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Original paper

A novel similarity measure based on the center of nine-point circle of the isosceles triangular fuzzy numbers and their applications

H. Cheng [1]

[1] *School of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, Henan, P. R. China*

chenghuiru316316@126.com

Abstract

A new similarity/distance measure based on the center of nine-point circle of the isosceles triangular fuzzy numbers is recommend in this paper. Extend the similarity/distance measure based on centroid, orthocenter, circumcenter, incenter and nine-point circle center of the isosceles triangles. It is proved that this general similarity/distance measure conforms to the properties of distance. Subsequently, some examples are presented to justify the superiority and validity of the proposed similarity/distance measure between IFSs based on the center of nine-point circle, which demonstrate that this measure overcomes the disadvantage of the existing similarity measures. The application of the proposed similarity measure to deal with pattern recognition problems is described, and the results are correlated with those reported in some prevailing studies. In addition, a clustering technique to classify objects based on the proposed similarity measure is discussed. Through a detailed comparative analysis of some existing measures, it is concluded that some of the existing measures fail to discriminate the results obtained under different circumstances, such as zero division or counter intuitive cases; in contrast, the proposed similarity measure successfully overcomes this weakness.

Keywords: Intuitionistic fuzzy set, similarity/distance measures, triangular fuzzy numbers, decision-making, nine-point circle.

1 Introduction

The intuitionistic fuzzy set(IFSs) is one of the most vigorous and reliable tools for depicting imprecise information using membership degree and non-membership degree. IFSs have been widely applied to some problems in cluster analysis, multi-attribute decision-making, pattern recognition, medical diagnosis, risk analysis etc. Similarity and distance measures are frequently used tools for treating imperfect and ambiguous information to achieve the final decision. The differences between membership and nonmembership are commonly used parameters in the similarity measure of IFSs. The degree of hesitation is also considered critical. The similarity/distance measures of IFSs have been researched in some studies, wherein attempts were made to utilize this parameter creatively. Some research on the distance measure claimed that their use of membership and hesitancy degree is effective. Some of the existing methods do not satisfy the axiomatic definitions of similarity measures. In Section 2, we review some of the existing similarity measures and discuss their limitations. This review motivated us to revisit the similarity measurement from this new perspective. Atanassov first proposed the IFSs [3]. Chen et al, proposed the notion of measuring the degree of similarity among vague sets [7]. Hong and Kim reported on similarity measures between vague sets and between elements [12]. Deng and Chuntian reported that Chen's measures are inadvisable for similarity measures in IFSs [19]. Mitchell examined the application of a measure based on Deng and Chuntian's similarity measure to pattern recognition [22]. In the same year, Liang and Shi proposed three new distance measures and described their properties and applications [20]. Szmidt and Kacprzyk derived a similarity measure for IFSs and discussed its application for supporting medical diagnostic reasoning [27]. A similarity measure of IFSs based on Hausdorff distance and Lp metric was developed by Hung and Yang [13], who applied

Corresponding Author: H. Cheng

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the distance measures of IFSs to pattern recognitions. New similarity measures between IFSs and between elements were developed by Liu [21]. Vlachos presented the application of intuitionistic fuzzy information in pattern recognition [28]. Ye proposed a cosine similarity measure, which is also used in pattern recognition problems and medical diagnosis [31]. Because of the potential of IFSs, many researchers studied the similarity and distance measures between IFSs, thereby promoting the development of the corresponding theory and its application [6, 18, 33]. An IFS similarity measure based on the centroid points of transformed fuzzy numbers and its application to pattern recognition was proposed by Chen and Cheng [9]. Ngan proposed a new metric H-max distance in an IFS environment [23]. In addition, many researchers proposed various formulae to calculate the distance and similarity between IFSs. Recently, similarity measures based on the centroid, orthocentre, circumcentre and incentre points of transformed triangular fuzzy numbers were proposed by Garg et al [11], and they were used for solving the problem of human decision-making. Based on the above works, a new similarity measure between IFSs based on the center of nine-point circle is proposed herein. The similarity measure is generalized with the centroid, orthocentre, circumcentre and incentre of the transformed triangular fuzzy numbers. According to the general formula, when and take different values, they correspond to different centers of the triangle. The rest of this paper is organized as follows. Section 2 provides a review some of the basic definitions and properties of IFSs. Furthermore, the previously proposed distance and similarity measures are especially noted. Section 3 introduces the similarity and distance measures based on the nine-point circle of the triangle. In addition, the general measure based on the centroid, orthocentre, circumcenter, incenter and nine-point circle of the transformed triangular numbers are reported, and their theories and properties are described. Section 4 presents the experimental results of the proposed similarity measure and compares them with the results obtained from existing similarity measures. Section 5 describes the applications of the proposed similarity measures for various purposes, including medical diagnosis, pattern recognition, and clustering analysis. In the end, the conclusions are declaration in Section 6.

2 Preliminaries

The work of this section involves a brief review of some essential concepts associated with intuitionistic fuzzy sets, as well as the recommendation of properties for similarity measures between IFSs, aiming to demonstrate the foundation of knowledge.

