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Fuzzy fractional calculus: A comprehensive overview with a focus on weighted Caputo-type generalized Hukuhara differentiability and analytical solutions for fuzzy fractional differential equations

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Abstract

This paper introduces a novel approach to obtaining analytical solutions for fuzzy fractional differential equations in the context of weighted Caputo-type generalized Hukuhara derivatives. The paper proposes the use of non-singular kernels to improve the accuracy of fractional calculus in fuzzy space and establishes the uniqueness of solutions for fuzzy fractional differential equations. The paper also introduces the concept of fuzzy Laplace transforms to facilitate the solution of these equations. Practical examples, such as the fuzzy fractional Newton's law of heating and cooling, are provided to demonstrate the effectiveness of the proposed method. Overall, this paper contributes to the development of practical solutions for real-world problems in fuzzy space and enhances the accuracy of fractional calculus in this context.

Keywords: Weighted Caputo-type generalized Hukuhara derivative, fuzzy fractional differential equations, fuzzy analytical solutions, uniqueness of solutions, Newton's law of heating and cooling.

1 Introduction

The appropriate type of fuzzy fractional derivative must be selected to ensure accurate qualitative analysis of fractional differential equations in fuzzy space. Two types of fuzzy fractional derivatives exist: classical ones with singular kernels, such as fuzzy Riemann-Liouville [1, 4, 8, 9, 25], and fuzzy Caputo generalized Hukuhara derivatives[2, 7, 17, 19, 27, 28], and non-singular ones, such as fuzzy Atangana-Baleanu [3, 6], and fuzzy Caputo-Fabrizio derivatives [3, 6], which should be chosen based on the problem's characteristics in fuzzy space.

The primary objective of this study is to put forth a novel definition of fractional derivative, the fuzzy generalized weighted Caputo derivative. This new definition extends the previously mentioned fractional derivatives that involve non-singular kernels, such as the fuzzy Atangana-Baleanu and fuzzy Caputo-Fabrizio derivatives for Caputo-type derivative. The concept of the generalized weighted Caputo derivative for real-valued functions was originally introduced in a prior work [15], and subsequent research has delved into the exploration of this form of differentiability [16, 22, 23].

Building upon the groundwork laid out in the aforementioned reference [22], for the first time we introduce the concepts of fuzzy generalized weighted fractional integrals and the weighted Caputo-type generalized Hukuhara derivative for fuzzy functions. Our proposed derivative represents a broader framework encompassing the fuzzy Caputo-Fabrizio fractional derivatives and Atangana-Baleanu fractional derivatives with non-singular kernels.

This paper primarily seeks to address the challenge of identifying an appropriate type of fractional derivative for conducting qualitative analyses of fractional differential equations within a fuzzy space. In this pursuit, we put forth the utilization of the weighted Caputo-type generalized Hukuhara derivative with non-singular kernels, specifically designed for fuzzy functions. The core objectives of this paper can be summarized as follows:

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Introduction and exploration : We introduce and delve into the concept of the weighted Caputo-type generalized Hukuhara derivative with non-singular kernels for fuzzy functions. We substantiate our approach by proving key properties associated with this type of fuzzy derivative.

Establishing uniqueness : We establish the uniqueness of solutions for fuzzy fractional differential equations, drawing on the various types of fuzzy weighted Caputo-type generalized Hukuhara derivatives.

Novel Fuzzy methodology : We propose an innovative fuzzy methodology employing the fuzzy Laplace transform to derive analytical solutions for fuzzy fractional differential equations. Notably, this methodology operates within the framework of the fuzzy weighted Caputo-type generalized Hukuhara derivative, eliminating the need for implicit embeddings into crisp equations.

Effectiveness demonstration : We demonstrate the effectiveness of our proposed approach through practical examples, including its application to the fuzzy fractional version of Newton's law of heating and cooling. Through these examples, we showcase the applicability and robustness of our methodology in real-world scenarios.

In essence, this paper aims to contribute to the understanding and application of fractional derivatives within the context of fuzzy space and their qualitative analysis, showcasing the utility of the weighted Caputo-type generalized Hukuhara derivative with non-singular kernels for fuzzy functions.

In this paper, we provide a comprehensive overview of fuzzy mathematics, with a focus on generalized Hukuhara differentiability, in Section 2. We then introduce a fuzzy generalized fractional integral and a weighted Caputo-type generalized Hukuhara derivative in Sections 3 and 4, respectively, and establish some of their key properties. To facilitate the solution of fuzzy fractional differential equations, we explain fuzzy Laplace transforms and their properties in Section 5. In Section 6, we consider the fuzzy fractional differential equation of the weighted Caputo-type generalized Hukuhara operator with an initial value problem and prove the existence of a unique solution. We then use the Laplace transform to obtain fuzzy analytical solutions for fractional fuzzy differential equations in terms of weighted Caputo-type $\mathtt{gH}-$differentiability in Section 7. Finally, we conclude the paper in Section 8 with some overall conclusions.

2 Fundamental backgrounds

In this section, we will explore various definitions and findings and introduce the notations that will be utilized throughout this paper.

The class of fuzzy sets on $\mathbb{R}$ includes fuzzy numbers. A fuzzy set on $\mathbb{R}$ is classified as a fuzzy number when every $\mathtt{r}-$cut in the set is both non-empty and bounded, and the set is closed [14]. Fuzzy numbers are denoted by the symbol $\Re_{\mathfrak{f}}$ in this paper. For $\breve{\vartheta} \in \Re_{\mathfrak{f}}$ and $\mathtt{r} \in [0,1]$, we use the symbol $[\vartheta_{\mathtt{r}}^{-}, \vartheta_{\mathtt{r}}^{+}]$ to represent $[\breve{\vartheta}]_{\mathtt{r}}$.

Suppose $\mathcal{I} := [a,b] \subset \mathbb{R}$ is a bounded interval. The map $\breve{\mathtt{u}} : \mathcal{I} \to \Re_{\mathfrak{f}}$ is a fuzzy-valued function, and the parametric form of this fuzzy function is a pair $[\breve{\mathtt{u}}(\mathfrak{k})]_{\mathtt{r}} = [\mathtt{u}_{\mathtt{r}}^{-}(\mathfrak{k}), \mathtt{u}_{\mathtt{r}}^{+}(\mathfrak{k})]$ for all $\mathtt{r} \in [0,1]$ and $\mathfrak{k} \in \mathcal{I}$. Our aim now is to examine some definitions and findings that are necessary for fuzzy calculus, utilizing definitions and results from [10, 11, 12].

Definition 2.1. (Fuzzy continuous function) *A fuzzy-valued function $\breve{\mathtt{u}} : \mathcal{I} \to \Re_{\mathfrak{f}}$ is said to be fuzzy continuous at $\mathfrak{k}_0 \in \mathcal{I}$ if for every $\varepsilon > 0$ and $t \in \mathcal{I}$, there exists a $\delta > 0$ such that $\mathcal{D}\left(\breve{\mathtt{u}}(\mathfrak{k}), \breve{\mathtt{u}}\left(\mathfrak{k}_0\right)\right) < \varepsilon$ whenever $|\mathfrak{k} - \mathfrak{k}_0| < \delta$, where $\mathcal{D}$ is the Hausdorff distance* [20].

Furthermore, if the fuzzy function is continuous for all $\mathfrak{k} \in \mathcal{I}$, it is said to be continuous on $\mathcal{I}$.

Definition 2.2. [24](Fuzzy piecewise continuous function) *A fuzzy piecewise continuous function $\breve{\mathtt{u}}$ defined on $[0,\infty)$ satisfies the following conditions:*

i. *The limit of $\breve{\mathtt{u}}$ as $\mathfrak{k}$ approaches 0 from the right is equal to $\breve{\mathtt{u}}(0^{+})$.*

ii. *$\breve{\mathtt{u}}$ is continuous on every finite interval $(0,b)$, except at a finite number of points $\mathfrak{k}_1, \mathfrak{k}_2, ..., \mathfrak{k}_n$ in $(0,b)$, where it may have jump discontinuities.*

Definition 2.3. (Fuzzy Riemann integral) *A function $\breve{\mathtt{u}}$ is said to be Riemann integrable on the interval $\mathcal{I}$ if there exists a fuzzy number $\mathbb{I}$ such that, for any given tolerance level $\varepsilon > 0$, there exists a positive $\delta > 0$ such that for any partition $d : a = \mathfrak{k}_0 < \ldots < \mathfrak{k}_n = b$ of norm $\nu(d) < \delta$ and any set of sample points $\xi_i \in [\mathfrak{k}_i, \mathfrak{k}_{i+1}]$ for $i = 0, \ldots, n-1$,*

$$\mathcal{D}\left(\sum_{i=0}^{n-1} \breve{\mathtt{u}}\left(\xi_i\right)\left(\mathfrak{k}_{i+1} - \mathfrak{k}_i\right), \mathbb{I}\right) < \varepsilon.$$

The Riemann integral of $\breve{\mathfrak{u}}$ *over the interval* $\mathcal{I}$ *is denoted by* $\int_a^b \breve{\mathfrak{u}}(\mathfrak{k})\mathrm{d}\mathfrak{k}$*, which is a fuzzy number* $\mathbb{I}$*.*

The lower and upper functions of $\breve{\mathfrak{u}}$*, denoted by* $\mathfrak{u}_{\mathtt{r}}^{-}(\mathfrak{k})$ *and* $\mathfrak{u}_{\mathtt{r}}^{+}(\mathfrak{k})$ *for all* $\mathtt{r} \in [0,1]$*, respectively, are used to define the fuzzy Riemann integral of* $\breve{\mathfrak{u}}$*. Specifically, the fuzzy Riemann integral over* $\mathcal{I}$ *is defined as*

$$\left[\int_a^b \breve{\mathfrak{u}}(\mathfrak{k})\mathrm{d}\mathfrak{k}\right]_{\mathtt{r}} = \left[\int_a^b \mathfrak{u}_{\mathtt{r}}^{-}(\mathfrak{k})\mathrm{d}\mathfrak{k}, \int_a^b \mathfrak{u}_{\mathtt{r}}^{+}(\mathfrak{k})\mathrm{d}\mathfrak{k}\right].$$

Definition 2.4. (Fuzzy generalized Hukuhara derivative) *The* gH*-derivative of a fuzzy-valued function* $\breve{\mathfrak{u}}(\mathfrak{k})$*, also known as the fuzzy generalized Hukuhara derivative, is defined as follows:*

$$\breve{\mathfrak{u}}'_{\mathtt{gH}}(\mathfrak{k}) = \lim_{\hbar \to 0} \frac{\breve{\mathfrak{u}}(\mathfrak{k}+\hbar) \ominus_{gH} \breve{\mathfrak{u}}(\mathfrak{k})}{\hbar}.$$

Provided that $\breve{\mathfrak{u}}'_{\mathtt{gH}}(\mathfrak{k}) \in \Re_{\mathfrak{f}}$*. Here,* $\breve{\mathfrak{u}}(\mathfrak{k}+\hbar) \ominus_{\mathtt{gH}} \breve{\mathfrak{u}}(\mathfrak{k})$ *represents the generalized Hukuhara difference of two fuzzy functions* $\breve{\mathfrak{u}}(\mathfrak{k}+\hbar)$ *and* $\breve{\mathfrak{u}}(\mathfrak{k})$ [10].

