Iranian Journal of Fuzzy Systems

Volume 21, Number 2, (2024), pp. 117-127

Original paper

University of Sistan and Baluchestan

The reflectivity of the category of stratified L-algebraic closure spaces

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Abstract

In this paper, for each commutative and integral quantale, we give the stratified Sierpinski L-algebraic closure space and a sobrification of stratified L-algebraic closure spaces. Furthermore, we show that $\mathbf{S}L\text{-}\mathbf{AC_0}$–the category of stratified S_0-L-algebraic closure spaces is epireflective in $\mathbf{S}L\text{-}\mathbf{AC}$–the category of stratified L-algebraic closure spaces, and $\mathbf{Sob}L\text{-}\mathbf{AC}$–the category of sober L-algebraic closure spaces is epireflective and $\mathcal{E}$-firm epireflective in the category $\mathbf{S}L\text{-}\mathbf{AC_0}$.

Keywords: Stratified L-algebraic closure space, Sierpinski space, sober, epireflective.

1 Introduction

Domain theory, a formal basis for the semantics of programming languages, originated in the work of Dana Scott [12, 13], plays an important role in theoretical computer science [3, 19]. As a branch of domain theory, the sobriety of topological spaces has undergone active research and many important results have been obtained(see [7, 8, 16, 17, 22, 23, 24]).

In classical setting, it is well known that the category of sober topological spaces is epireflective in the category of T_0-topological spaces, and it is the epireflective hull of the two-point Sierpinski topological space [8]. After, Höhle [6] pointed out that the category of sober L-topological spaces is reflective in the category of L-topological spaces, and in [9], Noor and Srivastava showed that the category of sober L-topological spaces is, in fact, the epireflective hull of the Sierpinski L-topological space in the category of T_0-L-topological space. Since there is difference whether one chooses to work with topological spaces on open sets or on closed sets in the fuzzy setting, Zhang [21] gave a sobrification of stratified quantale-valued cotopological spaces. Based on this, Zhao and Zhang [25] showed that the category $\mathbf{Sob}Q\text{-}\mathbf{CTop}$ of sober Q-cotopological spaces is the epireflective subcategory of $\mathbf{S}Q\text{-}\mathbf{CTop_0}$ of stratified T_0-Q-cotopological spaces. In particular, they proved that $\mathbf{Sob}Q\text{-}\mathbf{CTop}$ is the epireflective hull of Sierpinski Q-cotopological space in $\mathbf{S}Q\text{-}\mathbf{CTop_0}$, where Q is the unit interval endowed with a left continuous t-norm.

As we all know, in addition to the topological closure space, algebraic closure space is another frequently used concept of closure spaces. When the universal set X is finite, a topological closure space must be an algebraic closure space or a convex space, but the converse may be not true. Furthermore, these two kinds of closure spaces also have essential differences when they are generated by the subbases. For topological closure spaces, one has to form finite unions firstly and then form arbitrary intersections, while for algebraic closure spaces, one has to form arbitrary intersections firstly and then form directed unions [5]. Thus, there is a great difference between topological closure spaces and algebraic closure spaces. In [20], Yao and Zhou introduced the sobriety of quantale-valued fuzzy convex spaces and the algebraicness of fuzzy complete lattices, and they showed that the category of sober fuzzy convex spaces and the category of algebraic fuzzy complete lattices are dually equivalent to each other. Naturally, we will further consider the problems about "sobriety" and "reflectivity" in the category of L-algebraic closure spaces(also called L-convex spaces in [10]) in a proper lattice-valued environment.

The main contents are arranged as follows. Section 2 recalls some basic concepts about categories and stratified L-algebraic closure spaces. Section 3 considers the stratified Sierpinski L-algebraic closure space. Section 4 establishes a sober L-algebraic closure space, and it is called the sobrification of stratified L-algebraic closure spaces. Based on this,

Corresponding Author: Y. L. Yue
Received: November 2023; Accepted: April 2024.
https://doi.org/10.22111/IJFS.2024.47190.8314

section 5 proves that the category $\mathbf{S}L\text{-}\mathbf{AC_0}$ of stratified S_0-L-algebraic closure spaces is epireflective in the category $\mathbf{S}L\text{-}\mathbf{AC}$ of stratified L-algebraic closure spaces, and the category $\mathbf{Sob}L\text{-}\mathbf{AC}$ of sober L-algebraic closure spaces is epireflective and $\mathcal{E}$-firm epireflective in the category $\mathbf{S}L\text{-}\mathbf{AC_0}$ of stratified S_0-L-algebraic closure spaces.

2 Preliminaries

In this paper, if there is no further statement, $(L, *, \leq)$ is always a *commutative and integral quantale*, i.e., $(L, \leq)$ is a complete lattice with bottom element 0 and top element 1(also the unit), $(L, *)$ is a commutative semigroup and for all $a \in L$ and $\{b_i\}_{i\in I} \subseteq L$, it holds that

$$a * (\bigvee_{i\in I} b_i) = \bigvee_{i\in I} (a * b_i).$$

An *L-subset* on a set X is a map from X to L, and the family of all L-subsets on X will be denoted by L^X or $P_L X$, called the *L-power set* of X. Denote the constant map $\alpha : X \to L$ such that $\alpha(x) = \alpha$ for all $x \in X$ by α_X.

All algebraic operations on L can be extended to L^X in the pointwise way. That is, for all $A, B \in L^X$, $\alpha \in L$ and $x \in X$,

(1) $(A \vee B)(x) = A(x) \vee B(x)$;

(2) $(A \wedge B)(x) = A(x) \wedge B(x)$ and $(\alpha \wedge A)(x) = \alpha \wedge B(x)$;

(3) $(A \longrightarrow B)(x) = A(x) \longrightarrow B(x)$ and $(\alpha \longrightarrow B)(x) = \alpha \longrightarrow B(x)$.

For a map $f : X \longrightarrow Y$, we can define $f_L^{\rightarrow} : L^X \longrightarrow L^Y$ and $f_L^{\leftarrow} : L^Y \longrightarrow L^X$ by $f_L^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x)$ and $f_L^{\leftarrow}(B)(x) = B(f(x))$, respectively. Then $f_L^{\rightarrow}$ is the left adjoint of $f_L^{\leftarrow}$.

2.1 Categories

Now we shall give some basic definitions of the category theory [1, 2, 4, 9, 11]. We hope the readers familiar with the basic concepts in category, such as epimorphism, monomorphism, isomorphism, source and initial etc. For convenience, we only list some requisite definitions and results.

Throughout this section, without other statements, $(\mathcal{A}, U)$ denotes a concrete category over **Set**, and $\mathcal{S}$ denotes the subcategory of $\mathcal{A}$ which is always assumed to be isomorphism-closed. Furthermore, *epi* denotes the class of epimorphisms in $\mathcal{A}$. It is worth to note that for each $\mathcal{A}$-morphism f, f is an isomorphism if and only if f is a section and an epimorphism.