2.1 Basic definitions

Definition 2.1. [32] *Let Λ be a universe of discourse, then a fuzzy set* L *in Λ is defined as: $L = \{< \epsilon, \xi_L(\epsilon) > | \epsilon \in \Lambda\}$, where $\xi_L(\epsilon) \in [0, 1]$ is membership degree.*

Definition 2.2. [3] *An IFS $L \in \Lambda$ defined as $L = \{< \epsilon, \xi_L(\epsilon), \delta_L(\epsilon) > | \epsilon \in \Lambda\}$, where $\xi_L(\epsilon)$ is membership degree and $\delta_L(\epsilon)$ is non-membership degree, $0 \leq \xi_L(\epsilon) + \delta_L(\epsilon) \leq 1, \forall \epsilon \in \Lambda$, $\tau_A(\epsilon) = 1 - \xi_L(\epsilon) - \delta_L(\epsilon)$ is the degree of hesitation.*

Definition 2.3. [1] *For $L \in IFSs(\Lambda)$ and $N \in IFSs(\Lambda)$, some relations between L and N are defined as:*
(1) *$L \subseteq N$ if $\xi_L(\epsilon) \leq \xi_N(\epsilon)$ and $\delta_L(\epsilon) \geq \delta_N(\epsilon), \forall x \in \Lambda$;*
(2) *$L = N \Leftrightarrow L \subseteq N, N \subseteq L$.*

Definition 2.4. [29] *The distance/similarity measure $d(L, N) \to [0, 1]$ fulfill the following conditions:*
(1) *$0 \leq d(L, N) \leq 1$;*
(2) *$d(L, N) = 0$ if and only if $L = N$;*
(3) *$d(L, N) = d(N, L)$;*
(4) *If $L \subseteq N \subseteq P$, then $d(L, N) \leq d(L, P)$ and $d(N, P) \leq d(L, P)$.*

The properties of similarity/distance measures between IFSs are complementary, i.e., $s(L, N) = 1 - d(L, N)$. Therefore, similarity measures can be defined based on the definition of distance measures, and vice versa, distance measures can also be defined analogously based on similarity measures.

Definition 2.5. [4] *A matrix $Z = (z_{iq})_{m\times m}$ on IFSs $L_i (i = 1, 2, ...m)$ is called a similarity matrix if it satisfies:*
(1) *$0 \leq z_{iq} \leq 1$;*
(2) *$z_{ii} = 1$;*
(3) *$z_{iq} = z_{qi}$, where $i, q = 1, 2, ...m$.*
Where $z_{iq} = s(L_i, L_q)$ is the similarity measure.

Definition 2.6. [4] *A composition matrix defined as $Z^2 = Z \circ Z = (z_{iq})_{m\times m}$ where $\bar{z}_{iq} = \max_u(\min(z_{iu}, z_{uq}))$ is a similarity matrix. Additionally, if $\max_u(\min(z_{iu}, z_{uq})) \leq z_{iq}, \forall i, q = 1, 2, ..., m$ then Z^2 is called an equivalent similarity matrix.*

Definition 2.7. [4] *Let $Z = (z_{iq})_{m\times m}$ be a similarity matrix. Then after the finite times of compositions: $Z \to Z^2 \to Z^4 \to \cdots Z^{2^Z} \to \cdots$, there must exist a positive integer N such that $Z^{2^Z} = Z^{2^{Z+1}}$ and Z^{2^Z} is also an equivalent similarity matrix.*

Definition 2.8. [4] *A λ cutting matrix of an equivalent similarity matrix $Z = (z_{iq})_{m\times m}$ is denoted by $Z_\lambda = (z^\lambda_{iq})_{m\times m}$ where $z^\lambda_{iq} = \begin{cases} 0, & z_{iq} < \lambda \\ 1, & z_{iq} \geq \lambda \end{cases}$ and $\lambda \in [0, 1]$ is the confidence level.*

2.2 Existing similarity measure

In general, the rationality and accuracy of similarity/distance measures are achieved through comparing them with existing measurement results. For this purpose, Existing similarity/distance measures are also listed to prepare for the subsequent work.

(1) Atanassov's (1993) similarity measure s_{KA} [2]

$$s_{KA}(L,N) = 1 - \frac{\sum_{i=1}^{n} \left(\left| (\xi_L(\epsilon_i) - \xi_N(\epsilon_i)) - (\delta_L(\epsilon_i) - \delta_N(\epsilon_i)) \right| \right)}{2n}.$$

(2) Chen's (1995) similarity measure s_{chen} [30]

$$s_{chen}(L,N) = 1 - \frac{\sum_{i=1}^{n} \left(\left| (\xi_L(\epsilon_i) - \delta_L(\epsilon_i)) - (\xi_N(\epsilon_i) - \delta_N(\epsilon_i)) \right| \right)}{2n}.$$

(3) Hong and Kim's (1999) similarity measure s_{HK} [12]

$$s_{HK}(L,N) = 1 - \frac{\sum_{i=1}^{n} \left(|\xi_L(\epsilon_i) - \xi_N(\epsilon_i)| + |\delta_L(\epsilon_i) - \delta_N(\epsilon_i)| \right)}{2n}.$$

(4) Dengfeng and Chuntian's (2002) similarity measure s_{DC}[19]

$$s_{DC}(L,N) = 1 - \left(\frac{1}{n} \sum_{i=1}^{n} \left\{ \left| \frac{(\xi_L(\epsilon_i) - \delta_L(\epsilon_i)) - (\xi_N(\epsilon_i) - \delta_N(\epsilon_i))}{2} \right|^p \right\} \right)^{\frac{1}{p}}.$$

