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Null set concept for optimal solutions of fuzzy nonlinear optimization problems

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Abstract

In this work, we present a novel approach for resolving a fuzzy single-objective function with fuzzy constraints. The algorithm of the method is based on the null set concept and is focused on minimizing cases. With the null set concept, two partial subtraction orders for fuzzy numbers have been defined, namely simple subtraction and the Hukuhara difference. That allows us to define, respectively, optimal solutions and H-optimal solutions. First, the initial optimization problem is transformed into a deterministic, nonlinear, bi-objective optimization problem. Then, Karush Kuhn Tucker's (KKT) optimality conditions are applied to find deterministic optimal solutions. Finally, a few fuzzy algebraic operations are employed to transform deterministic optimal solutions into fuzzy optimal solutions for the initial solutions. In order to demonstrate the effectiveness of the approach, we have dealt with eleven test problems from the literature. Our method has been compared to those of other methods, and our method is at least the best in each instance.

Keywords: Fuzzy nonlinear optimization, null set, Hukuhara difference, ranking function, partial ordering.

1 Introduction

Fuzzy optimization is a very important part of operations research. Numerous real-life problems are modeled into a fuzzy optimization program due to the imprecise and subjective nature of the decision parameters. This model is more difficult to use than deterministic optimization, where the parameters are known with more precision. However, there are numerous works on modeling and solving fuzzy optimization problems in the literature. We can note that the application of linear fuzzy modeling[3, 5, 6, 9, 12, 16, 20, 21] is easier than nonlinear fuzzy modeling [2, 4, 6, 7, 15, 17, 22, 24] but both are difficult to solve. Among the many published methods on fuzzy optimization, it is not easy to choose one that will find the optimal solution to all fuzzy linear problems or fuzzy nonlinear problems, considering the preference of the decision maker. Therefore, it is necessary to have other works to complete the lack of existing methods, especially the solving of fuzzy nonlinear optimization problems that are widely used in many domains: economy, business, engineering, management and so on. There is a lot of work on this topic that has suggested some ways to solve it. For example, Hsien-Chung Wu [30] proposed the optimality conditions for the optimization problem with fuzzy-valued objective functions. Pathak et al. [18] established some necessary and sufficient optimality conditions for the nonlinear fuzzy optimization problem. Horng-Ren Tai et al. [24] proposed a work on fuzzy nonlinear programming approach for optimizing the performance of a four-objective fluctuation smoothing rule in a wafer fabrication factory. Sanjaya Kumar Behera et al. [7] investigated on optimal solution of a fuzzy nonlinear programming problem with linear constraints. N. Ravi Shankar et al. [22] proposed some technique for solving a fuzzy nonlinear optimization problem by genetic algorithm. S. H. Nasseri et al. [17] presented some works on fuzzy nonlinear optimization. Ali F. Jameel et al. [15] focused on solving nonlinear programming problems in a fuzzy environment.

The work of Hsien-Chung Wu [30] upon which our paper is based, employs the concept of continuous differentiability of fuzzy functions in order to derive sufficient optimality conditions for obtaining the optimal solution. This work is

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done for the case of a deterministic constraint, and it is limited when the constraints are fuzzy. On the one hand, we have planned to come up with an extension for the case of fuzzy constraints. Furthermore, the null set concept was introduced by Hsien-Chung Wu[32] and has been applied to fuzzy linear optimization problems with deterministic constraints. Indeed, the concept of a convex cone is derived from the concepts of fuzzy addition and multiplication by a scalar in fuzzy numbers. From the convex cone, partial orderings are defined on the space of fuzzy numbers, and from there, the solution concepts are naturally elucidated. However, in order to establish partial orderings via the convex cone, the difference between two fuzzy numbers must be considered. The ambiguity arises from the fact that the difference between a fuzzy number and its own value cannot be zero. Then, on the other hand, we have planned to use the null set concept to solve nonlinear optimization problems.

Let's remember that Hsien-Chung Wu's works are focused on solving a fuzzy linear optimization problem with deterministic constraints. Then, for the generalization proposed method, it is necessary to extend the null set concept to solve nonlinear optimization problems with fuzzy constraints. Therefore, the purpose of our work is to propose a new method that uses the null set concept to solve fuzzy nonlinear single-objective optimization problems with fuzzy constraints. Due to the use of simple subtraction and the Hukuhara difference, we have considered two fuzzy nonlinear single-objective optimization problems. That leads us to think about the concepts of optimal solutions and the concept of H-optimal solution. Each fuzzy nonlinear optimization problem is transformed into a deterministic bi-objective optimization problem. Through the optimality conditions of Karush-Kuhn-Tucker, we look for the optimal solution and the H-optimal solution. At each step of the method, some theorems are given to justify the convergence to an optimal solution. Also, two examples have been dealt with, and the results are compared with those of the method of Pathak et al.[18] on the one hand, and the method of Kumar Behera et al.[7] on the other hand. Using the ranking function, we can conclude that our proposed method is the best on these two examples.

The remainder of the present paper is presented in the following manner. Section 2 is devoted to preliminaries. We will talk about many properties of fuzzy numbers that will be used in the following. The main outcomes of this work will be presented in Section 3. The presentation of the method and algorithm, the numerical results and the discussion will be the focus. Finally, we will dedicate Section 4 to the conclusion.

2 Preliminaries

2.1 Fuzzy numbers

Definition 2.1. [8, 11] *Let $\mathcal{X}$ be a set. A fuzzy subset $\tilde{a}$ of $\mathcal{X}$ is characterized by a membership function $\mu_{\tilde{a}} : \mathcal{X} \to [0,1]$ and represented by a set of ordered pairs defined as follows:*

$$\tilde{a} = \{(x, \mu_{\tilde{a}}(x))/x \in \mathcal{X}\}.$$

The value $\mu_{\tilde{a}}(x) \in [0,1]$ represents the degree of membership of x to the fuzzy set and is interpreted as the extent to which x belongs to $\tilde{a}$.

Definition 2.2. [8, 11] *Let $\tilde{a}$ be a fuzzy set on $\mathcal{X}$ and $\alpha \in [0,1]$. The $\alpha-$level set of $\tilde{a}$ is the classical set noted $\tilde{a}_\alpha$ and is defined by:*

$$\tilde{a}_\alpha = \{x \in \mathcal{X}, \mu_{\tilde{a}}(x) \geq \alpha\}.$$

Definition 2.3. [29]
Let $\tilde{a}$ a fuzzy subset . Then, $\tilde{a}$ is called a fuzzy number if the following conditions are satisfied:

(i) *$\tilde{a}$ is normal, i.e, $\mu_{\tilde{a}}(x) = 1$, for some $x \in \mathbb{R}$;*

(ii) *$\tilde{a}$ is convex, i.e., the membership function $\mu_{\tilde{a}}(x)$ is quasi-concave;*

(iii) *$\mu_{\tilde{a}}(x)$ is upper semi-continuous, i.e., $\tilde{a}_\alpha$ is a closed subset of $\mathbb{R}$ for some $\alpha \in \mathbb{R}$;*

(iv) *the 0-level set $\tilde{a}_0$, is a compact subset of $\mathbb{R}$.*

We note by $\mathcal{F}(\mathbb{R})$, the set of all fuzzy numbers. Indeed, if $\tilde{a} \in \mathcal{F}(\mathbb{R})$, the α-level set of $\tilde{a}$ is a compact and convex subset of $\mathbb{R}$, i.e., $\tilde{a}_\alpha$ is a closed interval, denoted $\tilde{a}_\alpha$=$[\tilde{a}_\alpha^L, \tilde{a}_\alpha^U]$ for $\alpha \in [0,1]$.

Remark 2.4. *Let $\tilde{a} \in \mathcal{F}(\mathbb{R})$, then $\tilde{a}_\alpha^L$ and $\tilde{a}_\alpha^U$ are considered as functions of α.*

Definition 2.5. [29] *Let $\tilde{a} \in \mathcal{F}(\mathbb{R})$, $\tilde{a}$ is a canonical fuzzy number if the functions $\tilde{a}_\alpha^L$ and $\tilde{a}_\alpha^U$ are continuous with respect to α on $[0,1]$.*

Let $\tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R})$. Using the extension principle of Zadeh [33, 34, 35] and by referring to Puri and Ralescu [19], the membership function of the addition $\tilde{a} \oplus \tilde{b}$ is defined by

$$\mu_{\tilde{a}\oplus\tilde{b}}(z) = \sup_{\{(x,y):x+y=z\}} \min\{\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(x)\}, \tag{1}$$

and the membership function of the multiplication by a scalar $\lambda\tilde{a}$, is defined by

$$\mu_{\lambda\tilde{a}}(z) = \begin{cases} \mu_{\tilde{a}}(z/\lambda), & \text{if } \lambda \neq 0, \\ 0, & \text{if } \lambda = 0 \text{ and } z \neq 0, \\ 1 & \text{if } \lambda = 0 = z. \end{cases} \tag{2}$$

We say that $\tilde{1}_{\{a\}}$ is a crisp number with value a if and only if the membership function of $\tilde{1}_{\{a\}}$ is defined by

$$\mu_{\tilde{1}_{\{a\}}}(r) = \begin{cases} 1, & \text{if } r = a \\ 0, & r \neq a. \end{cases} \tag{3}$$

Indeed, for convenience we write $\lambda\tilde{a} = \tilde{1}_{\{a\}} \otimes \tilde{a}$.