The functor $U : \mathcal{A} \longrightarrow \mathbf{Set}$ is called *topological* if every U-structured source $(X, \{h_i : X \longrightarrow U(A_i)\}_i)$ has a unique U-initial lift $\{f_i : A \longrightarrow A_i\}_i$, where $X \in ob(\mathbf{Set})$ and $A, A_i \in ob(\mathcal{A})$ and "lift" means $U(f_i) = h_i$ for each $i \in I$. $(\mathcal{A}, U)$ is called *topological* provided that U is topological. In a topological construct $\mathcal{A}$, a $\mathcal{A}$-morphism $f : A \longrightarrow B$ is an epimorphism if and only if $f : U(A) \longrightarrow U(B)$ is surjective.

A $\mathcal{A}$-object A is called *injective* if for each inclusion map $e : Y \longrightarrow Z$ and each $\mathcal{A}$-morphism $f : Y \longrightarrow X$, there exists an $\mathcal{A}$-morphism $g : Z \longrightarrow X$ such that $g \circ e = f$. Moreover, A is called *a cogenerator*(or *a coseparator*) in $\mathcal{A}$ if for each pair of distinct $\mathcal{A}$-morphism $f, g : Y \longrightarrow Z$, there exists an $\mathcal{A}$-morphism $h : Z \longrightarrow X$ such that $h \circ f \neq h \circ g$.

Definition 2.1. *$\mathcal{S}$ is an* epireflective subcategory *of $\mathcal{A}$ provided that for each $B \in ob(\mathcal{A})$, there exists $A_B \in ob(\mathcal{S})$ and $\mathcal{A}$-epimorphism $r : B \longrightarrow A_B$ such that for each $\mathcal{A}$-morphism $f : B \longrightarrow C$, $C \in ob(\mathcal{S})$, there exists a unique $\mathcal{S}$-morphism $f^* : A_B \longrightarrow C$ with $f^* \circ r = f$.*

Let $\mathcal{H}$ be a class of $\mathcal{A}$-objects. Then the smallest epireflective subcategory $\mathcal{S}$ that contains $\mathcal{I}$ is said to be *the epireflective hull* in $\mathcal{A}$.

Definition 2.2. *An $\mathcal{A}$-morphism $f : A \longrightarrow B$ is* initial *provided that for each $\mathcal{A}$-object C, an $\mathbf{Set}$-morphism $g : U(C) \longrightarrow U(A)$ is an $\mathcal{A}$-morphism when $f \circ g : U(C) \longrightarrow U(B)$ is an $\mathcal{A}$-morphism. Moreover, an initial morphism $f : A \longrightarrow B$ that has a monomorphism underlying $\mathbf{Set}$-morphism $f : U(A) \longrightarrow U(B)$ is called an* embedding.

Let $A, B, C \in ob(\mathcal{A})$, then $g \circ f : A \longrightarrow C$ is an embedding if $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are embeddings. Moreover, $f : A \longrightarrow B$ is an embedding if $g \circ f : A \longrightarrow C$ is an embedding. Denote the class of $\mathcal{A}$-embeddings by $\mathcal{E}$.

Definition 2.3. *Let $\mathcal{S}$ be an epireflective subcategory of $\mathcal{A}$, then $\mathcal{S}$ is* $\mathcal{E}$-firm epireflective *in $\mathcal{A}$ if the reflection $r : B \longrightarrow A_B$ is in $epi \cap \mathcal{E}$ and for each morphism $f : B \longrightarrow C$ with $C \in ob(\mathcal{S})$, $f^* : A_B \longrightarrow C$ is an isomorphism when $f \in epi \cap \mathcal{E}$.*

2.2 L-algebraic closure space

Let us recall some concepts about L-algebraic closure spaces, where the definitions can be find in [10, 14, 15, 18]

Definition 2.4. *A subset $\mathcal{C}$ of L^X is called an* L-closure structure *on X if it satisfies the following conditions:*
(CS1) $0_X, 1_X \in \mathcal{C}$;
(CS2) *For any $\{A_i | i \in I\} \subseteq \mathcal{C}$, $\bigwedge_{i\in I} A_i \in \mathcal{C}$.*
A pair $(X, \mathcal{C})$ is called an L-closure space *if $\mathcal{C}$ is an L-closure structure on X.*

Definition 2.5. *An* L-algebraic closure structure *$\mathcal{C}$ on X is an L-closure structure that satisfies the following condition:*
(CS3) *If $\{A_i | i \in I\} \subseteq^d \mathcal{C}$ is directed, then $\bigvee_{i\in I} A_i \in \mathcal{C}$.*
The pair $(X, \mathcal{C})$ is said to be an L-algebraic closure space. *An L-algebraic closure space is also called an* L-convex space *in* [10]. *An L-algebraic closure space is* stratified *if it satisfies*
(S) *$a \longrightarrow C \in \mathcal{C}$ for each $a \in L$ and $C \in \mathcal{C}$.*

Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be two L-algebraic closure spaces and $f : X \longrightarrow Y$ be a map. Then f is called *L-continuous* provided that for all $B \in \mathcal{C}_Y$, $f_L^{\leftarrow}(B) \in \mathcal{C}_X$. f is called *L-algebraic* provided that for all $A \in \mathcal{C}_X$, $f_L^{\rightarrow}(A) \in \mathcal{C}_Y$. f is called *a homeomorphism* provided that f is L-continuous, L-algebraic, and a bijection. Let L-**AC** denote the category of L-algebraic closure spaces and L-continuous mappings, and let **S**L-**AC** denote the category of stratified L-algebraic closure spaces and L-continuous mappings.

For two L-algebraic closure spaces $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$, $f : X \longrightarrow Y$ is said to be an *embedding* if $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ is an L-continuous map, an injection and a homeomorphism onto image subspaces.

Remark 2.6. *The L-continuous map $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ between two L-algebraic closure spaces is an epimorphism if and only if $f : X \longrightarrow Y$ is surjective.*

Definition 2.7. *An L-algebraic closure space is called S_0 if for every distinct $x, y \in X$, there exists some $C \in \mathcal{C}$ such that $C(x) \neq C(y)$.*

Let L-**AC**$_0$ and **S**L-**AC**$_0$ denotes the subcategory of L-**AC**, whose objects are S_0-L-algebraic closure spaces and stratified S_0-L-algebraic closure spaces, respectively. Given an L-algebraic closure spaces $(X, \mathcal{C})$, its *hull operator* $^- : L^X \longrightarrow L^X$ is defined by $\overline{A} = \bigwedge\{C \in \mathcal{C} | A \leq C\}$ for all $A \in L^X$, and it is obvious that $\overline{A} \in \mathcal{C}$.