(5) Liang and Shi's (2003) similarity measure s_{LS1} and s_{LS2} [20]

$$\begin{aligned}
s_{LS1}(L,N) &= 1 - \tfrac{1}{n^{1/p}} \left(\sum_{i=1}^{n} \left(\tfrac{|\xi_L(\epsilon_i)-\xi_N(\epsilon_i)|+|\delta_L(\epsilon_i)-\delta_N(\epsilon_i)|}{2} \right)^p \right)^{\frac{1}{p}}, \\
s_{LS2}(L,N) &= 1 - \tfrac{1}{n^{1/p}} \left(\sum_{i=1}^{n} \left(\tfrac{|m_{L1}(\epsilon_i)-m_{N1}(\epsilon_i)|-|m_{L2}(\epsilon_i)-m_{N2}(\epsilon_i)|}{n} \right)^p \right)^{\frac{1}{p}}, \\
m_{L1}(\epsilon_i) &= \tfrac{\xi_L(\epsilon_i)-m_L(\epsilon_i)}{2}; m_{N1}(\epsilon_i) = \tfrac{\xi_N(\epsilon_i)-m_N(\epsilon_i)}{2}, \\
m_{L2}(\epsilon_i) &= \tfrac{m_L(\epsilon_i)+1-\delta_L(\epsilon_i)}{2}; m_{N2}(\epsilon_i) = \tfrac{m_N(\epsilon_i)+1-\delta_N(\epsilon_i)}{2}.
\end{aligned}$$

(6) Mitchell's (2003) similarity measure s_M [22]

$$s_M(L,N) = 1 - \frac{1}{2} \left(\frac{\sum_{i=1}^{n} \left(|\xi_L(\epsilon_i) - \xi_N(\epsilon_i)| \right)^p}{n} \right)^{\frac{1}{p}} - \frac{1}{2} \left(\frac{\sum_{i=1}^{n} \left(|\delta_A(\epsilon_i) - \delta_B(\epsilon_i)| \right)^p}{n} \right)^{\frac{1}{p}}.$$

(7) Szmidt and Kacprzyk's (2004) similarity measure s_{SK} [27]

$$s_{SK}(L,N) = \tfrac{1}{n} \sum_{i=1}^{n} \left(\tfrac{\min(|\xi_L(\epsilon_i)-\xi_N(\epsilon_i)|+|\delta_L(\epsilon_i)-\delta_N(\epsilon_i)|+|\tau_L(\epsilon_i)-\tau_N(\epsilon_i)|, |\xi_A(\epsilon_i)-\delta_N(\epsilon_i)|+|\delta_L(\epsilon_i)-\xi_N(\epsilon_i)|+|\tau_L(\epsilon_i)-\tau_N(\epsilon_i)|)}{\max(|\xi_L(\epsilon_i)-\xi_N(\epsilon_i)|+|\delta_L(\epsilon_i)-\delta_N(\epsilon_i)|+|\tau_L(\epsilon_i)-\tau_N(\epsilon_i)|, |\xi_L(\epsilon_i)-\delta_N(\epsilon_i)|+|\delta_L(\epsilon_i)-\xi_N(\epsilon_i)|+|\tau_L(\epsilon_i)-\tau_N(\epsilon_i)|)} \right)$$

(8) Hung and Yang's (2004) similarity measuress_{Hy1} and s_{Hy2} [13]

$$s_{Hy1}(L,N) = 1 - d_{Hy}(L,N), d_{Hy}(L,N) = \frac{\sum\limits_{i=1}^{n}(\max(|\xi_L(\epsilon_i)-\xi_N(\epsilon_i)|,|\xi_L(\epsilon_i)-\xi_N(\epsilon_i)|))}{n},$$
$$s_{Hy2}(L,N) = \frac{e^{-D_{Hy}}(L,N)-e^{-1}}{1-e^{-1}}, s_{Hy3}(L,N) = \frac{1-d_{Hy}(L,N))}{1+d_{Hy1}(L,N)}.$$

(9) Wang and Xin's (2005) similarity measure s_{WX} [29]

$$s_{WX}(L,N) = 1 - \sum_{i=1}^{n}\left(\frac{|\xi_L(\epsilon_i)-\xi_N(\epsilon_i)|+|\delta_L(\epsilon_i)-\delta_N(\epsilon_i)|}{4} + \frac{\max(|\xi_L(\epsilon_i)-\xi_N(\epsilon_i),|\,\delta_L(\epsilon_i)-\delta_N(\epsilon_i))}{2}\right).$$

(10) Liu's (2005) similarity measure s_{Liu} [21]

$$s_L(L,N) = 1 - \sqrt{\frac{\sum\limits_{i=1}^{n}(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))^2 + (\delta_L(\epsilon_i)-\delta_N(\epsilon_i))^2 + (\tau_L(\epsilon_i)-\tau_N(\epsilon_i))^2}{2n}}.$$

(11) Vlachos and Sergiadis's (2007) similarity measure s_{vs} [28]

$$s_{VS}(L,N) = 1 - I(L,N) - I(N,L),$$
$$I(L,N) = \sum_{i=1}^{n}\left(\xi_L(\epsilon_i)\ln\left(\frac{2\xi_L(\epsilon_i)}{\xi_L(\epsilon_i)+\xi_N(\epsilon_i)}\right) + \xi_L(\epsilon_i)\ln\left(\frac{2\delta_L(\epsilon_i)}{\delta_L(\epsilon_i)+\delta_N(\epsilon_i)}\right)\right).$$

(12) Hung and Yang's (2007) similarity measuress_{Hy4}, s_{Hy5} and s_{Hy6} [14]

$$s_{Hy4}(L,N) = \frac{2^{\frac{1}{p}} - d_P(L,N)}{2^{\frac{1}{p}}}; s_{Hy5}(L,N) = \frac{e^{-d_P}(L,N) - e^{-(2)^{\frac{1}{p}}}}{1-e^{-(2)^{\frac{1}{p}}}}; s_{Hy6}(L,N) = \frac{2^{\frac{1}{p}} - d_P(L,N)}{2^{\frac{1}{p}}(1+d_P(L,N))},$$
$$d_p(L,N) = \frac{1}{n}\sum_{i=1}^{n}(|\xi_L(\epsilon_i)-\xi_B(\epsilon_i)|^p + |\delta_L(\epsilon_i)-\delta_B(\epsilon_i)|^p)^{\frac{1}{p}}.$$