Let $\breve{\mathfrak{u}}(\mathfrak{k})$ *be a* $\mathtt{gH}-$*differentiable fuzzy function and* $\mathfrak{u}_{\mathtt{r}}^{-}(\mathfrak{k})$*,* $\mathfrak{u}_{\mathtt{r}}^{+}(\mathfrak{k})$ *both differentiable at* $\mathfrak{k} \in \mathcal{I}$ *for all* $\mathtt{r} \in [0,1]$*. This function is called* $[\mathtt{1.gH}]-$*differentiable over* $\mathcal{I}$ *(which is symbolized as* $\breve{\mathfrak{u}}'_{\mathtt{1.gH}}(\mathfrak{k})$ *), if*

$$\left[\breve{\mathfrak{u}}'_{\mathtt{1.gH}}(\mathfrak{k})\right]_{\mathtt{r}} = \left[\mathfrak{u}_{\mathtt{r}}^{-\prime}(\mathfrak{k}), \mathfrak{u}_{\mathtt{r}}^{+\prime}(\mathfrak{k})\right].$$

Similarly, a gH*-differentiable fuzzy function* $\breve{\mathfrak{u}}(\mathfrak{k})$ *is said to be* $[\mathtt{2.gH}]$*-differentiable over* $\mathcal{I}$*, denoted by* $\breve{\mathfrak{u}}'_{\mathtt{2.gH}}(\mathfrak{k})$*, if it satisfies:*

$$\left[\breve{\mathfrak{u}}'_{\mathtt{2.gH}}(\mathfrak{k})\right]_{\mathtt{r}} = \left[\mathfrak{u}_{\mathtt{r}}^{+\prime}(\mathfrak{k}), \mathfrak{u}_{\mathtt{r}}^{-\prime}(\mathfrak{k})\right].$$

Theorem 2.5. [13, 18] *Let* $\breve{\mathfrak{u}} : \mathcal{I} \to \Re_{\mathfrak{f}}$ *be a fuzzy generalized Hukuhara differentiable function on* $\mathcal{I}$*, where the type of* $\mathtt{gH}-$*differentiability remains the same on* $\mathcal{I}$*. Let* $\omega(\mathfrak{k})$ *be a real-valued continuous function such that* $\omega, \omega' > 0$ *for all* $t \in \mathcal{I}$*. Then, the following cases hold true:*

1. *If* $\breve{\mathfrak{u}}(\mathfrak{k})$ *is* $[\mathtt{1.gH}]-$*differentiable, then* $\omega(\mathfrak{k})\breve{\mathfrak{u}}(\mathfrak{k})$ *is* $[(i)-gH]-$*differentiable and*

$$\Big(\omega(\mathfrak{k})\breve{\mathfrak{u}}(\mathfrak{k})\Big)'_{\mathtt{1.gH}} = \omega(\mathfrak{k})\breve{\mathfrak{u}}'_{\mathtt{1.gH}}(\mathfrak{k}) \oplus \omega'(\mathfrak{k})\breve{\mathfrak{u}}(\mathfrak{k}).$$

2. *If* $\breve{\mathfrak{u}}(\mathfrak{k})$ *is* $[\mathtt{2.gH}]-$*differentiable, then* $\omega(\mathfrak{k})\breve{\mathfrak{u}}(\mathfrak{k})$ *is* $[(ii)-gH]-$*differentiable and*

$$\Big(\omega(\mathfrak{k})\breve{\mathfrak{u}}(\mathfrak{k})\Big)'_{\mathtt{2.gH}} = \omega(\mathfrak{k})\breve{\mathfrak{u}}'_{\mathtt{2.gH}}(\mathfrak{k}) \ominus (-1)\omega'(\mathfrak{k})\breve{\mathfrak{u}}(\mathfrak{k}).$$

Provided that the Hukuhara differences exists.

3 Generalizing weighted fractional integrals through fuzzy calculus techniques

In this section, we will introduce the concept of fuzzy generalized weighted fractional integrals. We consider the set denoted as $\mathcal{K}(\mathcal{I})$, which encompasses all fuzzy functions denoted as $\breve{\mathfrak{u}}$, defined on the interval $\mathcal{I}$. It is a requirement that both the fuzzy function $\breve{\mathfrak{u}}$ and its generalized H-differentiation, denoted as $\breve{\mathfrak{u}}'_{\mathtt{gH}}$, exhibit qualities of being fuzzy continuous and Riemann integrable functions on the interval $\mathcal{I}$. Furthermore, we stipulate that the type of their gH-differentiability should remain the same throughout their entire domain, with no switching points.

Definition 3.1. (Fuzzy generalized weighted fractional integral) *Let* $\wp$ *and* β *be parameters such that* $\wp \in [0,1)$*,* $\beta \in (0,1]$ *and* $\breve{\mathfrak{u}} \in \mathcal{K}(\mathcal{I})$*. The fuzzy generalized weighted fractional integral of* $\breve{\mathfrak{u}}(\mathfrak{k})$ *of order* $\wp$ *is given by:*

$$\mathbb{I}_{a,\omega}^{\wp,\beta}\breve{\mathfrak{u}}(\mathfrak{k}) = \frac{1-\wp}{\aleph(\wp)}\breve{\mathfrak{u}}(\mathfrak{k}) \oplus \frac{\wp}{\aleph(\wp)\Gamma(\beta)} \frac{1}{\omega(\mathfrak{k})} \int_a^{\mathfrak{k}} (\mathfrak{k}-\xi)^{\beta-1}\omega(\xi)\breve{\mathfrak{u}}(\xi)\mathrm{d}\xi. \tag{1}$$

where ω *is a real continuous function such that* $\omega, \omega' > 0$ *on* $\mathcal{I}$*, and* $\aleph(\wp)$ *is the normalization function that satisfies* $\aleph(0) = \aleph(1) = 1$*. The function* $\aleph(\wp)$ *can take two different forms, which are:*

$$\aleph(\wp) = 1, \qquad or \qquad \aleph(\wp) = 1 - \wp + \frac{\wp}{\Gamma(\wp)}.$$

Theorem 3.2. *Suppose $\breve{\mathfrak{u}} : \mathcal{I} \to \Re_{\mathfrak{f}}$ is a fuzzy function in $\mathcal{K}(\mathcal{I})$, and let $\wp \in [0,1)$ and $\beta \in (0,1]$. Then we have the following properties of the fuzzy generalized weighted fractional integral $\mathbb{I}^{\wp,\beta}_{a,\omega}\breve{\mathfrak{u}}(\mathfrak{k})$:*

i. *It is a fuzzy function.*

ii. $\left[\mathbb{I}^{\wp,\beta}_{a,\omega}\breve{\mathfrak{u}}(\mathfrak{k})\right]_{\mathfrak{r}} = \left[I^{\wp,\beta}_{a,\omega}\mathfrak{u}^{-}_{\mathfrak{r}}(\mathfrak{k}), I^{\wp,\beta}_{a,\omega}\mathfrak{u}^{+}_{\mathfrak{r}}(\mathfrak{k})\right].$

This means that the fuzzy generalized weighted fractional integral is a fuzzy number with a range of possible crisp values determined by the fuzzy lower and upper bounds of the input fuzzy function $\breve{\mathfrak{u}}$, and it satisfies the properties of the crisp weighted fractional integral for each possible crisp value.

Proof. Let us begin by demonstrating the status (i): Using $[\breve{\mathfrak{u}}(\mathfrak{k})]_{\mathfrak{r}} = [\mathfrak{u}^{-}_{\mathfrak{r}}(\mathfrak{k}), \mathfrak{u}^{+}_{\mathfrak{r}}(\mathfrak{k})]$, $\omega > 0$, and $(\mathfrak{k}-\xi)^{\beta-1} \geq 0$, we can show that for all $\wp \in [0,1)$ and $\beta \in (0,1]$:

$$\int_a^{\mathfrak{k}} (\mathfrak{k}-\xi)^{\beta-1}\omega(\xi)\mathfrak{u}^{-}_{\mathfrak{r}}(\xi)\mathrm{d}\xi \leq \int_a^{\mathfrak{k}} (\mathfrak{k}-\xi)^{\beta-1}\omega(\xi)\mathfrak{u}^{+}_{\mathfrak{r}}(\xi)\mathrm{d}\xi, \quad \forall \mathfrak{k} \in \mathcal{I}.$$

Next, we can use the coefficients for all $\wp \in [0,1)$ and $\beta \in (0,1]$ such that $\frac{1-\wp}{\aleph(\wp)}, \frac{\wp}{\aleph(\wp)\Gamma(\beta)}\frac{1}{\omega(\mathfrak{k})} \geq 0$. By inserting these coefficients, we obtain the following:

$$\begin{aligned}
&\frac{1-\wp}{\aleph(\wp)}\mathfrak{u}^{-}_{\mathfrak{r}}(\mathfrak{k}) + \frac{\wp}{\aleph(\wp)\Gamma(\beta)}\frac{1}{\omega(\mathfrak{k})}\int_a^{\mathfrak{k}} (\mathfrak{k}-\xi)^{\beta-1}\omega(\xi)\mathfrak{u}^{-}_{\mathfrak{r}}(\xi)\mathrm{d}\xi \\
&\qquad \leq \frac{1-\wp}{\aleph(\wp)}\mathfrak{u}^{+}_{\mathfrak{r}}(\mathfrak{k}) + \frac{\wp}{\aleph(\wp)\Gamma(\beta)}\frac{1}{\omega(\mathfrak{k})}\int_a^{\mathfrak{k}} (\mathfrak{k}-\xi)^{\beta-1}\omega(\xi)\mathfrak{u}^{+}_{\mathfrak{r}}(\xi)\mathrm{d}\xi.
\end{aligned}$$

For all $\mathfrak{k} \in \mathcal{I}$, $r \in [0,1]$, $\wp \in [0,1)$, and $\beta \in (0,1]$, we have $I^{\wp,\beta}_{a,\omega}\mathfrak{u}^{-}_{\mathfrak{r}}(\mathfrak{k}) \leq I^{\wp,\beta}_{a,\omega}\mathfrak{u}^{+}_{\mathfrak{r}}(\mathfrak{k})$, which implies that the set $\Omega_{\mathfrak{r}}(\mathfrak{k}) := \left[I^{\wp,\beta}_{a,\omega}\mathfrak{u}^{-}_{\mathfrak{r}}(\mathfrak{k}), I^{\wp,\beta}_{a,\omega}\mathfrak{u}^{+}_{\mathfrak{r}}(\mathfrak{k})\right]$ is a compact and convex subset of $\mathbb{R}$. Moreover, $\Omega_1(\mathfrak{k}) \neq \varnothing$, so we can conclude that $\Omega_{\mathfrak{r}}(\mathfrak{k})$ belongs to the set of real numbers $\Re_{\mathfrak{f}}$ [14].

Case (ii) can be demonstrated using the evidence presented below:

$$\begin{aligned}
\Omega_{\mathfrak{r}}(\mathfrak{k}) &= \left[I^{\wp,\beta}_{a,\omega}\mathfrak{u}^{-}_{\mathfrak{r}}(\mathfrak{k}), I^{\wp,\beta}_{a,\omega}\mathfrak{u}^{+}_{\mathfrak{r}}(\mathfrak{k})\right] \\
&= \Big[\frac{1-\wp}{\aleph(\wp)}\mathfrak{u}^{-}_{\mathfrak{r}}(\mathfrak{k}) + \frac{\wp}{\aleph(\wp)\Gamma(\beta)}\frac{1}{\omega(\mathfrak{k})}\int_a^{\mathfrak{k}} (\mathfrak{k}-\xi)^{\beta-1}\omega(\xi)\mathfrak{u}^{-}_{\mathfrak{r}}(\xi)\mathrm{d}\xi, \\
&\qquad \frac{1-\wp}{\aleph(\wp)}\mathfrak{u}^{+}_{\mathfrak{r}}(\mathfrak{k}) + \frac{\wp}{\aleph(\wp)\Gamma(\beta)}\frac{1}{\omega(\mathfrak{k})}\int_a^{\mathfrak{k}} (\mathfrak{k}-\xi)^{\beta-1}\omega(\xi)\mathfrak{u}^{+}_{\mathfrak{r}}(\xi)\mathrm{d}\xi\Big] \\
&= \frac{1-\wp}{\aleph(\wp)}\left[\mathfrak{u}^{-}_{\mathfrak{r}}(\mathfrak{k}), \mathfrak{u}^{+}_{\mathfrak{r}}(\mathfrak{k})\right] \oplus \frac{\wp}{\aleph(\wp)\Gamma(\beta)}\frac{1}{\omega(\mathfrak{k})}\left[\int_a^{\mathfrak{k}} (\mathfrak{k}-\xi)^{\beta-1}\omega(\xi)\mathfrak{u}^{-}_{\mathfrak{r}}(\xi)\mathrm{d}\xi, \int_a^{\mathfrak{k}} (\mathfrak{k}-\xi)^{\beta-1}\omega(\xi)\mathfrak{u}^{+}_{\mathfrak{r}}(\xi)\mathrm{d}\xi\right] \\
&= \frac{1-\wp}{\aleph(\wp)}\left[\breve{\mathfrak{u}}(\mathfrak{k})\right]_{\mathfrak{r}} \oplus \frac{\wp}{\aleph(\wp)\Gamma(\beta)}\frac{1}{\omega(\mathfrak{k})}\left[\int_a^{\mathfrak{k}} (\mathfrak{k}-\xi)^{\beta-1}\omega(\xi)\breve{\mathfrak{u}}(\xi)\mathrm{d}\xi\right]_{\mathfrak{r}} \\
&= \left[\frac{1-\wp}{\aleph(\wp)}\breve{\mathfrak{u}}(\mathfrak{k}) \oplus \frac{\wp}{\aleph(\wp)\Gamma(\beta)}\frac{1}{\omega(\mathfrak{k})}\int_a^{\mathfrak{k}} (\mathfrak{k}-\xi)^{\beta-1}\omega(\xi)\breve{\mathfrak{u}}(\xi)\mathrm{d}\xi\right]_{\mathfrak{r}} \\
&= \left[\mathbb{I}^{\wp,\beta}_{a,\omega}\breve{\mathfrak{u}}(\mathfrak{k})\right]_{\mathfrak{r}}.
\end{aligned}$$

Thus, we have achieved the desired result through the use of the above evidence. This completes the demonstration of case (ii). □

Example 3.3. *Consider the following computation for $\left[\mathbb{I}^{\wp,\beta}_{0,\omega}\breve{\mathfrak{u}}(\mathfrak{k})\right]_{\mathfrak{r}}$, where $\wp = \frac{1}{2}$, $\beta = \frac{1}{3}$, $\omega(\mathfrak{k}) = \mathfrak{k}^2$, and $[\breve{\mathfrak{u}}(\mathfrak{k})]_{\mathfrak{r}} = \left[(3+2\mathfrak{r})\mathfrak{k}^3, (9-4\mathfrak{r})\mathfrak{k}^3\right]$.*

$$\begin{aligned}
\left[\mathbb{I}^{\wp,\beta}_{0,\omega}\breve{\mathfrak{u}}(\mathfrak{k})\right]_{\mathfrak{r}} &= \left[I^{\wp,\beta}_{0,\omega}\left((3+2\mathfrak{r})\mathfrak{k}^3\right), I^{\wp,\beta}_{0,\omega}\left((9-4\mathfrak{r})\mathfrak{k}^3\right)\right] \\
&= \left[\frac{(3+2\mathfrak{r})\mathfrak{k}^3\left(1456+\frac{2184\mathfrak{k}^3}{\Gamma\left(\frac{1}{3}\right)}\right)}{2912}, \left(\frac{3}{2}+\mathfrak{r}\right)\mathfrak{k}^3 + \frac{2187(9-4\mathfrak{r})\mathfrak{k}^{\frac{10}{3}}}{2192\Gamma\left(\frac{1}{3}\right)}\right].
\end{aligned}$$

4 Key characteristics of weighted caputo gH−derivative

This section will present and demonstrate several important properties of a weighted Caputo gH−derivative of order $\wp$ with respect to the weight function $\omega(\mathfrak{k})$.