Definition 2.6. [30, 31, 32] *Let $\tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R})$ and the respective α-level $\tilde{a}_\alpha = [\tilde{a}_\alpha^L, \tilde{a}_\alpha^U]$ and $\tilde{b}_\alpha = [\tilde{b}_\alpha^L, \tilde{b}_\alpha^U]$ for $\alpha \in$ [0,1]. We have:*

(i) $\tilde{a} \oplus \tilde{b} \in \mathcal{F}(\mathbb{R})$ *and*

$$(\tilde{a} \oplus \tilde{b})_\alpha = \tilde{a}_\alpha \oplus \tilde{b}_\alpha = [\tilde{a}_\alpha^L + \tilde{b}_\alpha^L, \tilde{b}_\alpha^U + \tilde{b}_\alpha^U], \tag{4}$$

(ii) $\tilde{a} \ominus \tilde{b} \in \mathcal{F}(\mathbb{R})$ *and*

$$(\tilde{a} \ominus \tilde{b})_\alpha = (\tilde{a} \oplus (-\tilde{b}))_\alpha = \tilde{a}_\alpha \oplus (-\tilde{b})_\alpha = [\tilde{a}_\alpha^L - \tilde{b}_\alpha^U, \tilde{a}_\alpha^U - \tilde{b}_\alpha^L], \tag{5}$$

(iii) $\lambda\tilde{a} \in \mathcal{F}(\mathbb{R})$ *and*

$$(\lambda\tilde{a})_\alpha = \begin{cases} [\lambda\tilde{a}_\alpha^L, \lambda\tilde{a}_\alpha^U] \text{ if } \lambda \geq 0, \\ [\lambda\tilde{a}_\alpha^U, \lambda\tilde{a}_\alpha^L] \text{ if } \lambda < 0. \end{cases} \tag{6}$$

Let us approach the notion of the concept of null set defined by Wu [32]. It is inspired by the fact that if the difference of a real number by itself gives zero, this is not the case with fuzzy numbers, i.e., $\tilde{a} \ominus \tilde{a} \neq \{0\}$. This ambiguity requires the definition of a set which gathers all the differences of the fuzzy numbers by themselves.

Let $\tilde{a} \in \mathcal{F}(\mathbb{R})$ and $\tilde{a} \ominus \tilde{a} \in \mathcal{F}(\mathbb{R})$. We have:

$$(\tilde{a} \ominus \tilde{a})_\alpha = [\tilde{a}_\alpha^L - \tilde{a}_\alpha^U, \tilde{a}_\alpha^U - \tilde{a}_\alpha^L] = [-(\tilde{a}_\alpha^U - \tilde{a}_\alpha^L), \tilde{a}_\alpha^U - \tilde{a}_\alpha^L]. \tag{7}$$

Then, the α-level $(\tilde{a} \ominus \tilde{a})_\alpha$ is a set centered at 0. It is seen as an approximation of real number zero. Hence, $\tilde{a} \ominus \tilde{a} \simeq \tilde{0}$, which is a fuzzy zero. Let $\tilde{\Omega} = \{\tilde{a} \ominus \tilde{a} : \tilde{a} \in \mathcal{F}(\mathbb{R})\}$ such as for any $\tilde{\omega} \in \tilde{\Omega}$ we have $\tilde{\omega}_\alpha = [-\tilde{\omega}_\alpha^U, \tilde{\omega}_\alpha^U]$ and $\tilde{\omega}_\alpha^U \geq 0$ for some $\alpha \in [0,1]$. The subset $\tilde{\Omega}$ is called the null set in $\mathcal{F}(\mathbb{R})$.

Thus we have the following propositions.

Proposition 2.7. *If $\tilde{a} \in \tilde{\Omega}$, then $\tilde{a}_\alpha^L = -\tilde{a}_\alpha^U$.*

Proof. Suppose $\tilde{a} \in \tilde{\Omega}$, we have $\tilde{a}_\alpha = [-\tilde{a}_\alpha^U, \tilde{a}_\alpha^U] = [\tilde{a}_\alpha^L, \tilde{a}_\alpha^U]$, then $\tilde{a}_\alpha^L = -\tilde{a}_\alpha^U$. □

Proposition 2.8. [32] *The following propositions are true:*

(i) $-\tilde{\omega} = \tilde{\omega}$ *for some* $\tilde{\omega} \in \tilde{\Omega}$,

(ii) $\tilde{\Omega}$ *is a closed set under fuzzy addition, i.e.,* $\tilde{\omega}^{(1)} \oplus \tilde{\omega}^{(2)} \in \tilde{\Omega}$, $\forall\ \tilde{\omega}^{(1)}, \tilde{\omega}^{(2)} \in \tilde{\Omega}$,

(iii) $\lambda\tilde{\Omega} = \tilde{\Omega}$, *for* $\lambda \in \mathbb{R}^*$.

Definition 2.9. [32] *Let $\tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R})$, we say that $\tilde{a}$ and $\tilde{b}$ are almost identical if and only if there exists $\tilde{\omega}^{(1)}$, $\tilde{\omega}^{(2)} \in \tilde{\Omega}$ such as $\tilde{a} \oplus \tilde{\omega}^{(1)} \simeq \tilde{b} \oplus \tilde{\omega}^{(2)}$. In this case we write $\tilde{a} \overset{\tilde{\Omega}}{\simeq} \tilde{b}$.*

Proposition 2.10. [32] *Let $\tilde{a}$, $\tilde{b} \in \mathcal{F}(\mathbb{R})$. We have:*

(i) *if $\tilde{a} \ominus \tilde{b} \in \tilde{\Omega}$, then $\tilde{a} \overset{\tilde{\Omega}}{\simeq} \tilde{b}$,*

(ii) *if $\tilde{a} \overset{\tilde{\Omega}}{\simeq} \tilde{b}$, then there exists $\tilde{\omega} \in \tilde{\Omega}$ such as $\tilde{a} \ominus \tilde{b} \oplus \tilde{\omega} \in \tilde{\Omega}$.*

Indeed, if $\tilde{a} \ominus \tilde{a} = \{0\}$, then we use the Hukuhara difference which is given by the Definition 2.11.

Definition 2.11. [29, 31] *Let $\tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R})$, if there is a fuzzy number $\tilde{c} \in \mathcal{F}(\mathbb{R})$ satisfying $\tilde{a} = \tilde{b} \oplus \tilde{c}$, then $\tilde{c}$ is unique and is called the Hukuhara difference between $\tilde{a}$ and $\tilde{b}$ denoted by $\tilde{a} \ominus_H \tilde{b}$.*

Remark 2.12. *If the Hukuhara difference exists, then $\tilde{a} \ominus_H \tilde{a} = \{0\}$.*

Proof. Suppose that the Hukuhara difference exists for $\tilde{a} \in \mathcal{F}(\mathbb{R})$, i.e., there is $\tilde{c} \in \mathcal{F}(\mathbb{R})$ such as $\tilde{a} = \tilde{a} \oplus \tilde{c}$. We have

$$\begin{aligned} \tilde{a}_\alpha = (\tilde{a} \oplus \tilde{c})_\alpha = \tilde{a}_\alpha \oplus \tilde{c}_\alpha = [\tilde{a}_\alpha^L, \tilde{a}_\alpha^U] + [\tilde{c}_\alpha^L, \tilde{c}_\alpha^U] \\ = [\tilde{a}_\alpha^L + \tilde{c}_\alpha^L, \tilde{a}_\alpha^U + \tilde{c}_\alpha^U] \quad or\ \tilde{a}_\alpha = [\tilde{a}_\alpha^L, \tilde{a}_\alpha^U]. \end{aligned} \tag{8}$$

We have $a_\alpha^L = \tilde{a}_\alpha^L + \tilde{c}_\alpha^L \Rightarrow \tilde{c}_\alpha^L = 0$. Likewise $\tilde{a}_\alpha^U = \tilde{a}_\alpha^U + \tilde{c}_\alpha^U \Rightarrow \tilde{c}_\alpha^U = 0$.
So $\tilde{c}_\alpha = [0, 0] = 0$. Therefore, $\tilde{a} \ominus_H \tilde{a} = \{0\}$. □

Proposition 2.13. [29, 31] *Let $\tilde{a}$ and $\tilde{b}$ two fuzzy numbers. If the difference of Hukuhara $\tilde{c} = \tilde{a} \ominus_H \tilde{b}$ exists, then $\tilde{c}_\alpha^L = \tilde{a}_\alpha^L - \tilde{b}_\alpha^L$ and $\tilde{c}_\alpha^U = \tilde{a}_\alpha^U - \tilde{b}_\alpha^U$ for some $\alpha \in [0, 1]$.*

Definition 2.14. *Let θ, a vector space. The function $\eta : \mathcal{F}(\mathbb{R}) \to \Gamma$ is called additive and positively homogeneous function of degree k, if the following conditions are satisfied:*

$$\eta(\tilde{a} \oplus \tilde{b}) = \eta(\tilde{a}) + \eta(\tilde{b}) \quad and \quad \eta(\lambda^k \tilde{a}) = \lambda^k \eta(\tilde{a}), \quad \lambda \geq 0, \quad k > 0.$$

Indeed, the function η is a linear function if $k = 1$.

Remark 2.15. *If η is an additive and positively homogeneous function of degree k of $\mathcal{F}(\mathbb{R})$ to Γ, then its kernel, denoted* ker (η)*, and its image noted $Im\ (\eta)$ are defined by:*

$$\ker \eta = \{\tilde{a} : \eta(\tilde{a}) = \theta_\Gamma\} \ and \ Im\ (\eta) = \{\eta(\tilde{a}) : \tilde{a} \in \mathcal{F}(\mathbb{R})\}. \tag{9}$$

θ_Γ is the zero element of vector space Γ. In addition $\forall\ \tilde{\omega} \in \tilde{\Omega}$, $\eta(\tilde{\omega}) = \theta_\Gamma$, then we have $\tilde{\Omega} \subseteq$ ker (η).

