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Original paper

Study of the generalized hypothetical syllogism for some well known families of fuzzy implications with respect to strict t-norm

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Abstract

The generalized hypothetical syllogism (GHS) is an important property of fuzzy implications for its applications in approximate reasoning. Due to the complexity of the (GHS) and the variety of fuzzy implications, in this work, we study the (GHS) property with respect to a strict t-norm T for fuzzy implications which come from some well known families of fuzzy implications, viz., (S, N)-, QL-, g-, (U, N)-, (T, N)-implications. First, some results on the (GHS) for fuzzy implications are presented. Second, the (GHS) property of (S, N)-, QL-, g-, (U, N)-, and (T, N)-implications is studied. Finally, the (GHS) property for the fuzzy implications generated from old ones using the method of sup-T composition is also studied.

Keywords: Generalized hypothetical syllogism, fuzzy implications, strict t-norm, sup-T composition.

1 Introduction

Fuzzy implications are the generalization of classical implications in classical logic to the framework of fuzzy logic [2]. They have been verified to be useful in the fields like fuzzy control [10, 32], approximate reasoning [22], computing with words [24], fuzzy morphological [11], image processing [7, 18], etc. Due to this great quantity of applications, fuzzy implications have been extensively studied by many authors (see for instance the works [1, 5, 8, 12, 26, 27, 33, 38, 43], and the book [2]).

Among these studies the investigation on some properties of fuzzy implications has attracted the interest of many researchers for their specific applications. Some basic properties of fuzzy implications such as the left neutrality property, the ordering property, the identity principle, laws of contrapositive symmetrization, exchange principle and so on, have been minutely investigated (see for instance the works [4, 13, 34, 37, 43], and the book [2]). Moreover, some additional properties of fuzzy implications like the distributivity [5, 21, 31, 36, 39, 40], the law of importation [17, 25, 28], the generalized modus ponens [23, 30, 35], the generalization of hypothetical syllogism [42], etc, were studied in many works.

It is noticeable that the property of (GHS) for fuzzy implications plays an important role in fuzzy inference. Thus, many authors pay attention to this property. The (GHS) stems from the tautology in the classical logic

$$(A \to B) \wedge (B \to C) \to (A \to C), \tag{HS}$$

where A, B, C are propositions and $\to$ denotes the classical implication.

In the context of fuzzy logic, the generalization of (HS) is as follows:

$$\sup_{\tau \in [0,1]} T(I(x, \tau), I(\tau, y)) = I(x, y). \tag{GHS}$$

where I is a fuzzy implication, T is a t-norm, and $x, y \in [0, 1]$. However, (GHS) does not always hold for a given fuzzy implication [16, 29].

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Due to the (GHS) of fuzzy implications play an important role in approximate reasoning such as memory savings as well as increase in the speed of computation, many researchers conducted a study on (GHS). Mizumoto et al. [29] investigated the (GHS) property with respect to the minimum t-norm T_M for some implication operators. Temme et al. [41] studied the (GHS) of fuzzy implication for non-singleton inputs. Igel et al. [16] analyzed the validity of (GHS) in fuzzy systems.

Considering the investigation of fuzzy implications satisfying (GHS) for a general t-norm T is not an easy task, recently, Vemuri [42] investigated the (GHS) with respect to the minimum t-norm T_M for the (S, N)-, R-, QL-, and Yager's implications. Baczyński et al. [3] studied the (GHS) with respect to arbitrarily t-norm T for R-implications. Helbin et al. [15] studied the (GHS) with respect to arbitrarily t-norm T for Yager's implications. Chen [9] et al. investigated the (GHS) with respect to the product t-norm T_P for the (S, N)-, R-, QL-, and Yager's implications.

Although many works have done on (GHS), there is no study on the (GHS) respect to a strict t-norm T for (S, N)-, QL-, (U, N)-, (T, N)-implications. Moreover, The results on (GHS) for Yager's g-implications are not perfect. Hence, from the viewpoint of theory and practical application, as a necessary supplement, it is worth to study the (GHS) property with respect to a strict t-norm T for fuzzy implications.

In this paper, one task is to investigate the property of (GHS) with respect to a strict t-norm T for the family of (S, N)-, QL-, g-, (U, N)-, and (T, N)-implications. The other task is to investigate the property (GHS) for the fuzzy implication generated from old ones using the methods of sup-T composition.

The remainder of the paper is organized as follows. In Section 2, some definitions and results used in the paper are given. In Section 3, some necessary conditions of (GHS) with respect to a strict t-norm T for fuzzy implications are investigated. In Section 4, the properties of (GHS) with respect to a strict t-norm T for the family of (S, N)-, QL- and Yager's g-implication are given. In Section 5, the properties of (GHS) with respect to a strict t-norm T for the family of (U, N)- and (T, N)-implications are investigated. In Section 6, the properties of (GHS) with respect to a strict t-norm T for fuzzy implications generated from old ones using the sup-T composition method are studied. Finally, conclusions and further problems for future work are given in Section 7.

2 Preliminaries

In this section, the definitions and results to be used in the rest of the paper are presented. First, let Φ denote the family of all increasing bijections from [0, 1] to [0, 1].

Definition 2.1. [20] *A function $J : [0,1]^2 \to [0,1]$ is ψ-conjugate with a function $I : [0,1]^2 \to [0,1]$, if there exists a $\psi \in \Phi$ such that $J = I_\psi$, where $I_\psi(x,y) = \psi^{-1}(I(\psi(x), \psi(y)))$, $x, y \in [0,1]$.*

Definition 2.2. [19, Definition 1.1]. *An associative, commutative and increasing function $T : [0,1]^2 \to [0,1]$ is called a t-norm if it has a neutral element 1, i.e., $T(x,1) = x$ for all $x \in [0,1]$.*

- The product t-norm is T_P defined by $T_P(x,y) = xy$, $x, y \in [0,1]$.
- The minimum t-norm is T_M defined by $T_M(x,y) = \min(x,y)$, $x, y \in [0,1]$.

Definition 2.3. [2, 19] *A t-norm T is called a strict t-norm, if it is continuous and strictly monotone, i.e., $T(x,y) < T(x,z)$ whenever $x > 0$ and $y < z$.*

Theorem 2.4. [2, 19] *A t-norm T is a strict t-norm if and only if T is ψ-conjugate with the product t-norm T_P, i.e., there exists $\psi \in \Phi$, which is uniquely determined up to a positive constant exponent, such that*

$$T(x,y) = (T_P)_\psi(x,y) = \psi^{-1}(\psi(x) \cdot \psi(y)), x, y \in [0,1].$$

If a t-norm T is a strict t-norm, then we denote it by $T_{(\psi)}$.

Definition 2.5. [19, Definition 1.1]. *An associative, commutative and increasing function $S : [0,1]^2 \to [0,1]$ is called a t-conorm if it has a neutral element 0, i.e., $S(x,0) = x$ for all $x \in [0,1]$.*

Definition 2.6. [2, Definition 1.1.1] *A function $I : [0,1]^2 \to [0,1]$ is called a fuzzy implication if it satisfies, for all x, x_1, x_2, y, y_1, $y_2 \in [0,1]$, the following conditions:*

if $x_1 < x_2$, then $I(x_1,y) \geq I(x_2,y)$, i.e., $I(\cdot,y)$ is decreasing, (I1)

if $y_1 < y_2$, then $I(x,y_1) \leq I(x,y_2)$, i.e., $I(x,\cdot)$ is increasing, (I2)

$I(0,0) = 1$, $I(1,1) = 1$, $I(1,0) = 0$. (I3)

The set of all fuzzy implications will be denoted by FI.

Definition 2.7. [2, Definition 1.3.1] *An operator* $I : [0,1]^2 \to [0,1]$ *is said to satisfy*
(i) *the left neutrality property if* $I(1,y) = y$*, for all* $y \in [0,1]$. (NP)
(ii) *the identity principle, if* $I(x,x) = 1$*, for all* $x \in [0,1]$. (IP)
(iii) *the ordering property, if* $I(x,y) = 1 \Leftrightarrow x \le y$*, for all* $x, y \in [0,1]$. (OP)

Definition 2.8. [2, Definition 1.4.1] *A function* $N : [0,1] \to [0,1]$ *is called a fuzzy negation if* $N(0) = 1$*,* $N(1) = 0$*, and* N *is decreasing.*

- The least fuzzy negation is N_{D_1} defined by $N_{D_1}(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x > 0, \end{cases}$
- The classical fuzzy negation is N_C defined by $N_C(x) = 1 - x, x \in [0,1]$.

Definition 2.9. [2, Definition 1.4.15] *Let* $I \in FI$*. The function* N_I *defined by* $N_I(x) := I(x,0),\ x \in [0,1]$*, is called the natural negation of* I*.*

- The natural negation of Goguen implication $I_{GG} = \begin{cases} 1, & \text{if } x \le y, \\ \frac{y}{x}, & \text{if } x > y, \end{cases}$ is N_{D_1}.

Definition 2.10. [2, Definition 6.4.1] *Let* $I,\ J \in FI$ *and* T *be a t-norm. Then sup-*T *composition of* I*,* J *is given by*

$$(I \overset{T}{\circ} J)(x,y) = \sup_{\tau \in [0,1]} T(I(x,\tau), J(\tau,y)),\ \ x, y \in [0,1]. \tag{1}$$

In this work, the left-hand of equation (1) is denoted as $I \overset{T}{\circ} J$.

Proposition 2.11. [2, Proposition 6.4.19] *Let* T *be a t-norm. Then for all* $I,\ J \in FI$ *and any* $\psi \in \Phi$*,* $(I \overset{T}{\circ} J)_\psi = I_\psi \overset{T_\psi}{\circ} J_\psi$*.*

Definition 2.12. [16, 29, 42] *A fuzzy implication* I *is said to satisfy the generalized hypothetical syllogism with respect to a t-norm* T *if* $I \overset{T}{\circ} I = I$*.* (GHS($T$))

Proposition 2.13. *Let* $I \in FI$*,* $\psi \in \Phi$*, and* T *be a t-norm. Then the following statements are equivalent:*
(i) I *satisfies (GHS(*T*)).*
(ii) I_ψ *satisfies (GHS(*T_ψ*)) for any* $\psi \in \Phi$*.*

Proof. (i $\Rightarrow$ ii) Let I satisfy (GHS(T)). Then $I \overset{T}{\circ} I = I$. Thus $I_\psi = I_\varphi \overset{T_\psi}{\circ} I_\psi$ by Proposition 2.11. Therefore, I_ψ satisfies (GHS(T_ψ)).