Definition 4.1. (Fuzzy weighted Caputo-type generalized Hukuhara derivative) *Let $\wp \in [0,1)$, $\beta, \kappa \in (0,1]$ and $\breve{\mathfrak{u}} \in \mathcal{K}(\mathcal{I})$. The weighted Caputo-type generalized Hukuhara derivative of order $\wp$ with respect to the weight function $\omega(\mathfrak{k})$ (weighted Caputo gH−derivative) is defined as:*

$$ {}^{C}_{\mathtt{gH}}\mathbb{D}^{\wp,\beta,\kappa}_{a,\omega}\breve{\mathfrak{u}}(\mathfrak{k}) = \frac{\aleph(\wp)}{1-\wp}\frac{1}{\omega(\mathfrak{k})}\int_a^{\mathfrak{k}} \mathbb{E}_\beta\left(-\mu(\mathfrak{k}-\xi)^\kappa\right)\left(\omega(\xi)\breve{\mathfrak{u}}(\xi)\right)'_{\mathtt{gH}}\,\mathrm{d}\xi, $$

This definition is valid for ${}^{C}_{\mathtt{gH}}\mathbb{D}^{\wp,\beta,\kappa}_{a,\omega}\breve{\mathfrak{u}}(\mathfrak{k})$ is a fuzzy-valued function. Here, ω is a positive, increasing, and continuous real-valued function defined on the interval $\mathcal{I}$. The function $\aleph(\wp)$ represents the normalization function, μ is given by $\mu = \frac{\wp}{1-\wp}$, and $\mathbb{E}_\beta(z)$ denotes the Mittag-Leffler function with a single parameter β. It can be expressed as

$$ \mathbb{E}_\beta(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta k+1)}, $$

where Γ denotes the gamma function.

It should be noted that the definition presented above encompasses several other special cases of fuzzy fractional derivatives. For instance:

1. If we set $\omega(\mathfrak{k}) = 1$, $\beta = \kappa = 1$, we obtain the fuzzy Caputo-Fabrizio derivative in the sense of the fuzzy generalized Hukuhara derivative, given by:

$$ {}^{C}_{\mathtt{gH}}\mathbb{D}^{\wp,1}_{a,1}\breve{\mathfrak{u}}(\mathfrak{k}) = \frac{\aleph(\wp)}{1-\wp}\int_a^{\mathfrak{k}} e^{-\mu(t-\xi)}\mathfrak{u}'_{\mathtt{gH}}(\xi)\mathrm{d}\xi. $$

2. If we set $\omega(\mathfrak{k}) = 1$, $\beta = \kappa = \wp$, we get the fuzzy Atangana-Baleanu derivative in the sense of the fuzzy generalized Hukuhara derivative, given by:

$$ {}^{C}_{\mathtt{gH}}\mathbb{D}^{\wp,\wp,\wp}_{a,1}\breve{\mathfrak{u}}(\mathfrak{k}) = \frac{\aleph(\wp)}{1-\wp}\int_a^{\mathfrak{k}} \mathbb{E}_\wp(-\mu(t-\xi)^\wp)\breve{\mathfrak{u}}'_{\mathtt{gH}}(\xi)\mathrm{d}\xi. $$

Throughout this article, we assume that $\beta = \kappa$, and we denote ${}^{C}_{\mathtt{gH}}\mathbb{D}^{\wp,\beta,\beta}_{a,\omega}\breve{\mathfrak{u}}(\mathfrak{k})$ by ${}^{C}_{\mathtt{gH}}\mathbb{D}^{\wp,\beta}_{a,\omega}\breve{\mathfrak{u}}(\mathfrak{k})$.

Definition 4.2. *Let $\breve{\mathfrak{u}}(\mathfrak{k})$ be a function in $\mathcal{K}(\mathcal{I})$ that is differentiable using weighted Caputo-type gH−derivative of order $\wp$. Assume that ${}^{C}D^{\wp,\beta}_{a,\omega}u^{-}_{\mathtt{r}}(\mathfrak{k})$ and ${}^{C}D^{\wp,\beta}_{a,\omega}u^{+}_{\mathtt{r}}(\mathfrak{k})$ exist for all $\mathfrak{k} \in \mathcal{I}$. We say that $\breve{\mathfrak{u}}(\mathfrak{k})$ is (1)-weighted Caputo gH-differentiable of order $\wp$ with respect to the weight function $\omega(\mathfrak{k})$ over $\mathcal{I}$, denoted by ${}^{C}_{1.gH}\mathbb{D}^{\wp,\beta}_{a,\omega}\breve{\mathfrak{u}}(\mathfrak{k})$, if for all $\mathtt{r} \in [0,1]$, we have:*

$$ \left[\,{}^{C}_{1.gH}\mathbb{D}^{\wp,\beta}_{a,\omega}\breve{\mathfrak{u}}(\mathfrak{k})\right]_{\mathtt{r}} = \left[\,{}^{C}D^{\wp,\beta}_{a,\omega}u^{-}_{\mathtt{r}}(\mathfrak{k}),\ {}^{C}D^{\wp,\beta}_{a,\omega}u^{+}_{\mathtt{r}}(\mathfrak{k})\right]. \tag{2} $$

Similarly, we say that $\breve{\mathfrak{u}}(\mathfrak{k})$ is (2)-weighted Caputo gH-differentiable of order $\wp$ with respect to the weight function $\omega(\mathfrak{k})$ over $\mathcal{I}$, denoted by ${}^{C}_{2.gH}\mathbb{D}^{\wp,\beta}_{a,\omega}\breve{\mathfrak{u}}(\mathfrak{k})$, if for all $\mathtt{r} \in [0,1]$, we have:

$$ \left[\,{}^{C}_{2.gH}\mathbb{D}^{\wp,\beta}_{a,\omega}\breve{\mathfrak{u}}(\mathfrak{k})\right]_{\mathtt{r}} = \left[\,{}^{C}D^{\wp,\beta}_{a,\omega}u^{+}_{\mathtt{r}}(\mathfrak{k}),\ {}^{C}D^{\wp,\beta}_{a,\omega}u^{-}_{\mathtt{r}}(\mathfrak{k})\right]. \tag{3} $$

Here, ${}^{C}D^{\wp,\beta}_{a,\omega}u^{-}_{\mathtt{r}}(\mathfrak{k})$ and ${}^{C}D^{\wp,\beta}_{a,\omega}u^{+}_{\mathtt{r}}(\mathfrak{k})$ are the generalized Hattaf fractional derivatives for the real-valued functions $u^{-}_{\mathtt{r}}(\mathfrak{k})$ and $u^{+}_{\mathtt{r}}(\mathfrak{k})$, respectively [15, 22].

Example 4.3. *The function $\breve{\mathfrak{u}} : [0,2] \to \Re_{\mathfrak{f}}$ is defined by*

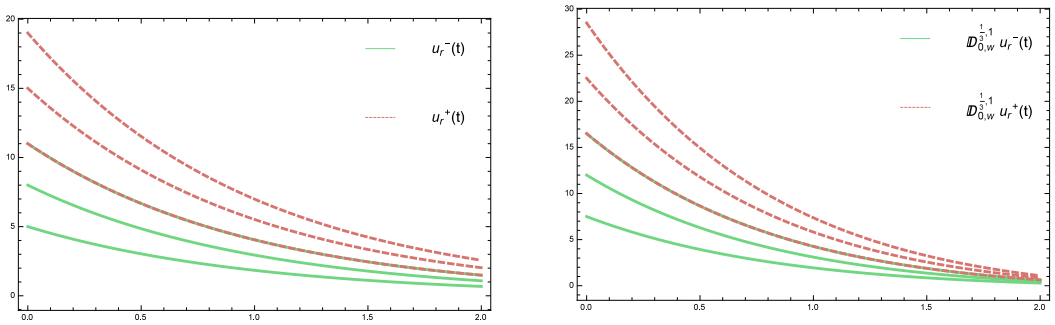
$$ \left[\breve{\mathfrak{u}}(\mathfrak{k})\right]_{\mathtt{r}} = \left[(5+6\mathtt{r})e^{-2\mathfrak{k}}, (19-8\mathtt{r})e^{-2\mathfrak{k}}\right]. $$

Using Definitions 4.1 and 4.2, we can compute ${}^{C}_{\mathtt{gH}}\mathbb{D}^{\wp,\beta}_{0,\omega}\breve{\mathfrak{u}}(\mathfrak{k})$ as:

$$ \left[\,{}^{C}_{1.gH}\mathbb{D}^{\wp,\beta}_{0,\omega}\breve{\mathfrak{u}}(\mathfrak{k})\right]_{\mathtt{r}} = \left[\frac{-3e^{-\mathfrak{k}}(5+6\mathrm{r})(e^{\frac{\mathfrak{k}}{2}}-\mathfrak{k}-1)}{\mathfrak{k}}, \frac{3e^{-\mathfrak{k}}(8\mathrm{r}-19)(e^{\frac{\mathfrak{k}}{2}}-\mathfrak{k}-1)}{\mathfrak{k}}\right], $$

where $\wp = \frac{1}{3}$, $\beta = 1$, *and* $\omega(\mathfrak{k}) = \mathfrak{k}$.

Figure 1 displays the r*-cuts for both* $\breve{\mathfrak{u}}(\mathfrak{k})$ *and* ${}^{C}_{\mathtt{gH}}\mathbb{D}^{\frac{1}{3},1}_{0,\omega}\breve{\mathfrak{u}}(\mathfrak{k})$. *These two functions exhibit the same lower (green) and upper (red) cuts, which implies that* $\breve{\mathfrak{u}}(\mathfrak{k})$ *is a (1)-weighted Caputo* gH*−differentiable function of order* $\frac{1}{3}$ *with respect to the weight function* $\omega(\mathfrak{k}) = \mathfrak{k}$.



(a).$\breve{\mathfrak{u}}(\mathfrak{k})$ (b). ${}^{C}_{\mathtt{gH}}\mathbb{D}^{\frac{1}{3},1}_{0,\omega}\breve{\mathfrak{u}}(\mathfrak{k})$

Figure 1: Visualizing the r-cuts of $\breve{\mathfrak{u}}(\mathfrak{k})$ and ${}^{C}_{\mathtt{gH}}\mathbb{D}^{\frac{1}{3},1}_{0,\omega}\breve{\mathfrak{u}}(\mathfrak{k})$ for Example 4.3 at $\mathtt{r} = 0, 0.5, 1$.