Proposition 2.8. *An stratified L-algebraic closure space $(X, \mathcal{C})$ is S_0 if and only if for every distinct $x, y \in X$, it holds that $\overline{1_{\{x\}}} \neq \overline{1_{\{y\}}}$.*

Proof. If $(X, \mathcal{C})$ is S_0, we have $C(x) \neq C(y)$ for some $C \in \mathcal{C}$, then $C(x) \longrightarrow C(y) \neq 1$ or $C(y) \longrightarrow C(x) \neq 1$. Here,we suppose that $C(x) \longrightarrow C(y) \neq 1$, it follows that $\overline{1_{\{x\}}}(y) \leq C(x) \to C(y) \neq 1$, thus $\overline{1_{\{x\}}} \neq \overline{1_{\{y\}}}$.
Conversely, if for every distinct $x, y \in X$, it holds that $\overline{1_{\{x\}}} \neq \overline{1_{\{y\}}}$, then we obtain that $\overline{1_{\{x\}}}(x) \neq \overline{1_{\{x\}}}(y)$ or $\overline{1_{\{y\}}}(x) \neq \overline{1_{\{y\}}}(y)$. □

Remark 2.9. *Let $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ be an epimorphism in* **S**L-**AC**$_0$, *then f is an embedding between $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ if and only if f is an embedding in* **S**L-**AC**$_0$.

Proposition 2.10. *Let $(X, \mathcal{C})$ be a stratified L-algebraic closure space. For each $A \in L^X$ and $C \in \mathcal{C}$, it holds that $S_X(A, C) = S_X(\overline{A}, C) = S_X(\overline{A}, \overline{C})$.*

Proof. For each $A, B \in L^X$, Since $A \leq S_X(A, B) \longrightarrow \overline{S_X(A, B) * A}$, then $\overline{A} \leq S_X(A, B) \longrightarrow \overline{S_X(A, B) * A}$, this forces that $\overline{A} * S_X(A, B) \leq \overline{A * S_X(A, B)} \leq \overline{B}$, so $S_X(A, B) \leq S_X(\overline{A}, \overline{B})$. Therefore, for each $C \in \mathcal{C}$, $S_X(\overline{A}, C) \leq S_X(A, C) \leq S_X(\overline{A}, \overline{C}) = S_X(\overline{A}, C)$, that is to say $S_X(A, C) = S_X(\overline{A}, C) = S_X(\overline{A}, \overline{C})$. □

Corollary 2.11. *When $(X, \mathcal{C})$ is a stratified L-algebraic closure space, for each $C \in \mathcal{C}$ and $x \in X$, it holds that $S_X(\overline{1_{\{x\}}}, C) = C(x)$.*

Definition 2.12. *Let $(X, \mathcal{C})$ be an L-algebraic closure space. An L-subset $F \in \mathcal{C}$ is called an* algebraic irreducible set *on X if*
(1) $\bigvee_{x\in X} F(x) = 1$;
(2) $S_X(F, \bigvee_{i\in I} C_i) = \bigvee_{i\in I} S_X(F, C_i)$ *for each directed family $\{C_i | i \in I\} \subseteq^d \mathcal{C}$.*

Let $ir(X)$ denote the set of all algebraic irreducible sets on X.

Definition 2.13. *A stratified L-algebraic closure space $(X, \mathcal{C})$ is* sober *if for each $F \in ir(X)$, there exits a unique $x \in X$ such that $F = \overline{1_{\{x\}}}$.*

It is easy to check that a sober L-algebraic closure space must be S_0. Let **Sob**L-**AC** denote the category of sober L-algebraic closure spaces and L-continuous mappings.

3 The stratified Sierpinski L-algebraic closure space

Definition 3.1. *Given a category $\mathcal{A} = (\mathcal{A}, U)$ of sets with structures. A family $\mathcal{F} = \{f_i : (X, s) \longrightarrow (Y_i, t_i) | i \in I\}$ of $\mathcal{A}$-morphisms is said to be* initial *if for each (Z, u) and a map $g : Z \longrightarrow X$, $g : (Z, u) \longrightarrow (X, s)$ is a $\mathcal{A}$-morphism if $f_i \circ g : (Z, u) \longrightarrow (Y_i, t_i)$ is a $\mathcal{A}$-morphism for each $i \in I$.*

Definition 3.2. *Given a category $\mathcal{A} = (\mathcal{A}, U)$ of sets with structures, an object S of $\mathcal{A}$ is called a* Sierpinski object *if for each $X = (X, s) \in ob\mathcal{A}$, the family of all $\mathcal{A}$-morphisms from X to S is initial.*

Consider the commutative and integral quantale L. Let $\mathcal{C}_S$ be the smallest stratified L-algebraic closure space that contains $\{id_L\}$, where id_L is the identity map on L. In this paper, $(L, \mathcal{C}_S)$ is called the *stratified Sierpinski L-algebraic closure space.*

Proposition 3.3. *Given an stratified L-algebraic closure space $(X, \mathcal{C}_X)$, then $C \in \mathcal{C}_X$ if and only if $C : (X, \mathcal{C}_X) \longrightarrow (L, \mathcal{C}_S)$ is an L-continuous map.*

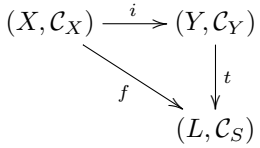
Proposition 3.4. *$(L, \mathcal{C}_S)$ is a Sierpinski object in the category $\mathbf{S}L$-$\mathbf{AC}$.*

Proof. Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be two stratified L-algebraic closure spaces, defined $F = \{f : (X, \mathcal{C}_X) \longrightarrow (L, \mathcal{C}_S) | f$ *is an* L$-$*continuous map*$\}$ and $g : Y \longrightarrow X$ is a morphism. For each $f \in F$, suppose that $f \circ g : (Y, \mathcal{C}_Y) \longrightarrow (L, \mathcal{C}_S)$ is an L-continuous map. Since for each $C \in \mathcal{C}_X$, $C : (X, \mathcal{C}_X) \longrightarrow (L, \mathcal{C}_S)$ is an L-continuous map, it follows that $C \in F$. Then we obtain that $g^{\leftarrow}_L(C) = C \circ g : (Y, \mathcal{C}_Y) \longrightarrow (L, \mathcal{C}_S)$ is an L-continuous map, that is to say $g^{\leftarrow}_L(C) \in \mathcal{C}_Y$. □

It is worth to note that the stratified Sierpinski L-algebraic closure space $(L, \mathcal{C}_S)$ is S_0, and the following explanations show that $(L, \mathcal{C}_S)$ is an injective cogenerator in the category $\mathbf{S}L$-$\mathbf{AC}_0$, and every stratified S_0-L-algebraic closure space can be embedded into the product space of copies of $(L, \mathcal{C}_S)$.