(ii $\Rightarrow$ i) Let I_ψ satisfy (GHS(T_ψ)). Then $I_\psi \overset{T_\psi}{\circ} I_\psi = I_\psi$. Since $\psi \in \Phi$, then $\psi^{-1} \in \Phi$. Therefore, by Proposition 2.11, we get

$$(I_\psi \overset{T_\psi}{\circ} I_\psi)_{\psi^{-1}} = (I_\psi)_{\psi^{-1}} \overset{(T_\psi)_{\psi^{-1}}}{\circ} (I_\psi)_{\psi^{-1}}. \tag{2}$$

From (2) we get $(I_\psi)_{\psi^{-1}} = (I_\psi)_{\psi^{-1}} \overset{(T_\psi)_{\psi^{-1}}}{\circ} (I_\psi)_{\psi^{-1}}$, i.e. $I = I \overset{T}{\circ} I$. Hence I satisfies (GHS(T)). □

Corollary 2.14. *Let* $I \in FI$*,* $\psi \in \Phi$*, and* $T_{(\psi)}$ *be a strict t-norm. Then the following statements are equivalent:*
(i) I *satisfies (GHS(*$T_{(\psi)}$*)).*
(ii) $I_{\psi^{-1}}$ *satisfies (GHS(*T_P*)).*

Proof. Obvious from Proposition 2.13. □

Theorem 2.15. [16, Theorem 6] *Let* I *be a fuzzy implication that satisfies (IP), and let* T *be a t-norm. If the pair* (T, I) *satisfies*

$$T(I(x,z), I(z,y)) \le I(x,y),\ \ for\ \ all\ \ x,\ y,\ z \in [0,1],$$

then I *satisfies (GHS(*T*)).*

Definition 2.16. [2, Definition 3.1.1] *Let* $f : [0,1] \to [0,\infty]$ *be a strictly decreasing and continuous function with* $f(1) = 0$*. An* f*-implication is a function* $I : [0,1]^2 \to [0,1]$ *defined as*

$$I(x,y) = f^{-1}(x \cdot f(y)),\ \ x, y \in [0,1],$$

with the understanding $0 \cdot \infty = 0$*. In such a case, we write* I_f *instead of* I*.*

Theorem 2.17. [15] *No* f*-implication satisfies (GHS(*T*)).*

3 The necessary conditions of (GHS($T_{(\psi)}$)) for fuzzy implications

Lemma 3.1. *Let I be a fuzzy implication and $T_{(\psi)}$ a strict t-norm. Then the following statements are equivalent:*
(i) *I satisfies (GHS($T_{(\psi)}$)).*
(ii) *$\psi(I)$ satisfies (GHS(T_P)).*

Proof. Let $x,\, y \in [0,1]$. Then I satisfies (GHS(T_ψ))

$$\begin{aligned}&\Leftrightarrow \sup_{\tau\in[0,1]} T_{(\psi)}(I(x,\tau), I(\tau,y)) = I(x,y)\\ &\Leftrightarrow \sup_{\tau\in[0,1]} \psi^{-1}\Big(\psi(I(x,\tau))\cdot\psi(I(\tau,y))\Big) = I(x,y)\\ &\Leftrightarrow \sup_{\tau\in[0,1]} \Big(\psi(I(x,\tau))\cdot\psi(I(\tau,y))\Big) = \psi(I(x,y))\\ &\Leftrightarrow \sup_{\tau\in[0,1]} T_P\Big(\psi(I(x,\tau)),\psi(I(\tau,y))\Big) = \psi(I(x,y))\\ &\Leftrightarrow \psi(I) \text{ satisfies } (\text{GHS}(T_P)).\end{aligned}$$

□

Proposition 3.2. *Let $I \in FI$ satisfy $I(1,x) > 0$ for all $x \in (0,1]$. If I satisfies (GHS) with respect to a strict t-norm $T_{(\psi)}$, then $N_I = N_{D_1}$.*

Proof. Let I satisfy (GHS) with respect to a strict t-norm $T_{(\psi)}$, then $\psi(I)$ satisfies (GHS(T_P)) by Lemma 3.1. Hence, for all $x,\, y \in [0,1]$, we have

$$\sup_{\tau\in[0,1]} \Big(\psi(I(x,\tau))\cdot\psi(I(\tau,y))\Big) = \psi(I(x,y)).$$

Let $x = 1$, $y = 0$. Then $\psi\big(I(1,\tau)\big)\cdot\psi\big(N_I(\tau)\big) = 0$ for all $\tau \in [0,1]$. Since $I(1,x) > 0$ for $x \in (0,1]$, then $\psi(I(x,\tau)) > 0$ for $\tau \in (0,1]$. Therefore, $\psi\big(N_I(\tau)\big) = 0$ for $\tau \in (0,1]$. Hence $N_I(\tau) = 0$ for $\tau \in (0,1]$, i.e., $N_I = N_{D_1}$. □

Corollary 3.3. *Let $I \in FI$ satisfy (NP). If I satisfies (GHS) with respect to a strict t-norm $T_{(\psi)}$, then $N_I = N_{D_1}$.*

Proposition 3.4. *Let $I \in FI$ satisfy (NP). If I satisfies (GHS) with respect to a strict t-norm $T_{(\psi)}$, then $I(x,y) \le \psi^{-1}\left(\frac{\psi(y)}{\psi(x)}\right)$ for all $x,\, y \in [0,1]$ with $x > y$.*

Proof. Assume that I satisfies (GHS) with respect to a strict t-norm $T_{(\psi)}$. Then, for $x = 1$, $y \in [0,1]$, we have

$$\sup_{\tau\in[0,1]} \big(\psi(I(1,\tau))\cdot\psi(I(\tau,y))\big) = \psi(I(1,y)).$$

Since I satisfies (NP), then $\sup_{\tau\in[0,1]} \big(\psi(\tau)\cdot\psi(I(\tau,y))\big) = \psi(y)$. Hence, $\psi(\tau)\cdot\psi(I(\tau,y)) \le \psi(y)$ for all $\tau \in [0,1]$. Thus, $I(x,y) \le \psi^{-1}\left(\frac{\psi(y)}{\psi(x)}\right)$ for all $x,\, y \in [0,1]$ with $x > y$. □

Remark 3.5. *The converse of Proposition 3.4 may not be true. To see this, consider the strict t-norm T_p, and the following fuzzy implication*

$$I(x,y) = \begin{cases} 1 - x + xy, & \text{if } x \le y,\\ y, & \text{if } x > y.\end{cases}$$

Obvious, I satisfies (NP) and $I(x,y) \le \frac{y}{x}$ for $x > y$. However, by calculations, we get $I(0.5,0.5) = 0.75$, and

$$\sup_{\tau\in[0,1]} (I(0.5,\tau)\cdot I(\tau,0.5)) = \max\left\{\begin{array}{c}\sup_{\tau\in[0,0.5)} (\tau\cdot I(\tau,0.5)),\\ I(0.5,0.5)\cdot I(0.5,0.5),\\ \sup_{\tau\in(0.5,1]} (I(0.5,\tau)\cdot 0.5)\end{array}\right\} \le \max(0.5\times 0.75, 0.75^2, 0.5) < 0.75.$$

This shows that I does not satisfy (GHS(T_P)).

4 The properties of (GHS($T_{(\psi)}$)) for the (S,N)-, QL-, Yager's g-implications

4.1 (GHS($T_{(\psi)}$)) of (S,N)-implications

Definition 4.1. [2, Definition 2.4.1] *An (S,N)-implication is a function $I : [0,1]^2 \to [0,1]$ defined as $I(x,y) = S(N(x),y)$, where S is a t-conorm, and N is a fuzzy negation. An (S,N)-implication I generated from a t-conorm S and a fuzzy negation N is denoted by $I_{S,N}$.*

Theorem 4.2. *An (S,N)-implication $I_{S,N}$ satisfies (GHS($T_{(\psi)}$)) if and only if $I_{S,N} = I_D$, where* $I_D(x,y) = \begin{cases} 1, & \text{if } x = 0, \\ y, & \text{if } x > 0. \end{cases}$

Proof. (Necessity) Let $I_{S,N}$ satisfy (GHS($T_{(\psi)}$)). Then $N_{I_{S,N}} = N_{D_1} = N$ by Corollary 3.3, since $I_{S,N}$ satisfies (NP). Hence

$$I_{S,N}(x,y) = S(N(x),y) = \begin{cases} 1, & \text{if } x = 0, \\ y, & \text{if } x > 0. \end{cases}.$$

Hence, $I_{S,N} = I_D$.

(Sufficiency) Obvious from calculations. □

4.2 (GHS($T_{(\psi)}$)) of QL-implications and QL-operations

Definition 4.3. [2, Definition 2.6.1] *A QL-operation is a function $I : [0,1]^2 \to [0,1]$ defined as*

$$I(x,y) = S(N(x), T(x,y)), \;\; x,y \in [0,1],$$

where S is a t-conorm, T is a t-norm, and N is a fuzzy negation. If I is a QL-operation generated from the triple (T,S,N), then we will often denote it by $I_{T,S,N}$. If $I_{T,S,N}$ is a fuzzy implication, then it is called QL-implication.

Theorem 4.4. *No QL-implication satisfies (GHS($T_{(\psi)}$)).*

Proof. Assume that a QL-implication $I_{T,S,N}$ satisfies (GHS($T_{(\psi)}$)). Then $N = N_{D_1}$ by Corollary 3.3, since $I_{T,S,N}$ satisfies (NP) and $N_{I_{T,S,N}} = N$ ([2], Proposition 2.6.2). Hence,

$$I_{T,S,N}(x,y) = \begin{cases} 1, & \text{if } x = 0, \\ T(x,y), & \text{if } x > 0. \end{cases}$$

However, $I_{T,S,N}(x,y)$ increasing with respect to x. This contradicts to that $I_{T,S,N}$ is a fuzzy implication. □

For a QL-operator, we have the following result.

Theorem 4.5. *Let $I_{T,S,N}$ be a QL-operation generated from the triple (T,S,N). Then $I_{T,S,N}$ satisfies (GHS($T_{(\psi)}$)) if and only if $N = N_{D_1}$ and $T = T_{(\psi)}$.*

Proof. (Necessity) Let $I_{T,S,N}$ satisfy (GHS($T_{(\psi)}$)). Then $N = N_{D_1}$ by Corollary 3.3, since $I_{T,S,N}$ satisfies (NP) and $N_{I_{T,S,N}} = N$ ([2, Proposition 2.6.2]). Hence,

$$I_{T,S,N}(x,y) = \begin{cases} 1, & \text{if } x = 0, \\ T(x,y), & \text{if } x > 0. \end{cases} \tag{3}$$

By Lemma 3.1, from $I_{T,S,N}$ satisfying (GHS($T_{(\psi)}$)) we get,

$$\sup_{\tau\in[0,1]} \big(\psi(I_{T,S,N}(x,\tau)) \cdot \psi(I_{T,S,N}(\tau,y))\big) = \psi(I_{T,S,N}(x,y)). \tag{4}$$

From (3) and (4) we get $\sup\limits_{\tau\in[0,1]} \big(\psi(T(x,\tau)) \cdot \psi(T(\tau,y))\big) = \psi(T(x,y)),$ for all $x \in (0,1]$. Since

$$\sup_{\tau\in[0,1]} \big(\psi(T(x,\tau)) \cdot \psi(T(\tau,y))\big) = \psi(T(x,1)) \cdot \psi(T(1,y)) = \psi(x) \cdot \psi(y),$$

then $T(x,y) = \psi^{-1}(\psi(x) \cdot \psi(y))$, i.e., $T = T_{(\psi)}$.

(Sufficiency) Obvious from calculations. □

4.3 (GHS($T_{(\psi)}$)) of Yager's g-implications

Definition 4.6. [2, Definition 3.2.1] *Let $g : [0,1] \to [0,\infty]$ be a strictly increasing and continuous function with $g(0) = 0$. The function $I : [0,1]^2 \to [0,1]$ defined by*

$$I(x,y) = g^{-1}\left(\min\left(\frac{1}{x} \cdot g(y), g(1)\right)\right), \ \ x, y \in [0,1],$$

with the understanding $\frac{1}{0} = \infty$ and $\infty \cdot 0 = \infty$, is called a Yager's g-implication, where g is called a g-generator. If I is a g-implication, then it is denoted by I_g.

Theorem 4.7. *Let $T_{(\psi)}$ be a strict t-norm, and I_g be a Yager's g-implication generated from a g-generator g. Then the following statements are equivalent:*
(i) *I_g satisfies (GHS($T_{(\psi)}$)).*
(ii) *$I_g = I_{GG}$, and the ψ satisfies $\psi(x) \cdot \psi(y) \leq \psi(xy)$, x, $y \in [0,1]$.*

Proof. (i $\Rightarrow$ ii) Let I_g satisfy (GHS($T_{(\psi)}$)). By Proposition 4.4.1 in [2] and Theorem 2.17 we get $g(1) < \infty$. Then the following function

$$g_1(x) = \frac{g(x)}{g(1)}, \ \ x \in [0,1],$$

is a g-generator of I_g ([2], Theorem 3.2.5), i.e., $I_g = I_{g_1}$.

Now, (GHS($T_{(\psi)}$)) implies that, for every $x \in (0,1)$, we have

$$\sup_{\tau \in [0,1]} \big(\psi(I_{g_1}(x,\tau)) \cdot \psi(I_{g_1}(\tau,x))\big) = \psi(I_{g_1}(x,x)). \tag{5}$$

From Theorem 3.2.8 (vi) in [2] we know that $\psi(I_{g_1}(x,\tau)) \cdot \psi(I_{g_1}(\tau,x))$ is continuous with respect to τ. Then equation (5) can be written as

$$\max_{\tau \in [0,1]} \big(\psi(I_{g_1}(x,\tau)) \cdot \psi(I_{g_1}(\tau,x))\big) = \psi(I_{g_1}(x,x)). \tag{6}$$

In the following, we prove that $I_{g_1}(x,x) = 1$ for all $x \in (0,1)$.