(13) Hung and Yang's(2008) similarity measuress_{Hy7}, s_{Hy8}, s_{Hy9} and s_{Hy10}[15]

$$s_{Hy7}(L,N) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\min(\xi_L(\epsilon_i),\xi_B(\epsilon_i)) + \min(\delta_L(\epsilon_i),\delta_B(\epsilon_i))}{\max(\xi_L(\epsilon_i),\xi_B(\epsilon_i)) + \max(\delta_L(\epsilon_i),\delta_B(\epsilon_i))}\right),$$
$$s_{Hy8}(L,N) = \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{1}{2}(|\xi_L(\epsilon_i)-\xi_B(\epsilon_i)| + + |\delta_L(\epsilon_i)-\delta_B(\epsilon_i)|)\right),$$
$$s_{Hy9}(L,N) = \frac{\sum\limits_{i=1}^{n}(\min(\xi_L(\epsilon_i),\xi_B(\epsilon_i)) + \min(\delta_L(\epsilon_i),\delta_B(\epsilon_i)))}{\sum\limits_{i=1}^{n}(\max(\xi_L(\epsilon_i),\xi_B(\epsilon_i)) + \min(\delta_L(\epsilon_i),\delta_B(\epsilon_i)))},$$
$$s_{Hy10}(L,N) = 1 - \frac{\sum\limits_{i=1}^{n}(|\xi_L(\epsilon_i)-\xi_B(\epsilon_i)| + |\delta_L(\epsilon_i)-\delta_B(\epsilon_i)|)}{\sum\limits_{i=1}^{n}(|\xi_L(\epsilon_i)+\xi_B(\epsilon_i)| + |\delta_L(\epsilon_i)+\delta_B(\epsilon_i)|)}.$$

(14) Ye's (2011) similarity measure s_Y [31]

$$s_y(L,N) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\xi_L(\epsilon_i)\xi_B(\epsilon_i) + \upsilon_L(\epsilon_i)\delta_B(\epsilon_i)}{\sqrt{\xi_L^2(\epsilon_i)+\xi_B^2(\epsilon_i)}\sqrt{\delta_L^2(\epsilon_i)+\delta_B^2(\epsilon_i)}}\right).$$

(15) Boran and Lkay's (2014) similarity measure s_{BL} [5]

$$s_{BA}(L,N) = 1 - \left(\sum_{i=1}^{n}\left(\frac{(|t(\xi_L(\epsilon_i)-\xi_B(\epsilon_i)) - (\delta_L(\epsilon_i)-\delta_B(\epsilon_i))| + |t(\delta_L(\epsilon_i)-\delta_B(\epsilon_i)) - (\xi_L(\epsilon_i)-\xi_B(\epsilon_i))|)}{2n(t+1)^P}\right)\right)^{\frac{1}{p}}.$$

(16) Song et al's (2014) similarity measure s_S [26]

$$s_S(L,N)=\frac{1}{2n}\sum_{i=1}^{n}\left(\sqrt{\xi_L(\epsilon_i)\xi_B(\epsilon_i)}+2\sqrt{\delta_L(\epsilon_i)\delta_B(\epsilon_i)}+\sqrt{\tau_L(\epsilon_i)\tau_B(\epsilon_i)}+\sqrt{(1-\delta_L(\epsilon_i))(1-\delta_B(\epsilon_i))}\right).$$

(17) Chen et al's (2016) similarity measure s_{CL} [8]

$$\begin{aligned}s_{CL}(L,N)=1&-\tfrac{|2(\xi_L(\epsilon_i)-\xi_B(\epsilon_i))-(\delta_L(\epsilon_i)-\delta_B(\epsilon_i))|}{3}\times(1-\tfrac{\tau_L(\epsilon_i)-\tau_B(\epsilon_i)}{2})\\&-\tfrac{|2(\delta_L(\epsilon_i)-\delta_B(\epsilon_i))-(\xi_L(\epsilon_i)-\xi_B(\epsilon_i))|}{3}\times(\tfrac{\tau_L(\epsilon_i)-\tau_B(\epsilon_i)}{2}).\end{aligned}$$

(18) Ngan et al's (2018) similarity measure s_N [24, 25]

$$s_N(L,N)=1-\tfrac{1}{3n}\left(|\xi_L(\epsilon)-\xi_B(\epsilon)|+|\delta_L(\epsilon)-\delta_B(\epsilon)|+|\max\{\xi_L(\epsilon)-\delta_B(\epsilon)\}-\max\{\xi_B(\epsilon),\delta_L(\epsilon)\}|\right)$$

(19) Garg and Kumar's (2018) similarity measure s_{GK} [10]

$$\begin{aligned}s_{GK}(L,N)&=\frac{1}{n}\sum_{i=1}^{n}\left(1-\frac{1}{3}\{|a_L(\epsilon_i)-a_B(\epsilon_i)|+|b_L(\epsilon_i)-b_B(\epsilon_i)|+|c_L(\epsilon_i)-c_B(\epsilon_i)|\}\right),\\a(\epsilon_i)&=\xi(\epsilon_i)\times(1-\delta_L(\epsilon_i)),b(\epsilon_i)=1-\xi(\epsilon_i)\times(1-\delta(\epsilon_i))-\delta(\epsilon_i)\times(1-\xi(\epsilon_i)),\\c(\epsilon_i)&=\delta(\epsilon_i)\times(1-\xi(\epsilon_i)).\end{aligned}$$

(20) Hwang,Yang and Hung's (2018) similarity measure s_H [16]

$$s_H(L,N)=\frac{1}{3n}\sum_{i=1}^{n}\begin{pmatrix}g(\xi_L(\epsilon_i),\xi_B(\epsilon_i))+g(1-\delta_L(\epsilon_i),1-\delta_B(\epsilon_i)),\\+g(\frac{1}{2}(1+\xi_L(\epsilon_i)-\delta_L(\epsilon_i)),\frac{1}{2}(1+\xi_B(\epsilon_i)-\delta_B(\epsilon_i)))\end{pmatrix}$$

$$where,g(m,n)=\begin{cases}1, & if\ m=n=0,\\ \frac{m\times n}{m^2+n^2-m\times n}, & if\ \ m>0\ and\ n>0\end{cases}$$

3 The proposed distance and similarity measure

In this section, a novel distance measure based on the center of the nine-point circle of isosceles triangular fuzzy numbers is introduced. Its general formula is presented, and some properties are proven.