Proposition 3.5. *$(L, \mathcal{C}_S)$ is an injective object of the category $\mathbf{S}L$-$\mathbf{AC}$.*

Proof. For the inclusion map $i : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ in the category $\mathbf{S}L$-$\mathbf{AC}$, and L-continuous map $f : (X, \mathcal{C}_X) \longrightarrow (L, \mathcal{C}_S)$, we have that $f \in \mathcal{C}_X$ and there exists $t \in \mathcal{C}_Y$ such that $i^{\rightarrow}_L(f) = t \wedge 1_X$ and $i^{\leftarrow}_L(t) = f$. Thus, there exists an L-continuous map $t : (Y, \mathcal{C}_Y) \longrightarrow (L, \mathcal{C}_S)$ such that the following diagram is commutative:



This completes the proof. □

Theorem 3.6. *$(L, \mathcal{C}_S)$ is a cogenerator in the category $\mathbf{S}L$-$\mathbf{AC}_0$.*

Proof. Let $f, g : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ be two L-continuous maps between stratified S_0-L-algebraic closure spaces and $f \neq g$, then there exists $x \in X$ such that $f(x) \neq g(x)$. Since $(Y, \mathcal{C}_Y)$ is S_0, it holds that there exists $C \in \mathcal{C}_Y$ such that $C(f(x)) \neq C(g(x))$. Therefore, there exists an L-continuous map $C : (Y, \mathcal{C}_Y) \longrightarrow (L, \mathcal{C}_S)$ such that $C \circ f \neq C \circ g$, that is to say $(L, \mathcal{C}_S)$. □

Theorem 3.7. *Each stratified S_0-L-algebraic closure space can be embedded into the product space of copies of $(L, \mathcal{C}_S)$.*

Proof. Let $(X, \mathcal{C}_X)$ be a stratified S_0-L-algebraic closure space. Define $m_X : (X, \mathcal{C}_X) \longrightarrow \Pi_{\mathcal{C}_X}(L, \mathcal{C}_S)$ by $m_X(x)(C) = C(x)$ for all $x \in X$ and $C \in \mathcal{C}_X$. Since $p_C \circ m_X = C$ and $C : (X, \mathcal{C}_X) \longrightarrow (L, \mathcal{C}_S)$ is an L-continuous map, it follows that m_X is an L-continuous map. Furthermore, as $(X, \mathcal{C}_X)$ is S_0, then for each $x_1 \neq x_2 \in X$, there exists $C \in \mathcal{C}_X$ such that $C(x_1) \neq C(x_2)$. Thus, by the definition of m_X, it is an injection. Finally, for each $C \in \mathcal{C}_X$ and $x \in X$, we have $(m_X)^{\rightarrow}_L(C)(m(x)) = \bigvee_{m(z)=m(x)} C(z) = C(x) = m_X(x)(C) = p_C(m(x))$, it holds that $(m_X)^{\rightarrow}_L(C) = (p_C)^{\leftarrow}_L(id_L) \wedge 1_{m^{\rightarrow}_X(X)}$, then m_X is a homeomorphic map. Therefore, $m_X : (X, \mathcal{C}_X) \longrightarrow \Pi_{\mathcal{C}_X}(L, \mathcal{C}_S)$ is an embedding map. □

4 The sobrification of L-algebraic closure spaces

In [21], a sober L-cotopological space is given by the fuzzy inclusion order. Following the idea in [21], we can also construct a sober L-algebraic closure space in the framework of L-algebraic closure spaces.

Given a stratified L-algebraic closure space $(X,\mathcal{C})$ and let $C\in\mathcal{C}$, define $t(C):ir(X)\longrightarrow L$, for each $F\in ir(X)$ by

$$t(C)(F)=S_X(F,C).$$

Denote $T(X)=\{t(C)|C\in\mathcal{C}\}$, then the following propositions show that $T(X)$ is a sober stratified L-algebraic closure structure on $ir(X)$. In addition, $(ir(X),T(X))$ is called the *sobrification* of $(X,\mathcal{C})$.

Proposition 4.1. *Let $\mathcal{C}$ be a stratified L-algebraic closure structure on X, then $T(X)=\{t(C)|C\in\mathcal{C}\}$ is a stratified L-algebraic closure structure on $ir(X)$.*

Proof. From the definition of the stratified L-algebraic closure structure, (CS1)(CS2) and (S) are trivial, we only check (CS3).

Let $\{t(C_i)\}_{i\in I}\subseteq^d T(X)$ be a directed family. Then for each C_i,C_j, there exists $t(C_k)$ such that $t(C_i)\vee t(C_j)\leq t(C_k)$, it follows that for each $F\in ir(X)$, $S_X(F,C_i)\vee S_X(F,C_j)\leq S_X(F,C_k)$. Specially, for each $x\in X$, let $F=\overline{1_{\{x\}}}$, then we have $C_i(x)\vee C_j(x)\leq C_k(x)$, that is to say $\{C_i\}_{i\in I}$ is directed. So from the definition of algebraic irreducible set, we know that for all $F\in ir(X)$,

$$\bigvee_{i\in I}t(C_i)(F)=\bigvee_{i\in I}S_X(F,C_i)=S_X(F,\bigvee_{i\in I}C_i)=t(\bigvee_{i\in I}C_i)(F),$$

that is to say $\bigvee_{i\in I}t(C_i)\in T(X)$. □

Proposition 4.2. *Let $\mathcal{C}$ be a stratified L-algebraic closure structure on X, then $(ir(X),T(X))$ is sober.*

Proof. For each algebraic irreducible set $F\in ir(X)$, we have $\overline{1_{\{F\}}}=\bigwedge\{t(C)\in T(X)|C\in\mathcal{C},1_{\{F\}}\leq t(C)\}=t(F)$, so it suffices to check that C is algebraic irreducible for each algebraic irreducible set $t(C)\in ir(ir(X))$. In fact,

(1) For each $F\in ir(X)$ and $x\in X$, we know that $F(x)*S_X(F,C)\leq C(x)$, then

$$\begin{aligned}\bigvee_{x\in X}C(x)&\geq\bigvee_{x\in X}\bigvee_{F\in ir(X)}F(x)*S_X(F,C)\\&=\bigvee_{F\in ir(X)}(\bigvee_{x\in X}F(x))*S_X(F,C)\\&=\bigvee_{F\in ir(X)}S_X(F,C)\\&=\bigvee_{F\in ir(X)}t(C)(F)\\&=1.\end{aligned}$$

(2) We firstly show that for each $C_1,C_2\in\mathcal{C}$,

$$S_X(C_1,C_2)=S_{ir(X)}(t(C_1),t(C_2)).$$

On the one hand,

$$\begin{aligned}S_{ir(X)}(t(C_1),t(C_2))&=\bigwedge_{F\in ir(X)}S_X(F,C_1)\longrightarrow S_X(F,C_2)\\&\leq\bigwedge_{x\in X}S_X(\overline{1_{\{x\}}},C_1)\longrightarrow S_X(\overline{1_{\{x\}}},C_2)\\&=\bigwedge_{x\in X}C_1(x)\longrightarrow C_2(x)\\&=S_X(C_1,C_2),\end{aligned}$$

the other hand is trivial.