Definition 4.4. *Let* $\wp, \beta \in [0, 1)$ *and* $\breve{\mathfrak{u}}$ *is a weighted Caputo* gH*−differentiable function of order* $\wp$ *with respect to the weight function* $\omega(\mathfrak{k})$. *We say that a point* $\xi_0 \in \mathcal{I}$ *is a switching point for the weighted Caputo* gH*−derivative, if in any neighborhood* $\mathcal{V}$ *of* ξ_0 *there exist points* $\xi_1 < \xi_0 < \xi_2$ *such that*

Type I. *at* ξ_1 *(2) holds while (3) does not hold and at* ξ_2 *(3) holds and (2) does not hold, or*

Type II. *at* ξ_1 *(3) holds while (2) does not hold and at* ξ_2 *(2) holds and (3) does not hold.*

Example 4.5. *Consider the fuzzy function* $\breve{\mathfrak{u}} : [0, \pi] \to \Re_{\mathfrak{f}}$ *as defined by the following expression:*

$$\breve{\mathfrak{u}}(\mathfrak{k}) = \Big[(2 + 3\mathtt{r})\sin(\mathfrak{k}), (9 - 4\mathtt{r})\sin(\mathfrak{k})\Big].$$

For $\mathfrak{k} \in [0, 2.2841]$, *the function* $\breve{\mathfrak{u}}(\mathfrak{k})$ *is a (1)-weighted Caputo* gH*−differentiable function of order* $\frac{1}{2}$ *with respect to the weight function* $\omega(\mathfrak{k}) = 1$. *The weighted Caputo* gH*−derivative is given by:*

$${}^{C}_{\mathtt{i.gH}}\mathbb{D}^{\frac{1}{2},1}_{0,\omega}\breve{\mathfrak{u}}(\mathfrak{k}) = [(2 + 3\mathtt{r}), (9 - 4\mathtt{r})]\left(e^{\mathfrak{k}}(\cos(\mathfrak{k}) + \sin(\mathfrak{k})) - e^{-\mathfrak{k}}\right),$$

For $\mathfrak{k} \in [2.2841, \pi]$, *the function* $\breve{\mathfrak{u}}(\mathfrak{k})$ *is a (2)-weighted Caputo* gH*−differentiable function of order* $\frac{1}{2}$ *with respect to the weight function* $\omega(\mathfrak{k}) = 1$. *The weighted Caputo* gH*−derivative in this interval is given by:*

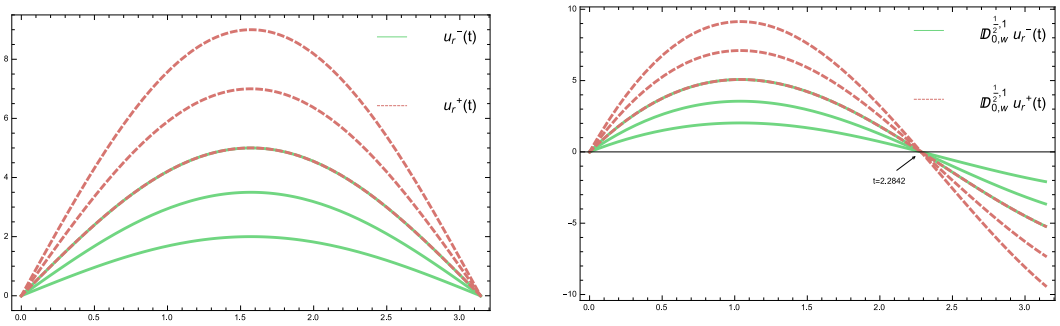
$${}^{C}_{\mathtt{ii.gH}}\mathbb{D}^{\frac{1}{2},1}_{0,\omega}\breve{\mathfrak{u}}(\mathfrak{k}) = [(9 - 4\mathtt{r}), (2 + 3\mathtt{r})]\left(e^{\mathfrak{k}}(\cos(\mathfrak{k}) + \sin(\mathfrak{k})) - e^{-\mathfrak{k}}\right).$$

Hence, the point $\mathfrak{k} = 2.2841$ *serves as a switching point of* ***Type I*** *for the weighted Caputo* gH*−differentiability of* $\breve{\mathfrak{u}}(\mathfrak{k})$. *Figures 2(a) and 2(b) illustrate the fuzzy functions* $\breve{\mathfrak{u}}(\mathfrak{k})$ *and* ${}^{C}_{\mathtt{gH}}\mathbb{D}^{\frac{1}{2},1}_{0,\omega}\breve{\mathfrak{u}}(\mathfrak{k})$ *respectively. In Figure 2(b), one can observe the change in the positions of the lower cut (green color) and upper cut (red color) of the fuzzy function* ${}^{C}_{\mathtt{gH}}\mathbb{D}^{\frac{1}{2},1}_{0,\omega}\breve{\mathfrak{u}}(\mathfrak{k})$ *at* $\mathfrak{k} = 2.2841$, *indicating a switching point of* ***Type I***.

Theorem 4.6. *Let* $\breve{\mathfrak{u}} : \mathcal{I} \to \Re_{\mathfrak{f}}$ *be a fuzzy function and* $\breve{\mathfrak{u}} \in \mathcal{K}(\mathcal{I})$. *We can establish the following principles for* $\breve{\mathfrak{u}}$*:*

i. *If* $\breve{\mathfrak{u}}$ *is* [1.gH]*−differentiable for all* $\mathfrak{k} \in \mathcal{I}$, *then it is a (1)-weighted Caputo* gH*−differentiable function of order* $\wp$ *with respect to the weight function* $\omega(\mathfrak{k})$.

ii. *If* $\breve{\mathfrak{u}}$ *is* [2.gH]*−differentiable for all* $\mathfrak{k} \in \mathcal{I}$, *then it is a (2)-weighted Caputo* gH*−differentiable function of order* $\wp$ *with respect to the weight function* $\omega(\mathfrak{k})$.



(a).$\breve{\mathtt{u}}(\mathfrak{k})$ (b). ${}^{C}_{\mathtt{gH}}\mathbb{D}^{\frac{1}{2},1}_{0,\omega}\breve{\mathtt{u}}(\mathfrak{k})$

Figure 2: Visualizing the $\mathtt{r}$-cuts of $\breve{\mathtt{u}}(\mathfrak{k})$ and ${}^{C}_{\mathtt{gH}}\mathbb{D}^{\frac{1}{2},1}_{0,\omega}\breve{\mathtt{u}}(\mathfrak{k})$ for Example 4.5 at $\mathtt{r}=0, 0.5, 1$.

Proof. Assuming that $\breve{\mathtt{u}}(\mathfrak{k})$ is a fuzzy function that is $[\mathtt{1.gH}]-$differentiable, we can use Theorem 2.5 to show that $\omega(\mathfrak{k})\breve{\mathtt{u}}(\mathfrak{k})$ is also $[\mathtt{1.gH}]-$differentiable. From this, we can derive the inequality

$$\omega'(\mathfrak{k})\mathtt{u}_r^-(\mathfrak{k})+\omega(\mathfrak{k})u_r^{-\prime}(t)\leq\omega'(\mathfrak{k})\mathtt{u}_r^+(\mathfrak{k})+\omega(\mathfrak{k})u_r^{+\prime}(\mathfrak{k}).$$

where $\mathtt{u}_r^-(\mathfrak{k})$ and $\mathtt{u}_r^+(\mathfrak{k})$ are the left- and right-sided limits of $\breve{\mathtt{u}}'_{\mathtt{gH}}(\mathfrak{k})$, respectively. Using the fact that the Mittag-Lefler function $\mathbb{E}_\beta(-z)$ is completely monotone for $z>0$ [21], we can obtain the following inequality by multiplying both sides of the above equation by $\mathbb{E}_\beta\left(-\mu(\mathfrak{k}-\xi)^\beta\right)$ and integrating over ξ from a to $\mathfrak{k}$:

$$\mathbb{E}_\beta\left(-\mu(\mathfrak{k}-\xi)^\beta\right)\left(\omega'(\mathfrak{k})\mathtt{u}_r^-(\mathfrak{k})+\omega(\mathfrak{k})u_r^{-\prime}(t)\right)\leq\mathbb{E}_\beta\left(-\mu(\mathfrak{k}-\xi)^\beta\right)\left(\omega'(\mathfrak{k})\mathtt{u}_r^+(\mathfrak{k})+\omega(\mathfrak{k})u_r^{+\prime}(\mathfrak{k})\right).$$

where $\aleph(\wp)$ is a weight function, $\wp\in[0,1)$, and $\beta\in(0,1]$. Using integral properties and inserting coefficients, we can derive the following inequality:

$$\begin{aligned}&\frac{\aleph(\wp)}{1-\wp}\frac{1}{\omega(\mathfrak{k})}\int_a^{\mathfrak{k}}\mathbb{E}_\beta\left(-\mu(\mathfrak{k}-\xi)^\beta\right)\left(\omega'(\xi)\mathtt{u}_r^-(\xi)+\omega(\xi)u_r^{-\prime}(\xi)\right)\mathrm{d}\xi\\&\leq\frac{\aleph(\wp)}{1-\wp}\frac{1}{\omega(\mathfrak{k})}\int_a^{\mathfrak{k}}\mathbb{E}_\beta\left(-\mu(\mathfrak{k}-\xi)^\beta\right)\left(\omega'(\xi)\mathtt{u}_r^+(\xi)+\omega(\xi)u_r^{+\prime}(\xi)\right)\mathrm{d}\xi.\end{aligned}$$

Therefore, we can express:

$$\begin{aligned}\left[{}^{C}_{\mathtt{gH}}\mathbb{D}^{\wp,\beta}_{a,\omega}\breve{\mathtt{u}}(\mathfrak{k})\right]_{\mathtt{r}}&=\left[\frac{\aleph(\wp)}{1-\wp}\frac{1}{\omega(\mathfrak{k})}\int_a^{\mathfrak{k}}\mathbb{E}_\beta\left(-\mu(\mathfrak{k}-\xi)^\beta\right)\left(\omega(\xi)\breve{\mathtt{u}}(\xi)\right)'_{\mathtt{gH}}\mathrm{d}\xi\right]_r\\&=\Big[\frac{\aleph(\wp)}{1-\wp}\frac{1}{\omega(\mathfrak{k})}\int_a^{\mathfrak{k}}\mathbb{E}_\beta\left(-\mu(\mathfrak{k}-\xi)^\beta\right)\left(\omega'(\xi)\mathtt{u}_r^-(\xi)+\omega(\xi)u_r^{-\prime}(\xi)\right)\mathrm{d}\xi,\\&\qquad\frac{\aleph(\wp)}{1-\wp}\frac{1}{\omega(\mathfrak{k})}\int_a^{\mathfrak{k}}\mathbb{E}_\beta\left(-\mu(\mathfrak{k}-\xi)^\beta\right)\left(\omega'(\xi)\mathtt{u}_r^+(\xi)+\omega(\xi)u_r^{+\prime}(\xi)\right)\mathrm{d}\xi\Big]\\&=\Big[{}^{C}D^{\wp,\beta}_{a,\omega}u^-_{\mathtt{r}}(\mathfrak{k}),\ {}^{C}D^{\wp,\beta}_{a,\omega}u^+_{\mathtt{r}}(\mathfrak{k})\Big].\end{aligned}$$

Therefore, we can conclude that $\breve{\mathtt{u}}$ is a function that is differentiable using the Caputo $\mathtt{gH}$ derivative of order $\wp$, with a weight function $\omega(\mathfrak{k})$ that has a weight of 1. If we have a different case, we can apply a similar process to demonstrate the same conclusion. □ □

Theorem 4.7. *Let* $\breve{\mathtt{u}}:\mathcal{I}\to\Re_{\mathtt{f}}$ *and* $\breve{\mathtt{u}}\in\mathcal{K}(\mathcal{I})$.

1. *If* $\breve{\mathtt{u}}(\mathfrak{k})$ *is a (1)-weighted Caputo* $\mathtt{gH}-$*differentiable function of order* $\wp$ *with respect to the weight function* $\omega(\mathfrak{k})$*, then*

$$\mathbb{I}^{\wp,\beta}_{a,\omega}\Big({}^{C}_{1.gH}\mathbb{D}^{\wp,\beta}_{a,\omega}\breve{\mathtt{u}}(\mathfrak{k})\Big)=\frac{1}{\omega(\mathfrak{k})}\Big(\omega(\mathfrak{k})\breve{\mathtt{u}}(\mathfrak{k})\ominus\omega(a)\breve{\mathtt{u}}(a)\Big).$$

2. *If $\breve{\mathtt{u}}(\mathfrak{k})$ is a (2)-weighted Caputo $\mathtt{gH}-$differentiable function of order $\wp$ with respect to the weight function $\omega(\mathfrak{k})$, then*

$$\mathbb{I}_{a,\omega}^{\wp,\beta}\Big({}_{2.gH}^{C}\mathbb{D}_{a,\omega}^{\wp,\beta}\breve{\mathtt{u}}(\mathfrak{k})\Big) = \frac{1}{\omega(\mathfrak{k})}\Big((-1)\omega(a)\breve{\mathtt{u}}(a) \ominus (-1)\omega(\mathfrak{k})\breve{\mathtt{u}}(\mathfrak{k})\Big).$$