Proposition 2.16. *Let $\tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R})$. Suppose that η is an additive and positively homogeneous function of degree k defined on $\mathcal{F}(\mathbb{R})$. If the difference of Hukuhara $\tilde{b} \ominus_H \tilde{a}$ exists, then $\eta(\tilde{b} \ominus_H \tilde{a}) = \eta(\tilde{b}) - \eta(\tilde{a})$.*

Proof. Let $\tilde{c} = \tilde{b} \ominus_H \tilde{a}$, i.e., $\tilde{b} = \tilde{a} \oplus \tilde{c}$. We have

$$\eta(\tilde{b}) = \eta(\tilde{a} \oplus \tilde{c}) = \eta(\tilde{a}) + \eta(\tilde{c}), \tag{10}$$

then $\eta(\tilde{c}) = \eta(\tilde{b}) - \eta(\tilde{a})$. Therefore $\eta(\tilde{b} \ominus_H \tilde{a}) = \eta(\tilde{b}) - \eta(\tilde{a})$. □

Definition 2.17. *A subset $\mathcal{C}$ of $\mathcal{F}(\mathbb{R})$ is a convex cone if $\forall \tilde{a}, \tilde{b} \in \mathcal{C}$, $\lambda\tilde{a} \oplus \beta\tilde{b} \in \mathcal{C}$, for $\lambda, \beta > 0$.*

From the Proposition 2.8, we remark that $\forall\ \tilde{a}, \tilde{b} \in \tilde{\Omega}$, we have $\lambda\tilde{a} \oplus \beta\tilde{b} \in \tilde{\Omega}$, with $\lambda, \beta > 0$. What it means that the null set $\tilde{\Omega}$ is a convex cone.

Proposition 2.18. *Let $\mathcal{C}$ a convex cone of $\mathcal{N}(\mathbb{R})$. If the function η is additive and positively homogeneous of degree k, then the set $\eta(\mathcal{C}) = \{\eta(\tilde{a}) : \tilde{a} \in \mathcal{C}\}$ is a convex cone in the vector space Γ.*

Proof. Assume that $\mathcal{C}$ is a convex cone, i.e., if $\forall \tilde{a}, \tilde{b} \in \mathcal{C}$, $\lambda\tilde{a} \oplus \beta\tilde{b} \in \mathcal{C}$, for $\lambda, \beta > 0$. Consider the function η which is an additive and positively homogeneous of degree k. We have $\eta(\lambda\tilde{a} \oplus \beta\tilde{b}) \in \eta(\mathcal{C})$, then $\lambda\eta(\tilde{a}) + \beta\eta(\tilde{b}) \in \eta(\mathcal{C})$. Therefore $\pi(\mathcal{C})$ is a convex cone in the vector space Γ. □

Suppose $\Gamma = \mathbb{R}^2$, we define the function η by

$$\eta(\tilde{a}) = (-\tilde{a}_\alpha^L - \tilde{a}_\alpha^U; \tilde{a}_\alpha^L + \tilde{a}_\alpha^U) \text{ with } \tilde{a}_\alpha^L + \tilde{a}_\alpha^U \geq 0. \tag{11}$$

Let $\mathcal{C}$ is a convex cone of $\mathcal{F}(\mathbb{R})$, we have:
$\eta(\mathcal{C}) = \{(-\tilde{c}_\alpha^L - \tilde{c}_\alpha^U; \tilde{c}_\alpha^L + \tilde{c}_\alpha^U) \in \mathbb{R}^2 / \tilde{c}_\alpha^L + \tilde{c}_\alpha^U \geq 0 \text{ for some } \alpha \in [0,1]\} \subseteq \{(-x, x) \in \mathbb{R}^2, x \geq 0\}$. Indeed, if $\tilde{\omega} \in \tilde{\Omega}$, we have $\tilde{\omega}_\alpha^L = -\tilde{\omega}_\alpha^U$, $\forall \in [0,1]$, then, $\eta(\tilde{\omega}) = (0,0)$ is the zero element of $\mathbb{R}^2$ with respect to the addition, i.e., $\tilde{\Omega} \subseteq \ker\ \eta$.

So, if $\tilde{\Omega} \subseteq \ker\ \eta$, then, $\tilde{a} \in \tilde{\Omega}$ and $\tilde{a}_\alpha^L + \tilde{a}_\alpha^U = 0$. Let $\mathcal{C}$ a convex cone of $\mathcal{F}(\mathbb{R})$ and $\prod = \left\{\tilde{a} \in \mathcal{F}(\mathbb{R}) : \exists\ \tilde{b} \in \mathcal{C} \text{ such as } \tilde{a} \overset{\tilde{\Omega}}{\simeq} \tilde{b}\right\}$.
Based on the notion of the convex cone, we define the partial orderings on the space of fuzzy numbers $\mathcal{F}(\mathbb{R})$ and the vector space $\eta(\mathcal{F}(\mathbb{R})) \subseteq \Gamma$.

Definition 2.19. [32] *The binary relations $\preccurlyeq$ and $\preccurlyeq_H$ are called partial orderings on $\mathcal{F}(\mathbb{R})$ if the following conditions are satisfied:*

$$\forall\ \tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R}), \quad \tilde{a} \preccurlyeq \tilde{b} \text{ if and only if } \tilde{b} \ominus \tilde{a} \in \prod, \tag{12}$$

and

$$\tilde{a} \preccurlyeq_H \tilde{b} \text{ if and only if } \tilde{b} \ominus_H \tilde{a} \text{ exist and } (\tilde{b} \ominus_H \tilde{a}) \oplus \tilde{\omega} \in \mathcal{C} \text{ for } \tilde{\omega} \in \tilde{\Omega}. \tag{13}$$

Definition 2.20. [32] *The binary relations $\leq$ and $\leq_H$ are called partial orderings on $\eta(\mathcal{F}(\mathbb{R}))$ if the following conditions are satisfied:*

$$\eta(\tilde{a}) \leq \eta(\tilde{b}) \text{ if and only if } \eta(\tilde{b}) - \eta(\tilde{a}) \in \eta(\mathcal{C}), \tag{14}$$

and

$$\eta(\tilde{a}) \leq_H \eta(\tilde{b}) \text{ if and only if } \eta(\tilde{b}) - \eta(\tilde{a}) \in \eta(\mathcal{C}) \text{ and } \tilde{b} \ominus_H \tilde{a} \text{ exist.} \tag{15}$$

Proposition 2.21. [27, 29, 32] *Let partial ordering $\preccurlyeq$ on $\mathcal{F}(\mathbb{R})$. The following axioms are satisfied:*

(i) *if $\tilde{\Omega} \subseteq \mathcal{C}$, then $\tilde{a} \preccurlyeq \tilde{a}$ for some $\tilde{a} \in \mathcal{F}(\mathbb{R})$ (reflexivity),*

(ii) *if $\tilde{a} \preccurlyeq \tilde{b}$ and $\tilde{b} \preccurlyeq \tilde{c}$, then $\tilde{a} \preccurlyeq \tilde{c}$ for some $\tilde{a}, \tilde{b}, \tilde{c} \in \mathcal{F}(\mathbb{R})$ (transitivity),*

(iii) *if $\tilde{a} \preccurlyeq \tilde{b}$ and $\lambda > 0$, then $\lambda\tilde{a} \preccurlyeq \lambda\tilde{b}$ for some $\tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R})$,*

(iv) *if $\tilde{a} \preccurlyeq \tilde{b}$ and $\tilde{d} \preccurlyeq \tilde{e}$, then $\tilde{a} \oplus \tilde{d} \preccurlyeq \tilde{b} \oplus \tilde{e}$ for some $\tilde{a}, \tilde{b}, \tilde{d}, \tilde{e} \in \mathcal{F}(\mathbb{R})$.*

Proposition 2.22. [27, 29, 32] *Let the partial ordering $\preccurlyeq_H$ on $\mathcal{F}(\mathbb{R})$. The following axioms are satisfied:*

(i) *if $\tilde{1}_{\{0\}} \in \mathcal{C}$, then $\tilde{a} \preccurlyeq_H \tilde{a}$ for some $\tilde{a} \in \mathcal{F}(\mathbb{R})$ (reflexivity),*

(ii) *if $\tilde{a} \preccurlyeq_H \tilde{b}$ and $\tilde{b} \preccurlyeq_H \tilde{c}$, then $\tilde{a} \preccurlyeq_H \tilde{c}$ for some $\tilde{a}, \tilde{b}, \tilde{c} \in \mathcal{F}(\mathbb{R})$ (transitivity),*

(iii) *if $\tilde{a} \preccurlyeq_H \tilde{b}$ and $\lambda > 0$, then $\lambda\tilde{a} \preccurlyeq_H \lambda\tilde{b}$ for some $\tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R})$,*

(iv) *if $\tilde{a} \preccurlyeq_H \tilde{b}$ and $\tilde{d} \preccurlyeq_H \tilde{e}$, then $\tilde{a} \oplus \tilde{d} \preccurlyeq_H \tilde{b} \oplus \tilde{e}$ for some $\tilde{a}, \tilde{b}, \tilde{d}, \tilde{e} \in \mathcal{F}(\mathbb{R})$.*

Proposition 2.23. [27, 29, 32] *Let partial ordering $\leq$ on $\eta(\mathcal{F}(\mathbb{R}))$ and $\mathcal{C}$ a convex cone of $\mathcal{F}(\mathbb{R})$. Suppose the function $\eta : \mathcal{F}(\mathbb{R}) \to \Gamma$ is additive and positively homogeneous of degree k. Then the following axioms are satisfied:*

(i) *if $\theta_\Gamma \in \eta(\mathcal{C})$, then $x \leq x$ for some $x \in \eta(\mathcal{F}(\mathbb{R}))$ (reflexivity),*

(ii) *if $x \leq y$ and $y \leq z$, then $x \leq z$ for some $x, y, z \in \eta(\mathcal{F}(\mathbb{R}))$ (transitivity),*

(iii) *if $x \leq y$ and $\lambda > 0$, then $\lambda x \leq \lambda y$ for some $x, y \in \eta(\mathcal{F}(\mathbb{R}))$,*

(iv) *if $x \leq y$ and $a \leq b$, then $x + a \leq y + b$ for some $x, y, a, b \in \eta(\mathcal{F}(\mathbb{R}))$.*

Proposition 2.24. [27, 29, 32] *Let partial ordering $\leq_H$ on $\eta(\mathcal{F}(\mathbb{R}))$ and $\mathcal{C}$ a convex cone of $\mathcal{F}(\mathbb{R})$. Suppose the function $\eta : \mathcal{F}(\mathbb{R}) \to \Gamma$ is additive and positively homogeneous of degree k. Then the following axioms are satisfied:*

(i) *if $\tilde{1}_{\{0\}} \in \mathcal{C}$, then $x \leq_H x$ for some $x \in \eta(\mathcal{F}(\mathbb{R}))$ (reflexivity),*

(ii) *if $x \leq_H y$ and $y \leq_H z$, then $x \leq_H z$ for some $x, y, z \in \eta(\mathcal{F}(\mathbb{R}))$ (transitivity),*

(iii) *if* $x \leq_H y$ *and* $\lambda > 0$, *then* $\lambda x \leq_H \lambda y$ *for some* $x, y \in \eta(\mathcal{F}(\mathbb{R}))$,

(iv) *if* $x \leq_H y$ *and* $a \leq_H b$, *then* $x + a \leq_H y + b$ *for some* $x, y, a, b \in \eta(\mathcal{F}(\mathbb{R}))$.