So for each directed family $\{A_i\}_{i\in I} \subseteq^d \mathcal{C}$, it follows that

$$\begin{aligned} S_X(C, \bigvee_{i\in I} A_i) &= S_{ir(X)}(t(C), t(\bigvee_{i\in I} A_i)) = S_{ir(X)}(t(C), \bigvee_{i\in I} t(A_i)) \\ &= \bigvee_{i\in I} S_{ir(X)}(t(C), t(A_i)) = \bigvee_{i\in I} S_X(C, A_i). \end{aligned}$$

This completes the proof. □

Lemma 4.3. *Let $(X, \mathcal{C})$ be a stratified L-algebraic closure space, define $\eta_X : X \longrightarrow ir(X)$ by $\eta_X(x) = \overline{1_{\{x\}}}$ for each $x \in X$, then*

(1) *$\eta_X : (X, \mathcal{C}) \longrightarrow (ir(X), T(X))$ is an L-continuous map;*
(2) *$(X, \mathcal{C})$ is sober if and only if η_X is a homeomorphism;*
(3) *For each $F \in ir(X)$, $\overline{(\eta_X)^{\rightarrow}_L(F)} = t(F)$.*

Proof. (1) For each $C \in \mathcal{C}$, $t(C) \in T(X)$ and $x \in X$, since

$$(\eta_X)^{\leftarrow}_L(t(C))(x) = t(C)(\eta_X(x)) = t(C)(\overline{1_{\{x\}}}) = S_X(\overline{1_{\{x\}}}, C) = C(x),$$

it follows that $\eta_X : (X, \mathcal{C}) \longrightarrow (ir(X), T(X))$ is an L-continuous map.

(2) Necessity: From (1), we only need to check that $\eta_X : (X, \mathcal{C}) \longrightarrow (ir(X), T(X))$ is a bijection and an L-algebraic map. Since $(X, \mathcal{C})$ is sober, it follows that it is S_0, and from Proposition 2.8, we know that for each $x, y \in X$, if $x \neq y$, then $\eta_X(x) \neq \eta_X(y)$. Furthermore, for each $F \in ir(X)$, there exits a unique $x \in X$ such that $F = \overline{1_{\{x\}}} = \eta_X(x)$. Thus, η_X is a bijection.

Next, we show that $\eta_X : (X, \mathcal{C}) \longrightarrow (ir(X), T(X))$ is an L-algebraic map, it suffices to check that for each $C \in \mathcal{C}$ and $F \in ir(X)$, $(\eta_X)^{\rightarrow}_L(C)(F) = t(C)(F)$. In fact, since $(X, \mathcal{C})$ is sober, for a given $F \in ir(X)$, there exits a unique $x \in X$ such that $F = \overline{1_{\{x\}}} = \eta_X(x)$. Furthermore,

$$\begin{aligned} (\eta_X)^{\rightarrow}_L(C)(\overline{1_{\{x\}}}) &= \bigvee_{\eta_X(z) = \overline{1_{\{x\}}}} C(z) = \bigvee \{C(z) | \eta_X(z) = \overline{1_{\{x\}}}\} \\ &= C(x) = S_X(\overline{1_{\{x\}}}, C) = t(C)(\overline{1_{\{x\}}}). \end{aligned}$$

Thus $(\eta_X)^{\rightarrow}_L(C)(F) = t(C)(F)$.

The sufficiency is trivial.

(3) For each $F \in ir(X)$, it holds that

$$\begin{aligned} \overline{(\eta_X)^{\rightarrow}_L(F)} &= \bigwedge \{t(C) \in T(X) | (\eta_X)^{\rightarrow}_L(F) \leq t(C)\} \\ &= \bigwedge \{t(C) \in T(X) | F \leq t(C) \circ \eta_X\} \\ &= \bigwedge \{t(C) \in T(X) | F \leq C\} \\ &= t(F). \end{aligned}$$

□

Lemma 4.4. *Let $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ be an L-continuous map between stratified L-algebraic closure spaces, then for each algebraic irreducible set F on X, $\overline{f^{\rightarrow}_L(F)}$ is an algebraic irreducible set on Y.*

Proof. For each $F \in ir(X)$, we obtain that $\bigvee_{y\in Y} \overline{f^{\rightarrow}_L(F)}(y) \geq \bigvee_{y\in Y} f^{\rightarrow}_L(F)(y) = \bigvee_{y\in Y} \bigvee_{f(x)=y} F(x) \geq \bigvee_{x\in X} F(x) = 1$. Furthermore, for each directed family $\{A_i\}_{i\in I} \subseteq^d \mathcal{C}_Y$, it follows that

$$S_Y(\overline{f^{\rightarrow}_L(F)}, \bigvee_{i\in I} A_i) = S_Y(f^{\rightarrow}_L(F), \bigvee_{i\in I} A_i) = S_X(F, \bigvee_{i\in I} f^{\leftarrow}_L(A_i)) = \bigvee_{i\in I} S_X(F, f^{\leftarrow}_L(A_i)) = \bigvee_{i\in I} S_X(\overline{f^{\rightarrow}_L(F)}, A_i).$$

□

5 The reflectivity of Sob$L$-AC and SL-AC$_0$

In this section, based on the approaches in [21, 25], we also can easily study the categorical relations between **Sob**L-**AC** and **S**L-**AC**$_0$.

Theorem 5.1. **Sob**L-**AC** *is a reflective subcategory of* **S**L-**AC**.

Proof. It suffices to check that for each stratified L-algebraic closure space $(X,\mathcal{C}_X)$, each sober L-algebraic closure space $(Y,\mathcal{C}_Y)$, and each L-continuous map $f:(X,\mathcal{C}_X)\longrightarrow(Y,\mathcal{C}_Y)$, there exits a unique L-continuous map $f^*:(ir(X),T(X))\longrightarrow(Y,\mathcal{C}_Y)$ such that $f^*\circ\eta_X=f$.