Definition 3.1. *Let $L=<\xi_L(\epsilon),\delta_L(\epsilon)>$ and $N=<\xi_N(\epsilon),\delta_N(\epsilon)>$ be two intuitionistic fuzzy numbers. The mapping $s(L,N)/d(L,N):IFNs\times IFNs\to[0,1]$, the similarity/distance measures based on the center of nine-point circle is defined as follows:*

$$\begin{aligned}d_{9C}(L,N)=&\sum_{\mathrm{i=1}}^{\mathrm{n}}\psi_{\mathrm{i}}\left(\tfrac{|(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))-(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))|}{2}\right)\times(1-\tfrac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2})\\&+\sum_{\mathrm{i=1}}^{\mathrm{n}}\psi_{\mathrm{i}}\left(\tfrac{|11(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))-5(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))|}{16}\right)\times\tfrac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2}\end{aligned}$$

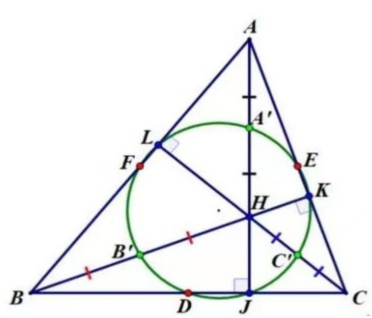

Figure 1: Nine-point circle

where, $s_{9C}(L,N)=1-d_{9C}(L,N)$ is called as similarity measure.

Let $\psi_i > 0, \sum_{\mathrm{i=1}}^{\mathrm{n}} \psi_i = 1$ represent the weight allocated to the element of $U = \{\epsilon_1, \epsilon_2, ..., \epsilon_n\}$. The midpoint of three sides of a triangle, three high perpendicular feet and three Euler points (The midpoint of the three line segments obtained by connecting the vertices and the vertical center of the triangle) Nine points in a circle(Figure 1). The center of the nine point circle is on the Euclidean line and is exactly the midpoint of the line connecting the vertical center and the outer center.

Let $O_{L(\epsilon_i)}$ and $O_{N(\epsilon_i)}$ be the center of nine point circle of TFNs $L(\epsilon_i)$ and $N(\epsilon_i)$. Using the coordinates of the orthocenter and circumcenter of the isosceles TFNs. we can determine the coordinates of $O_{L(\epsilon_i)}$ and $O_{N(\epsilon_i)}$ by calculation.

$$O_{L(\epsilon_i)} = (\frac{1 + \xi_L(\epsilon_i) - \delta_L(\epsilon_i)}{2}, \frac{5 + 11\delta_L(\epsilon_i) - 5\xi_L(\epsilon_i)}{16}) \text{ and } O_{N(\epsilon_i)} = (\frac{1 + \xi_N(\epsilon_i) - \delta_N(\epsilon_i)}{2}, \frac{5 + 11\delta_N(\epsilon_i) - 5\xi_N(\epsilon_i)}{16}).$$

The distance d_{9C} between $L(\epsilon_i)$ and $N(\epsilon_i)$ is given as:

$$\begin{aligned} d_{9C}(L,N) = &\sum_{\mathrm{i=1}}^{\mathrm{n}} \psi_{\mathrm{i}} \left(\tfrac{|(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))-(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))|}{2} \right) \times (1 - \tfrac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2}) \\ &+ \sum_{\mathrm{i=1}}^{\mathrm{n}} \psi_{\mathrm{i}} \left(\tfrac{|11(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))-5(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))|}{16} \right) \times \tfrac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2}. \end{aligned}$$

Definition 3.2. *Let $L =< \xi_L(\epsilon), \delta_L(\epsilon) >$ and $N =< \xi_N(\epsilon), \delta_N(\epsilon) >$ be two intuitionistic fuzzy numbers. The mapping $s(L,N) : IFNs \times IFNs \to [0,1]$, The general distance measures is defined as follows:*

$$\begin{aligned} d_{GC}(L,N) = &\sum_{\mathrm{i=1}}^{\mathrm{n}} \psi_{\mathrm{i}} \left(\tfrac{|(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))-(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))|}{2} \right) \times (1 - \tfrac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2}) \\ &+ \sum_{\mathrm{i=1}}^{\mathrm{n}} \psi_{\mathrm{i}} \left(\tfrac{|k_1(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))-k_2(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))|}{k_1+k_2} \right) \times \tfrac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2}. \end{aligned}$$

where, $k_1 > k_2 > 0$.

Property 3.3. (p1) $0 \le d(L,N) \le 1$.