Proof. To begin, we establish the case 1. Consider a function $\breve{\mathtt{u}}(\mathfrak{k})$ that is (1)-weighted Caputo $\mathtt{gH}-$differentiable of order $\wp$ with respect to the weight function $\omega(\mathfrak{k})$. Employing Definition 4.2 and Theorem 3.2, we proceed to demonstrate this case:

$$\begin{aligned}
\Big[\mathbb{I}_{a,\omega}^{\wp,\beta}\Big({}_{1.gH}^{C}\mathbb{D}_{a,\omega}^{\wp,\beta}\breve{\mathtt{u}}(\mathfrak{k})\Big)\Big]_{\mathtt{r}} &= \Big[I_{a,\omega}^{\wp,\beta}\Big({}^{C}D_{a,\omega}^{\wp,\beta}\mathtt{u}_{\mathtt{r}}^{-}(\mathfrak{k})\Big), I_{a,\omega}^{\wp,\beta}\Big({}^{C}D_{a,\omega}^{\wp,\beta}\mathtt{u}_{\mathtt{r}}^{+}(\mathfrak{k})\Big)\Big] \\
&= \left[\frac{1}{\omega(\mathfrak{k})}\Big(\omega(\mathfrak{k})\mathtt{u}_{\mathtt{r}}^{-}(\mathfrak{k}) - \omega(a)\mathtt{u}_{\mathtt{r}}^{-}(a)\Big), \frac{1}{\omega(\mathfrak{k})}\Big(\omega(\mathfrak{k})\mathtt{u}_{\mathtt{r}}^{+}(\mathfrak{k}) - \omega(a)\mathtt{u}_{\mathtt{r}}^{+}(a)\Big)\right] \\
&= \frac{1}{\omega(\mathfrak{k})}\left[\omega(\mathfrak{k})\mathtt{u}_{\mathtt{r}}^{-}(\mathfrak{k}), \omega(\mathfrak{k})\mathtt{u}_{\mathtt{r}}^{+}(\mathfrak{k})\right] \ominus \frac{1}{\omega(\mathfrak{k})}\left[\omega(a)\mathtt{u}_{\mathtt{r}}^{-}(a), \omega(a)\mathtt{u}_{\mathtt{r}}^{+}(a)\right] \\
&= \frac{1}{\omega(\mathfrak{k})}\Big(\omega(\mathfrak{k})\breve{\mathtt{u}}(\mathfrak{k}) \ominus \omega(a)\breve{\mathtt{u}}(a)\Big).
\end{aligned}$$

In the next case, we proceed to find the Caputo fractional derivative of order $\wp$ with respect to the weight function $\omega(\mathfrak{k})$ for a (2)-weighted Caputo $\mathtt{gH}-$differentiable function $\breve{\mathtt{u}}(\mathfrak{k})$. We apply Definition 4.2 and Theorem 3.2 once again to derive the following expression:

$$\begin{aligned}
\Big[\mathbb{I}_{a,\omega}^{\wp,\beta}\Big({}_{2.gH}^{C}\mathbb{D}_{a,\omega}^{\wp,\beta}\breve{\mathtt{u}}(\mathfrak{k})\Big)\Big]_{\mathtt{r}} &= \Big[I_{a,\omega}^{\wp,\beta}\Big({}^{C}D_{a,\omega}^{\wp,\beta}\mathtt{u}_{\mathtt{r}}^{+}(\mathfrak{k})\Big), I_{a,\omega}^{\wp,\beta}\Big({}^{C}D_{a,\omega}^{\wp,\beta}\mathtt{u}_{\mathtt{r}}^{-}(\mathfrak{k})\Big)\Big] \\
&= \left[\frac{1}{\omega(\mathfrak{k})}\Big(\omega(\mathfrak{k})\mathtt{u}_{\mathtt{r}}^{+}(\mathfrak{k}) - \omega(a)\mathtt{u}_{\mathtt{r}}^{+}(a)\Big), \frac{1}{\omega(\mathfrak{k})}\Big(\omega(\mathfrak{k})\mathtt{u}_{\mathtt{r}}^{-}(\mathfrak{k}) - \omega(a)\mathtt{u}_{\mathtt{r}}^{-}(a)\Big)\right] \\
&= \frac{1}{\omega(\mathfrak{k})}\left[\omega(\mathfrak{k})\mathtt{u}_{\mathtt{r}}^{+}(\mathfrak{k}), \omega(\mathfrak{k})\mathtt{u}_{\mathtt{r}}^{-}(\mathfrak{k})\right] \ominus \frac{1}{\omega(\mathfrak{k})}\left[\omega(a)\mathtt{u}_{\mathtt{r}}^{+}(a), \omega(a)\mathtt{u}_{\mathtt{r}}^{-}(a)\right] \\
&= \frac{1}{\omega(\mathfrak{k})}\Big((-1)\omega(a)\breve{\mathtt{u}}(a) \ominus (-1)\omega(\mathfrak{k})\breve{\mathtt{u}}(\mathfrak{k})\Big).
\end{aligned}$$

Thus, we have achieved the intended result. □

5 Fuzzy Laplace transform

In this section, we will explore the concept of fuzzy Laplace transform, which plays a significant role in fuzzy calculus. We begin by considering a fuzzy function $\breve{\mathtt{u}}(\mathfrak{k})$ that is of exponential order on $0 \leq \mathfrak{k} < \infty$ and defined on the interval $[a, \infty)$. The fuzzy Laplace transform of this function, denoted by $\mathscr{L}_a\left\{\breve{\mathtt{u}}(\mathfrak{k})\right\}(s)$, is defined as

$$\mathscr{L}_a\left\{\breve{\mathtt{u}}(\mathfrak{k})\right\}(s) = \int_a^{\infty} e^{-s(\xi-a)}\breve{\mathtt{u}}(\xi)\mathrm{d}\xi = \lim_{\Re\to\infty}\int_a^{\Re} e^{-s(\xi-a)}\breve{\mathtt{u}}(\xi)\mathrm{d}\xi,$$

The limit of this integral as the upper bound approaches infinity is also considered, provided that the limit exists. This definition is in accordance with [25] and is a fundamental concept in fuzzy calculus.

Definition 5.1. (Convolution)[25] *Consider a real-valued piecewise continuous function $\omega(\mathfrak{k}) : [a, \infty) \to \mathbb{R}$ and a fuzzy piecewise continuous function $\breve{\mathtt{u}}(\mathfrak{k}) : [a, \infty) \to \Re_{\mathfrak{f}}$. Their convolution, denoted as $(\omega * \breve{\mathtt{u}})(\mathfrak{k})$, is defined as the integral over $[a, \mathfrak{k}]$ of the product of ω and a shifted version of $\breve{\mathtt{u}}$:*

$$(\omega * \breve{\mathtt{u}})(\mathfrak{k}) = \int_a^{\mathfrak{k}} \omega(\mathfrak{k} - \xi + a)\breve{\mathtt{u}}(\xi)\mathrm{d}\xi,$$

Theorem 5.2. [26] *Let $\breve{\mathtt{u}}(\mathfrak{k})$ be a fuzzy piecewise continuous function on $[a, \infty)$ with exponential order γ, and $\omega(\mathfrak{k})$ be a piecewise continuous real-valued function on $[a, \infty)$. Then, for $Re(s) > \gamma$,*

$$\mathscr{L}_a\left\{(\omega * \breve{\mathtt{u}})(\mathfrak{k})\right\}(s) = \mathscr{L}_a\left\{\omega(\mathfrak{k})\right\}(s)\mathscr{L}_a\left\{\breve{\mathtt{u}}(\mathfrak{k})\right\}(s), \qquad (Re(s) > \gamma).$$

Theorem 5.3. [24] *Let $\breve{\mathtt{u}} : [a, \infty) \to \Re_{\mathfrak{f}}$ be a fuzzy continuous function that is $\mathtt{gH}-$differentiable with exponential order γ. Furthermore, suppose that $\breve{\mathtt{u}}'_{\mathtt{gH}}(\mathfrak{k})$ is piecewise continuous on every finite closed interval of $[a, \infty)$, and the type of $\mathtt{gH}-$differentiability is constant on each interval. Then, for $Re(s) > \gamma$:*

$\mathscr{L}_a\left\{\breve{\mathfrak{u}}'_{1.\mathtt{gH}}(\mathfrak{k})\right\}(s) = s\mathscr{L}_a\left\{\breve{\mathfrak{u}}(\mathfrak{k})\right\}(s) \ominus \breve{\mathfrak{u}}(a).$

$\mathscr{L}_a\left\{\breve{\mathfrak{u}}'_{2.\mathtt{gH}}(\mathfrak{k})\right\}(s) = (-1)\breve{\mathfrak{u}}(a) \ominus (-1)s\mathscr{L}_a\left\{\breve{\mathfrak{u}}(\mathfrak{k})\right\}(s).$

Based on the concepts and theorems outlined above, we will now explore the fuzzy Laplace transform for the fuzzy weighted Caputo-type generalized Hukuhara derivative. Referring to Definitions 4.1 and 5.1, we can conclude that:

$$ {}^{C}_{\mathtt{gH}}\mathbb{D}^{\wp,\beta}_{a,\omega}\breve{\mathfrak{u}}(\mathfrak{k}) = \frac{\aleph(\wp)}{1-\wp}\frac{1}{\omega(\mathfrak{k})}\Big(\mathbb{E}_\beta\left(-\mu(\mathfrak{k}-a)^\beta\right) * (\omega(\mathfrak{k})\breve{\mathfrak{u}}(\mathfrak{k}))'_{\mathtt{gH}}\Big). \tag{4}$$

where $*$ denotes the convolution of two functions, and $(\omega(\mathfrak{k})\breve{\mathfrak{u}}(\mathfrak{k}))'_{\mathtt{gH}}$ denotes the gH-derivative of $\omega(\mathfrak{k})\breve{\mathfrak{u}}(\mathfrak{k})$ with respect to $\mathfrak{k}$.

Theorem 5.4. *Assuming $\breve{\mathfrak{u}}$ is a fuzzy continuous function, $\mathtt{gH}$-differentiable and has exponential γ with $\breve{\mathfrak{u}}'_{\mathtt{gH}}(\mathfrak{k})$ being piecewise continuous in every finite closed interval of $[a,\infty)$ and the type of $\mathtt{gH}$-differentiability does not change in the interval, then for $\wp \in [0,1)$ and $\beta \in (0,1]$, based on the types of $\mathtt{gH}$-differentiability, we have:*

1. *If $\breve{\mathfrak{u}}(\mathfrak{k})$ is $[1.\mathtt{gH}]$-differentiable function, then*

$$\mathscr{L}_a\left\{\omega(\mathfrak{k})\ {}^{C}_{1.gH}\mathbb{D}^{\wp,\beta}_{a,\omega}\breve{\mathfrak{u}}(\mathfrak{k})\right\}(s) = \aleph(\wp)\left(\frac{s^\beta\mathscr{L}_a\left\{\omega(\mathfrak{k})\breve{\mathfrak{u}}(\mathfrak{k})\right\}(s) \ominus s^{\beta-1}\omega(a)\breve{\mathfrak{u}}(a)}{s^\beta(1-\wp)+\wp}\right).$$

2. *If $\breve{\mathfrak{u}}(\mathfrak{k})$ is $[2.\mathtt{gH}]$-differentiable function, then*

$$\mathscr{L}_a\left\{\omega(\mathfrak{k})\ {}^{C}_{2.gH}\mathbb{D}^{\wp,\beta}_{a,\omega}\breve{\mathfrak{u}}(\mathfrak{k})\right\}(s) = \aleph(\wp)\left(\frac{(-1)s^{\beta-1}\omega(a)\breve{\mathfrak{u}}(a) \ominus (-1)s^\beta\mathscr{L}_a\left\{\omega(\mathfrak{k})\breve{\mathfrak{u}}(\mathfrak{k})\right\}(s)}{s^\beta(1-\wp)+\wp}\right).$$

Proof. By applying Equation (4) and Theorem 5.2, we obtain the following expression:

$$\mathscr{L}_a\left\{\omega(\mathfrak{k})\ {}^{C}_{\mathtt{gH}}\mathbb{D}^{\wp,\beta}_{a,\omega}\breve{\mathfrak{u}}(\mathfrak{k})\right\}(s) = \frac{\aleph(\wp)}{1-\wp}\mathscr{L}_a\left\{\left(\mathbb{E}_\beta\left(-\mu(\mathfrak{k}-a)^\beta\right)\right\}(s)\mathscr{L}_a\left\{(\omega(\mathfrak{k})\breve{\mathfrak{u}}(\mathfrak{k}))'_{\mathtt{gH}}\right)\right\}(s). \tag{5}$$

Assuming that $\breve{\mathfrak{u}}(\mathfrak{k})$ is a $[1.\mathtt{gH}]-$differentiable fuzzy function, we can conclude that it is a (1)-weighted Caputo $\mathtt{gH}-$differentiable function according to Theorem 4.6. Using Theorem 5.3 and the fact that $\mathscr{L}_a\left\{\left(\mathbb{E}_\beta\left(-\mu(\mathfrak{k}-a)^\beta\right)\right\}(s) = \frac{s^{\beta-1}}{(s^\beta+\mu)}$, we get:

$$\begin{aligned}\mathscr{L}_a\left\{\omega(\mathfrak{k})\ {}^{C}_{1.\mathtt{gH}}\mathbb{D}^{\wp,\beta}_{a,\omega}\breve{\mathfrak{u}}(\mathfrak{k})\right\}(s) &= \frac{\aleph(\wp)}{1-\wp}\left(\frac{s^\beta\mathscr{L}_a\left\{\omega(\mathfrak{k})\breve{\mathfrak{u}}(\mathfrak{k})\right\}(s) \ominus s^{\beta-1}\omega(a)\breve{\mathfrak{u}}(a)}{s^\beta+\frac{\wp}{1-\wp}}\right)\\ &= \aleph(\wp)\left(\frac{s^\beta\mathscr{L}_a\left\{\omega(\mathfrak{k})\breve{\mathfrak{u}}(\mathfrak{k})\right\}(s) \ominus s^{\beta-1}\omega(a)\breve{\mathfrak{u}}(a)}{s^\beta(1-\wp)+\wp}\right).\end{aligned}$$