Suppose that there exists an $x_0 \in (0,1)$ such that $I_{g_1}(x_0,x_0) < 1$. Obvious, $I_{g_1}(x_0,x_0) \neq 0$. Then from (6) we get

$$0 < \max_{\tau \in [0,1]} \big(\psi(I_{g_1}(x_0,\tau)) \cdot \psi(I_{g_1}(\tau,x_0))\big) = \psi(I_{g_1}(x_0,x_0)) < \psi(1) = 1.$$

Since $\psi(I_{g_1}(x_0,\tau)) \cdot \psi(I_{g_1}(\tau,x_0))$ is continuous with respect to $\tau \in [0,1]$, then there exists a $\tau_0 \in [0,1]$ such that

$$0 < \psi(I_{g_1}(x_0,x_0)) = \psi(I_{g_1}(x_0,\tau_0)) \cdot \psi(I_{g_1}(\tau_0,x_0)) < 1. \tag{7}$$

Obvious, $\tau_0 \neq x_0$. Moveover, $\tau_0 \neq 0$ and $\tau_0 \neq 1$. In fact, if $\tau_0 = 0$, we get

$$\psi(I_{g_1}(x_0,0)) \cdot \psi(I_{g_1}(0,x_0)) = \psi(0) \cdot \psi(1) = 0 < \psi(I_{g_1}(x_0,x_0)),$$

a contradiction. If $\tau_0 = 1$, from $I_{g_1}(x_0,x_0) < 1$ we get $g_1^{-1}\left(\min\left(\frac{g_1(x_0)}{x_0}, 1\right)\right) < 1$. Hence, $\frac{g_1(x_0)}{x_0} < 1$. Thus

$$\begin{aligned}\psi(I_{g_1}(x_0,1)) \cdot \psi(I_{g_1}(1,x_0)) = \psi(I_{g_1}(1,x_0)) = \psi\left(g_1^{-1}\left(\min\left(\frac{g_1(x_0)}{1}, 1\right)\right)\right) &< \psi\left(g_1^{-1}\left(\min\left(\frac{g_1(x_0)}{x_0}, 1\right)\right)\right) \\ &= \psi\big(I_{g_1}(x_0,x_0)\big).\end{aligned}$$

A contradiction. Therefore, $\tau_0 \in (0,x_0) \cup (x_0,1)$. On the one hand, if $\tau_0 \in (0,x_0)$, then $g_1(\tau_0) < g_1(x_0)$. Thus,

$$\frac{g_1(\tau_0)}{x_0} < \frac{g_1(x_0)}{x_0} < 1.$$

Hence, we have

$$\begin{aligned}\psi(I_{g_1}(x_0,\tau_0)) \cdot \psi(I_{g_1}(\tau_0,x_0)) &\leq \psi(I_{g_1}(x_0,\tau_0)) \quad [\text{since } I_{g_1}(\tau_0,x_0) \leq 1] \\ &= \psi\left(g_1^{-1}\Big(\min\big(\frac{g_1(\tau_0)}{x_0}, 1\big)\Big)\right) \\ &< \psi\left(g_1^{-1}\Big(\min\big(\frac{g_1(x_0)}{x_0}, 1\big)\Big)\right) \\ &= \psi\big(I_{g_1}(x_0,x_0)\big).\end{aligned}$$

This contradicts to (7).

Similarly, on the other hand, if $\tau_0 \in (x_0, 1)$, then $\frac{g_1(x_0)}{\tau_0} < \frac{g_1(x_0)}{x_0} < 1$. Hence, we have

$$\psi(I_{g_1}(x_0,\tau_0)) \cdot \psi(I_{g_1}(\tau_0,x_0)) \le \psi(I_{g_1}(\tau_0,x_0)) = \psi\left(g_1^{-1}\Big(\min\big(\frac{g_1(x_0)}{\tau_0},1\big)\Big)\right) < \psi\left(g_1^{-1}\Big(\min\big(\frac{g_1(x_0)}{x_0},1\big)\Big)\right) = \psi\left(I_{g_1}(x_0,x_0)\right).$$

This contradicts to (7).

From above discussion, we get that $I_{g_1}(x,x) = 1$ for all $x \in (0,1)$. Therefore, $I_{g_1}(x,y) = 1$ for all $x, y \in (0,1)$ with $x \le y$. That is, implication I_{g_1} satisfies (IP). Also, by Proposition 3.4 we have

$$I_{g_1}(x,y) \le \psi^{-1}\left(\frac{\psi(y)}{\psi(x)}\right) < 1 \text{ for all } x,y \in [0,1] \text{ with } x > y.$$

Hence, I_{g_1} satisfies (OP), i.e., I_g satisfies (OP). Thus, by Theorem 3.2.9 in [2], we obtain that $g(x) = cx$, $x \in [0,1]$, $c > 0$, i.e., $I_g = I_{GG}$.

Next, we prove that ψ satisfies $\psi(x)\cdot\psi(y) \le \psi(xy)$, x, $y \in [0,1]$.

It suffices to prove that ψ satisfies $\psi(x)\cdot\psi(y) \le \psi(xy)$, x, $y \in (0,1)$. From I_g satisfying (GHS($T_{(\psi)}$)), for x, $y \in [0,1]$ with $x > y$, we get $\sup_{\tau\in[0,1]} \big(\psi(I_g(x,\tau))\cdot\psi(I_g(\tau,y))\big) = \psi(I_g(x,y))$. Hence,

$$\max\Big\{\sup_{\tau\le y}\big(\psi(I_g(x,\tau))\cdot\psi(I_g(\tau,y))\big), \sup_{y<\tau<x}\big(\psi(I_g(x,\tau))\cdot\psi(I_g(\tau,y))\big), \sup_{\tau\le x}\big(\psi(I_g(x,\tau))\cdot\psi(I_g(\tau,y))\big)\Big\} = \psi\left(\frac{y}{x}\right),$$

i.e.,

$$\max\left\{\sup_{\tau\le y}\psi\left(\frac{\tau}{x}\right), \sup_{y<\tau<x}\left(\psi(\frac{\tau}{x})\cdot\psi(\frac{y}{\tau})\right), \sup_{\tau\ge x}\psi\left(\frac{y}{\tau}\right)\right\} = \psi\left(\frac{y}{x}\right),$$

i.e.,

$$\max\left\{\psi\left(\frac{y}{x}\right), \sup_{y<\tau<x}\left(\psi(\frac{\tau}{x})\cdot\psi(\frac{y}{\tau})\right), \psi\left(\frac{y}{x}\right)\right\} = \psi\left(\frac{y}{x}\right).$$

Thus, $\sup_{y<\tau<x}\left(\psi(\frac{\tau}{x})\cdot\psi(\frac{y}{\tau})\right) \le \psi\left(\frac{y}{x}\right)$. Therefore, $\psi\left(\frac{\tau}{x}\right)\cdot\psi\left(\frac{y}{\tau}\right) \le \psi\left(\frac{y}{x}\right)$ for $\tau \in (y,x)$. Hence, $\psi(u)\cdot\psi(v) \le \psi(uv)$ for all $u, v \in (0,1)$. Indeed, let us assume that $\psi(u_0)\cdot\psi(v_0) > \psi(u_0v_0)$ for some $u_0, v_0 \in (0,1)$. Taking a $\tau_0 \in (0,1)$. Then there exist x_0, $y_0 \in (0,1)$ such that $v_0 = \frac{y_0}{\tau_0}$, and $u_0 = \frac{\tau_0}{x_0}$. Since $u_0, v_0 \in (0,1)$, then $x_0 > \tau_0 > y_0$. Therefore,

$$\psi(\frac{\tau_0}{x_0})\cdot\psi(\frac{y_0}{\tau_0}) > \psi(\frac{y_0}{x_0}).$$

This contradicts the fact that $\psi\left(\frac{\tau}{x}\right)\cdot\psi\left(\frac{y}{\tau}\right) \le \psi\left(\frac{y}{x}\right)$ for $\tau \in (y,x)$, $x, y \in (0,1)$.

(ii ⇒ i) Obvious from calculations. □

Remark 4.8. *For a strict t-norm $T_{(\psi)}$, if there exists some $x_0, y_0 \in (0,1)$ such that $\psi(x_0)\cdot\psi(y_0) > \psi(x_0\cdot y_0)$, then there is no Yager's g-implication satisfies (GHS($T_{(\psi)}$)).*

5 The properties of GHS($T_{(\psi)}$)) for (U,N)- and (T,N)-implications

5.1 (GHS($T_{(\psi)}$)) of (U,N)-implications

Definition 5.1. [2, Definition 5.1.1] *A function $U : [0,1]^2 \to [0,1]$ is called a uninorm if it is an associative, commutative and increasing operation, and has a neutral element $e \in [0,1]$, i.e., $U(e,x) = x$, for all $x \in [0,1]$.*

Definition 5.2. [2, Remark 5.1.3(iv)] *A uninorm U is disjunctive, if $U(0,1) = 1$.*

Definition 5.3. [2, Theorem 5.1.4] *Let T be a t-norm, S a t-conorm and $e \in (0,1)$. A uninorm U is called a $U_{\max}$ uninorm if it is defined by*

$$U(x,y) = \begin{cases} e\cdot T(\frac{x}{e},\frac{y}{e}), & \text{if } x,y \in [0,e],\\ e + (1-e)\cdot S(\frac{x-e}{1-e},\frac{y-e}{1-e}), & \text{if } x,y \in [e,1],\\ \max(x,y), & \text{otherwise.}\end{cases}$$

Definition 5.4. [2, Definition 5.3.1] *A (U, N)-implication is a function $I : [0,1]^2 \to [0,1]$ defined as*

$$I(x, y) = U(N(x), y), \ \ x, y \in [0,1],$$

where U is a disjunctive uninorm, and N is a fuzzy negation. If I is a (U, N)-implication, then it is denoted by $I_{U,N}$.

Proposition 5.5. *Let $I_{U,N}$ be a (U, N)-implication that satisfies (GHS($T_{(\psi)}$)), where U is a disjunctive uninorm with a neutral element $e \in (0,1)$, and N is a fuzzy negation. Then $\psi(U(0,x)) \cdot \psi(N(x)) = 0$ for all $x \in [0,1]$.*

Proof. Let $I_{U,N}$ satisfy (GHS($T_{(\psi)}$)). Then $\sup\limits_{x\in[0,1]} \Big(\psi(I_{U,N}(1,x)) \cdot \psi(I_{U,N}(x,e))\Big) = \psi(I_{U,N}(1,e))$. Hence,

$$\sup_{x\in[0,1]} \Big(\psi(U(0,x)) \cdot \psi(U(N(x),e))\Big) = \psi(U(0,e)).$$

Therefore, $\psi(U(0,x)) \cdot \psi(N(x)) = 0$ for all $x \in [0,1]$. □

Proposition 5.6. *Let $I_{U,N}$ be a (U, N)-implication that satisfies (GHS($T_{(\psi)}$)), where U is a disjunctive uninorm with a neutral element $e \in (0,1)$, and N is a fuzzy negation. If $U = U_{\max}$, then*
(i) *$N(x) = 0$ for all $x > e$.*
(ii) *$N(x) > e$ or $N(x) = 0$ for $x \in (0, e]$.*

Proof. (i) Assume that $U = U_{\max}$, then $U(0,x) = \max(0,x) = x$ for $x \in (e,1]$. Let $I_{U,N}$ satisfy (GHS($T_{(\psi)}$)). Then from Proposition 5.5 we get $\psi(x) \cdot \psi(N(x)) = 0$ for all $x \in (e,1]$. Hence $N(x) = 0$ for all $x \in (e,1]$.

(ii) *To prove (ii), we first prove that there is no $x \in [0,e]$ such that $N(x) = e$.*

Assume that there exists an $x_0 \in [0,e]$ such that $N(x_0) = e$. Obvious, $x_0 \in (0,e]$.