Proof. Let $L = \{< \epsilon_i, \xi_L(\epsilon_i), \delta_L(\epsilon_i) > | i = 1,2,...n\}$ and $N = \{< \epsilon_i, \xi_N(\epsilon_i), \delta_N(\epsilon_i) > | i = 1,2,...n\}$ be be two IFSs defined on X. Since $0 \le \xi_L(\epsilon_i), \delta_L(\epsilon_i), \xi_N(\epsilon_i), \delta_N(\epsilon_i) \le 1$, $k_1, k_2 > 0$. Therefore, $-1 \le \xi_L(\epsilon_i) - \xi_N(\epsilon_i) \le 1$ and $-1 \le \delta_L(\epsilon_i) - \delta_N(\epsilon_i) \le 1 \Rightarrow -2 \le (\xi_L(\epsilon_i) - \xi_N(\epsilon_i)) - (\delta_L(\epsilon_i) - \delta_N(\epsilon_i)) \le 2$ and $-(k_1 + k_2) \le k_1(\delta_L(\epsilon_i) - \delta_N(\epsilon_i)) - k_2(\xi_L(\epsilon_i) - \xi_N(\epsilon_i)) \le k_1 + k_2$, So, $0 \le \frac{|(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))-(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))|}{2} \le 1$ and $0 \le \frac{|k_1(\delta_L(x_i)-\delta_N(\epsilon_i))-k_2(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))|}{k_1+k_2} \le 1$. Hence, by Definition 3.2, we are able to get

$$\begin{aligned} d_{GC}(L,N) = &\sum_{\mathrm{i=1}}^{\mathrm{n}} \psi_{\mathrm{i}} \left(\tfrac{|(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))-(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))|}{2} \right) \times (1 - \tfrac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2}) \\ &+ \sum_{\mathrm{i=1}}^{\mathrm{n}} \psi_{\mathrm{i}} \left(\tfrac{|k_1(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))-k_2(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))|}{k_1+k_2} \right) \times \tfrac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2} \\ &\le \sum_{\mathrm{i=1}}^{\mathrm{n}} \psi_{\mathrm{i}} \left(1 - \tfrac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2} + \tfrac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2} \right) = \sum_{\mathrm{i=1}}^{\mathrm{n}} \psi_{\mathrm{i}} = 1. \end{aligned}$$

Thus, $d_{GC}(L,N) \le 1$. Also, it is evident from that $d_{GC}(L,N) \ge 0$. So, $0 \le d_{GC}(L,N) \le 1$.

□

Property 3.4. (p2) $d(L,N) = 0 \Leftrightarrow L = N$.

Proof. For $L = N$, we have $\xi_L(\epsilon_i) = \xi_N(\epsilon_i), \delta_L(\epsilon_i) = \delta_N(\epsilon_i), \tau_L(\epsilon_i) = \tau_N(\epsilon_i)$ $\forall \epsilon = 1,2,3,...n$. Then,

$$\begin{aligned} d_{GC}(L,N) = &\sum_{\mathrm{i=1}}^{\mathrm{n}} \psi_{\mathrm{i}} \left(\tfrac{|(\xi_L(x_i)-\xi_N(x_i))-(\delta_L(x_i)-\delta_N(x_i))|}{2} \right) \times (1 - \tfrac{\tau_L(x_i)+\tau_N(x_i)}{2}) \\ &+ \sum_{\mathrm{i=1}}^{\mathrm{n}} \psi_{\mathrm{i}} \left(\tfrac{|k_1(\delta_L(x_i)-\delta_N(x_i))-k_2(\xi_L(x_i)-\xi_N(x_i))|}{k_1+k_2} \right) \times \tfrac{\tau_L(x_i)+\tau_N(x_i)}{2} = 0. \end{aligned}$$

Thus, $L = N$, implies that $d_{GC}(L,N) = 0$. Now, we shall prove that $d_{GC}(L,N) = 0 \Rightarrow L = N$.

$$\begin{aligned} d_{GC}(L,N) = 0 \Rightarrow &\sum_{\mathrm{i=1}}^{\mathrm{n}} \psi_{\mathrm{i}} \left(\frac{|(\xi_L(\epsilon_i) - \xi_N(\epsilon_i)) - (\delta_L(\epsilon_i) - \delta_N(\epsilon_i))|}{2} \right) \times (1 - \frac{\tau_L(\epsilon_i) + \tau_N(\epsilon_i)}{2}) \\ &+ \sum_{\mathrm{i=1}}^{\mathrm{n}} \psi_{\mathrm{i}} \left(\frac{|k_1(\delta_L(\epsilon_i) - \delta_N(\epsilon_i)) - k(\xi_L(\epsilon_i) - \xi_N(\epsilon_i))|}{k_1 + k_2} \right) \times \frac{\tau_L(\epsilon_i) + \tau_N(\epsilon_i)}{2} = 0 \end{aligned}$$

$$\begin{aligned}
&\Rightarrow \frac{|(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))-(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))|}{2}\times(1-\frac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2})\\
&\quad+\frac{|k_1(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))-k_2(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))|}{k_1+k_2}\times\frac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2}=0\\
&\Rightarrow \frac{|(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))-(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))|}{2}\times(1-\frac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2})=0\\
&\quad\text{and}\quad \frac{|k_1(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))-k_2(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))|}{k_1+k_2}\times\frac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2}=0
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \frac{(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))-(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))}{2}=0\quad\text{or}\quad(1-\frac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2})=0\\
&\quad\text{and}\quad\frac{k_1(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))-k_2(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))}{k_1+k_2}=0\ or\ \frac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2}=0.
\end{aligned}$$

Now we have to start splitting in the following situation.