Step 1: For each algebraic irreducible set F on X, from Lemma 4.4, we know that $\overline{f_L^{\rightarrow}(F)}$ is an algebraic irreducible set on Y. Since $(Y,\mathcal{C}_Y)$ is sober, there exits $y_0\in Y$ such that $\overline{f_L^{\rightarrow}(F)}=\overline{1_{\{y_0\}}}$. Define $f^*:(ir(X),T(X))\longrightarrow(Y,\mathcal{C}_Y)$ by $f^*(F)=y_0$ for all $F\in ir(X)$. For each $C\in\mathcal{C}_Y$ and $F\in ir(X)$,

$$(f^*)_L^{\leftarrow}(C)(F)=C(f^*(F))=S_Y(\overline{1_{\{f*(F)\}}},C)=S_Y(\overline{1_{\{y_0\}}},C)=S_Y(\overline{f_L^{\rightarrow}(F)},C)=S_X(F,f_L^{\leftarrow}(C))=t(f_L^{\leftarrow}(C))(F),$$

it follows that $(f^*)_L^{\leftarrow}(C)=t(f_L^{\leftarrow}(C))\in T(X)$ for each $C\in\mathcal{C}_Y$, that is to say $f^*:(ir(X),T(X))\longrightarrow(Y,\mathcal{C}_Y)$ is an L-continuous map.

Step 2: For all $x\in X$, since $1_{\{f(x)\}}\leq f_L^{\rightarrow}(\overline{1_{\{x\}}})\leq\overline{f_L^{\rightarrow}(1_{\{x\}})}=\overline{1_{\{f(x)\}}}$, it follows that $\overline{f_L^{\rightarrow}(\eta_X(x))}=\overline{f_L^{\rightarrow}(\overline{1_{\{x\}}})}=\overline{1_{\{f(x)\}}}$. Thus, from the definition of f^*, we obtain that $f^*\circ\eta_X=f$.

Step 3: The definition of f^* is unique. Suppose there is an L-continuous map $g:(ir(X),T(X))\longrightarrow(Y,\mathcal{C}_Y)$ such that $g\circ\eta_X=f$. It suffices to show that $f^*(F)=g(F)$ for each $F\in ir(X)$. For this, we need to check that $\overline{f_L^{\rightarrow}(F)}=\overline{1_{\{g(F)\}}}$.

On the one hand, since $g_L^{\leftarrow}(\overline{1_{\{g(F)\}}})\in T(X)$, it holds that there exists $C_0\in\mathcal{C}_X$ such that $g_L^{\leftarrow}(\overline{1_{\{g(F)\}}})=t(C_0)$. So, $F\leq C_0$. Furthermore, for each $x\in X$, since

$$\overline{1_{\{g(F)\}}}(f(x))=\overline{1_{\{g(F)\}}}(g(\eta_X(x)))=\overline{1_{\{g(F)\}}}(g(\overline{1_{\{x\}}}))=g_L^{\leftarrow}(\overline{1_{\{g(F)\}}})(\overline{1_{\{x\}}})=t(C_0)(\overline{1_{\{x\}}})=C_0(x),$$

this immediately forces $F\leq\overline{1_{\{g(F)\}}}\circ f=f_L^{\leftarrow}(\overline{1_{\{g(F)\}}})$, then we obtain that $f_L^{\rightarrow}(F)\leq\overline{1_{\{g(F)\}}}$. Thus, $\overline{f_L^{\rightarrow}(F)}\leq\overline{1_{\{g(F)\}}}$.

On the other hand, since

$$\begin{aligned}\overline{f_L^{\rightarrow}(F)}(g(F))&=\overline{g_L^{\rightarrow}\circ(\eta_X)_L^{\rightarrow}(F)}(g(F))\\&\geq g_L^{\rightarrow}(\overline{(\eta_X)_L^{\rightarrow}(F)})(g(F))\\&=g_L^{\leftarrow}(g_L^{\rightarrow}(\overline{(\eta_X)_L^{\rightarrow}(F)}))(F)\\&\geq\overline{(\eta_X)_L^{\rightarrow}(F)}(F)\\&=t(F)(F)\\&=1,\end{aligned}$$

it holds that $1_{\{g(F)\}}\leq\overline{f_L^{\rightarrow}(F)}$, then $\overline{1_{\{g(F)\}}}\leq\overline{f_L^{\rightarrow}(F)}$.

Therefore, we obtain that $\overline{1_{\{g(F)\}}}=\overline{f_L^{\rightarrow}(F)}$. This ends the proof. □

Since it is easy to check that **Sob**L-**AC** is a subcategory of **S**L-**AC**$_0$, from Theorem 5.1, we know that **Sob**L-**AC** is also reflective in **S**L-**AC**$_0$. In the following, we show that **Sob**L-**AC** is still epireflective in **S**L-**AC**$_0$. For this, a characterization of epimorphisms in **S**L-**AC**$_0$ is given.

Lemma 5.2. *A morphism $f:(X,\mathcal{C}_X)\longrightarrow(Y,\mathcal{C}_Y)$ in* **S**L-**AC**$_0$ *is an epimorphism if and only if for each $C_1,C_2\in\mathcal{C}_Y$, $f_L^{\leftarrow}(C_1)=f_L^{\leftarrow}(C_2)$ implies that $C_1=C_2$.*

Proof. Let $g,h:(Y,\mathcal{C}_Y)\longrightarrow(Z,\mathcal{C}_Z)$ be two morphisms in **S**L-**AC**$_0$, which satisfies $g\circ f=h\circ f$. Suppose $g\neq h$, then there exits $y_0\in Y$ such that $g(y_0)\neq h(y_0)$. Since $(Z,\mathcal{C}_Z)$ is S_0, it follows that $\overline{1_{\{g(y_0)\}}}\neq\overline{1_{\{h(y_0)\}}}$, then $\overline{1_{\{g(y_0)\}}}(h(y_0))\neq 1$ or $\overline{1_{\{h(y_0)\}}}(g(y_0))\neq 1$. Without loss of generality, let $\overline{1_{\{g(y_0)\}}}(h(y_0))\neq 1$. Since $f_L^{\leftarrow}(g_L^{\leftarrow}(\overline{1_{\{g(y_0)\}}}))=f_L^{\leftarrow}(h_L^{\leftarrow}(\overline{1_{\{g(y_0)\}}}))$, it holds that $g_L^{\leftarrow}(\overline{1_{\{g(y_0)\}}})=h_L^{\leftarrow}(\overline{1_{\{g(y_0)\}}})$, then we obtain that $\overline{1_{\{g(y_0)\}}}(h(y_0))=\overline{1_{\{g(y_0)\}}}(g(y_0))=1$, a contradiction. Therefore, $g\circ f=h\circ f$ implies $g=h$, then $f:(X,\mathcal{C}_X)\longrightarrow(Y,\mathcal{C}_Y)$ is an epimorphism.

Conversely, it is trivial. □

Theorem 5.3. **Sob**L-**AC** *is an epireflective subcategory of* **S**L-**AC**$_0$.