Since $I_{U,N}$ satisfies (GHS($T_{(\psi)}$)), then, for x, $y \in [0,1]$, we get $\sup\limits_{\tau\in[0,1]} \Big(\psi(I_{U,N}(x,\tau)) \cdot \psi(I_{U,N}(\tau,y))\Big) = \psi(I_{U,N}(x,y))$, i.e.,

$$\sup_{\tau\in[0,1]} \Big(\psi(U(N(x),\tau)) \cdot \psi(U(N(\tau),y))\Big) = \psi(U(N(x),y)). \tag{8}$$

Take $x = x_0$ and $y = e$ in equation (8), we get $\sup\limits_{\tau\in[0,1]} \big(\psi(\tau) \cdot \psi(N(\tau))\big) = \psi(e)$. Hence,

$$\max\left\{\sup_{\tau\in[0,e]} \Big(\psi(\tau) \cdot \psi(N(\tau))\Big), \sup_{\tau\in(e,1]} \Big(\psi(\tau) \cdot \psi(N(\tau))\Big)\right\} = \psi(e).$$

By (i), we have

$$\sup_{\tau\in[0,e]} \Big(\psi(\tau) \cdot \psi(N(\tau))\Big) = \psi(e). \tag{9}$$

Therefore,

$$\max\left\{\sup_{\tau\in[0,x_0]} \Big(\psi(\tau) \cdot \psi(N(\tau))\Big), \sup_{\tau\in(x_0,e]} \Big(\psi(\tau) \cdot \psi(N(\tau))\Big)\right\} = \psi(e). \tag{10}$$

However, for the case of $x_0 < e$, since $\sup\limits_{\tau\in[0,x_0]} \Big(\psi(\tau) \cdot \psi(N(\tau))\Big) \leq \sup\limits_{\tau\in[0,x_0]} \psi(\tau) = \psi(x_0) < \psi(e)$, and

$$\sup_{\tau\in(x_0,e]} \Big(\psi(\tau) \cdot \psi(N(\tau))\Big) \leq \sup_{\tau\in(x_0,e]} \Big(\psi(e) \cdot \psi(N(\tau))\Big) = \psi(e) \cdot \psi(N(x_0)) = \psi(e)^2 < \psi(e).$$

Then

$$\max\left\{\sup_{\tau\in[0,x_0]} \Big(\psi(\tau) \cdot \psi(N(\tau))\Big), \sup_{\tau\in(x_0,e]} \Big(\psi(\tau) \cdot \psi(N(\tau))\Big)\right\} < \psi(e).$$

This contradicts equation (10).

For the case of $x_0 = e$, since $\psi(x_0) \cdot \psi(N(x_0)) = \psi(e) \cdot \psi(N(e)) = \psi(e)^2 < \psi(e)$. Then from (9) we get

$$\sup_{\tau\in[0,e)} \Big(\psi(\tau) \cdot \psi(N(\tau))\Big) = \psi(e).$$

Thus $N(x) = 1$ for $x \in [0, e)$. In fact, assume that there exists a $\tau_0 \in [0, e)$ such that $N(\tau_0) < 1$. Obviously $\tau_0 \neq 0$, i.e., $\tau_0 \in (0, e)$. Then

$$\max\left\{\sup_{\tau\in[0,\tau_0]} \Big(\psi(\tau) \cdot \psi(N(\tau))\Big), \sup_{\tau\in(\tau_0,e)} \Big(\psi(\tau) \cdot \psi(N(\tau))\Big)\right\} = \psi(e). \tag{11}$$

Since $\sup_{\tau\in[0,\tau_0)} \Big(\psi(\tau) \cdot \psi(N(\tau))\Big) \leq \sup_{\tau\in[0,\tau_0)} \psi(\tau) = \psi(\tau_0) < \psi(e)$, and $\sup_{\tau\in(\tau_0,e)} \Big(\psi(\tau) \cdot \psi(N(\tau))\Big) \leq \psi(e) \cdot \psi(N(\tau_0)) < \psi(e)$. This leads to a contradiction. Hence $N(x) = 1$ for $x \in [0, e)$.

Take $x = x_0 = e$ and $y < e$ in equation (8). Then we get $\sup_{\tau\in[0,1]} \Big(\psi(\tau) \cdot \psi(U(N(\tau), y))\Big) = \psi(y)$, i.e.,

$$\max\left\{\sup_{\tau\in[0,e]} \Big(\psi(\tau) \cdot \psi(U(N(\tau), y))\Big), \sup_{\tau\in(e,1]} \Big(\psi(\tau) \cdot \psi(U(N(\tau), y))\Big)\right\} = \psi(y).$$

By (i) we have

$$\max\left\{\sup_{\tau\in[0,e]} \Big(\psi(\tau) \cdot \psi(U(N(\tau), y))\Big), \sup_{\tau\in(e,1]} \Big(\psi(\tau) \cdot \psi(U(0, y))\Big)\right\} = \psi(y). \tag{12}$$

Since $\sup_{\tau\in(e,1]} (\psi(\tau)\cdot\psi(U(0, y))) = \sup_{\tau\in(e,1]} (\psi(\tau)\cdot\psi(0)) = 0$ for $y < e$, then from (12) we get $\sup_{\tau\in[0,e]} \Big(\psi(\tau)\cdot\psi(U(N(\tau), y))\Big) = \psi(y)$, i.e., $\max\Big\{\sup_{\tau\in[0,e)} (\psi(\tau) \cdot \psi(U(1, y))), \psi(e)\psi(y)\Big\} = \psi(y)$. However, since $\sup_{\tau\in[0,e)} (\psi(\tau) \cdot \psi(U(1, y))) = \sup_{\tau\in[0,e)} \psi(\tau) = \psi(e) > \psi(y)$, and $\psi(e) \cdot \psi(y) \neq \psi(y)$, a contradiction. That is, $N(e) \neq e$.

Second, we prove that there is no $x \in (0, e]$ *such that* $0 < N(x) < e$.

Assume that there exists an $x_1 \in (0, e]$ such that $0 < N(x_1) < e$.

Take $x = x_1$ and $y = 0$ in equation (8). Then $\sup_{\tau\in[0,1]} \Big(\psi\big(U(N(x_1), \tau)\big) \cdot \psi\big(U(N(\tau), 0)\big)\Big) = \psi\big(U(N(x_1), 0)\big)$. Hence,

$$\max\left\{\sup_{\tau\in[0,e]} \Big(\psi(U(N(x_1), \tau)) \cdot \psi(U(N(\tau), 0))\Big), \sup_{\tau\in(e,1]} \Big(\psi(U(N(x_1), \tau)) \cdot \psi(U(N(\tau), 0))\Big)\right\} = 0.$$

By (i) we have $\sup_{\tau\in[0,e]} \Big(\psi(U(N(x_1), \tau))\cdot\psi(U(N(\tau), 0))\Big) = 0$. Hence, $\sup_{\tau\in[0,e],\, N(\tau)>e} \Big(\psi(U(N(x_1), \tau))\cdot\psi(U(N(\tau), 0))\Big) = 0$, i.e., $\sup_{\tau\in[0,e],\, N(\tau)>e} \Big(\psi(U(N(x_1), \tau)) \cdot \psi(N(\tau))\Big) = 0$, i.e.,

$$\sup_{\tau\in[0,e],\, N(\tau)>e} \psi(U(N(x_1), \tau)) = 0. \tag{13}$$

On the other hand, take $x = x_1$ and $y = e$ in equation (8), we get $\sup_{\tau\in[0,1]} \Big(\psi(U(N(x_1), \tau)) \cdot \psi(U(N(\tau), e))\Big) = \psi(U(N(x_1), e))$. Hence, $\sup_{\tau\in[0,1]} \Big(\psi(U(N(x_1), \tau)) \cdot \psi(N(\tau))\Big) = \psi(N(x_1))$. Thus,

$$\max\left\{\sup_{\tau\in[0,e]} \Big(\psi(U(N(x_1), \tau)) \cdot \psi(N(\tau))\Big), \sup_{\tau\in(e,1]} \Big(\psi(U(N(x_1), \tau)) \cdot \psi(N(\tau))\Big)\right\} = \psi(N(x_1)).$$

By (i) we have $\sup_{\tau\in[0,e]} \Big(\psi(U(N(x_1), \tau)) \cdot \psi(N(\tau))\Big) = \psi(N(x_1))$. By (13), we have

$$\sup_{\tau\in[0,e],\, N(\tau)\leq e} \Big(\psi(U(N(x_1), \tau)) \cdot \psi(N(\tau))\Big) = \psi(N(x_1)).$$

However, since $\sup_{\tau\in[0,e],\, N(\tau)\leq e} \Big(\psi(U(N(x_1), \tau))\cdot\psi(N(\tau))\Big) \leq \psi(N(x_1))\cdot\psi(e) < \psi(N(x_1))$. This leads a contradiction. □

Theorem 5.7. *Let $I_{U,N}$ be a (U, N)-implication, and let $U = U_{\max}$. Then $I_{U,N}$ satisfies (GHS($T_{(\psi)}$)) if and only if there exists a connected set $K \subset [0,1]$, such that the following conditions hold:*
(a) *$N(x) = 0$ for $x \in K$, $N(x) > e$ for $x \in [0,1]\backslash K$, and $(e,1] \subseteq K$.*
(b) *$I_{U,N}(x,y) = \max(N(x), y)$ for $x \in [0,1]\backslash K$ and $y > e$.*

Proof. (Necessity). Let $I_{U,N}$ satisfy (GHS($T_{(\psi)}$)). From Proposition 5.6, it is easy to see that (a) holds. Also, (GHS($T_{(\psi)}$)) shows that, for all x, $y \in [0,1]$, $\sup_{\tau\in[0,1]}\Big(\psi(U(N(x),\tau))\cdot\psi(U(N(\tau),y))\Big) = \psi(U(N(x),y))$, i.e.,

$$\max\left\{\sup_{\tau\in[0,1]\backslash K}\Big(\psi(U(N(x),\tau))\cdot\psi(U(N(\tau),y))\Big), \sup_{\tau\in K}\Big(\psi(U(N(x),\tau))\cdot\psi(U(N(\tau),y))\Big)\right\} = \psi(U(N(x),y)).$$

Taking $x \in [0,1]\backslash K$ and $y > e$, we get $N(x) > e$, since $\tau \in [0,1]\backslash K \Rightarrow \tau \le e$, then

$$\max\left\{\sup_{\tau\in[0,1]\backslash K}\Big(\psi(N(x))\cdot\psi(U(N(\tau),y))\Big), \sup_{\tau\in K}\Big(\psi(U(N(x),\tau))\cdot\psi(U(0,y))\Big)\right\} = \psi(U(N(x),y)).$$

Hence,

$$\max\left\{\psi(N(x))\cdot\sup_{\tau\in[0,1]\backslash K}\psi(U(N(\tau),y)), \psi(U(0,y))\cdot\sup_{\tau\in K}\psi(U(N(x),\tau))\right\} = \psi(U(N(x),y)). \tag{14}$$

Notice the fact that $0 \notin K$, and $1 \in K$. Then

$$\sup_{\tau\in[0,1]\backslash K}\psi(U(N(\tau),y)) = \psi(U(N(0),y)) = \psi(U(1,y)) = 1, \sup_{\tau\in K}\psi(U(N(x),\tau)) = \psi(U(N(x),1)) = 1.$$

Hence, from (14) we get $\max\{\psi(N(x)),\psi(y)\} = \psi(U(N(x),y))$. Hence, $\max\{N(x),y\} = U(N(x),y)$, i.e., (b) holds.

(Sufficiency) From (a) and (b) we get

$$I_{U,N}(x,y) = \begin{cases} \max(N(x),y), & \text{if } x\in[0,1]\backslash K \text{ and } y>e,\\ N(x), & \text{if } x\in[0,1]\backslash K \text{ and } y\le e,\\ y, & \text{if } x\in K \text{ and } y>e,\\ 0, & \text{if } x\in K \text{ and } y\le e.\end{cases}$$

The rest of the proof is only a calculation. □

If we consider the fuzzy negation N defined as $N(x) = 0$ for $x \in K$, and $N(x) = 1$ for $x \in [0,1]\backslash K$, then we get the following result.

Corollary 5.8. *Let $I_{U,N}$ be a (U, N)-implication defined as*

$$I_{U,N}(x,y) = \begin{cases} 1, & \text{if } x\in[0,1]\backslash K,\\ y, & \text{if } x\in K \text{ and } y>e,\\ 0, & \text{if } x\in K \text{ and } y\le e,\end{cases}$$

where $e \in (0,1)$, K is a connected set such that $(e,1] \subseteq K \subset [0,1]$. Then $I_{U,N}$ satisfies (GHS($T_{(\psi)}$)).