Proposition 2.25. [32] *Let* $\mathcal{C}$ *a convex cone of* $\mathcal{F}(\mathbb{R})$. *Suppose the function* $\eta : \mathcal{F}(\mathbb{R}) \to \Gamma$ *is additive and positively homogeneous of degree* k. *Suppose that* $\tilde{\Omega} \subseteq \ker \eta \subseteq \mathcal{C}$. *Then we have the following results:*

(i) *if* $\tilde{a} \preccurlyeq \tilde{b}$, *then* $\eta(\tilde{a}) \leq \eta(\tilde{b})$,

(ii) *if* $\eta(\tilde{a}) \leq \eta(\tilde{b})$, *then* $\tilde{a} \preccurlyeq \tilde{b}$.

Proposition 2.26. [32] *Let* $\mathcal{C}$ *a convex cone of* $\mathcal{F}(\mathbb{R})$. *Suppose the function* $\eta : \mathcal{F}(\mathbb{R}) \to \Gamma$ *is additive and positively homogeneous of degree* k. *Suppose that* $\tilde{\Omega} \subseteq \ker \eta \subseteq \mathcal{C}$. *Then we have the following results:*

(i) *if* $\tilde{a} \preccurlyeq_H \tilde{b}$, *then* $\eta(\tilde{a}) \leq_H \eta(\tilde{b})$,

(ii) *if* $\eta(\tilde{a}) \leq_H \eta(\tilde{b})$, *then* $\tilde{a} \preccurlyeq_H \tilde{b}$.

Below we present the triangular fuzzy numbers and its algebraic operations that we will use in this paper.

Definition 2.27. [22, 28, 33] *A triangular fuzzy number* $\tilde{a}$ *can be defined by a triple* (a^L, a, a^U) *whose membership function* $\mu_{\tilde{a}}(x)$ *given below:*

$$\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{x - a^L}{a - a^L}, & \text{if } \ a^L \leq x < a; \\ \dfrac{a^U - x}{a^U - a}, & \text{if } \ a \leq x < a^U; \\ 0, & \text{else.} \end{cases} \tag{16}$$

Then, the α*-level set* $\tilde{a}$ *is*

$$\tilde{a}_\alpha = \left[(1-\alpha)a^L + \alpha a, (1-\alpha)a^U + \alpha a\right]; \quad \alpha \in [0,1]. \tag{17}$$

such as,

$$\tilde{a}^L(\alpha) = (1-\alpha)a^L + \alpha a \ \text{ and } \ \tilde{a}^U(\alpha) = (1-\alpha)a^U + \alpha a. \tag{18}$$

Definition 2.28. [22, 28] *A triangular fuzzy number* $\tilde{a} = (a^L, a, a^U)$, *is assumed to be non negative triangular fuzzy number if and only if* $a^L \geq 0$.

Definition 2.29. [22, 28] *Let* $\tilde{a} = (a^L, a, a^U)$ *and* $\tilde{b} = (b^L, b, b^U)$ *be any two triangular fuzzy numbers, then fuzzy algebraic operations are as follows:*

(i) *addition:* $\tilde{a} \oplus \tilde{b} = (a^L + b^L, a + b, a^U + b^U)$,

(ii) *subtraction:* $\tilde{a} \ominus \tilde{b} = (a^L - b^U, a - b, a^U - b^L)$,

(iii) *multiplication:* $\tilde{b}$ *to be nonnegative*

$$\tilde{a} \otimes \tilde{b} = \begin{cases} (a^L b^L, ab, a^U b^U), & si\ a^L \geq 0, \\ (a^L b^U, ab, a^U b^U), & si\ a^L < 0, a^U \geq 0, \\ (a^L b^U, ab, a^U b^L), & si\ a^U < 0, \end{cases}$$

and

$$k\tilde{a} = \begin{cases} (ka^L, ka, ka^U) \ \text{if } k \geq 0, \\ \\ (ka^U; ka; ka^L) \ \text{if } k < 0, \end{cases}$$

(iv) *division:* $\tilde{a} \oslash \tilde{b} = (\dfrac{a^L}{b^U}; \dfrac{a}{b}; \dfrac{a^U}{b^L})$.

Definition 2.30. [12, 16] *A ranking function is an application* $\mathcal{R} : \mathcal{F}(\mathbb{R}) \to \mathbb{R}$, *from the set of fuzzy numbers to the set of real numbers such as* $\tilde{a}$ *and* $\tilde{b} \in \mathcal{F}(\mathbb{R})$, *we have the following relations:*

(i) $\tilde{a} \approx_{\mathcal{R}} \tilde{b}$ *if and only if* $\mathcal{R}(\tilde{a}) = \mathcal{R}(\tilde{b})$,

(ii) $\tilde{a} \succeq_{\mathcal{R}} \tilde{b}$ *if and only if* $\mathcal{R}(\tilde{a}) \geq \mathcal{R}(\tilde{b})$,

(ii) $\tilde{a} \preceq_{\mathcal{R}} \tilde{b}$ *if and only if* $\mathcal{R}(\tilde{a}) \leq \mathcal{R}(\tilde{b})$,

(iv) $\tilde{a} \succ_{\mathcal{R}} \tilde{b}$ *if and only if* $\mathcal{R}(\tilde{a}) > \mathcal{R}(\tilde{b})$.

Mathematically the ranking function is defined by the formula

$$\mathcal{R}(\tilde{a}) = \frac{1}{2}\int_0^1 (\inf \tilde{a}_\alpha + \sup \tilde{a}_\alpha)\, d\alpha. \tag{19}$$

For the case of triangular fuzzy numbers $\tilde{a} = (a^L, a, a^U) \in \mathcal{F}(\mathbb{R})$, we have

$$\mathcal{R}(\tilde{a}) = \frac{a^L + 2a + a^U}{4}. \tag{20}$$

2.2 Karush-Kuhn-Tucker optimality conditions

Let f a real-valued function defined on $\mathbb{R}^n$. Consider the following deterministic nonlinear optimization problem with inequality constraints:

$$\begin{cases} \min\ f(x), \\ \\ S = \left\{x \in \mathbb{R}^n : g_j(x) \leq 0, j = \overline{1,m}\right\}. \end{cases} \tag{21}$$

Suppose that the constrained functions g_j are convex on $\mathbb{R}^n$ for each $j = 1, 2, \ldots, m$. Then the feasible set S is a convex set of $\mathbb{R}^n$. The optimality conditions for the problem (21) (see [6] and [14]) is stated below:

Theorem 2.31. *Suppose that the constrained functions $g_j : \mathbb{R}^n \to \mathbb{R}$ are convex on $\mathbb{R}^n$ for $j = 1, 2, \ldots, m$. Let $S = \{x : g_j(x) \leq 0, j = 1, 2, \ldots, m\}$ is a feasible set and a point $\overline{x} \in S$. Suppose the objective function $f : \mathbb{R}^n \to \mathbb{R}$ is convex in $\overline{x}$ and f, g_j, $j = 1, 2, \ldots, m$, are continuously differentiable in $\overline{x}$. If there are (Lagrange) multipliers $0 \leq \mu_j \in \mathbb{R}$, $j = 1, 2, \ldots, m$ such as:*

(i) $\nabla f(\overline{x}) + \sum_{j=1}^{n} \mu_j \nabla g_j(\overline{x}) = 0$;

(ii) $\mu_j g_j(\overline{x}) = 0$ *for some* $j = 1, 2, \ldots, m$.

Then, $\overline{x}$ is an optimal solution of the problem (21).

3 Main results

3.1 Method and algorithm

Let $\mathcal{X}$, a real vector space and $\tilde{f} : \mathcal{X} \to \mathcal{F}(\mathbb{R})$ a fuzzy nonlinear function. Consider the following two fuzzy nonlinear optimization problems:

$$\begin{cases} \min \quad \tilde{f}(x), \\ st \\ \quad \tilde{g}_j(x) \preccurlyeq \tilde{0}, \quad j = \overline{1,m}, \\ \quad x \in \mathbb{R}^n_+, \end{cases} \tag{22}$$

and

$$\begin{cases} \min \quad \tilde{f}(x), \\ st \\ \quad \tilde{g}_j(x) \preccurlyeq_H \tilde{0}, \quad j = \overline{1,m}, \\ \quad x \in \mathbb{R}^n_+, \end{cases} \tag{23}$$

where $\tilde{f}$ and $\tilde{g}_i$, $i = 1, 2, \ldots, m$ are nonlinear canonical fuzzy functions defined on the vector space $\mathcal{X}$. Each of these problems will be solved according to its corresponding solution concept. Note $\tilde{S} = \{x \in \mathbb{R}^n_+ / \tilde{g}_j(x) \preccurlyeq \tilde{0},\ i = \overline{1,m}\}$ and $\tilde{S}^H = \{x \in \mathbb{R}^n_+ / \tilde{g}_j(x) \preccurlyeq_H \tilde{0},\ i = \overline{1,m}\}$ respectively the feasible set of problems (22) and (23).
Consider $\mathcal{C}$ a convex cone of $\mathcal{F}(\mathbb{R})$ and $\forall x \in \tilde{S}$ (or $\forall x \in \tilde{S}^H$), $\tilde{f}(x) \in \mathcal{C}$. Based on the concept of solutions employed by Wu [26], we note by

$$MIN\left(\tilde{f}, \tilde{S}\right) = \left\{\tilde{f}(\overline{x}) :\ \text{there is no } x(\neq \overline{x}) \in \tilde{S} \text{ such as } \tilde{f}(x) \preccurlyeq \tilde{f}(\overline{x})\right\}. \tag{24}$$

the set of all non-dominated values of problem (22) and

$$MIN\left(\tilde{f},\tilde{S}^H\right)^H=\left\{\tilde{f}(\overline{x}):\text{there is no } x(\neq\overline{x})\in\tilde{S}^H \text{ such as } \tilde{f}(x)\preccurlyeq_H\tilde{f}(\overline{x})\right\}. \tag{25}$$

the set of all non-dominated values of problem (23)

Proposition 3.1. (i) *$\overline{x}$ is an optimal solution of problem (22) if and only if $\tilde{f}(\overline{x})\in MIN(\tilde{f},\tilde{S})$.*

(ii) *$\overline{x}$ is an H-optimal solution of problem (23) if and only if $\tilde{f}(\overline{x})\in MIN(\tilde{f},\tilde{S}^H)^H$.*

Proof. (i) Suppose $\overline{x}$ be an optimal solution of problem (22), then there is no $x\in\tilde{S}$ such as $\tilde{f}(x)\preccurlyeq\tilde{f}(\overline{x})$, i.e, $\tilde{f}(\overline{x})\in MIN(\tilde{f},\tilde{S})$.