Proof. From Theorem 5.1, it suffices to check that $\eta_X : (X, \mathcal{C}_X) \longrightarrow (ir(X), T(X))$ is an epimorphism. For each $t(C_1), t(C_2) \in T(X)$ with $(\eta_X)_L^{\leftarrow}(t(C_1)) = (\eta_X)_L^{\leftarrow}(t(C_2))$ and $x \in X$, we have

$$(\eta_X)_L^{\leftarrow}(t(C_1))(x) = t(C_1)(\eta_X(x)) = t(C_1)(\overline{1_{\{x\}}}) = S_X(\overline{1_{\{x\}}}, C_1) = C_1(x),$$

$$(\eta_X)_L^{\leftarrow}(t(C_2))(x) = t(C_2)(\eta_X(x)) = t(C_2)(\overline{1_{\{x\}}}) = S_X(\overline{1_{\{x\}}}, C_2) = C_2(x),$$

it follows that $C_1 = C_2$. Therefore η_X is an epimorphism by lemma 5.2. □

Let *epi* and $\mathcal{E}$ denote the classes of epimorphisms and embeddings in $\mathbf{S}L\text{-}\mathbf{AC}_0$, respectively. In the following, we will show that $\mathbf{S}L\text{-}\mathbf{AC}_0$ is an epireflective subcategory of $\mathbf{S}L\text{-}\mathbf{AC}$, and $\mathbf{Sob}L\text{-}\mathbf{AC}$ is, in fact, $\mathcal{E}$-firm epireflective in $\mathbf{S}L\text{-}\mathbf{AC}_0$.

Proposition 5.4. *Let $(X, \mathcal{C})$ be a stratified L-algebraic closure space, $\eta_X : (X, \mathcal{C}) \longrightarrow (ir(X), T(X))$ is an embedding if $(X, \mathcal{C})$ is S_0.*

Proof. According to Lemma 4.3 and $(X, \mathcal{C})$ is S_0, it holds that $\eta_X : (X, \mathcal{C}) \longrightarrow (ir(X), T(X))$ is an L-continuous map and an injection. For each $C \in \mathcal{C}$ and $x \in X$, since

$$(\eta_X)_L^{\rightarrow}(C)(\overline{1_{\{x\}}}) = \bigvee_{\eta_X(z) = \overline{1_{\{x\}}}} C(z) = C(x) = S_x(\overline{1_{\{x\}}}, C) = t(C)(\overline{1_{\{x\}}}),$$

then $\eta_X : (X, \mathcal{C}) \longrightarrow (ir(X), T(X))$ is a homemorphism. □

Lemma 5.5. $\mathbf{S}L\text{-}\mathbf{AC}$ *is topological.*

Proof. Trivial and omitted. □

Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be stratified S_0-L-algebraic closure spaces, and $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y) \in epi \cap \mathcal{E}$, then for each $C \in \mathcal{C}_X$, there exists $C_f \in \mathcal{C}_Y$ such that $f_L^{\rightarrow}(C) = C_f \wedge 1_{f(X)}$ and $f_L^{\leftarrow}(C_f) = C$. It is worth to note that C_f is unique. Furthermore, from Lemma 5.2, it is routine to check that for each $a \in L$, $C_1, C_2 \in \mathcal{C}_X$, $\{C_i\}_{i \in I} \subseteq \mathcal{C}_X$ and directed family $\{D_i\}_{i \in I} \subseteq^d \mathcal{C}_X$, it holds that

$$(\bigwedge_{i \in I} C_i)_f = \bigwedge_{i \in I} (C_i)_f,$$

$$(\bigvee_{i \in I} D_i)_f = \bigvee_{i \in I} (D_i)_f,$$

$$(a \longrightarrow C_1)_f = a \longrightarrow (C_1)_f,$$

$$S_X(C_1, C_2) = S_Y((C_1)_f, (C_2)_f).$$

Theorem 5.6. $\mathbf{S}L\text{-}\mathbf{AC}_0$ *is an epireflective subcategory of* $\mathbf{S}L\text{-}\mathbf{AC}$.

Proof. Let $(X, \mathcal{C}_X)$ be a stratified L-algebraic closure space and $(Y, \mathcal{C}_Y)$ be an S_0-stratified L-algebraic closure space. Define for each $x, y \in X, x \sim y \Leftrightarrow \overline{1_{\{x\}}} = \overline{1_{\{y\}}}$, then $\sim$ is an equivalence relation. Define $q : X \longrightarrow X/\sim$ by for each $x \in X$, $q(x) = [x]$, then $(X/\sim, \mathcal{C}_X/\sim)$ is an S_0-stratified L-algebraic closure space, where $\mathcal{C}_X/\sim = \{\phi \in L^{X/\sim} \mid q_L^{\leftarrow}(\phi) \in \mathcal{C}_X\}$. Furthermore, $q : X \longrightarrow X/\sim$ is a surjection, and $q : (X, \mathcal{C}_X) \longrightarrow (X/\sim, \mathcal{C}_X/\sim)$ is an L-continuous map. Thus, it is an epimorphism. Then it suffices to check that for each L-continuous map $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$, there exists a unique L-continuous map $f^* : (X/\sim, \mathcal{C}_X/\sim) \longrightarrow (Y, \mathcal{C}_Y)$ such that $f^* \circ q = f$, that is to say the following diagram is commutative:

$$\begin{array}{ccc} (X, \mathcal{C}_X) & \xrightarrow{q} & (X/\sim, \mathcal{C}_X/\sim) \\ & \searrow^{f} & \downarrow{\scriptstyle f^*} \\ & & (Y, \mathcal{C}_Y) \end{array}$$

In fact, define f^* by for each $[x] \in \mathcal{C}_X/\sim$, $f^*([x]) = f(x)$, then it is obvious that $f^* \circ q = f$. Since for each $C \in \mathcal{C}_Y$ and $[x] \in \mathcal{C}_X/\sim$, we have $(f^*)_L^{\leftarrow}(C)([x]) = C \circ f^*([x]) = C(f(x))$, it follows that $(f^*)^{\leftarrow}(C) = f_L^{\leftarrow}(C) \in \mathcal{C}_X$, then $f^* : (X/\sim, \mathcal{C}_X/\sim) \longrightarrow (Y, \mathcal{C}_Y)$ is an L-continuous map. Further, if there exists an L-continuous map $h : (X/\sim, \mathcal{C}_X/\sim) \longrightarrow (Y, \mathcal{C}_Y)$ such that $h \circ q = f$, then $h \circ q = f^* \circ q$, thus $h = f^*$ considered q is an epimorphism. □

Theorem 5.7. **Sob**L-**AC** *is a* $\mathcal{E}$*-firm epireflective subcategory of* $\mathbf{S}L$-$\mathbf{AC}_0$.