5.2 (GHS($T_{(\psi)}$)) of (T,N)-implications

Definition 5.9. [6] *A function $I : [0,1]^2 \to [0,1]$ is said to be a (T, N)-implication if there exists a t-norm T and a fuzzy negation N, such that*

$$I(x,y) = N(T(x,N(y))), \ \ x,y\in[0,1].$$

If I is a (T, N)-implication, then it is denoted by I_T^N.

Proposition 5.10. *Let I_T^N be a (T, N)-implication, where T is a t-norm and N is a fuzzy negation. If I_T^N satisfies (GHS($T_{(\psi)}$)). Then*
(i) *N has no fixed point. That is, there is no $x \in [0,1]$ such that $N(x) = x$.*
(ii) *there exists a connected set $K \subseteq (0,1]$ such that $N(x) = 0$ for all $x \in K$, and $N(x) \in K$ for all $x \in [0,1] \setminus K$, $T(x,N(y)) \in K$ for all $x \in K$ and $y \in [0,1] \setminus K$.*

Proof. Let I_T^N satisfy (GHS($T_{(\psi)}$)). Then, for a, $b \in [0,1]$, we have

$$\sup_{x\in[0,1]} \big(\psi(I_T^N(a,x)) \cdot \psi(I_T^N(x,b))\big) = \psi(I_T^N(a,b)). \tag{15}$$

Take $a = 1$ and $b = 0$ in (15), we get

$$\psi(N(N(x))) \cdot \psi(N(x)) = 0 \text{ for all } x \in [0,1]. \tag{16}$$

(i) Assume that there exists an $x_0 \in [0,1]$ such that $N(x_0) = x_0$. Obvious, $x_0 \in (0,1)$. However, from (16) we get $(\psi(x_0))^2 = 0$. Hence $x_0 = 0$, a contradiction.

(ii) Let $\{x \in [0,1]|N(x) = 0\} = K$. Then $K \neq \emptyset$, since $N(1) = 0$. From N is decreasing, we get that K is a connected set, and $K \subseteq (0,1]$.

In the following, we prove that $N(x) \in K$ for all $x \in [0,1] \setminus K$.

Assume that there exists an $x_1 \in [0,1] \setminus K$ such that $N(x_1) \notin K$. Then $N(x_1) > 0$ and $N(N(x_1)) > 0$. Thus $\psi(N(x_1)) > 0$ and $\psi(N(N(x_1))) > 0$. Hence $\psi(N(N(x_1))) \cdot \psi(N(x_1)) > 0$. This contradicts (16).

In the following, we prove that $T(x, N(y)) \in K$ for all $x \in K$ and $y \in [0,1] \setminus K$.

Let $a = 1$ in (15), we get

$$\sup_{x\in[0,1]} \Big(\psi\big(N(N(x))\big) \cdot \psi\big(N(T(x,N(b)))\big)\Big) = \psi(N(N(b))). \tag{17}$$

Let $b \in [0,1] \setminus K$ in (17). Then $N(N(b)) = 0$. Then we get $\sup_{x\in[0,1]} \Big(\psi\big(N(N(x))\big) \cdot \psi\big(N(T(x,N(b)))\big)\Big) = 0$. Hence,

$$\max\left\{\sup_{x\in K} \Big(\psi\big(N(N(x))\big) \cdot \psi\big(N(T(x,N(b)))\big)\Big), \sup_{x\in[0,1]\setminus K} \Big(\psi\big(N(N(x))\big) \cdot \psi\big(N(T(x,N(b)))\big)\Big)\right\} = 0.$$

Since $N(N(x)) = 1$ for $x \in K$ and $N(N(x)) = 0$ for $x \in [0,1] \setminus K$, then $\sup_{x\in K} \psi\big(N(T(x,N(b)))\big) = 0$. Hence, $N(T(x,N(b))) = 0$ for all $x \in K$ and $b \in [0,1] \setminus K$. Thus, $T(x,N(b)) \in K$ for all $x \in K$ and $b \in [0,1] \setminus K$. That is, $T(x,N(y)) \in K$ for all $x \in K$ and $y \in [0,1] \setminus K$. □

Proposition 5.11. *Let I_T^N be a (T,N)-implication, where T is a t-norm. If there exists a connected set $K \subseteq (0,1]$ such that $N(x) = 0$ for all $x \in K$, and $N(x) = 1$ for all $x \in [0,1] \setminus K$, i.e.,*

$$N(x) = \begin{cases} 0, & \text{if } x \in K, \\ 1, & \text{if } x \in [0,1]\backslash K, \end{cases}$$

then I_T^N satisfies (GHS($T_{(\psi)}$)).

Proof. Let $N(x) = 0$ for all $x \in K$, and $N(x) = 1$ for all $x \in [0,1] \setminus K$. Then $I_T^N(x,y) = \begin{cases} 1, & \text{if } y \in K, \\ N(x), & \text{if } y \in [0,1]\backslash K, \end{cases}$

If $y \in K$, then $\sup_{\tau\in[0,1]} \Big(\psi(I_T^N(x,\tau)) \cdot \psi(I_T^N(\tau,y))\Big) = 1 = \psi(I_T^N(x,y))$.

If $y \in [0,1]\backslash K$, then

$$\begin{aligned}
\sup_{\tau\in[0,1]} \Big(\psi(I_T^N(x,\tau)) \cdot \psi(I_T^N(\tau,y))\Big) &= \sup_{\tau\in[0,1]} \Big(\psi(I_T^N(x,\tau)) \cdot \psi(N(\tau))\Big) \\
&= \max\left\{\sup_{\tau\in K} \Big(\psi(I_T^N(x,\tau)) \cdot \psi(N(\tau))\Big), \sup_{\tau\in[0,1]\setminus K} \Big(\psi(I_T^N(x,\tau)) \cdot \psi(N(\tau))\Big)\right\} \\
&= \sup_{\tau\in[0,1]\setminus K} \Big(\psi(I_T^N(x,\tau)) \cdot \psi(N(\tau))\Big) \\
&= \sup_{\tau\in[0,1]\setminus K} \Big(\psi(N(x)) \cdot \psi(N(\tau))\Big) \\
&= \psi(N(x)) \\
&= \psi(I_T^N(x,y)).
\end{aligned}$$

Hence, I_T^N satisfies (GHS($T_{(\psi)}$)) by Lemma 3.1. □

Proposition 5.12. *Let I_T^N be a (T,N)-implication. If $T=T_M$ and there exists a connected set $K\subseteq(0,1]$ such that $N(x)=0$ for all $x\in K$, $N(x)\in K$ for all $x\in[0,1]\setminus K$, then I_T^N satisfies (GHS($T_{(\psi)}$)).*

Proof. Let x, $y\in[0,1]$ and $T=T_M$. Then $I_T^N(x,y)=N(\min(x,N(y)))=\max(N(x),N(N(y)))$. Since $N(x)=0$ for all $x\in K$, $N(x)\in K$ for all $x\in[0,1]\setminus K$, then

$$\begin{aligned}\sup_{\tau\in[0,1]}\Big(\psi(I_T^N(x,\tau))\cdot\psi(I_T^N(\tau,y))\Big)&=\sup_{\tau\in[0,1]}\Big(\max\big(\psi(N(x)),\psi(N(N(\tau)))\big)\cdot\max\big(\psi(N(\tau)),\psi(N(N(y)))\big)\Big)\\&=\max\left\{\begin{aligned}&\sup_{\tau\in K}\Big(\max\big(\psi(N(x)),\psi(N(N(\tau)))\big)\cdot\psi\big(N(N(y))\big)\Big),\\&\sup_{\tau\in[0,1]\setminus K}\Big(\psi(N(x))\cdot\max\big(\psi(N(\tau)),\psi(N(N(\tau)))\big)\Big)\end{aligned}\right\}\\&=\max\big\{\psi(N(N(y))),\psi(N(x))\big\}\\&=\psi(I_T^N(x,y)).\end{aligned}$$

Therefore, I_T^N satisfies (GHS($T_{(\psi)}$)) by Lemma 3.1. □

6 Preservation of (GHS($T_{(\psi)}$)) by Sup-$T_{(\psi)}$ composition method

Problem 6.1. *Let $I,\hat{I}\in FI$ satisfy (GHS($T_{(\psi)}$)), does $I\overset{T_{(\psi)}}{\circ}\hat{I}$ satisfy (GHS($T_{(\psi)}$)) ?*

Unfortunately, the answer is negative. To see this, consider the following example.

Example 6.2. *Consider the following two fuzzy implications:*

$$I(x,y)=\begin{cases}1, & \text{if } y=1,\\ 1-x, & \text{if } y<1,\end{cases}\quad I_D(x,y)=\begin{cases}1, & \text{if } x=0,\\ y, & \text{if } x>0.\end{cases}$$

It is easy to verify that I and I_D respectively satisfy (GHS($T_{(\psi)}$)). However, from Definition 2.10 we get

$$\begin{aligned}(I\overset{T_{(\psi)}}{\circ}I_D)(x,y)&=\sup_{\alpha\in[0,1]}\psi^{-1}\Big(\psi(I(x,\alpha))\cdot\psi(I_D(\alpha,y))\Big)\\&=\max\left\{\sup_{\alpha\in\{0\}}\psi^{-1}\Big(\psi(I(x,\alpha))\cdot\psi(I_D(\alpha,y))\Big),\sup_{\alpha\in(0,1]}\psi^{-1}\Big(\psi(I(x,\alpha))\cdot\psi(I_D(\alpha,y))\Big)\right\}\\&=\max\{1-x,y\}.\end{aligned}$$

To simplify the symbol $I\overset{T_{(\psi)}}{\circ}I_D$, now we let $I\overset{T_{(\psi)}}{\circ}I_D=\widetilde{I}$. Obviously, $\widetilde{I}$ is a fuzzy implication.

Since $\widetilde{I}(1,y)=y$, and $N_{\widetilde{I}}(x)=1-x\neq N_{D_1}$ then $\widetilde{I}$ does not satisfy (GHS($T_{(\psi)}$)) by Corollary 3.3, i.e., $I\overset{T_{(\psi)}}{\circ}I_D$ does not satisfy (GHS($T_{(\psi)}$)).

Even though one of the fuzzy implications I, $\hat{I}$ does not satisfy (GHS($T_{(\psi)}$)), $I\overset{T_{(\psi)}}{\circ}\hat{I}$ may satisfy (GHS($T_{(\psi)}$)). To see this, consider the following example.

Example 6.3. Consider the strict t-norm T_P, and the following two fuzzy implications

$$I_0(x,y)=\begin{cases}1, & \text{if } x=0 \text{ or } y=1,\\ 0, & \text{otherwise},\end{cases}\qquad I(x,y)=\begin{cases}1, & \text{if } x\leq y,\\ \sin\left(\frac{\pi y}{2x}\right), & \text{if } x>y.\end{cases}$$

It is easy to verify that I_0 satisfies (GHS(T_P)). For fuzzy implication I, Since

$$\sup_{\nu\in[0,1]}\big(I(0.9,\nu)\cdot I(\nu,0.3)\big)\geq I(0.9,0.6)\cdot I(0.6,0.3)=\frac{\sqrt{6}}{4}>\frac{1}{2}=I(0.9,0.3),$$

then I does not satisfy (GHS(T_P)). However, by calculations we get

$$\big(I\overset{T_P}{\circ}I_0\big)(x,y)=\begin{cases}1, & \text{if } x=0 \text{ or } y=1,\\ 0, & \text{otherwise}\end{cases}$$

i.e., $I\overset{T_P}{\circ}I_0=I_0$. Hence, $I\overset{T_P}{\circ}I_0$ satisfies (GHS(T_P)).

Examples 6.2 and 6.3 show that the condition "I and $\hat{I}$ satisfy (GHS($T_{(\psi)}$))" is not sufficient and necessary. In the following, we characterize the (GHS($T_{(\psi)}$)) property of implication $I \overset{T_{(\psi)}}{\circ} \hat{I}$.