Conversely, let $\tilde{f}(\overline{x})\in MIN(\tilde{f},\tilde{S})$, then, there is no $x\in\tilde{S}$ such as $\tilde{f}(x)\preccurlyeq\tilde{f}(\overline{x})$, i.e, $\overline{x}$ is an optimal solution of problem (22).

(ii) Likewise that the proof (i), suppose $\overline{x}$ be a H-optimal solution of problem (23), that is to say that there is no $x\in\tilde{S}^H$ such as $\tilde{f}(x)\preccurlyeq_H\tilde{f}(\overline{x})$, Which means that, $\tilde{f}(\overline{x})\in MIN(\tilde{f},\tilde{S}^H)^H$.

Conversely, let $\tilde{f}(\overline{x})\in MIN(\tilde{f},\tilde{S}^H)^H$, then, there is no $x\in\tilde{S}^H$ such as $\tilde{f}(x)\preccurlyeq_H\tilde{f}(\overline{x})$, i.e, $\overline{x}$ is a H-optimal solution of problem (23).

□

Proposition 3.2. [10, 28, 30] *Let $\tilde{f}:\mathcal{X}\to\mathcal{F}(\mathbb{R})$, a fuzzy-valued function defined on an open set $\mathcal{X}$ of $\mathbb{R}^n$ with α-level $\tilde{f}_\alpha=[\tilde{f}_\alpha^L;\tilde{f}_\alpha^U]$. Then, $\tilde{f}$ is convex at $\overline{x}$ if and only if $\tilde{f}_\alpha^L$ and $\tilde{f}_\alpha^U$ are convex at $\overline{x}$ for some $\alpha\in[0,1]$.*

We are now able to present our new approach to solve any type of fuzzy nonlinear problems. This new approach is based on the concept of null set and the Karush-Kunh-Tucker optimality conditions. The main steps can be summarized as follows.

Step 1 Defuzzification.

It consists of the transformation the fuzzy nonlinear optimization problem into a deterministic nonlinear optimization problem. For this purpose we use the function η defined in equation (11) for this operation.

Using this function η from equation (11), then we consider the following two optimization problems:

$$\begin{cases}\min\quad \eta o\tilde{f}(x),\\ st\\ (\eta o\tilde{g}_j(x))\leq\pi(\tilde{0})\quad j=\overline{1,m},\\ x\in\mathbb{R}^n_+.\end{cases} \tag{26}$$

and

$$\begin{cases}\min\quad \eta o\tilde{f}(x),\\ st\\ (\eta o\tilde{g}_j(x))\leq_H\pi(\tilde{0})\quad j=\overline{1,m},\\ x\in\mathbb{R}^n_+.\end{cases} \tag{27}$$

where $\eta(\tilde{0})=(0,0)$, the zero element of $\mathbb{R}^2$ with

$$\eta\circ\tilde{f}(x)=\Big(-\tilde{f}_\alpha^L(x)-\tilde{f}_\alpha^U(x),\tilde{f}_\alpha^L(x)+\tilde{f}_\alpha^U(x)\Big)=\Big(\underline{\tilde{f}_\alpha}(x),\overline{\tilde{f}_\alpha}(x)\Big), \tag{28}$$

and

$$\eta\circ\tilde{g}_j(x)=\Big(-\tilde{g}_{j\alpha}^L(x)-\tilde{g}_{j\alpha}^U(x),\tilde{g}_{j\alpha}^L(x)+\tilde{g}_{j\alpha}^U(x)\Big)=\Big(\underline{\tilde{g}_\alpha}(x),\overline{\tilde{g}_\alpha}(x)\Big),\ j=\overline{1,m}. \tag{29}$$

The functions $\tilde{f}$ and $\tilde{g}_j$, $j=\overline{1,m}$ being canonical fuzzy functions, then $\underline{\tilde{f}_\alpha}$, $\overline{\tilde{f}_\alpha}$, $\underline{\tilde{g}_{j\alpha}}$ and $\overline{\tilde{g}_{j\alpha}}$, $j=\overline{1,m}$, are continuous functions with respect to $\alpha\in[0,1]$. Thus, we can consider the following new functions:

$$\underline{f}(x)=\int_0^1\underline{\tilde{f}_\alpha}(x)d\alpha\ \text{ and }\ \overline{f}(x)=\int_0^1\overline{\tilde{f}_\alpha}d\alpha, \tag{30}$$

and

$$\underline{g}_j(x)=\int_0^1\underline{\tilde{g}_{j\alpha}}(x)d\alpha\ \text{ and }\ \overline{g}_j(x)=\int_0^1\overline{\tilde{g}_{j\alpha}}d\alpha,\ j=\overline{1,m}. \tag{31}$$

Remark 3.3. *From the Proposition 2.25, each constraint* $\tilde{g}_j(x) \preccurlyeq \tilde{0}$ *implies that* $\eta(\tilde{g}_j(x)) \leq \eta(\tilde{0})$. *We have* $(\underline{g}_j(x), \overline{g}_j(x)) \leq (0,0)$ *i.e.* $\underline{g}_j(x) \leq 0$ *and* $\overline{g}_j(x) \leq 0$. *As* $\underline{g}_j(x) \leq \overline{g}_j(x)$, *then* $\underline{g}_j(x) \leq \overline{g}_j(x) \leq 0$. *So, let's simply consider the constraint* $\overline{g}_j(x) \leq 0$, $j = \overline{1,m}$. *Similarly, consider the constraint* $\overline{g}_j(x) \leq_H 0$, $j = \overline{1,m}$.

Based on the equations (30), (31) and the Remark 3.3, the problems (26) and (27) are rewritten into the following deterministic nonlinear bi-objective optimization problems:

$$\begin{cases} \min \quad \left(\underline{f}(x), \overline{f}(x)\right) \\ st \\ \overline{g}_j(x) \leq 0, \quad j = \overline{1,m}, \\ x \in \mathbb{R}^n_+. \end{cases} \tag{32}$$

and

$$\begin{cases} \min \quad \left(\underline{f}(x), \overline{f}(x)\right) \\ st \\ \overline{g}_j(x) \leq_H 0, \quad j = \overline{1,m}, \\ x \in \mathbb{R}^n_+. \end{cases} \tag{33}$$

Theorem 3.4. *Suppose* $\tilde{\Omega} \subseteq \ker(\eta) \subseteq \mathcal{C}$. *If* $\overline{x}$ *is a Pareto optimal solution of the problem (32), then* $\overline{x}$ *is an optimal solution of the problem (22).*

Proof. Let $\overline{x}$ a feasible solution of the problem (22). From the remark 3.3, $\overline{x}$ is a feasible solution of the problem (32). Furthermore, suppose that $\overline{x}$ is a Pareto optimal solution of the problem (32). Then, there is no x $(x \neq \overline{x})$ such as $\eta(\tilde{f}(x)) \leq \eta(\tilde{f}(\overline{x}))$. Using the Proposition 2.25, then there is no x $(x \neq \overline{x})$ such as $\tilde{f}(x) \preccurlyeq \tilde{f}(\overline{x})$ i.e., $\tilde{f}(\overline{x}) \in MIN(\tilde{f}, \tilde{S})$. Therefore $\overline{x}$ is an optimal solution of the problem (22). □

Theorem 3.5. *Suppose* $\tilde{\Omega} \subseteq \ker(\eta) \subseteq \mathcal{C}$. *If* $\overline{x}$ *is a Pareto H-optimal solution of the problem (33), then* $\overline{x}$ *is an optimal solution of the problem (23).*

Proof. Let $\overline{x}$ a feasible solution of the problem (23). From the remark 3.3, $\overline{x}$ is a feasible solution of the problem (33). Furthermore, suppose that $\overline{x}$ is a Pareto H-optimal solution of the problem (33). Then, there is no x $(x \neq \overline{x})$ such as $\eta(\tilde{f}(x)) \leq_H \eta(\tilde{f}(\overline{x}))$. Using the Proposition 2.26, then there is no x $(x \neq \overline{x})$ such as $\tilde{f}(x) \preccurlyeq_H \tilde{f}(\overline{x})$ i.e., $\tilde{f}(\overline{x}) \in MIN(\tilde{f}, \tilde{S}^H)^H$. Therefore $\overline{x}$ is an H-optimal solution of the problem (23). □

Step 2 Résolution.

This step consists of using the Karush-Kunh-Tucker optimality conditions to find the optimal or H-optimal solutions of fuzzy nonlinear problems. To do this we introduce the following propositions.