Case 1. When

$$\frac{(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))-(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))}{2}=0 \text{ and } \frac{k_1(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))-k_2(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))}{k_1+k_2}=0,$$

since $\frac{(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))-(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))}{2}=0$, it means

$$(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))-(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))=0,$$

which gives that $\xi_L(\epsilon_i)-\xi_N(\epsilon_i)=\delta_L(\epsilon_i)-\delta_N(\epsilon_i)$. Using this relation $\xi_L(\epsilon_i)-\xi_N(\epsilon_i)=\delta_L(\epsilon_i)-\delta_N(\epsilon_i)$ in the equation

$$\frac{k_1(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))-k_2(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))}{k_1+k_2}=0,$$

we get $(k_1-k_2)(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))=0$, due to $k_1\neq k_2$, which gives that $\delta_L(\epsilon_i)-\delta_N(\epsilon_i)=0$. Further, utilizing this equality $\xi_L(\epsilon_i)-\xi_N(\epsilon_i)=\delta_L(\epsilon_i)-\delta_N(\epsilon_i)$, it can be seen that $\xi_L(\epsilon_i)=\xi_N(\epsilon_i)$. Hence, $L=N$.

Case 2. When $\frac{(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))-(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))}{2}=0$ and $\frac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2}=0$. Since

$$\frac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2}=0\Rightarrow\xi_L(\epsilon_i)+\delta_L(\epsilon_i)+\xi_N(\epsilon_i)+\delta_N(\epsilon_i)=2\Rightarrow\xi_L(\epsilon_i)+\delta_L(\epsilon_i)=1,$$

and

$$\xi_N(\epsilon_i)+\delta_N(\epsilon_i)=1\Rightarrow\xi_L(\epsilon_i)-\xi_N(\epsilon_i)=-(\delta_L(\epsilon_i)-\delta_N(\epsilon_i)).$$

Using this relation in the equation $\frac{(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))-(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))}{2}=0$, it can be seen that $\delta_L(\epsilon_i)=\delta_N(\epsilon_i)$. Further, using this equality $\delta_L(\epsilon_i)=\delta_N(\epsilon_i)$ in $\xi_L(\epsilon_i)-\xi_N(\epsilon_i)=-(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))$, we get $\xi_L(\epsilon_i)=\xi_N(\epsilon_i)$. Hence, $L=N$.

Case 3. When $1-\frac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2}=0$ and $\frac{k_1(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))-k_2(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))}{k_1+k_2}=0$, since

$$\begin{aligned}
1-\frac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2}=0&\Rightarrow\tau_L(\epsilon_i)+\tau_N(\epsilon_i)=2\\
&\Rightarrow\xi_L(\epsilon_i)+\xi_N(\epsilon_i)+\delta_L(\epsilon_i)+\delta_N(\epsilon_i)=0\\
&\Rightarrow\xi_L(\epsilon_i)=\xi_N(\epsilon_i)=\delta_L(\epsilon_i)=\delta_N(\epsilon_i)=0.
\end{aligned}$$

Hence, $L=N$. □

Property 3.5. (p3) $d(L,N)=d(N,L)$.

Proof. By Definition 3.2,

$$\begin{aligned}
d_{GC}(L,N)&=\sum_{\mathrm{i}=1}^{\mathrm{n}}\psi_{\mathrm{i}}\left(\frac{|(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))-(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))|}{2}\right)\times(1-\frac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2})\\
&\quad+\sum_{\mathrm{i}=1}^{\mathrm{n}}\psi_{\mathrm{i}}\left(\frac{|k_1(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))-k_2(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))|}{k_1+k_2}\right)\times\frac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2}\\
&=\sum_{\mathrm{i}=1}^{\mathrm{n}}\psi_{\mathrm{i}}\left(\frac{|(\xi_N(\epsilon_i)-\xi_L(\epsilon_i))-(\delta_N(\epsilon_i)-\delta_L(\epsilon_i))|}{2}\right)\times(1-\frac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2})\\
&\quad+\sum_{\mathrm{i}=1}^{\mathrm{n}}\psi_{\mathrm{i}}\left(\frac{|k_1(\delta_N(\epsilon_i)-\delta_L(\epsilon_i))-k_2(\xi_N(\epsilon_i)-\xi_L(\epsilon_i))|}{k_1+k_2}\right)\times\frac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2}=d_{GC}(N,L).
\end{aligned}$$

□

Property 3.6. *(p4) If $L \subseteq N \subseteq P$ then $d(L,P) \geq d(L,N)$ and $d(L,P) \geq d(N,P)$.*

Proof. Let $P = \{<\epsilon_i, \xi_P(\epsilon_i), \delta_P(\epsilon_i)> | i = 1,2,3,...,n\}$, since, $L \subseteq N \subseteq P$, therefore, by Definition 3.2 $\xi_L(\epsilon_i) \leq \xi_N(\epsilon_i) \leq \xi_P(\epsilon_i)$ and $\delta_L(\epsilon_i) \geq \delta_N(\epsilon_i) \geq \delta_P(\epsilon_i)$. It gives that $\xi_L(\epsilon_i) - \xi_N(\epsilon_i) \leq 0$, $\delta_L(\epsilon_i) - \delta_N(\epsilon_i) \geq 0$, so,

$$\begin{aligned}&(\xi_L(\epsilon_i) - \xi_N(\epsilon_i)) - (\delta_L(\epsilon_i) - \delta_N(\epsilon_i)) \leq 0\\ &\Rightarrow |(\xi_L(\epsilon_i) - \xi_N(\epsilon_i)) - (\delta_L(\epsilon_i) - \delta_N(\epsilon_i))| = (\delta_L(\epsilon_i) - \delta_N(\epsilon_i)) - (\xi_L(\epsilon_i) - \xi_N(\epsilon_i)).\end{aligned}$$

Proceeding in a similar manner, it can be obtained that

$$|k_1(\delta_L(\epsilon_i) - \delta_N(\epsilon_i)) - k_2(\xi_L(\epsilon_i) - \xi_N(\epsilon_i))| = k_1(\delta_L(\epsilon_i) - \delta_N(\epsilon_i)) - k_2(\xi_L(\epsilon_i) - \xi_N(\epsilon_i)).$$