Theorem 6.4. [2, Theorem 6.4.4.] *Let $I, \hat{I} \in FI$. Then $I \overset{T}{\circ} \hat{I} \in FI \Leftrightarrow (I \overset{T}{\circ} \hat{I})(1,0) = 0$.*

Proposition 6.5. *Let fuzzy implications I, $\hat{I}$ satisfy (NP) such that $I \overset{T_{(\psi)}}{\circ} \hat{I} \in FI$. If $I \overset{T_{(\psi)}}{\circ} \hat{I}$ satisfies (GHS($T_{(\psi)}$)), then $N_I = N_{\hat{I}} = N_{D_1}$*

Proof. Let $I \overset{T_{(\psi)}}{\circ} \hat{I} \in FI$. Then $(I \overset{T_{(\psi)}}{\circ} \hat{I})(1,0) = 0$. That is, $\sup\limits_{\nu\in[0,1]} \Big(\psi(I(1,\nu)) \cdot \psi(\hat{I}(\nu,0))\Big) = \psi(0) = 0$. Since I satisfies (NP), then $\sup\limits_{\nu\in[0,1]} \Big(\psi(\nu) \cdot \psi(\hat{I}(\nu,0))\Big) = 0$. Hence, $\hat{I}(\nu,0) = 0$ for all $\nu > 0$. Therefore, $N_{\hat{I}} = N_{D_1}$.

Let $I \overset{T_{(\psi)}}{\circ} \hat{I}$ satisfy (GHS($T_{(\psi)}$)). Then $\left((I \overset{T_{(\psi)}}{\circ} \hat{I}) \overset{T_{(\psi)}}{\circ} (I \overset{T_{(\psi)}}{\circ} \hat{I})\right)(1,0) = (I \overset{T_{(\psi)}}{\circ} \hat{I})(1,0) = 0$. That is,

$$\sup_{\omega\in[0,1]} \left(\psi\Big((I \overset{T_{(\psi)}}{\circ} \hat{I})(1,\omega)\Big) \cdot \psi\Big((I \overset{T_{(\psi)}}{\circ} \hat{I})(\omega,0)\Big)\right) = \psi(0) = 0. \tag{18}$$

Since $\psi\Big((I \overset{T_{(\psi)}}{\circ} \hat{I})(1,\omega)\Big) = \sup\limits_{t_1\in[0,1]} \Big(\psi(I(1,t_1)) \cdot \psi(\hat{I}(t_1,\omega))\Big)$, and $\psi\Big((I \overset{T_{(\psi)}}{\circ} \hat{I})(\omega,0)\Big) = \sup\limits_{t_2\in[0,1]} \Big(\psi(I(\omega,t_2)) \cdot \psi(\hat{I}(t_2,0))\Big)$, then from (18) we have

$$\sup_{\omega\in[0,1]} \left(\sup_{t_1\in[0,1]} \Big(\psi(I(1,t_1)) \cdot \psi(\hat{I}(t_1,\omega))\Big) \cdot \sup_{t_2\in[0,1]} \Big(\psi(I(\omega,t_2)) \cdot \psi(\hat{I}(t_2,0))\Big)\right) = 0.$$

From $N_{\hat{I}} = N_{D_1}$, we get $\sup\limits_{t_2\in[0,1]} \big(\psi(I(\omega,t_2)) \cdot \psi(\hat{I}(t_2,0))\big) = \psi(I(\omega,0)) = \psi(N_I(\omega))$. Hence, we have

$$\sup_{\omega\in[0,1]} \left(\sup_{t_1\in[0,1]} \big(\psi(I(1,t_1)) \cdot \psi(\hat{I}(t_1,\omega))\big) \cdot \psi(N_I(\omega))\right) = 0.$$

Therefore, $\sup\limits_{\omega\in[0,1]} \left(\sup\limits_{t_1\in[0,1]} \big(\psi(t_1) \cdot \psi(\hat{I}(t_1,\omega))\big) \cdot \psi(N_I(\omega))\right) = 0$. Hence, $\sup\limits_{t_1\in[0,1]} \big(\psi(t_1) \cdot \psi(\hat{I}(t_1,\omega))\big) \cdot \psi(N_I(\omega)) = 0$ for all $\omega \in [0,1]$.

Consider $\omega > 0$, since $\sup\limits_{t_1\in[0,1]} \big(\psi(t_1) \cdot \psi(\hat{I}(t_1,\omega))\big) \geq \psi(1) \cdot \psi(\hat{I}(1,\omega)) = \psi(\omega) > 0$, then $\psi(N_I(\omega)) = 0$ for all $\omega > 0$. Hence, $N_I(\omega) = 0$ for all $\omega > 0$, i.e., $N_I = N_{D_1}$. □

6.1 One of I and $\hat{I}$ is an (S,N)-implication

Theorem 6.6. *Let $I \in FI$ satisfy (NP) and $\hat{I}$ be an (S,N)-implication such that $I \overset{T_{(\psi)}}{\circ} \hat{I} \in FI$. Then the following statements are equivalent:*

(i) *$I \overset{T_{(\psi)}}{\circ} \hat{I}$ satisfies (GHS($T_{(\psi)}$)).*

(ii) *$N_I = N_{D_1}$, $\hat{I} = I_D$. In this case, $I \overset{T_{(\psi)}}{\circ} \hat{I} = I_D$.*

Proof. (i ⇒ ii) it is Obvious form Proposition 6.5.

(ii ⇒ i) Let $\hat{I} = I_D$, $N_I = N_{D_1}$. Then

$$\begin{aligned}(I \overset{T_{(\psi)}}{\circ} \hat{I})(x,y) &= \sup_{\nu\in[0,1]} \psi^{-1}\Big(\psi(I(x,\nu)) \cdot \psi(\hat{I}(\nu,y))\Big) \\ &= \max\Big\{\sup_{\nu\in\{0\}} \psi^{-1}\Big(\psi(I(x,\nu)) \cdot \psi(\hat{I}(\nu,y))\Big), \sup_{\nu\in(0,1]} \psi^{-1}\Big(\psi(I(x,\nu)) \cdot \psi(\hat{I}(\nu,y))\Big)\Big\} \\ &= \max\{N_I(x), y\} \\ &= \begin{cases} 1, & \text{if } x = 0, \\ y, & \text{if } x > 0. \end{cases} \\ &= I_D(x,y).\end{aligned}$$

Since I_D satisfies (GHS($T_{(\psi)}$)) ([42], Theorem 3.3), then $I \overset{T_{(\psi)}}{\circ} \hat{I}$ satisfies (GHS($T_{(\psi)}$)). □

Remark 6.7. (i) *Let I and $\hat{I}$ be (S,N)-implications such that $I \overset{T_{(\psi)}}{\circ} \hat{I} \in FI$. Then $I \overset{T_{(\psi)}}{\circ} \hat{I}$ satisfies (GHS($T_{(\psi)}$)) if and only if $I = \hat{I} = I_D$.*

(ii) *Let I be an R-implication and $\hat{I}$ an (S,N)-implication such that $I \overset{T_{(\psi)}}{\circ} \hat{I} \in FI$. Then $I \overset{T_{(\psi)}}{\circ} \hat{I}$ satisfies (GHS($T_{(\psi)}$)) if and only if $N_I = N_{D_1}$ and $\hat{I} = I_D$.*

(iii) *Let I be an f-implication and $\hat{I}$ an (S,N)-implication such that $I \overset{T_{(\psi)}}{\circ} \hat{I} \in FI$. Then $I \overset{T_{(\psi)}}{\circ} \hat{I}$ satisfies (GHS($T_{(\psi)}$)) if and only if $f(0) = \infty$ and $\hat{I} = I_D$.*

(iv) *Let I be a g-implication and $\hat{I}$ an (S,N)-implication such that $I \overset{T_{(\psi)}}{\circ} \hat{I} \in FI$. Then $I \overset{T_{(\psi)}}{\circ} \hat{I}$ satisfies (GHS($T_{(\psi)}$)) if and only if $\hat{I} = I_D$.*

Theorem 6.8. *Let I be an (S,N)-implication and $\hat{I} \in FI$ satisfy (NP). Then the following statements are equivalent:*

(i) *$I \overset{T_{(\psi)}}{\circ} \hat{I}$ is a fuzzy implication such that $I \overset{T_{(\psi)}}{\circ} \hat{I}$ satisfies (GHS($T_{(\psi)}$)).*

(ii) *$I = I_D$, $N_{\hat{I}} = N_{D_1}$.*

Proof. (i ⇒ ii) it is Obvious form Proposition 6.5.

(ii ⇒ i) Let $I = I_D$. Then

$$\big(I \overset{T_{(\psi)}}{\circ} \hat{I}\,\big)(x,y) = \sup_{\nu\in[0,1]} \psi^{-1}\Big(\psi(I_D(x,\nu))\cdot\psi(\hat{I}(\nu,y))\Big) = \begin{cases} 1, & \text{if } x=0,\\ \sup\limits_{\nu\in[0,1]} \psi^{-1}\Big(\psi(\nu)\cdot\psi(\hat{I}(\nu,y))\Big), & \text{if } x>0. \end{cases}$$

Thus, for $x > 0$,
$$\begin{aligned}\big(I \overset{T_{(\psi)}}{\circ} \hat{I}\,\big)(x,0) &= \sup_{\nu\in[0,1]} \psi^{-1}\Big(\psi(\nu)\cdot\psi(\hat{I}(\nu,0))\Big)\\ &= \sup_{\nu\in(0,1]} \psi^{-1}\Big(\psi(\nu)\cdot\psi(\hat{I}(\nu,0))\Big) \text{ [since } N_{\hat{I}} = N_{D_1}]\\ &= 0\end{aligned}$$

Hence, $I \overset{T_{(\psi)}}{\circ} \hat{I}$ is a fuzzy implication.

In the following, we simplify the symbol $I \overset{T_{(\psi)}}{\circ} \hat{I}$. Let $I \overset{T_{(\psi)}}{\circ} \hat{I} = \widetilde{I}$.

For $x = 0$, $\big(\widetilde{I} \overset{T_{(\psi)}}{\circ} \widetilde{I}\big)(0,y) = \sup\limits_{w\in[0,1]} \psi^{-1}\Big(\psi(\widetilde{I}(0,w))\cdot\psi(\widetilde{I}(w,y))\Big) = \sup\limits_{w\in[0,1]} \psi^{-1}\Big(\psi(\widetilde{I}(w,y))\Big) = \sup\limits_{w\in[0,1]} \widetilde{I}(w,y) = \widetilde{I}(0,y)$.

For $x > 0$,

$$\begin{aligned}\big(\widetilde{I} \overset{T_{(\psi)}}{\circ} \widetilde{I}\big)(x,y) &= \sup_{w\in[0,1]} \psi^{-1}\Big(\psi(\widetilde{I}(x,w))\cdot\psi(\widetilde{I}(w,y))\Big)\\ &= \sup_{w\in[0,1]} \psi^{-1}\Big(\sup_{\nu_1\in[0,1]}\big(\psi(\nu_1)\cdot\psi(\hat{I}(\nu_1,w))\big)\cdot\psi(\widetilde{I}(w,y))\Big)\\ &= \sup_{w\in(0,1]} \psi^{-1}\Big(\sup_{\nu_1\in[0,1]}\big(\psi(\nu_1)\cdot\psi(\hat{I}(\nu_1,w))\big)\cdot\psi(\widetilde{I}(w,y))\Big)\\ &= \psi^{-1}\left(\sup_{w\in(0,1]}\Big(\sup_{\nu_1\in[0,1]}\big(\psi(\nu_1)\cdot\psi(\hat{I}(\nu_1,w))\big)\cdot\sup_{\nu_2\in[0,1]}\big(\psi(\nu_2)\cdot\psi(\hat{I}(\nu_2,y))\big)\Big)\right)\\ &= \psi^{-1}\left(\sup_{\nu_2\in(0,1]}\Big(\psi(\nu_2)\cdot\psi(\hat{I}(\nu_2,y))\Big)\right)\\ &= \widetilde{I}(x,y).\end{aligned}$$

Thus, $\big(\widetilde{I} \overset{T_{(\psi)}}{\circ} \widetilde{I}\,\big)(x,y) = \widetilde{I}(x,y)$ for x, $y \in [0,1]$. Therefore, $I \overset{T_{(\psi)}}{\circ} \hat{I}$ satisfies (GHS($T_{(\psi)}$)). □

Inspired by the proof of (ii ⇒ i) in Theorem 6.8, a new result is obtained.