Proposition 3.6. *Let* $\mathcal{X}$ *is an open interval. If* $\underline{\tilde{f}_\alpha}$ *and* $\overline{\tilde{f}_\alpha}$ *are continuous on* $[0;1] \times \mathcal{X}$ *and admit partial derivatives* $\dfrac{\partial \underline{\tilde{f}_\alpha}}{\partial x_i}$ *et* $\dfrac{\partial \overline{\tilde{f}_\alpha}}{\partial x_i}$ *respectively for* $i = 1, ..., n$, *themselves continuous on* $[0;1] \times \mathcal{X}$. *So the functions* $\underline{f}$ *and* $\overline{f}$ *are well defined on* $\mathcal{X}$, *they are of class* C^1 *and*

$$\forall x \in \mathcal{X}, \frac{\partial \underline{f}(x)}{\partial x_i} = \int_0^1 \frac{\partial \underline{\tilde{f}_\alpha}(x)}{\partial x_i} d\alpha \quad and \quad \frac{\partial \overline{f}(x)}{\partial x_i} = \int_0^1 \frac{\partial \overline{\tilde{f}_\alpha(x)}}{\partial x_i} d\alpha \quad for\ i = 1, ..., n. \tag{34}$$

Proposition 3.7. *Let* $\tilde{f}$ *be a fuzzy-valued function defined on an open subset* $\mathcal{X}$ *of* $\mathbb{R}^n$. *If* $\tilde{f}$ *is continuously differentiable in the neighborhood of* $\overline{x}$. *Then, the real-valued functions* $\underline{f}$ *and* $\overline{f}$ *are continuously differentiable in* $\overline{x}$ *and*

$$\frac{\partial \underline{f}}{\partial x_i}(\overline{x}) = \int_0^1 \frac{\partial \underline{\tilde{f}_\alpha}}{\partial x_i}(\overline{x}) d\alpha \quad et \quad \frac{\partial \overline{f}}{\partial x_i}(\overline{x}) = \int_0^1 \frac{\partial \overline{\tilde{f}_\alpha}}{\partial x_i}(\overline{x}) d\alpha. \tag{35}$$

Proof. Let us show that the partial derivatives $\dfrac{\partial \underline{f}}{\partial x_i}$ and $\dfrac{\partial \overline{f}}{\partial x_i}$ exists in the neighborhood of $\overline{x}$ and are continuous at $\overline{x}$ for some $i = 1, ..., n$. Indeed, the proof will be done only for the $\dfrac{\partial \underline{f}}{\partial x_i}$ and that of $\dfrac{\partial \overline{f}}{\partial x_i}$ is done in the same

way. $\tilde{f}$ being a canonical fuzzy function and assume $\tilde{f}$ continuously differentiable at the neighborhood of $\overline{x}$, i.e. $\underline{\tilde{f}_\alpha}$ and $\overline{\tilde{f}_\alpha}$ are continuously differentiable at $\overline{x}$ for $\alpha \in [0,1]$. Through the Proposition 3.6, $\dfrac{\partial \underline{\tilde{f}_\alpha}}{\partial x_i}$ is continuous at $\overline{x}$ i.e., $\forall \epsilon > 0$, there is a $\delta > 0$ such as
$|x - \overline{x}| < \delta \Rightarrow \left| \dfrac{\partial \underline{\tilde{f}_\alpha}}{\partial x_i}(x) - \dfrac{\partial \underline{\tilde{f}_\alpha}}{\partial x_i}(\overline{x}) \right| < \epsilon$ for each $\alpha \in [0,1]$. From Proposition 3.6, we have: $\mid x - \overline{x} \mid < \delta$ implies that $\forall\ i = 1, ..., n$,

$$\begin{aligned} \left| \frac{\partial \underline{f}}{\partial x_i}(x) - \frac{\partial \underline{f}}{\partial x_i}(\overline{x}) \right| &= \left| \int_0^1 \frac{\partial \underline{\tilde{f}_\alpha}}{\partial x_i}(x) - \int_0^1 \frac{\partial \underline{\tilde{f}_\alpha}}{\partial x_i}(\overline{x}) \right| \\ &= \left| \int_0^1 [\frac{\partial \underline{\tilde{f}_\alpha}}{\partial x_i}(x) - \frac{\partial \underline{\tilde{f}_\alpha}}{\partial x_i}(\overline{x})] d\alpha \right| \\ &\leq \int_0^1 \left| \frac{\partial \underline{\tilde{f}_\alpha}}{\partial x_i}(x) - \frac{\partial \underline{\tilde{f}_\alpha}}{\partial x_i}(\overline{x}) \right| d\alpha < \epsilon. \end{aligned} \tag{36}$$

Therefore $\dfrac{\partial \underline{f}}{\partial x_i}$is continuous $\forall\ i = 1, ..., n$. □

Proposition 3.8. *Let $\tilde{f}$ be a fuzzy-valued function defined on an open subset $\mathcal{X}$ of $\mathbb{R}^n$. If $\tilde{f}$ is continuously H-differentiable in the neighborhood of $\overline{x}$. Then, the real-valued functions $\underline{f}$ and $\overline{f}$ are continuously differentiable in $\overline{x}$ with*

$$\frac{\partial \underline{f}}{\partial x_i}(\overline{x}) = \int_0^1 \frac{\partial \underline{\tilde{f}_\alpha}}{\partial x_i}(\overline{x}) d\alpha \quad et \quad \frac{\partial \overline{f}}{\partial x_i}(\overline{x}) = \int_0^1 \frac{\partial \overline{\tilde{f}_\alpha}}{\partial x_i}(\overline{x}) d\alpha. \tag{37}$$

Proof. The proof is similar to the proof of Proposition 3.7 assuming that the function $\tilde{f}$ continuously H-differentiable in the neighborhood of $\overline{x}$. □

Remark 3.9. *Propositions 3.6, 3.7 and 3.8 are valid for constrained functions $\tilde{g}_j$, $\forall\ j = 1, 2, \ldots, m$.*

Now we present the Karush-Kuhn-Tucker optimality conditions for optimization problems (22) and (23).

Theorem 3.10. *Let $\tilde{\Omega} \subseteq \ker(\pi) \subseteq \mathcal{C}$. If the constrained functions with fuzzy value $\tilde{g}_j : \mathcal{X} \to \mathcal{F}(\mathbb{R})$ are convex on $\mathcal{X}$ and continuously differentiable at $\overline{x} \in \mathcal{X}$ for $i = 1, 2, \ldots, m$, and the objective function $\tilde{f}$ is convex and continuously differentiable at $\overline{x}$. If in addition, there are multipliers (Lagrange) $\mu_j \geq 0$, $j = 1, 2, \ldots, m$ and reals $\lambda_1, \lambda_2 > 0$ $(\lambda_1 \neq \lambda_2)$ such as:*

(i) $\lambda_1 \nabla \underline{f}(\overline{x}) + \lambda_2 \nabla \overline{f}(\overline{x}) + \sum_{i=1}^{m} \mu_i \nabla \overline{g}_j(\overline{x}) = 0, \quad j = 1, 2, \ldots, m$

(ii) $\mu_i \overline{g}_j(\overline{x}) = 0, \quad j = 1, 2, \ldots, m.$

then $\overline{x}$ is an optimal solution of the problem (22).

Proof. We will prove by contradiction. Suppose that $\overline{x}$ is not an optimal solution of the problem (22), then there exists $x(\neq \overline{x}) \in \tilde{S}$ such as $\tilde{f}(x) \preccurlyeq \tilde{f}(\overline{x})$. We have $\eta(\tilde{f}(x)) \leq \eta(\tilde{f}(\overline{x})) \Rightarrow (\underline{f}(x), \overline{f}(x)) \leq (\underline{f}(\overline{x}), \overline{f}(\overline{x}))$.
Let $\lambda_1, \lambda_2 > 0$ with $\lambda_1 \neq \lambda_2$, then

$$\lambda_1 \underline{f}(x) + \lambda_2 \overline{f}(x) \leq \lambda_1 \underline{f}(\overline{x}) + \lambda_2 \overline{f}(\overline{x}) \Rightarrow F(x) \leq F(\overline{x})$$

with $F(x) = \lambda_1 \underline{f}(x) + \lambda_2 \overline{f}(x)$ and $F(\overline{x}) = \lambda_1 \underline{f}(\overline{x}) + \lambda_2 \overline{f}(\overline{x})$.

Suppose $\tilde{f}$ be convex and continuously differentiable in $\overline{x}$, then the functions $\underline{f}$ and $\overline{f}$ are also convex and continuously differentiable in $\overline{x}$ according to the Proposition 3.7. Which implies F is a convex and continuously differentiable function. Moreover $\overline{x} \in \tilde{S}$, we obtain by the conditions of KKT:

(i) $\lambda_1 \nabla \underline{f}(\overline{x}) + \lambda_2 \nabla \overline{f}(\overline{x}) + \sum_{i=1}^{m} \mu_i \nabla \overline{g}_j(\overline{x}) = 0, \quad j = 1, 2, \ldots, m$

(ii) $\mu_i \overline{g}_j(\overline{x}) = 0, \quad j = 1, 2, \ldots, m.$

According to the Theorem 2.31, $\overline{x}$ is an optimal solution of the objective function F under the constraints $\overline{g}_j$, $j = 1, 2, \ldots, m$. As a result $\overline{x}$ is an optimal solution of the problem (22). This contradicts the initial hypothesis. Therefore $\overline{x}$ is an optimal solution of the problem (22). □

Theorem 3.11. *Let $\tilde{\Omega} \subseteq \ker(\eta) \subseteq \mathcal{C}$. If the fuzzy-valued constraint functions $\tilde{g}_j : \mathcal{X} \to \mathcal{F}(\mathbb{R})$ are convex on $\mathcal{X}$ and continuously H-differentiable at $\overline{x} \in \mathcal{X}$ for $j = 1, 2, \ldots, m$ and the objective function $\tilde{f}$ is convex and continuously H-differentiable at $\overline{x}$. If, in addition, there are multipliers of Lagrange $\mu_j \geq 0$, $j = 1, 2, \ldots, m$ and reals $\lambda_1, \lambda_2 > 0$ $(\lambda_1 \neq \lambda_2)$ such as:*

1. $\lambda_1 \nabla \underline{f}(\overline{x}) + \lambda_2 \nabla \overline{f}(\overline{x}) + \sum_{i=1}^{m} \mu_i \nabla \overline{g}_j(\overline{x}) = 0, \quad j = 1, 2, \ldots, m,$
2. $\mu_i \overline{g}_j(\overline{x}) = 0, \quad j = 1, 2, \ldots, m.$

then $\overline{x}$ is an H-optimal solution of the problem (23).