Following the procedure outlined above, we can deduce, based on Theorem 4.6, that if $\breve{\mathfrak{u}}(\mathfrak{k})$ is a $[2.\mathtt{gH}]-$differentiable fuzzy function, it can be established as a (2)-weighted Caputo $\mathtt{gH}-$differentiable function. Consequently, employing Eq.(5) and Theorem 5.3, we can express the result as:

$$\begin{aligned}\mathscr{L}_a\left\{\omega(\mathfrak{k})\ {}^{C}_{2.\mathtt{gH}}\mathbb{D}^{\wp,\beta}_{a,\omega}\breve{\mathfrak{u}}(\mathfrak{k})\right\}(s) &= \frac{\aleph(\wp)}{1-\wp}\left(\frac{(-1)s^{\beta-1}\omega(a)\breve{\mathfrak{u}}(a) \ominus (-1)s^\beta\mathscr{L}_a\left\{\omega(\mathfrak{k})\breve{\mathfrak{u}}(\mathfrak{k})\right\}(s)}{s^\beta+\frac{\wp}{1-\wp}}\right)\\ &= \aleph(\wp)\left(\frac{(-1)s^{\beta-1}\omega(a)\breve{\mathfrak{u}}(a) \ominus (-1)s^\beta\mathscr{L}_a\left\{\omega(\mathfrak{k})\breve{\mathfrak{u}}(\mathfrak{k})\right\}(s)}{s^\beta(1-\wp)+\wp}\right).\end{aligned}$$

This concludes the demonstration. □

Proposition 5.5. [5] *For $\beta \in (0,1]$, the following holds true.*

i. $\mathscr{L}_0\left\{\mathbb{E}_\beta\left(\lambda\mathfrak{k}^\beta\right)\right\}(s) = \frac{s^{\beta-1}}{s^\beta-\lambda}, \qquad \left|\frac{\lambda}{s^\beta}\right| < 1.$

ii. $\mathscr{L}_0\left\{\frac{d}{d\mathfrak{k}}(\mathbb{E}_\beta\left(\lambda\mathfrak{k}^\beta\right))\right\}(s) = \frac{\lambda}{s^\beta-\lambda}, \qquad \left|\frac{\lambda}{s^\beta}\right| < 1.$

6 Existence and uniqueness of solutions for fuzzy fractional differential equations with weighted Caputo-Type generalized Hukuhara operator

We are considering a fuzzy fractional differential equation, which involves the weighted Caputo-type generalized Hukuhara operator and an initial value problem. The equation is given by:

$$\begin{cases} {}^{C}_{\mathtt{gH}}\mathbb{D}^{\wp,\beta}_{a,\omega}\breve{\mathfrak{u}}(\mathfrak{k}) = \mathfrak{h}(\mathfrak{k},\breve{\mathfrak{u}}(\mathfrak{k})), \\ \breve{\mathfrak{u}}(a) = \breve{\mathfrak{u}}_0, \end{cases} \tag{6}$$

Here, $\mathfrak{h} : \mathcal{I} \times \Re_{\mathfrak{f}} \to \Re_{\mathfrak{f}}$, $\breve{\mathfrak{u}}$ is a fuzzy continuous function that belongs to the space of fuzzy continuous functions over $\mathcal{I}$, $\wp \in [0,1)$, $\beta \in (0,1]$ and $\breve{\mathfrak{u}}_0$ is a given fuzzy number.

We have used the Banach fixed point theorem to prove the existence and uniqueness of a solution to Equation (6). As a result, we can conclude that a unique solution exists.

Lemma 6.1. *Suppose we have a fuzzy initial value problem given by Equation (6), where $\wp \in [0,1)$, $\beta \in (0,1]$, $\breve{\mathfrak{u}}(\mathfrak{k}) \in \mathcal{K}(\mathcal{I})$, and $\mathfrak{h} : \mathcal{I} \times \Re_{\mathfrak{f}} \to \Re_{\mathfrak{f}}$ is a fuzzy continuous function. Let $\breve{\mathfrak{u}}(\mathfrak{k})$ be a solution to this problem. Then, depending on the type of weighted Caputo $\mathtt{gH}-$differentiability, we have the following expressions for $\breve{\mathfrak{u}}(\mathfrak{k})$:*

1. *If $\breve{\mathfrak{u}}(\mathfrak{k})$ is a (1)-weighted Caputo $\mathtt{gH}-$differentiable function of order $\wp$ with respect to the weight function $\omega(\mathfrak{k})$ over $\mathcal{I}$, then*

$$\breve{\mathfrak{u}}(\mathfrak{k}) = \frac{\omega(a)\breve{\mathfrak{u}}(a)}{\omega(\mathfrak{k})} \oplus \frac{1-\wp}{\aleph(\wp)}\mathfrak{h}(\mathfrak{k},\breve{\mathfrak{u}}(\mathfrak{k})) \oplus \frac{\wp}{\aleph(\wp)\Gamma(\beta)}\frac{1}{\omega(\mathfrak{k})}\int_a^{\mathfrak{k}} (\mathfrak{k}-\xi)^{\beta-1}\omega(\xi)\mathfrak{h}\left(\xi,\breve{\mathfrak{u}}(\xi)\right) \mathrm{d}\xi. \tag{7}$$

2. *If $\breve{\mathfrak{u}}(\mathfrak{k})$ is a (2)-weighted Caputo $\mathtt{gH}-$differentiable function of order $\wp$ with respect to the weight function $\omega(\mathfrak{k})$ over $\mathcal{I}$, then*

$$\breve{\mathfrak{u}}(\mathfrak{k}) = \frac{\omega(a)\breve{\mathfrak{u}}(a)}{\omega(\mathfrak{k})} \ominus (-1)\left(\frac{1-\wp}{\aleph(\wp)}\mathfrak{h}(\mathfrak{k},\breve{\mathfrak{u}}(\mathfrak{k})) \oplus \frac{\wp}{\aleph(\wp)\Gamma(\beta)}\frac{1}{\omega(\mathfrak{k})}\int_a^{\mathfrak{k}} (\mathfrak{k}-\xi)^{\beta-1}\omega(\xi)\mathfrak{h}\left(\xi,\breve{\mathfrak{u}}(\xi)\right) \mathrm{d}\xi\right). \tag{8}$$

Proof. In the following analysis, we will focus on status (1). However, the same analysis can be applied to status (2). To both sides of Equation (6), we apply fuzzy generalized weighted fractional integral, resulting in:

$$\mathbb{I}^{\wp,\beta}_{a,\omega}\left({}^{C}_{\mathtt{gH}}\mathbb{D}^{\wp,\beta}_{a,\omega}\breve{\mathfrak{u}}(\mathfrak{k})\right) = \mathbb{I}^{\wp,\beta}_{a,\omega}\left(\mathfrak{h}\left(\mathfrak{k},\breve{\mathfrak{u}}(\mathfrak{k})\right)\right).$$

Using Theorem 4.7 for (1)-weighted Caputo $\mathtt{gH}-$differentiable functions and Definition 3.1, we obtain:

$$\breve{\mathfrak{u}}(\mathfrak{k}) \ominus \frac{\omega(a)\breve{\mathfrak{u}}(a)}{\omega(\mathfrak{k})} = \frac{1-\wp}{\aleph(\wp)}\mathfrak{h}(\mathfrak{k},\breve{\mathfrak{u}}(\mathfrak{k})) \oplus \frac{\wp}{\aleph(\wp)\Gamma(\beta)}\frac{1}{\omega(\mathfrak{k})}\int_a^{\mathfrak{k}} (\mathfrak{k}-\xi)^{\beta-1}\omega(\xi)\mathfrak{h}\left(\xi,\breve{\mathfrak{u}}(\xi)\right) \mathrm{d}\xi.$$

The result has been achieved. □

Theorem 6.2. *Let $\wp \in [0,1)$, $\beta \in (0,1]$, and $\mathfrak{h} : \mathcal{I} \times \Re_{\mathfrak{f}} \to \Re_{\mathfrak{f}}$ be a fuzzy continuous function satisfying the inequality*

$$\mathcal{D}\left(\mathfrak{h}(\mathfrak{k},\breve{\mathfrak{u}_1}(\mathfrak{k})),\mathfrak{h}(\mathfrak{k},\breve{\mathfrak{u}_2}(\mathfrak{k}))\right) \leq \varepsilon\mathcal{D}\left(\breve{\mathfrak{u}_1},\breve{\mathfrak{u}_2}\right).$$

where ε is a constant. If

$$\varepsilon\left(\frac{(1-\wp)}{\aleph(\wp)} + \frac{\wp}{\aleph(\wp)\Gamma(\beta)}\frac{\omega(b)}{\omega(a)}\frac{(b-a)^{\beta}}{\beta}\right) < 1,$$

then the fuzzy initial value problem (6) has a unique solution in $\mathcal{I}$ depending on the weighted Caputo $\mathtt{gH}-$differentiability type.

Proof. To begin, we introduce a continuous map T given by:

$$(T\breve{\mathfrak{u}})(\mathfrak{k}) = \frac{\omega(a)\breve{\mathfrak{u}}(a)}{\omega(\mathfrak{k})} \oplus \frac{1-\wp}{\aleph(\wp)}\mathfrak{h}(\mathfrak{k}, \breve{\mathfrak{u}}(\mathfrak{k})) \oplus \frac{\wp}{\aleph(\wp)\Gamma(\beta)}\frac{1}{\omega(\mathfrak{k})}\int_a^{\mathfrak{k}}(\mathfrak{k}-\xi)^{\beta-1}\omega(\xi)\mathfrak{h}\left(\xi, \breve{\mathfrak{u}}(\xi)\right)\mathrm{d}\xi,$$

which applies to the (1)-weighted Caputo $\mathtt{gH}-$differentiable function $\breve{\mathfrak{u}}(\mathfrak{k})$. Using this, Equation (7) provides a fuzzy solution to the initial value problem in Equation (6). In other words, T exists as a fixed point that satisfies Equation (6).

For any $\breve{\mathfrak{u}_1}, \breve{\mathfrak{u}_2} \in \mathcal{K}(\mathcal{I})$ and $\mathfrak{k} \in \mathcal{I}$, the following inequality holds:

$$\begin{aligned}
\mathcal{D}\left((T\breve{\mathfrak{u}_1})(\mathfrak{k}), (T\breve{\mathfrak{u}_2})(\mathfrak{k})\right) &\leq \mathcal{D}\left(\frac{1-\wp}{\aleph(\wp)}\mathfrak{h}(\mathfrak{k}, \breve{\mathfrak{u}_1}(\mathfrak{k})), \frac{1-\wp}{\aleph(\wp)}\mathfrak{h}(\mathfrak{k}, \breve{\mathfrak{u}_2}(\mathfrak{k}))\right) \\
&+\mathcal{D}\left(\frac{\wp}{\aleph(\wp)\Gamma(\beta)}\frac{1}{\omega(\mathfrak{k})}\int_a^{\mathfrak{k}}(\mathfrak{k}-\xi)^{\beta-1}\omega(\xi)\mathfrak{h}\left(\xi, \breve{\mathfrak{u}_1}(\xi)\right)\mathrm{d}\xi, \frac{\wp}{\aleph(\wp)\Gamma(\beta)}\frac{1}{\omega(\mathfrak{k})}\int_a^{\mathfrak{k}}(\mathfrak{k}-\xi)^{\beta-1}\omega(\xi)\mathfrak{h}\left(\xi, \breve{\mathfrak{u}_2}(\xi)\right)\mathrm{d}\xi\right) \\
&\leq \frac{1-\wp}{\aleph(\wp)}\mathcal{D}\left(\mathfrak{h}(\mathfrak{k}, \breve{\mathfrak{u}_1}(\mathfrak{k})), \mathfrak{h}(\mathfrak{k}, \breve{\mathfrak{u}_2}(\mathfrak{k}))\right) + \frac{\wp}{\aleph(\wp)\Gamma(\beta)}\frac{1}{\omega(\mathfrak{k})}\int_a^{\mathfrak{k}}(\mathfrak{k}-\xi)^{\beta-1}\omega(\xi)\mathcal{D}\left(\mathfrak{h}\left(\xi, \breve{\mathfrak{u}_1}(\xi)\right), \mathfrak{h}\left(\xi, \breve{\mathfrak{u}_2}(\xi)\right)\mathrm{d}\xi\right) \\
&\leq \frac{\varepsilon(1-\wp)}{\aleph(\wp)}\mathcal{D}\left(\breve{\mathfrak{u}_1}, \breve{\mathfrak{u}_2}\right) + \frac{\wp}{\aleph(\wp)\Gamma(\beta)}\frac{\varepsilon}{\omega(\mathfrak{k})}\int_a^{\mathfrak{k}}(\mathfrak{k}-\xi)^{\beta-1}\omega(\xi)\mathcal{D}\left(\breve{\mathfrak{u}_1}, \breve{\mathfrak{u}_2}\right)\mathrm{d}\xi \\
&\leq \frac{\varepsilon(1-\wp)}{\aleph(\wp)}\mathcal{D}\left(\breve{\mathfrak{u}_1}, \breve{\mathfrak{u}_2}\right) + \frac{\wp}{\aleph(\wp)\Gamma(\beta)}\frac{\varepsilon}{\omega(\mathfrak{k})}\mathcal{D}\left(\breve{\mathfrak{u}_1}, \breve{\mathfrak{u}_2}\right)\int_a^{\mathfrak{k}}(\mathfrak{k}-\xi)^{\beta-1}\omega(\xi)\mathrm{d}\xi.
\end{aligned}$$

