S. Kalaiselvan, J. V. Kureethara, V. Annapoorani, D. Kang, *A congruent approach to normal wiggly interval-valued hesitant Pythagorean fuzzy set for thermal energy storage technique selection applications*, International Journal of Fuzzy Systems, **23** (2021), 1581-1599. https://doi.org/10.1007/s40815-021-01057-2

[28] M. Sahin, I. Deli, V. Ulucay, *Extension principle based on neutrosophic multi-fuzzy sets and algebraic operations*, Journal of Mathematical Extension, **12**(1) (2018), 69-90.

[29] M. Sahin, V. Ulucay, M. Menekşe, *Some new operations of* (α, β, γ) *interval cut set of interval valued neutrosophic sets*, Journal of Mathematical and Fundamental Sciences, **50**(2) (2018), 103-120. https://doi.org/10.5614/J.MATH.FUND.SCI.2018.50.2.1

[30] S. Sebastian, T. V. Ramakrishnan, *Multi-fuzzy sets*, International Mathematical Forum, **5**(50) (2010), 2471-2476.

[31] S. Sebastian, T. V. Ramakrishnan, *Multi-fuzzy subgroups*, International Journal of Contemporary Mathematical Sciences, **6**(8) (2011), 365-372.

[32] T. Senapati, R. R. Yager, *Fermatean fuzzy weighted averaging/geometric operators and its application in multi-criteria decision-making methods*, Engineering Applications of Artificial Intelligence, **85** (2019), 112-121. https://doi.org/10.1016/j.engappai.2019.05.012

[33] M. Shakeel, S. Abdullah, M. Aslam, M. Jamil, *Ranking methodology of induced Pythagorean trapezoidal fuzzy aggregation operators based on Einstein operations in group decision making*, Soft Computing, **24**(10) (2020), 7319-7334. https://doi.org/10.1007/s00500-019-04356-x

[34] A. Syropoulos, *On generalized fuzzy multisets and their use in computation*, Iranian Journal of Fuzzy Systems, **9**(2) (2012), 115-127.

[35] V. Uluçay, I. Deli, M. Sahin, *Trapezoidal fuzzy multi-number and its application to multi-criteria decision-making problems*, Neural Computing and Applications, **30**(5) (2018), 1469-1478. https://doi.org/10.1007/s00521-016-2760-3

[36] V. Uluçay, I. Deli, M. Sahin, *Similarity measures of bipolar neutrosophic sets and their application to multiple criteria decision making*, Neural Computing and Applications, **29**(3) (2018), 739-748. https://doi.org/10.1007/s00521-016-2479-1

[37] V. Uluçay, A. Kilic, I. Yildiz, M. Sahin, *A new approach for multi-attribute decision-making problems in bipolar neutrosophic sets*, Neutrosophic Sets and Systems, **23**(1) (2018), 12-22. https://doi.org/10.5281/zenodo.2154873

[38] V. Uluçay, M. Sahin, N. Hassan, *Generalized neutrosophic soft expert set for multiple-criteria decision-making*, Symmetry, **10**(10) (2018), 437-447. https://doi.org/10.3390/sym10100437

[39] C. Veeramani, R. Venugopal, S. A. Edalatpanah, *Neutrosophic DEMATEL approach for financial ratio performance evaluation of the NASDAQ exchange*, Neutrosophic Sets and Systems, **51**(1) (2022), 766-782. https://digitalrepository.unm.edu/cgi/viewcontent.cgi?article=2160&context=nss_journal

[40] G. W. Wei, *Pythagorean fuzzy Hamacher power aggregation operators in multiple attribute decision making*, Fundamenta Informaticae, **166**(1) (2019), 57-85. https://doi.org/10.1002/int.21946

[41] Z. Xu, H. Liao, *A survey of approaches to decision making with intuitionistic fuzzy preference relations*, Knowledge-Based Systems, **80** (2015), 131-142. https://doi.org/10.1016/j.knosys.2014.12.034

[42] R. R. Yager, *On the theory of bags*, International Journal of General System, **13** (1986), 23-37. https://doi.org/10.1080/03081078608934952

[43] R. R. Yager, *Generalized OWA aggregation operators*, Fuzzy Optimization and Decision Making, **3**(1) (2004), 93-107. https://doi.org/10.1023/B:FODM.0000013074.68765.97

[44] J. Ye, *Some weighted aggregation operators of trapezoidal neutrosophic numbers and their multiple attribute decision making method*, Informatica, **28**(2) (2017), 387-402. https://doi.org/10.15388/Informatica.2017.108

[45] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8**(3) (1965), 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X

[46] J. Zhan, W. Wang, J. C. R. Alcantud, J. Zhan, *A three-way decision approach with prospect-regret theory via fuzzy set pair dominance degrees for incomplete information systems*, Information Sciences, **617** (2022), 310-330. https://doi.org/10.1016/j.ins.2022.10.107

[47] J. Zhan, J. Wang, W. Ding, Y. Yao, *Three-way behavioral decision making with hesitant fuzzy information systems: Survey and challenges*, IEEE/CAA Journal of Automatica Sinica, **10**(2) (2023), 330-350. https://doi.org/10.1109/JAS.2022.106061

[48] J. Zhu, X. Ma, L. Martínez, J. Zhan, *A probabilistic linguistic three-way decision method with regret theory via fuzzy C-means clustering algorithm*, IEEE Transactions on Fuzzy Systems, (2023). https://doi.org/10.1109/TFUZZ.2023.3236386

[49] H. J. Zimmermann, *Fuzzy set theory and its applications*, Kluwer Academic Publishers, 1993. https://doi.org/10.1007/978-94-010-0646-0