Proof. We prove this theorem 3.11 by absurd. Suppose that $\overline{x}$ is not an H-optimal solution of the problem (23), then there exists $x(\neq \overline{x}) \in \tilde{S}^H$ such as $\tilde{f}(x) \preccurlyeq_H \tilde{f}(\overline{x})$. We have $\eta(\tilde{f}(x)) \leq_H \eta(\tilde{f}(\overline{x})) \Rightarrow (\underline{f}(x), \overline{f}(x)) \leq_H (\underline{f}(\overline{x}), \overline{f}(\overline{x}))$.

Let $\lambda_1, \lambda_2 > 0$ with $\lambda_1 \neq \lambda_2$, then

$$\lambda_1 \underline{f}(x) + \lambda_2 \overline{f}(x) \leq_H \lambda_1 \underline{f}(\overline{x}) + \lambda_2 \overline{f}(\overline{x}) \Rightarrow F(x) \leq_H F(\overline{x}),$$

with $F(x) = \lambda_1 \underline{f}(x) + \lambda_2 \overline{f}(x)$ and $F(\overline{x}) = \lambda_1 \underline{f}(\overline{x}) + \lambda_2 \overline{f}(\overline{x})$.

Let $\tilde{f}$ be convex and continuously H-differentiable in $\overline{x}$, then the functions $\underline{f}$ and $\overline{f}$ are also convex and continuously H-differentiable in $\overline{x}$ according to the Proposition 3.7. Which implies F is a convex and continuously H-differentiable function. Moreover $\overline{x} \in \tilde{S}^H$, we obtain by the conditions of KKT:

(i) $\lambda_1 \nabla \underline{f}(\overline{x}) + \lambda_2 \nabla \overline{f}(\overline{x}) + \sum_{i=1}^{m} \mu_i \nabla \overline{g}_j(\overline{x}) = 0, \quad j = 1, 2, \ldots, m,$

(ii) $\mu_i \overline{g}_j(\overline{x}) = 0, \quad j = 1, 2, \ldots, m.$

From Theorem 2.31, $\overline{x}$ is a H-optimal solution of the objective function F under the constraints $\overline{g}_j$, $j = 1, 2, \ldots, m$. As a result $\overline{x}$ is a H-optimal solution of the problem (23). Which is absurd according to the initial hypothesis. Therefore $\overline{x}$ is a H-optimal solution of the problem (23). □

Step 3 Solution initialization.

In this last step, we use the fuzzy arithmetic operations to transform the obtained deterministic solution to the solution of initial problem which is fuzzy.

Let (P) be the problem (22) or (23), then, the algorithm of the method can be summarized as follows.

Algorithm 1 : Pseudo-code

Data (P), $\lambda_1, \lambda_2 > 0$, $\lambda_1 \neq \lambda_2$, $\mu_j \geq 0, j = 1, ..., m$;

Defuzzify (P) and obtain $\tilde{f} = \big(\underline{f}(x), \overline{f}(x)\big)$, $\tilde{g}_i = \big(\underline{g}_i(x), \overline{g}_i(x)\big)$;

Determine $\nabla \underline{f}(x)$; $\nabla \overline{f}(x)$ and $\nabla \overline{g}_j(x)$, $j = 1, ..., m$

Formulate (P') by $\begin{cases} \lambda_1 \nabla \underline{f}(\overline{x}) + \lambda_2 \nabla \overline{f}(\overline{x}) + \sum_{i=1}^{m} \mu_j \nabla \overline{g}_j(\overline{x}) = 0, \ j = \overline{j, m}; \\ \mu_j \overline{g}_j(\overline{x}) = 0, \ j = \overline{j, m}. \end{cases}$

Repeat

Choose μ_j, λ_1, λ_2

Determine $\overline{x}$ solution of (P') by KKT

check if $\overline{x} \in \tilde{S}$,

Convert $\overline{x}$ to fuzzy value and save it;

else change the values of μ_j, λ_1, λ_2;

End repeat

End

3.2 Numerical examples

In this subsection we solve 11 fuzzy nonlinear problems of different types. For this purpose, we simply consider the partial order $\preccurlyeq$. These problems are summarized in the following table.

Table 1: Problems Tests

Code	Problem formulation	Characteristics	source
P_1	$\begin{cases} \min \ \tilde{f}(x_1, x_2) = (1,2,3) \odot x_1^2 \oplus ((0,1,2) \odot x_2^2, \\ st \\ \quad (0,1,2)(x_1 - 2)^2) \oplus (0,1,2)(x_2 - 2)^2) \preccurlyeq (0,2,4), \\ \quad x_1 \geq 0, \ x_2 \geq 0, \end{cases}$	Fuzzy nonlinear objective, fuzzy nonlinear constraints and deterministic variables	[18]
P_2	$\begin{cases} \min \ \tilde{f}(x_1, x_2) = (1,3,4)x_1^2 \oplus (1,2,3)x_2^2, \\ st \\ \quad (0,1,3)x_1^2 \oplus (2,3,5)x_2^2 \leq (3,4,6), \\ \quad (1,2,4)x_1^2 \ominus (0,1,2)x_2^2 \leq (1,2,5), \\ \quad x_1 \geq 0, \ x_2 \geq 0. \end{cases}$	Fuzzy nonlinear objective, fuzzy nonlinear constraints and deterministic variables	[7]
P_3	$\begin{cases} \min \tilde{f}(x) = (1,2,3) \odot x_1^2 \oplus (0,1,2) \odot x_2^2 \\ st \\ \quad (x_1 - 2)^2 + (x_2 - 2)^2) \leq 1 \\ \quad x_1, x_2 \geq 0. \end{cases}$	Fuzzy nonlinear objective, deterministic nonlinear constraints and deterministic variables	[18]

To modeling the problem P_5, we must assume that unit cost and sale's price of product A and B are set by $UC_1 = c_1$;

P_4	$\begin{cases} \min \tilde{f}(x) = \tilde{3}x_1 \oplus \tilde{2}x_2^2 \\ st \\ \quad (x_1-2)^2 + x_2^2 \leq 4 \\ \quad x_1, x_2 \geq 0. \end{cases}$	Fuzzy nonlinear objective, deterministic nonlinear constraints and deterministic variables	[10]
P_5	A manufacturing factory is going to produce 2 kinds of products A and B in a period (such as one month). The production of A and B requires 3 kinds of resources R_1, R_2 and R_3. The requirements of each kind of resource to produce each product A are 2, 4 and 3 units respectively.To produce each product B, 3, 2 and 2 units are to be required respectively.The planed available resource of R_1 and R_2 are 50 and 44 units respectively but there are additional 30 and 20 units safety store of material which are administrated by the general manager. The estimate value of available quantity of resource R_3 is 36 units, estimate error are 5units. Assuming that the planned production quantity of A and B is x_1 and x_2 respectively.	Deterministic nonlinear objective, fuzzy linear constraints and deterministic variables	[23]
P_6	$\begin{cases} \min \tilde{f}(x) = (1,3,4)x_1^2 \oplus (1,2,3)x_2^2 \\ st \\ \quad (0,1,3)x_1 \oplus (2,3,5)x_2 \preccurlyeq (3,4,6) \\ \quad (1,2,4)x_1 \ominus (0,1,2)x_2 \preccurlyeq (1,2,5) \\ \quad x_1, x_2 \geq 0. \end{cases}$	Fuzzzy nonlinear objective, fuzzy linear constraints and deterministic variables	[7]
P_7	$\begin{cases} \max \tilde{f}(\tilde{x}) = \tilde{x}_1^2 \oplus \tilde{x}_2^2 \\ st \\ \quad (0,1,2)\tilde{x}_1 \oplus (1,2,3)\tilde{x}_2 \preccurlyeq (1,10,27) \\ \quad (1,2,3)\tilde{x}_1 \oplus (0,1,2)\tilde{x}_2 \preccurlyeq (2,11,28) \end{cases}$	Fuzzy nonlinear objective, fuzzy linear constraints and triangular fuzzy variables	[7]
P_8	$\begin{cases} \max \tilde{f}(\tilde{x}) = (1,3,4)\tilde{x}_1^2 \oplus (1,2,3)\tilde{x}_2^2 \\ st \\ \quad (0,1,3)\tilde{x}_1 \oplus (2,3,5)\tilde{x}_2 \preccurlyeq (3,4,6) \\ \quad (1,2,4)\tilde{x}_1 \ominus (0,1,2)\tilde{x}_2 \preccurlyeq (1,2,5) \end{cases}$	Fuzzy nonlinear objective, fuzzy linear constraints and triangular fuzzy variables	[25]
P_9	$\begin{cases} \max \tilde{f}(\tilde{x}) = (0,1,2)\tilde{x}_1 \oplus (0,1,3)\tilde{x}_2^4 \\ st \\ \quad (2,3,4)\tilde{x}_1 \oplus (1,2,3)\tilde{x}_2^2 \preccurlyeq (8,9,10) \\ \quad \tilde{x}_1, \tilde{x}_2 \succcurlyeq \tilde{0} \end{cases}$	Fuzzy nonlinear objective, fuzzy nonlinear constraints and triangular fuzzy variables	[13]
P_{10}	$\begin{cases} \min \tilde{f} = (1,2,3)x^2 \\ st \\ \quad -1 \leq x \leq 2 \end{cases}$	Fuzzy nonlinear objective, deterministic nonlinear constraints and deterministic variables	[10]
P_{11}	$\begin{cases} \max \tilde{f}(x) = (1,3,4)x_1^2 \oplus (1,2,3)x_2^2 \\ st \\ \quad (0,1,3)x_1 \oplus (2,3,5)x_2 \preccurlyeq (3,4,6) \\ \quad (1,2,4)x_1 \ominus (0,1,2)x_2 \preccurlyeq (1,2,5) \\ \quad x_1, x_2 \geq 0. \end{cases}$	Fuzzy nonlinear objective, fuzzy linear constraints and deterministic variables	[1]