By applying the mean value theorem and using the fact that $(\mathfrak{k}-\xi)^{\beta-1} > 0$ and $\omega > 0$, we obtain ζ such that $a < \zeta < b$ and

$$\begin{aligned}
\int_a^{\mathfrak{k}}(\mathfrak{k}-\xi)^{\beta-1}\omega(\xi)\mathrm{d}\xi &= \omega(\zeta)\int_a^{\mathfrak{k}}(\mathfrak{k}-\xi)^{\beta-1}\mathrm{d}\xi \\
&= \omega(\zeta)\frac{(t-a)^{\beta}}{\beta} \\
&\leq \omega(b)\frac{(b-a)^{\beta}}{\beta}.
\end{aligned}$$

Thus, we have

$$\mathcal{D}\left((T\breve{\mathfrak{u}_1})(\mathfrak{k}), (T\breve{\mathfrak{u}_2})(\mathfrak{k})\right) \leq \varepsilon\left(\frac{(1-\wp)}{\aleph(\wp)} + \frac{\wp}{\aleph(\wp)\Gamma(\beta)}\frac{\omega(b)}{\omega(a)}\frac{(b-a)^{\beta}}{\beta}\right)\mathcal{D}\left(\breve{\mathfrak{u}_1}, \breve{\mathfrak{u}_2}\right).$$

This implies that T is a contraction operator since $\varepsilon\left(\frac{(1-\wp)}{\aleph(\wp)} + \frac{\wp}{\aleph(\wp)\Gamma(\beta)}\frac{\omega(b)}{\omega(a)}\frac{(b-a)^{\beta}}{\beta}\right) < 1$. By the Banach contraction mapping principle, we can conclude that T has a unique fixed point which is the (1)-weighted Caputo $\mathtt{gH}-$differentiable solution of problem (6) on $\mathcal{I}$.

Now, assuming $\breve{\mathfrak{u}}(\mathfrak{k})$ is a (2)-weighted Caputo $\mathtt{gH}-$differentiable function, we can define a continuous map T as:

$$(T\breve{\mathfrak{u}})(\mathfrak{k}) = \frac{\omega(a)\breve{\mathfrak{u}}(a)}{\omega(\mathfrak{k})} \ominus (-1)\left(\frac{1-\wp}{\aleph(\wp)}\mathfrak{h}(\mathfrak{k}, \breve{\mathfrak{u}}(\mathfrak{k})) \oplus \frac{\wp}{\aleph(\wp)\Gamma(\beta)}\frac{1}{\omega(\mathfrak{k})}\int_a^{\mathfrak{k}}(\mathfrak{k}-\xi)^{\beta-1}\omega(\xi)\mathfrak{h}\left(\xi, \breve{\mathfrak{u}}(\xi)\right)\mathrm{d}\xi\right).$$

This kind of differentiability can be shown in the same manner, so we omit it from the proof. □

7 Fuzzy Laplace transform method for analyzing fractional fuzzy differential equations: Practical examples and applications

This section of the paper presents practical examples demonstrating the proposed approach for obtaining analytical solutions to the fuzzy fractional differential equation using the fuzzy Laplace transform. The examples include the fuzzy fractional Newton's law of heating and cooling, a commonly studied problem in applied mathematics. The section illustrates how the method can be applied to real-world problems and provides a clear understanding of the

steps involved in solving such problems using the proposed approach. By presenting concrete examples, the section enhances the practical value of the proposed method and strengthens the validity of the results obtained through its application.

To obtain analytical solutions for specific cases of the fractional fuzzy differential equation (6), we will utilize the Laplace transform. The approach involves the following steps: First, we multiply both sides of Equation (6) by a function $\omega(\mathfrak{k})$ and apply the fuzzy Laplace transform using relevant theorems and properties outlined in Section 5. Next, we incorporate the initial conditions into the transformed equation and solve for the output variable. Finally, we apply the fuzzy inverse Laplace transform to obtain the solution in the original time domain.

In the subsequent theorems, we delve into the fuzzy solution for specific types of the fractional fuzzy differential equation (6). These theorems demonstrate the practical application of the aforementioned methodology and provide a deeper understanding of the solution process for a variety of real-world problems.

Theorem 7.1. *Suppose $\wp \in [0,1)$, $\beta \in (0,1]$, $\breve{\mathrm{u}}(\mathfrak{k})$ is a fuzzy function defined on $\mathcal{I}$, and $\breve{\nu}$ is a positive fuzzy constant. If $\lambda\breve{\mathrm{u}}(0) \ominus \breve{\nu} = 0$, then the unique (1)-weighted Caputo $\mathtt{gH}-$differentiable solution of the fuzzy linear fractional differential equation*

$$\begin{cases} {}^{C}_{\mathtt{gH}}\mathbb{D}^{\wp,\beta}_{0,\omega}\breve{\mathrm{u}}(\mathfrak{k}) = \lambda\breve{\mathrm{u}}(\mathfrak{k}) \ominus \breve{\nu}, \quad \mathfrak{k} \in \mathcal{I},\ \lambda \in \mathbb{R}^{+} \\ \breve{\mathrm{u}}(0) = \breve{\mathrm{u}_0} \in \Re_{\mathfrak{f}}, \end{cases} \tag{9}$$

is obtained as follows:

$$\breve{\mathrm{u}}(\mathfrak{k}) = \frac{\aleph(\wp)\omega(0)\breve{\mathrm{u}_0}}{\varphi\omega(\mathfrak{k})}\mathbb{E}_{\beta}\left(\frac{\lambda\wp}{\varphi}\mathfrak{k}^{\beta}\right) \ominus \frac{\breve{\nu}(1-\wp)}{\phi} \ominus \frac{\breve{\nu}\aleph(\wp)}{\lambda\phi\omega(\mathfrak{k})}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{k}}\mathbb{E}_{\beta}\left(\frac{\lambda\wp}{\varphi}\mathfrak{k}^{\beta}\right) * \omega(\mathfrak{k}). \tag{10}$$

Where $\varphi = \aleph(\wp) - \lambda(1-\wp) \neq 0$.

Proof. After multiplying both sides of Equation (9) by the function $\omega(\mathfrak{k})$, we take the Laplace transform of the resulting equation. Using the fuzzy Laplace properties [25] and Theorem 5.4, we can write:

$$\mathscr{L}_0\left\{\omega(\mathfrak{k})\breve{\mathrm{u}}(\mathfrak{k})\right\}(s) = \frac{\aleph(\wp)\omega(0)\breve{\mathrm{u}}(0)s^{\beta-1}}{[\aleph(\wp)-\lambda(1-\wp)]s^{\beta}-\lambda\wp} \ominus \left(\frac{\breve{\nu}(1-\wp)s^{\beta} \oplus \breve{\nu}\wp}{[\aleph(\wp)-\lambda(1-\wp)]s^{\beta}-\lambda\wp}\right)\mathscr{L}_0\left\{\omega(\mathfrak{k})\right\}(s).$$

Here, we let $\varphi = \aleph(\wp) - \lambda(1-\wp)$. Simplifying further, we get:

$$\mathscr{L}_0\left\{\omega(\mathfrak{k})\breve{\mathrm{u}}(\mathfrak{k})\right\}(s) = \frac{\aleph(\wp)\omega(0)\breve{\mathrm{u}}(0)}{\varphi}\frac{s^{\beta-1}}{s^{\beta}-\frac{\lambda\wp}{\varphi}} \ominus \frac{\breve{\nu}(1-\wp)}{\varphi}\frac{s^{\beta-1}}{s^{\beta}-\frac{\lambda\wp}{\varphi}}s\mathscr{L}_0\left\{\omega(\mathfrak{k})\right\}(s) \ominus \frac{\breve{\nu}\wp}{\varphi}\frac{1}{s^{\beta}-\frac{\lambda\wp}{\varphi}}\mathscr{L}_0\left\{\omega(\mathfrak{k})\right\}(s).$$

Finally, we apply Proposition 5.5 to obtain:

$$\begin{aligned} \mathscr{L}_0\left\{\omega(\mathfrak{k})\breve{\mathrm{u}}(\mathfrak{k})\right\}(s) &= \frac{\aleph(\wp)\omega(0)\breve{\mathrm{u}}(0)}{\varphi}\mathscr{L}_0\left\{\mathbb{E}_{\beta}\left(\frac{\lambda\wp}{\varphi}\mathfrak{k}^{\beta}\right)\right\}(s) \\ &\ominus \frac{\breve{\nu}(1-\wp)}{\varphi}\mathscr{L}_0\left\{\mathbb{E}_{\beta}\left(\frac{\lambda\wp}{\varphi}\mathfrak{k}^{\beta}\right)\right\}\left(\mathscr{L}_0\left\{\omega'(\mathfrak{k})\right\}(s) + \omega(0)\right) \\ &\ominus \frac{\breve{\nu}}{\lambda}\mathscr{L}_0\left\{\frac{\mathrm{d}}{\mathrm{d}\mathfrak{k}}\mathbb{E}_{\beta}\left(\frac{\lambda\wp}{\varphi}\mathfrak{k}^{\beta}\right)\right\}\mathscr{L}_0\left\{\omega(\mathfrak{k})\right\}(s). \end{aligned}$$

The fuzzy inverse Laplace method yields the following expression:

$$\begin{aligned} \omega(\mathfrak{k})\breve{\mathrm{u}}(\mathfrak{k}) &= \frac{\aleph(\wp)\omega(0)\breve{\mathrm{u}}(0)}{\varphi}\mathbb{E}_{\beta}\left(\frac{\lambda\wp}{\varphi}\mathfrak{k}^{\beta}\right) \ominus \frac{\breve{\nu}(1-\wp)}{\varphi}\mathbb{E}_{\beta}\left(\frac{\lambda\wp}{\varphi}\mathfrak{k}^{\beta}\right) * \omega'(\mathfrak{k}) \\ &\ominus \frac{\breve{\nu}(1-\wp)}{\varphi}\mathbb{E}_{\beta}\left(\frac{\lambda\wp}{\varphi}\mathfrak{k}^{\beta}\right)\omega(0) \ominus \frac{\breve{\nu}}{\lambda}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{k}}\mathbb{E}_{\beta}\left(\frac{\lambda\wp}{\varphi}\mathfrak{k}^{\beta}\right) * \omega(\mathfrak{k}). \end{aligned} \tag{11}$$

Next, we can use integration by parts, as shown in [15], to simplify the term $\frac{\breve{\nu}(1-\wp)}{\varphi}\mathbb{E}_{\beta}\left(\frac{\lambda\wp}{\varphi}\mathfrak{k}^{\beta}\right) * \omega'(\mathfrak{k})$:

$$\frac{\breve{\nu}(1-\wp)}{\varphi}\mathbb{E}_{\beta}\left(\frac{\lambda\wp}{\varphi}\mathfrak{k}^{\beta}\right) * \omega'(\mathfrak{k}) = \frac{\breve{\nu}(1-\wp)}{\phi}\omega(\mathfrak{k}) \ominus \frac{\breve{\nu}(1-\wp)}{\varphi}\mathbb{E}_{\beta}\left(\frac{\lambda\wp}{\varphi}\mathfrak{k}^{\beta}\right)\omega(0) \oplus \frac{\breve{\nu}(1-\wp)}{\phi}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{k}}\mathbb{E}_{\beta}\left(\frac{\lambda\wp}{\varphi}\mathfrak{k}^{\beta}\right) * \omega(\mathfrak{k}). \tag{12}$$

Finally, by substituting Equation (12) into Equation (11), we arrive at the desired result. □

Example 7.2. *(Newton's Law of Heating and Cooling) Energy transfer into or out of a system causes a temperature difference. Heating results from energy transfer into the system, while cooling results from energy transfer out of the system. The precise temperature value is difficult to determine in real-world situations since they differ from laboratory conditions. Fuzzy models, which account for uncertainty and vagueness, are better suited to comprehensively address such issues. According to Newton's law of heating, $\breve{\mathfrak{u}}(\mathfrak{k})$, the rate of temperature change is directly proportional to the temperature difference between an object and its surrounding environment. As a consequence of this law, $\breve{\mathfrak{u}}(\mathfrak{k})$ satisfies the following differential equation:*

$$\overset{C}{\mathtt{gH}}\mathbb{D}_{0,\omega}^{\wp,\beta}\breve{\mathfrak{u}}(\mathfrak{k}) = \lambda\breve{\mathfrak{u}}(\mathfrak{k}) \ominus \breve{\mathcal{M}},$$

where λ is a positive real proportionality constant, and $\breve{\mathcal{M}}$ is the ambient temperature, which is a positive fuzzy constant.