Proof. For each S_0-stratified L-algebraic closure space $(X, \mathcal{C}_X)$ and sober L-algebraic closure space $(Y, \mathcal{C}_Y)$, from Theorem 5.1, Theorem 5.3 and Proposition 5.4, it suffices to prove that $f^* : (ir(X), T(X)) \longrightarrow (Y, \mathcal{C}_Y)$ is an isomorphism if $f \in epi \cap \mathcal{E}$. Since $f, \eta_X \in epi \cap \mathcal{E}$ and $f^* \circ \eta_X = f$, it follows that f^* is an epimorphism. So we only need to check that f^* has a left inverse.

Step 1: Define $g : Y \longrightarrow ir(X)$ by $g(y) = \bigwedge_{C \in \mathcal{C}_X} C_f(y) \longrightarrow C$ for each $y \in Y$. Then $g(y) \in ir(X)$.

(1) $\bigvee_{x \in X} g(y)(x) = \bigvee_{x \in X} \bigwedge_{C \in \mathcal{C}_X} C_f(y) \longrightarrow C(x) = \bigvee_{x \in X} \bigwedge_{C \in \mathcal{C}_X} C_f(y) \longrightarrow f_L^{\leftarrow}(C_f)(x) = \bigvee_{x \in X} \bigwedge_{C \in \mathcal{C}_X} C_f(y) \longrightarrow (C_f)(f(x)) \geq \bigwedge_{C \in \mathcal{C}_X} C_f(y) \longrightarrow C_f(y) = 1$.

(2) For each directed family $\{C_i\}_{i \in I} \subseteq^d \mathcal{C}_X$, it is obvious that $S_X(g(y), \bigvee_{i \in I} C_i) \geq \bigvee_{i \in I} S_X(g(y), C_i)$. Conversely, for each $y \in Y$, there always exists $x_0 \in X$ such that $f(x_0) = y$, then

$$g(y)(x_0) = \bigwedge_{C \in \mathcal{C}_X} C_f(y) \longrightarrow C(x_0) = \bigwedge_{C \in \mathcal{C}_X} C_f(y) \longrightarrow C_f(f(x_0)) = \bigwedge_{C \in \mathcal{C}_X} C_f(y) \longrightarrow C_f(y) = 1.$$

Thus,

$$\begin{aligned}
S_X(g(y), \bigvee_{i \in I} C_i) &= \bigwedge_{x \in X} (g(y)(x) \longrightarrow \bigvee_{i \in I} C_i(x)) \\
&\leq g(y)(x_0) \longrightarrow \bigvee_{i \in I} C_i(x_0) \\
&= \bigvee_{i \in I} C_i(x_0) \\
&= \bigvee_{i \in I} (C_i)_f(f(x_0)) \\
&= \bigvee_{i \in I} (C_i)_f(y) \\
&\leq \bigvee_{i \in I} S_X((C_i)_f(y) \longrightarrow C_i, C_i) \\
&\leq \bigvee_{i \in I} S_X(\bigwedge_{C \in \mathcal{C}_X} C_f(y) \longrightarrow C, C_i) \\
&= \bigvee_{i \in I} S_X(g(y), C_i).
\end{aligned}$$

Thus, $S_X(g(y), \bigvee_{i \in I} C_i) = \bigvee_{i \in I} S_X(g(y), C_i)$.

Step 2: Since for each $t(C) \in T(X)$ and $y \in Y$, we have

$$\begin{aligned}
g_L^{\leftarrow}(t(C))(y) &= S_X(\bigwedge_{D \in \mathcal{C}_X} D_f(y) \longrightarrow D, C) \\
&= S_Y((\bigwedge_{D \in \mathcal{C}_X} D_f(y) \longrightarrow D)_f, C_f) \\
&= S_Y(\bigwedge_{D \in \mathcal{C}_X} D_f(y) \longrightarrow D_f, C_f) \\
&= C_f(y),
\end{aligned}$$

it follows that $g_L^{\leftarrow}(t(C)) = C_f \in \mathcal{C}_Y$, that is to say $g : (Y, \mathcal{C}_Y) \longrightarrow (ir(X), T(X))$ is an L-continuous map.

Step 3: Since for each $F \in ir(X)$, it holds that

$$\begin{aligned} g \circ f^*(F) &= g(y_0) \\ &= \bigwedge_{C \in \mathcal{C}_X} C_f(y_0) \longrightarrow C \\ &= \bigwedge_{C \in \mathcal{C}_X} S_Y(\overline{1_{\{y_0\}}}, C_f) \longrightarrow C \\ &= \bigwedge_{C \in \mathcal{C}_X} S_Y(\overline{f_L^{\rightarrow}(F)}, C_f) \longrightarrow C \\ &= \bigwedge_{C \in \mathcal{C}_X} S_Y(f_L^{\rightarrow}(F), C_f) \longrightarrow C \\ &= \bigwedge_{C \in \mathcal{C}_X} S_Y(F, f_L^{\leftarrow}(C_f)) \longrightarrow C \\ &= \bigwedge_{C \in \mathcal{C}_X} S_Y(F, C) \longrightarrow C \\ &= F, \end{aligned}$$

then we have g is the left inverse of f^*.

The proof is completed. □

6 Conclusions

In this paper, we study the reflectivity of stratified L-algebraic closure spaces. In detail, the stratified Sierpinski L-algebraic closure space is considered, and a sobrification of stratified L-algebraic closure spaces is given. We prove that $\mathbf{S}L\text{-}\mathbf{AC_0}$ is epireflective in $\mathbf{S}L\text{-}\mathbf{AC}$, and $\mathbf{Sob}L\text{-}\mathbf{AC}$ is epireflective and $\mathcal{E}$-firm epireflective in $\mathbf{S}L\text{-}\mathbf{AC_0}$. However, what is the epireflective hull of stratified Sierpinski L-algebraic closure space in $\mathbf{S}$ $L\text{-}\mathbf{AC_0}$? We leave it for future study.

Following the methods in [21, 25], we can easily study the sober L-algebraic closure spaces and many results in [21, 25] are also valid in the frame of L-algebraic closure spaces. Why is this happening? We guess that one of the reasons is as follows: When proving the continuity of a map, we only need to check the preimage of subbase elements, and it is clear that the Sierpinski objects in the category of L-algebraic closure spaces and of L-cotopological spaces are exactly both generated by the same subbase $\{id_L\}$. In the future, we will focus on the relationship between L-algebraic closure spaces and L-cotopological spaces.

Acknowledgement

The authors wish to express their appreciation for several excellent suggestions for improvements in this paper made by the referees. This work is supported by National Natural Science Foundation of China (No. 12371467, 11971448), and the Fundamental Research Funds for the Central Universities (No. 202261098, 202264007).

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