Proposition 6.9. *Let $\varphi \in \Phi$ and $T_{(\psi)}$ be a strict t-norm, and let I_φ be a fuzzy implication that φ-conjugate with fuzzy implication I_D, i.e.,*

$$I_\varphi(x,y) = \begin{cases} 1, & \text{if } x=0,\\ \varphi(y), & \text{if } x>0.\end{cases}$$

Then operator $I_\varphi \overset{T_{(\psi)}}{\circ} \hat{I}$ *satisfies (GHS(*$T_{(\psi)}$*)) for fuzzy implication* $\hat{I}$ *such that* $N_{\hat{I}} = N_{D_1}$.

Proof. Analogues to the proof of Theorem 6.8. □

6.2 The case of I and $\hat{I}$ are R-implications

Proposition 6.10. [14, Proposition 1.6] *Let* I_T *be an R-implication generated from a left-continuous t-norm* T. *Then* $T(I_T(x,y), I_T(y,z)) \le I_T(x,z)$ *for all* $x, y, z \in [0,1]$.

Proposition 6.11. *Let* I_{T_1} *be an R-implication generated from a left-continuous t-norm* T_1, *and* I_{T_2} *an R-implication generated from a left-continuous t-norm* T_2. *Let* $T_{(\psi)}$ *be a strict t-norm,* $\psi \in \Phi$. *If* T_1, $T_2 \ge T_{(\psi)}$, *and*

$$\psi(I_{T_2}(x,w)) \cdot \psi(I_{T_1}(w,y)) \le \max\{\psi(I_{T_1}(x,y)), \psi(I_{T_2}(x,y))\}$$

for all x, w, $y \in [0,1]$ *with* $x > w > y$, *then* $I_{T_1} \overset{T_{(\psi)}}{\circ} I_{T_2}$ *is a fuzzy implication, and it satisfies (GHS(*$T_{(\psi)}$*)).*

Proof. First, we prove that $I_{T_1} \overset{T_{(\psi)}}{\circ} I_{T_2}$ *is a fuzzy implication.*

Let T_1, $T_2 \ge T_{(\psi)}$, then $I_{T_1}(x,y) \le I_{T_{(\psi)}}(x,y)$, $I_{T_2}(x,y) \le I_{T_{(\psi)}}(x,y)$. Hence, $N_{I_{T_1}} = N_{D_1}$, $N_{I_{T_2}} = N_{D_1}$. Therefore,

$$\begin{aligned}
(I_{T_1} \overset{T_{(\psi)}}{\circ} I_{T_2})(1,0) &= \sup_{\nu\in[0,1]} \Big(\psi^{-1}\big(\psi(I_{T_1}(1,\nu)) \cdot \psi(I_{T_2}(\nu,0))\big)\Big)\\
&= \max\left\{\sup_{\nu\in\{0\}} \psi^{-1}\Big(\psi(I_{T_1}(1,\nu)) \cdot \psi(I_{T_2}(\nu,0))\Big), \sup_{\nu\in(0,1]} \psi^{-1}\Big(\psi(I_{T_1}(1,\nu)) \cdot \psi(I_{T_2}(\nu,0))\Big)\right\}\\
&= \max\{I_{T_1}(1,0), 0\}\\
&= 0.
\end{aligned}$$

Hence, $I_{T_1} \overset{T_{(\psi)}}{\circ} I_{T_2}$ is a fuzzy implication by Theorem 6.4.

Second , we prove that $I_{T_1} \overset{T_{(\psi)}}{\circ} I_{T_2}$ *satisfies (IP).*

Since I_{T_1} and I_{T_2} respectively satisfy (OP) (Theorem 2.5.7 in [2]), then $I_{T_1}(x,x) = I_{T_2}(x,x) = 1$ for all $x \in [0,1]$. Thus

$$\begin{aligned}
(I_{T_1} \overset{T_{(\psi)}}{\circ} I_{T_2})(x,x) &= \sup_{\nu\in[0,1]} \psi^{-1}\Big(\psi(I_{T_1}(x,\nu)) \cdot \psi(I_{T_2}(\nu,x))\Big)\\
&= \max\left\{\sup_{0\le\nu\le x} \psi^{-1}\Big(\psi(I_{T_1}(x,\nu)) \cdot \psi(I_{T_2}(\nu,x))\Big), \sup_{1\ge\nu>x} \psi^{-1}\Big(\psi(I_{T_1}(x,\nu)) \cdot \psi(I_{T_2}(\nu,x))\Big)\right\}\\
&= \max\big\{I_{T_1}(x,x), \sup_{1\ge\nu>x} I_{T_2}(\nu,x)\big\}\\
&= 1.
\end{aligned}$$

Hence $I_{T_1} \overset{T_{(\psi)}}{\circ} I_{T_2}$ satisfies (IP).

Next, we prove that $I_{T_1} \overset{T_{(\psi)}}{\circ} I_{T_2}$ *satisfies (GHS(*$T_{(\psi)}$*)).*

From Proposition 6.10, for all x, ν, $y \in [0,1]$, we get

$$T_1\left(I_{T_1}(x,\nu_1), I_{T_1}(\nu_1,y)\right) \le I_{T_1}(x,y),\ T_2\left(I_{T_2}(x,\nu_2), I_{T_2}(\nu_2,y)\right) \le I_{T_2}(x,y).$$

Let T_1, $T_2 \ge T_{(\psi)}$, then $T_{(\psi)}(I_{T_1}(x,\nu_1), I_{T_1}(\nu_1,y)) \le I_{T_1}(x,y)$, $T_{(\psi)}(I_{T_2}(x,\nu_2), I_{T_2}(\nu_2,y)) \le I_{T_2}(x,y)$. Thus, we have

$$\psi(I_{T_1}(x,\nu_1)) \cdot \psi(I_{T_1}(\nu_1,y))) \le \psi(I_{T_1}(x,y)),\ \psi(I_{T_2}(x,\nu_2)) \cdot \psi(I_{T_2}(\nu_2,y))) \le \psi(I_{T_2}(x,y)),$$

Since $\psi(I_{T_2}(x,w)) \cdot \psi(I_{T_1}(w,y)) \le \max\{\psi(I_{T_1}(x,y)), \psi(I_{T_2}(x,y))\}$ for all $x, w, y \in [0,1]$ with $x > w > y$, then, for all $\nu_1, w, \nu_2 \in [0,1]$, we have

$$\psi(I_{T_2}(\nu_1,w)) \cdot \psi(I_{T_1}(w,\nu_2)) \le \max\left\{\psi(I_{T_2}(\nu_1,\nu_2)), \psi(I_{T_1}(\nu_1,\nu_2))\right\}. \tag{19}$$

In fact, if $\nu_1 \le \nu_2$, obviously inequality (19) holds, since the right of (19) is equal to 1 by (OP). If $\nu_1 > \nu_2 \ge w$, then $\psi(I_{T_2}(\nu_1,w)) \cdot \psi(I_{T_1}(w,\nu_2)) = \psi(I_{T_2}(\nu_1,w)) \le \psi(I_{T_2}(\nu_1,\nu_2)) \le \max\left\{\psi(I_{T_2}(\nu_1,\nu_2)), \psi(I_{T_1}(\nu_1,\nu_2))\right\}$. Thus inequality

(19) holds. If $\nu_1 > w > \nu_2$, obviously inequality (19) holds. If $w > \nu_1 > \nu_2$, then $\psi(I_{T_2}(\nu_1, w)) \cdot \psi(I_{T_1}(w, \nu_2)) = \psi(I_{T_1}(w, \nu_2)) \leq \psi(I_{T_1}(\nu_1, \nu_2)) \leq \max\{\psi(I_{T_2}(\nu_1, \nu_2)), \psi(I_{T_1}(\nu_1, \nu_2))\}$, i.e., inequality (19) holds.

Therefore,
$$\begin{aligned}
&\big(\psi(I_{T_1}(x,\nu_1)) \cdot \psi(I_{T_2}(\nu_1, w))\big) \cdot \big(\psi(I_{T_1}(w,\nu_2)) \cdot \psi(I_{T_2}(\nu_2, y))\big)\\
&= \psi(I_{T_1}(x,\nu_1)) \cdot \big(\psi(I_{T_2}(\nu_1, w)) \cdot \psi(I_{T_1}(w,\nu_2))\big) \cdot \psi(I_{T_2}(\nu_2, y))\\
&\leq \psi(I_{T_1}(x,\nu_1)) \cdot \max\Big\{\psi(I_{T_2}(\nu_1,\nu_2)), \psi(I_{T_1}(\nu_1,\nu_2))\Big\} \cdot \psi(I_{T_2}(\nu_2, y))\\
&= \max\big\{\psi(I_{T_1}(x,\nu_1)) \cdot \psi(I_{T_2}(\nu_1,\nu_2)) \cdot \psi(I_{T_2}(\nu_2, y)), \psi(I_{T_1}(x,\nu_1)) \cdot \psi(I_{T_1}(\nu_1,\nu_2)) \cdot \psi(I_{T_2}(\nu_2, y))\big\}\\
&\leq \max\Big\{\psi(I_{T_1}(x,\nu_1)) \cdot \psi(I_{T_2}(\nu_1, y)), \psi(I_{T_1}(x,\nu_2)) \cdot \psi(I_{T_2}(\nu_2, y))\Big\}
\end{aligned}$$

Hence,
$$\begin{aligned}
&\sup_{\nu_1\in[0,1]} (\psi(I_{T_1}(x,\nu_1)) \cdot \psi(I_{T_2}(\nu_1, w))) \cdot \sup_{\nu_2\in[0,1]} (\psi(I_{T_1}(w,\nu_2)) \cdot \psi(I_{T_2}(\nu_2, y)))\\
&\leq \max\Big\{\sup_{\nu_1\in[0,1]} \big(\psi(I_{T_1}(x,\nu_1)) \cdot \psi(I_{T_2}(\nu_1, y))\big), \sup_{\nu_2\in[0,1]} \big(\psi(I_{T_1}(x,\nu_2)) \cdot \psi(I_{T_2}(\nu_2, y))\big)\Big\}\\
&= \sup_{\nu_1\in[0,1]} \big(\psi(I_{T_1}(x,\nu_1)) \cdot \psi(I_{T_2}(\nu_1, y))\big),
\end{aligned}$$

i.e., $T_{(\psi)}\Big((I_{T_1} \overset{T_{(\psi)}}{\circ} I_{T_2})(x,w), (I_{T_1} \overset{T_{(\psi)}}{\circ} I_{T_2})(w,y)\Big) \leq (I_{T_1} \overset{T_{(\psi)}}{\circ} I_{T_2})(x,y)$. Thus $I_{T_1} \overset{T_{(\psi)}}{\circ} I_{T_2}$ satisfies (GHS($T_{(\psi)}$)) by Theorem 2.15. □

Corollary 6.12. *Let I_{T_1} be an R-implication generated from a left-continuous t-norm T_1, and I_{T_2} an R-implication generated from a left-continuous t-norm T_2. If $T_1 \geq T_2 \geq T_{(\psi)}$, then $I_{T_1} \overset{T_{(\psi)}}{\circ} I_{T_2}$ and $I_{T_2} \overset{T_{(\psi)}}{\circ} I_{T_1}$ are fuzzy implications, and they satisfy (GHS($T_{(\psi)}$))*

Proof. By Proposition 6.11, to prove that $I_{T_1} \overset{T_{(\psi)}}{\circ} I_{T_2}$ satisfies (GHS($T_{(\psi)}$)), it suffices to prove that

$$\psi(I_{T_2}(x,w)) \cdot \psi(I_{T_1}(w,y)) \leq \max\{\psi(I_{T_1}(x,y)), \psi(I_{T_2}(x,y))\}$$

for all x, w, $y \in [0,1]$ with $x > w > y$. In fact, since $T_1 \geq T_2 \geq T_{(\psi)}$, then $I_{T_1}(x,y) \leq I_{T_2}(x,y) \leq I_{T_\psi}(x,y)$ for all $x, y \in [0,1]$. Since $T_{(\psi)}(I_{T_2}(x,\nu_1), I_{T_2}(\nu_1,y)) \leq T_2(I_{T_2}(x,\nu_1), I_{T_2}(\nu_1,y)) \leq I_{T_2}(x,y)$, then, $\psi(I_{T_2}(x,w)) \cdot \psi(I_{T_2}(w,y)) \leq \psi(I_{T_2}(x,y))$. Thus,

$$\psi(I_{T_2}(x,w)) \cdot \psi(I_{T_1}(w,y)) \leq \psi(I_{T_2}(x,w)) \cdot \psi(I_{T_2}(w,y)) \leq \psi(I_{T_2}(x,y)) \leq \max\{\psi(I_{T_1}(x,y)), \psi(I_{T_2}(x,y))\}.$$

In the same way, we can prove that $I_{T_2} \overset{T_{(\psi)}}{\circ} I_{T_1}$ satisfies (GHS($T_{(\psi)}$)). □

6.3 One of I and $\hat{I}$ is a special class fuzzy implication

Definition 6.13. [42, Definition 4.1] *Let N be a fuzzy negation. Define I_N, $L_N : [0,1]^2 \to [0,1]$ respectively as follows:*

$$I_N(x,y) = \begin{cases} 1, & \text{if } y = 1, \\ N(x), & \text{if } y < 1, \end{cases} \quad L_N(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ N(x), & \text{if } x > y. \end{cases}$$

Lemma 6.14. [42, Lemma 4.2] *The functions I_N, L_N are fuzzy implications.*

Proposition 6.15. *Operator $I \overset{T_{(\psi)}}{\circ} I_N$ satisfies (GHS($T_{(\psi)}$)) for any $I \in FI$.*

Proof. Let $G = I \overset{T_{(\psi)}}{\circ} I_N$. Then $G(x,y) = \begin{cases} 1, & \text{if } y = 1, \\ \psi^{-1}\Big(\sup_{\nu\in[0,1]} \big(\psi(I(x,\nu)) \cdot \psi(N(\nu))\big)\Big), & \text{if } y < 1. \end{cases}$

Thus, for $y = 1$, $\sup_{w\in[0,1]} (\psi(G(x,w)) \cdot \psi(G(w,y)) = 1 = \psi(G(x,y))$.