$UC_2 = c_2$ and $US_1 = \dfrac{k_1}{x_1^{1/\alpha_1}}$; $US_2 = \dfrac{k_2}{x_2^{1/\alpha_2}}$ respectively. It can be described as

$$\begin{cases} \max f(x) = k_1 x_1^{1-1/\alpha_1} - c_1 x_1 + k_2 x_2^{1-1/\alpha_2} - c_2 x_2 \\ st \\ 2x_1 + 3x_2 \leq \tilde{50} \\ 4x_1 + 2x_2 \leq \tilde{44} \\ 3x_1 + 2x_2 = \tilde{36} \\ x_1, x_2 \geq 0 \\ r = 1,\ k_1 = 50 \\ c_1 = 8.0,\ k_2 = 45,\ c_2 = 10,\ \alpha_1 = \alpha_2 = 2. \end{cases} \tag{38}$$

First, let us solve the problem P_1 of the table 1. This resolution follows the following three (3) steps

Step 1: Defuzzification

For converting of the fuzzy nonlinear program into a deterministic bi-objective we apply the function η positively homogeneous additive of degree 2, we have

$$\begin{cases} \min\ \eta\big((1,2,)\odot x_1^2 \oplus (0,1,2)\odot x_2^2\big), \\ st \\ \qquad \eta\big((0,1,2)\odot (x_1-2)^2) \oplus (0,1,2)\odot (x_2-2)^2\big) \preccurlyeq \eta\big((0,2,4)\big), \\ \qquad x_1 \geq 0,\ x_2 \geq 0, \end{cases} \tag{39}$$

By using Remark 3.3 the problem (39) is rewritten into a bi-objective nonlinear deterministic problem as follows:

$$\begin{cases} \min\ \big(\underline{f}, \overline{f}\big) = (-4x_1^2 - 2x_2^2,\ 4x_1^2 + 2x_2^2\big), \\ st \\ \qquad 2(x_1-2)^2 + 2(x_2-2)^2 \leq 4, \\ \qquad x_1 \geq 0,\ x_2 \geq 0. \end{cases} \tag{40}$$

Step 2: Resolution

Karush-Kuhn- Tucker optimality condition is applied according to the Theorem 3.10. Let us set $\lambda_1 = 1$, $\lambda_2 = 2$, then we have the following Karush-Kuhn-Tucker conditions:

$$\begin{cases} 8x_1 + 4\mu(x_1 - 2) = 0 \\ 4x_2 + 4\mu(x_2 - 2) = 0 \\ \mu(2(x_1-2)^2 + 2(x_2-2)^2 - 4) = 0. \end{cases} \tag{41}$$

By solving the equation 41, we get the solution $(x_1, x_2) = (0.8432874, 1.186356)$ and $\mu = 1.458077$. Through the Theorem 3.10, the solution $\overline{x} = (\overline{x}_1, \overline{x}_2) = (0.8432874, 1.186356)$ is an optimal solution of the problem (39)

Step 3: Initilization

Using fuzzy algebraic operations, we transform the deterministic solution obtained into a fuzzy solution, thus we have $\tilde{f}_{\min} = (0.71, 2.83, 4.94)$.

Secondly, either to solve the problem P_8 of the table. We have fuzzy variables in this problem. So we pose $\tilde{x}_1 = (x_1, y_1, z_1)$ and $\tilde{x}_2 = (x_2, y_2, z_2)$.

We have

$$\begin{cases} \max \tilde{f}(\tilde{x}) = (1,3,4)(x_1,y_1,z_1) \otimes (x_1,y_1,z_1) \oplus (1,2,3)(x_2,y_2,z_2) \otimes (x_2,y_2,z_2) \\ st \\ \qquad (0,1,3)(x_1,y_1,z_1) \oplus (2,3,5)(x_2,y_2,z_2) \preccurlyeq (3,4,6) \\ \qquad (1,2,4)(x_1,y_1,z_1) \ominus (0,1,2)(x_2,y_2,z_2) \preccurlyeq (1,2,5) \end{cases} \tag{42}$$

Using fuzzy algebraic operations, problem (42) is rewritten as follows:

$$\begin{cases} \max \tilde{f}(\tilde{x}) = (x_1^2, 3y_1^2, 4z_1^2) \oplus (x_2^2, 2y_2^2, 3z_2^2) \\ st \\ \qquad (2x_2, y_1 + 3y_2, 3z_1 + 5z_2) \preccurlyeq (3,4,6) \\ (x_1 - 2x_2, 2y_1 - y_2, \qquad 4z_1) \preccurlyeq (1,2,5) \\ x_1, x_2, y_1, y_2, z_1, z_2 \geq 0 \end{cases} \tag{43}$$

By transforming problem (43) into a minimization problem and after defuzzification using the function η, we obtain:

$$\begin{cases} \min f(x) = \left(-\frac{1}{2}x_1^2 - 3y_1^2 - 2z_1^2 - \frac{1}{2}x_2^2 - 2y_2^2 - \frac{3}{2}z_2^2, \frac{1}{2}x_1^2 + 3y_1^2 + 2z_1^2 + \frac{1}{2}x_2^2 + 2y_2^2 + \frac{3}{2}z_2^2\right) \\ st \\ \quad y_1 + \frac{3}{2}z_1 + x_2 + 3y_2 + \frac{5}{2}z_2 \leq 8.5 \\ \quad \frac{1}{2}x_1 + 2y_1 + 2z_1 - x_2 - y_2 \leq 5 \\ \quad x_1, x_2, y_1, y_2, z_1, z_2 \geq 0 \end{cases} \tag{44}$$

By the conditions of KKT and taking $\lambda_1 = 1$, $\lambda_2 = 2$, we have the following equations:

$$\begin{cases} x_1 + \mu_2 = 0 \\ 6y_1 + \mu_1 + 2\mu_2 = 0 \\ 4z_1 + \frac{3}{2}\mu_1 + 2\mu_2 = 0 \\ x_2 + \mu_1 - \mu_2 = 0 \\ 4y_2 + 3\mu_1 - \mu_2 = 0 \\ 3z_2 + \frac{5}{2}\mu_1 = 0 \\ \mu_1(y_1 + \frac{3}{2}z_1 + x_2 + 3y_2 + \frac{5}{2}z_2 - 8,5) = 0 \\ \mu_2(\frac{1}{2}x_1 + 2y_1 + 2z_1 - x_2 - y_2 - 5) = 0 \end{cases} \tag{45}$$

After solving equation (45) we obtain as a solution $\tilde{x}_1 = (x_1, y_1, y_1) = (0, 0, 0)$, $\tilde{x}_2 = (x_2, y_2, y_2) = (0, 0, 0)$, $\mu_1 = \mu_2$ and $\tilde{f}_{\min} = 0$.

Thus, the other problems in table 1 are resolved, the results obtained and others from existing methods in the literature are summarized in the following table:

Table 2: Results of test problems

	Methods	**Solutions**	**Ranking function value**	**best method**
P_1	Our method	$(0.71, 2.833, 4.94)$	**2.82**	Our method
	Pathak et al's method	$(0.84, 3.28, 5.71)$	3.27	
P_2	Our method	$(1.83, 5.07, 6.89)$	**6.95**	Our method
	Kumar et al's method	$(2.65, 7.55, 10.05)$	6.95	
P_3	Our method	$(1.44, 5.13, 8.82)$	**5.13**	Both
	Pathak et al's method	$(1.44, 5.13, 8.82)$	5.13	
P_4	Our method	$(0, 0, 0)$	**0**	Both
	Chalco-Cano et al's method.	$(0, 0, 0)$	0	
P_5	Our method	128.75	128.75	Both
	Jiafu Tang et al's method	128.75	128.75	
P_6	Our method	$(0, 0, 0)$	**0**	Our method
	Kumar et al's method	$(2.65, 7.53, 10.05)$	6.87	
P_7	Our method	$(0, 0, 0)$	**0**	Our method
	Kumar et al's method	$(2.77, 32.00, 68, 11)$	$23, 62$	
	Ganesan method	$(25, 9 - 9r, 17 - 17r), \ \ r \in [0, 1]$	$10.62, \ \ r = 0.5$	
P_8	Our method	$(0, 0, 0)$	**0**	Our method
	Ganesan method	$(7.59, 2 - 2r, 3 - 3r), \ r \in [0, 1]$	$2.11 \ r \in 0.5$	
P_9	Our method	$(0, 0, 0)$	**0**	Our method
	Bharathi method	$(19.45, 1 - k, 1 - k), \ \ k \in [0, 1]$	$5.6, \ \ k = 0.5$	
P_{10}	Our method	$(0, 0, 0)$	0	Both
	Chalco-Cano method	$(0, 0, 0)$	0	
P_{11}	Our method	$(0, 0, 0)$	**0**	Our method
	Akrami method	$(0.36, 7.59, 5.11)$	5.16	

3.3 Discussion

Table 2, presents the results of a comparison between our method and seven other methods on the eleven test problems presented above. We have the following results:

- our method gives the best solution for the problems P_1, P_2, P_6, P_7, P_8, P_9 and P_{11};
- both methods gives the same solution for the problems P_3, P_4, P_5 and P_{11}.

According to the characteristics of test problems given in the table 1, we can formulate that our method is best than other when the test problem owns deterministic variable, or objective function or constraint function. Therefore, we can say that our method is the best option for solving nonlinear, single-objective, fuzzy optimization problems.

4 Conclusions

This work studied the combination of the null concept with the Karush-Kuhn-Tucker's optimality conditions for minimizing a constrained objective function. The new method has been used to solve fuzzy nonlinear optimization problems and find the best solutions. In practice, any fuzzy nonlinear optimization problem is transformed into a deterministic bi-objective nonlinear optimization problem. The optimality conditions of Karush-Kuhn-Tucker are then directly used to find optimal solutions. Eleven test problems have been treated in order to highlight the theoretical and numerical aspects, and the found results have been compared with others of existing methods in the literature. This comparison has allowed us to conclude that our method is an adequate for solving fuzzy nonlinear optimization problems with fuzzy constraint functions.

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