To analyze the model based on previous results, consider the following example: a turkey is initially at a temperature range of $[40.5 + 3.25\mathtt{r}, 44.75 - \mathtt{r}]^{\circ}\, F$, and is placed in an oven with a temperature range of $[324 + 26\mathtt{r}, 358.9 - 8.9\mathtt{r}]^{\circ}\, F$.

Let's consider the given parameters: $\lambda = 8$, $\wp = \frac{1}{2}$, $\beta = 1$, and $\omega(\mathfrak{k}) = 2\mathfrak{k} + 1$. Initially, we find that $\breve{\mathfrak{u}}(0) = \frac{1}{\lambda}\breve{\mathcal{M}}$. As the turkey is cooking, its temperature gradually increases, and we can calculate this temperature evolution using Theorem 7.1. Employing Equation 10, we can express the temperature evolution as follows:

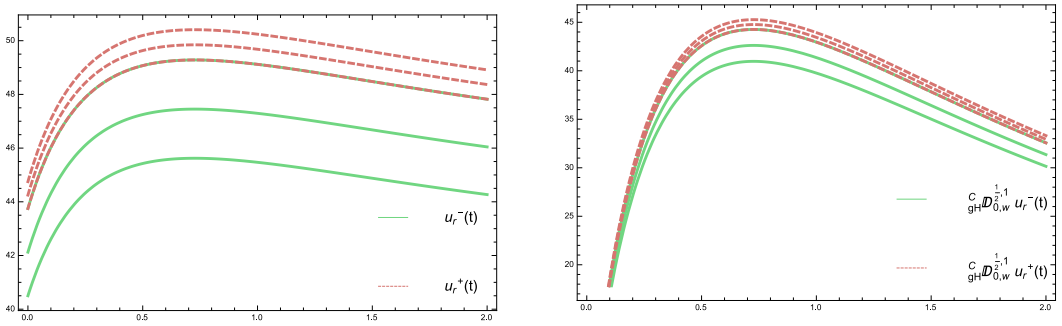
$$\begin{aligned}\breve{\mathfrak{u}}(\mathfrak{k}) \;=\; & \frac{[324 + 26\mathtt{r}, 358.9 - 8.9\mathtt{r}]}{24(1+2\mathfrak{k})} e^{\frac{-4\mathfrak{k}}{3}} \ominus \left[\frac{-324 - 26\mathtt{r}}{6}, \frac{-358 + 8\mathtt{r}}{6}\right] \\ & \ominus \left[\frac{324 + 26\mathtt{r}}{24(1+2\mathfrak{k})}, \frac{358 - 8\mathtt{r}}{24(1+2\mathfrak{k})}\right] \int_0^{\mathfrak{k}} \frac{4}{3} e^{-\frac{4}{3}(\mathfrak{k}-\xi)}(1+2\xi)\mathtt{d}\xi.\end{aligned}$$

This expression can be further simplified to:

$$\mathfrak{u}(\mathfrak{k}) = [162 + 13\mathtt{r}, 179 - 4\mathtt{r}] \left(\frac{e^{\frac{-4\mathfrak{k}}{3}}(e^{\frac{4\mathfrak{k}}{3}}(4\mathfrak{k}+3) - 1)}{8(2\mathfrak{k}+1)}\right). \tag{13}$$

To simplify, we have assumed that $\aleph(\wp) = 1$.

In Figure 3, we can observe the $\mathtt{r}$-cuts of both $\breve{\mathfrak{u}}(\mathfrak{k})$ and $\overset{C}{\mathtt{gH}}\mathbb{D}_{0,\omega}^{\frac{1}{2},1}\breve{\mathfrak{u}}(\mathfrak{k})$. Both functions have the same lower (green) and upper (red) cuts, indicating that $\breve{\mathfrak{u}}(\mathfrak{k})$ is a (1)-weighted Caputo $\mathtt{gH}-$differentiable function of order $\wp$ with respect to the weight function $\omega(\mathfrak{k})$.



(a).$[\breve{\mathfrak{u}}(\mathfrak{k})]_{\mathtt{r}}$ (b). $\overset{C}{\mathtt{gH}}\mathbb{D}_{0,\omega}^{\frac{1}{2},1}\breve{\mathfrak{u}}(\mathfrak{k})$

Figure 3: Plots of $[\breve{\mathfrak{u}}(\mathfrak{k})]_{\mathtt{r}}$ and $\overset{C}{\mathtt{gH}}\mathbb{D}_{0,\omega}^{\frac{1}{2},1}\breve{\mathfrak{u}}(\mathfrak{k})$ of Example 7.2 at $\mathtt{r} = 0, 0.5, 1$.

Theorem 7.3. *Let $\wp \in [0,1)$, $\beta \in (0,1]$, $\breve{\mathfrak{u}}(\mathfrak{k}) \in \mathcal{K}(\mathcal{I})$, and $\breve{\nu}$ is a positive fuzzy constant. If $\lambda\breve{\mathfrak{u}}(0) \ominus \breve{\nu} = 0$, then the unique (2)-weighted Caputo $\mathtt{gH}-$differentiable solution of the fuzzy linear fractional differential equation*

$$\begin{cases} \overset{C}{\mathtt{gH}}\mathbb{D}_{0,\omega}^{\wp,\beta}\breve{\mathfrak{u}}(\mathfrak{k}) = (-1)\left(\lambda\breve{\mathfrak{u}}(\mathfrak{k}) \ominus \breve{\nu}\right), \quad \mathfrak{k} \in \mathcal{I},\ \lambda \in \mathbb{R}^+ \\ \breve{\mathfrak{u}}(0) = \breve{\mathfrak{u}}_0 \in \Re_{\mathfrak{f}}, \end{cases} \tag{14}$$

is given by

$$\breve{\mathtt{u}}(\mathfrak{k}) = \frac{\aleph(\wp)\omega(0)\breve{\mathtt{u}_0}}{\varphi\omega(\mathfrak{k})}\mathbb{E}_\beta\left(-\frac{\lambda\wp}{\varphi}\mathfrak{k}^\beta\right) \oplus \frac{\breve{\nu}(1-\wp)}{\phi} \oplus (-1)\frac{\breve{\nu}\aleph(\wp)}{\lambda\phi\omega(\mathfrak{k})}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{k}}\mathbb{E}_\beta\left(-\frac{\lambda\wp}{\varphi}\mathfrak{k}^\beta\right) * \omega(\mathfrak{k}). \tag{15}$$

Where $\varphi = \aleph(\wp) + \lambda(1-\wp) \neq 0$.

Proof. One can employ a comparable approach to establish this theorem, akin to the method used for Theorem 7.1. □

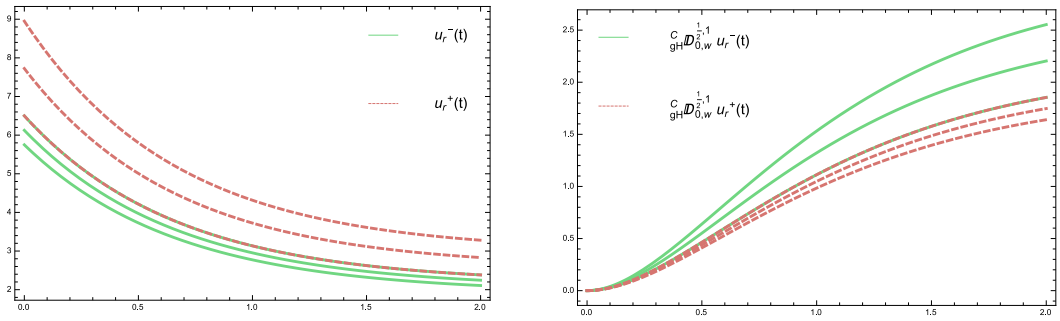
Example 7.4. *Let us consider the fuzzy fractional differential equation expressed in terms of weighted Caputo* gH*−differentiability given below:*

$$\begin{cases} {}^{C}_{\mathtt{gH}}\mathbb{D}^{\frac{1}{2},1}_{0,\omega}\breve{\mathtt{u}}(\mathfrak{k}) = (-1)\left(2\breve{\mathtt{u}}(\mathfrak{k}) \ominus [11.5+1.5\mathtt{r}, 17.9-4.9\mathtt{r}]\right), \\ \breve{\mathtt{u}}(0) = [5.75+0.75\mathtt{r}, 8.95-2.45\mathtt{r}], \end{cases}$$

With $\wp = \frac{1}{2}$, $\beta = 1$, *and* $\aleph(\wp) = 1$, *we can employ Equation (15) to derive the following solution:*

$$\begin{aligned} \breve{\mathtt{u}}(\mathfrak{k}) &= \frac{1}{4}[11.5+1.5\mathtt{r}, 17.9-4.9\mathtt{r}]\, e^{-\frac{3\mathfrak{k}}{2}} \oplus \frac{1}{4}[11.5+1.5\mathtt{r}, 17.9-4.9\mathtt{r}] \\ &\quad \oplus(-1)\frac{e^{-\mathfrak{k}}}{4}[11.5+1.5\mathtt{r}, 17.9-4.9\mathtt{r}]\int_0^{\mathfrak{k}}\left(\frac{-1}{2}e^{\xi+\frac{1}{2}(\xi-\mathfrak{k})}\mathrm{d}\xi\right) \\ &= \left[1.91667 + e^{\frac{-3\mathfrak{k}}{3}}(3.83333+0.5\mathtt{r}) + 0.25\mathtt{r}, 2.98333 + e^{\frac{-3\mathfrak{k}}{3}}(5.96667-1.63333\mathtt{r}) - 0.816667\mathtt{r}\right]. \end{aligned}$$

In Figure 4, the r*-cuts of* $\breve{\mathtt{u}}(\mathfrak{k})$ *and* ${}^{C}_{\mathtt{gH}}\mathbb{D}^{\frac{1}{2},1}_{0,\omega}\breve{\mathtt{u}}(\mathfrak{k})$ *are depicted. The change in the positions of the lower cut (Green) and upper cut (Red) shown in Figure 4 indicate that* $\breve{\mathtt{u}}(\mathfrak{k})$ *is a function of order* $\wp$ *with respect to the weight function* $\omega(\mathfrak{k})$ *and is (2)-weighted Caputo* gH*−differentiable.*



(a).$[\breve{\mathtt{u}}(\mathfrak{k})]_{\mathtt{r}}$. (b). ${}^{C}_{\mathtt{gH}}\mathbb{D}^{\frac{1}{2},1}_{0,\omega}\breve{\mathtt{u}}(\mathfrak{k})$.

Figure 4: Plots of $[\breve{\mathtt{u}}(\mathfrak{k})]_{\mathtt{r}}$ and ${}^{C}_{\mathtt{gH}}\mathbb{D}^{\frac{1}{2},1}_{0,\omega}\breve{\mathtt{u}}(\mathfrak{k})$ of Example 7.4 at $\mathtt{r} = 0, 0.5, 1$.

8 Conclusion

This paper introduced novel concepts of generalized weighted fractional integrals and weighted Caputo-type generalized Hukuhara derivatives for fuzzy functions. The paper also presented a proof for the existence and uniqueness theorem for fuzzy fractional differential equations in terms of fuzzy weighted Caputo gH−derivatives. Fuzzy analytical solutions for such equations were obtained using fuzzy Laplace method, which proved to be a convenient and effective approach for solving these types of equations. This research opens up new avenues for future investigations in the field of fuzzy mathematics and fractional calculus. Further studies can be conducted based on the findings of this paper to extend the scope of fuzzy fractional differential equations.

As for future plans, it would be interesting to investigate the applications of the generalized weighted fractional integrals and weighted Caputo-type generalized Hukuhara derivatives in other areas of mathematics and science. Additionally, it may be worthwhile to explore further extensions and generalizations of these concepts, such as applying them to fuzzy partial differential equations or exploring different types of fuzzy derivatives. Finally, it would be beneficial to conduct numerical simulations and experiments to validate the effectiveness and accuracy of our proposed method for solving fuzzy fractional differential equations.

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