For $y < 1$, since
$$\begin{aligned}
\sup_{w\in[0,1]} \big(\psi(G(x,w)) \cdot \psi(G(w,y))\big) &= \max\Big\{\sup_{w\in[0,1)} \big(\psi(G(x,w)) \cdot \psi(G(w,y))\big), \sup_{w\in\{1\}} \big(\psi(G(x,w)) \cdot \psi(G(w,y))\big)\Big\}\\
&= \max\Bigg\{\sup_{w\in[0,1)} \Big(\sup_{\nu\in[0,1]} (\psi(I(x,\nu)) \cdot \psi(N(\nu))) \cdot \sup_{\nu\in[0,1]} (\psi(I(w,\nu)) \cdot \psi(N(\nu)))\Big), \sup_{\nu\in[0,1]} \Big(\psi(I(1,\nu)) \cdot \psi(N(\nu))\Big)\Bigg\}\\
&= \max\Big\{\sup_{\nu\in[0,1]} \big(\psi(I(x,\nu)) \cdot \psi(N(\nu))\big) \cdot \sup_{\nu\in[0,1]} \psi(N(\nu)), \sup_{\nu\in[0,1]} \big(\psi(I(1,\nu)) \cdot \psi(N(\nu))\big)\Big\}
\end{aligned}$$

$$= \max\Big\{ \sup_{\nu\in[0,1]} \big(\psi(I(x,\nu))\cdot\psi(N(\nu))\big), \sup_{\nu\in[0,1]} \big(\psi(I(1,\nu))\cdot\psi(N(\nu))\big)\Big\}$$
$$= \sup_{\nu\in[0,1]} \big(\psi(I(x,\nu))\cdot\psi(N(\nu))\big).$$

Therefore, $\sup_{w\in[0,1]} \big(\psi(G(x,w))\cdot\psi(G(w,y)\big) = \psi(G(x,y))$. Hence, $I \overset{T_{(\psi)}}{\circ} I_N$ satisfies (GHS($T_{(\psi)}$)) by Lemma 3.1. □

Proposition 6.16. *Let $\hat{I}\in FI$, and $\psi\in\Phi$. Then the following statements are equivalent:*

(i) $I_N \overset{T_{(\psi)}}{\circ} \hat{I}$ *satisfies (GHS($T_{(\psi)}$)).*

(ii) $\psi(\hat{I}(1,t))\cdot\psi(N(t)) = 0$ *for all* $t\in[0,1]$.

Proof. Let $G = I_N \overset{T_{(\psi)}}{\circ} \hat{I}$. Then $G(x,y)=\max\{N(x),\hat{I}(1,y)\}$ by calculations. Thus

$$\begin{aligned}
\sup_{w\in[0,1]} \big(\psi(G(x,w))\cdot\psi(G(w,y))\big) &= \sup_{w\in[0,1]} \Big(\max\{\psi(N(x)),\psi(\hat{I}(1,w))\}\cdot\max\{\psi(N(w)),\psi(\hat{I}(1,y))\}\Big)\\
&= \sup_{w\in[0,1]} \left(\max\begin{Bmatrix}\psi(N(x))\cdot\max\{\psi(N(w)),\psi(\hat{I}(1,y))\},\\ \psi(\hat{I}(1,w))\cdot\max\{\psi(N(w)),\psi(\hat{I}(1,y))\}\end{Bmatrix}\right)\\
&= \sup_{w\in[0,1]} \left(\max\begin{Bmatrix}\max\Big\{\psi(N(x))\cdot\psi(N(w)),\psi(N(x))\cdot\psi(\hat{I}(1,y))\Big\},\\ \max\Big\{\psi(\hat{I}(1,w))\cdot\psi(N(w)),\psi(\hat{I}(1,w))\cdot\psi(\hat{I}(1,y))\Big\}\end{Bmatrix}\right)\\
&= \sup_{w\in[0,1]} \left(\max\begin{Bmatrix}\psi(N(x))\cdot\psi(N(w)),\psi(N(x))\cdot\psi(\hat{I}(1,y)),\\ \psi(\hat{I}(1,w))\cdot\psi(N(w)),\psi(\hat{I}(1,w))\cdot\psi(\hat{I}(1,y))\end{Bmatrix}\right)\\
&= \max\begin{pmatrix}\sup_{w\in[0,1]}(\psi(N(x))\cdot\psi(N(w))),\\ \sup_{w\in[0,1]}(\psi(N(x))\cdot\psi(\hat{I}(1,y))),\\ \sup_{w\in[0,1]}(\psi(\hat{I}(1,w))\cdot\psi(N(w))),\\ \sup_{w\in[0,1]}(\psi(\hat{I}(1,w))\cdot\psi(\hat{I}(1,y)))\end{pmatrix} \Big[\text{since } \sup_{w\in[0,1]}(\psi(N(x))\cdot\psi(\hat{I}(1,y)))\le\psi(N(x))\Big]\\
&= \max\Big(\psi(N(x)),\psi(\hat{I}(1,y)), \sup_{w\in[0,1]}\big(\psi(\hat{I}(1,w))\cdot\psi(N(w))\big)\Big). \qquad (20)
\end{aligned}$$

In the following, we prove that (i ⇒ ii).

Let $I_N \overset{T_{(\psi)}}{\circ} \hat{I}$ satisfy (GHS($T_{(\psi)}$)), i.e. $G(x,y)$ satisfies (GHS($T_{(\psi)}$)). Then

$$\sup_{w\in[0,1]} \big(\psi(G(x,w))\cdot\psi(G(w,y))\big) = \psi(G(x,y)) = \max\big(\psi(N(x)),\psi(\hat{I}(1,y))\big).$$

Thus, from (20) we get $\sup_{w\in[0,1]} \big(\psi(\hat{I}(1,w))\cdot\psi(N(w))\big) \le \max\big(\psi(\hat{I}(1,y)),\psi(N(x))\big)$ for all $x,\ y\in[0,1]$. Therefore, $\psi(\hat{I}(1,w))\cdot\psi(N(w)) \le \max\big(\psi(\hat{I}(1,0)),\psi(N(1))\big)$ for all $w\in[0,1]$. Hence, $\psi(\hat{I}(1,w))\cdot\psi(N(w))=0$ for all $w\in[0,1]$.

Next, we prove that (ii ⇒ i).

Let $\psi(\hat{I}(1,t))\cdot\psi(N(t))=0$ for all $t\in[0,1]$. Then $\sup_{w\in[0,1]} \big(\psi(\hat{I}(1,w))\cdot\psi(N(w))\big)=0$. Thus, from (20) we get

$$\sup_{w\in[0,1]} \big(\psi(G(x,w))\cdot\psi(G(w,y))\big) = \max\big(\psi(N(x)),\psi(\hat{I}(1,y))\big).$$

Hence, $\sup_{w\in[0,1]} \big(\psi(G(x,w))\cdot\psi(G(w,y))\big) = \psi(G(x,y))$, i.e, $I_N \overset{T_{(\psi)}}{\circ} \hat{I}$ satisfies (GHS($T_{(\psi)}$)). □

Corollary 6.17. *Let $\hat{I}$ be an S-implication (R-, f-, g-implication, respectively). Then $I_N \overset{T_{(\psi)}}{\circ} \hat{I}$ satisfies (GHS($T_{(\psi)}$)) if and only if $N = N_{D_1}$.*

Proposition 6.18. *Let $\hat{I}\in FI$, and let L_N be a fuzzy implication defined by Definition 6.13. If $N_{\hat{I}}\ge N$, then $L_N \overset{T_{(\psi)}}{\circ} \hat{I}$ satisfies (GHS($T_{(\psi)}$)) if and only if $\hat{I}$ satisfies (GHS($T_{(\psi)}$)).*

Proof. Let $x,\ y \in [0,1]$. Then $(L_N \overset{T_{(\psi)}}{\circ} \hat{I})(x,y) = \sup_{t\in[0,1]} \psi^{-1}\big(\psi(L_N(x,t)) \cdot \psi(\hat{I}(t,y))\big)$

$$= \max\left(\hat{I}(x,y),\ \sup_{t\in[0,x)} \psi^{-1}(\psi(N(x)) \cdot \psi(\hat{I}(t,y)))\right)$$
$$= \max(\hat{I}(x,y), N(x))$$
$$= \hat{I}(x,y).\ [\text{Since } N_{\hat{I}} \geq N \Rightarrow \hat{I}(x,y) \geq N(x)].$$

Hence, $L_N \overset{T_{(\psi)}}{\circ} \hat{I}$ satisfies (GHS($T_{(\psi)}$)) $\Leftrightarrow$ $\hat{I}$ satisfies (GHS($T_{(\psi)}$)). □

7 Conclusions

In this paper, we characterize the (GHS) with respect to a strict t-norm $T_{(\psi)}$ for fuzzy implications. First, we study the (GHS($T_{(\psi)}$)) property of the families of (S,N)-, QL-, Yager's g-, (U,N)-, (T,N)-implications. Next, we investigate the (GHS($T_{(\psi)}$)) property of fuzzy implications generated from old ones using the method of sup-$T_{(\psi)}$ composition. So far, for a summary, studies on (GHS) of fuzzy implications are shown in Table 1.

Table 1: Studies on (GHS) of fuzzy implications.

Researcher	Year	Research contents
Mizumoto et al.	1982, [29]	(GHS) w.r.t. T_M for operators I_{LK}, I_{RS}, I_{GD}, I_{GG}, I_{KD}. etc.
Igel et al.	2004, [16]	The validity of (GHS) in fuzzy systems.
Vemuri	2017, [42]	(GHS) w.r.t. T_M for (S,N)-, R-, QL-, g-, and f-implications and the preservation of (GHS) for fuzzy implications generated by some methods.
Baczýnski et al.	2018, [3]	(GHS) w.r.t. T for R-implications generated from left continuous t-norm.
Chen et al.	2018, [9]	(GHS) w.r.t. T_P for (S,N)-, R-, QL-implications.
Helbin et al.	2019, [15]	(GHS) w.r.t. T for Yager's f-, g-implications.
This paper.		(GHS) w.r.t. strict t-norm T for (S,N)-, QL-, Yager's g-, (U,N)-, (T,N)-implications and the (GHS) of fuzzy implications generated by sup-$T_{(\psi)}$ composition method.

From Table 1, it is easy to see that there exists many unstudied contents on (GHS) of fuzzy implications. For instance, the (GHS) of fuzzy implications with respect to a nilpotent t-norm T, the (GHS($T_{(\psi)}$)) property of RU-implications, the (GHS(T)) property of R-implication generated from non-left continuous t-norm, etc. These are our future works.

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