Hence,

$$\begin{aligned}d_{GP}(L,N) = &\sum_{\mathrm{i}=1}^{\mathrm{n}} \psi_{\mathrm{i}} \left(\tfrac{(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))-(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))}{2}\right) \times (1 - \tfrac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2})\\ &+ \sum_{\mathrm{i}=1}^{\mathrm{n}} \psi_{\mathrm{i}} \left(\tfrac{k_1(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))-k_2(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))}{k_1+k_2}\right) \times \tfrac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}d_{GC}(L,P) = &\sum_{\mathrm{i}=1}^{\mathrm{n}} \psi_{\mathrm{i}} \left(\tfrac{(\delta_L(\epsilon_i)-\delta_P(\epsilon_i))-(\xi_L(\epsilon_i)-\xi_P(\epsilon_i))}{2}\right) \times (1 - \tfrac{\tau_L(\epsilon_i)+\tau_P(\epsilon_i)}{2})\\ &+ \sum_{\mathrm{i}=1}^{\mathrm{n}} \psi_{\mathrm{i}} \left(\tfrac{k_1(\delta_L(\epsilon_i)-\delta_P(\epsilon_i))-k_2(\xi_L(\epsilon_i)-\xi_P(\epsilon_i))}{k_1+k_2}\right) \times \tfrac{\tau_L(\epsilon_i)+\tau_P(\epsilon_i)}{2}.\end{aligned}$$

$$\begin{aligned}d_{GC}(N,P) = &\sum_{\mathrm{i}=1}^{\mathrm{n}} \psi_{\mathrm{i}} \left(\tfrac{(\delta_N(\epsilon_i)-\delta_P(\epsilon_i))-(\xi_N(\epsilon_i)-\xi_P(\epsilon_i))}{2}\right) \times (1 - \tfrac{\tau_N(\epsilon_i)+\tau_P(\epsilon_i)}{2})\\ &+ \sum_{\mathrm{i}=1}^{\mathrm{n}} \psi_{\mathrm{i}} \left(\tfrac{k_1(\delta_N(\epsilon_i)-\delta_P(\epsilon_i))-k_2(\xi_N(\epsilon_i)-\xi_P(\epsilon_i))}{k_1+k_2}\right) \times \tfrac{\tau_N(\epsilon_i)+\tau_P(\epsilon_i)}{2}.\end{aligned}$$

Now, in order to prove that $d_{GC}(L,P) \geq d_{GC}(L,N)$, it is sufficient to show that $d_{GC}(L,P) - d_{GC}(L,N) \geq 0$. Therefore,

$$\begin{aligned}&d_{GC}(L,P) - d_{GC}(L,N)\\ = &\sum_{\mathrm{i}=1}^{\mathrm{n}} \psi_i \left(\frac{(\delta_L(\epsilon_i) - \delta_P(\epsilon_i)) - (\xi_L(\epsilon_i) - \xi_P(\epsilon_i))}{2}\right) \times (1 - \frac{\tau_L(\epsilon_i) + \tau_P(\epsilon_i)}{2})\\ &+ \sum_{\mathrm{i}=1}^{\mathrm{n}} \psi_i \left(\frac{k_1(\delta_L(\epsilon_i) - \delta_P(\epsilon_i)) - k_2(\xi_L(\epsilon_i) - \xi_P(\epsilon_i))}{k_1 + k_2}\right) \times \frac{\tau_L(\epsilon_i) + \tau_P(\epsilon_i)}{2}\\ &- \sum_{\mathrm{i}=1}^{\mathrm{n}} \psi_i \left(\frac{(\delta_L(\epsilon_i) - \delta_N(\epsilon_i)) - (\xi_L(\epsilon_i) - \xi_N(\epsilon_i))}{2}\right) \times (1 - \frac{\tau_L(\epsilon_i) + \tau_N(\epsilon_i)}{2})\\ &- \sum_{\mathrm{i}=1}^{\mathrm{n}} \psi_{\mathrm{i}} \left(\frac{k_1(\delta_L(\epsilon_i) - \delta_N(\epsilon_i)) - k_2(\xi_L(\epsilon_i) - \xi_N(\epsilon_i))}{k_1 + k_2}\right) \times \frac{\tau_L(\epsilon_i) + \tau_N(\epsilon_i)}{2}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \psi_i \frac{(\delta_N(\epsilon_i) - \delta_P(\epsilon_i)) - (\xi_N(\epsilon_i) - \xi_P(\epsilon_i))}{2} + \sum_{i=1}^{n} \psi_i \frac{(k_1 - k_2)(\delta_L(\epsilon_i) - \delta_P(\epsilon_i) + \xi_L(\epsilon_i) - \xi_P(\epsilon_i))}{2(k_1 + k_2)} \times \frac{\tau_L(\epsilon_i) + \tau_P(\epsilon_i)}{2} \\
&\quad + \sum_{i=1}^{n} \psi_i \frac{(k_1 - k_2)(\delta_N(\epsilon_i) - \delta_L(\epsilon_i) + \xi_N(\epsilon_i) - \xi_L(\epsilon_i))}{2(k_1 + k_2)} \times \frac{\tau_L(\epsilon_i) + \tau_N(\epsilon_i)}{2} \\
&= \sum_{i=1}^{n} \psi_i \frac{(\delta_N(\epsilon_i) - \delta_P(\epsilon_i)) - (\xi_N(\epsilon_i) - \xi_P(\epsilon_i))}{2} + \sum_{i=1}^{n} \psi_i \frac{k_1 - k_2}{2(k_1 + k_2)} \times (\tau_P(\epsilon_i) - \tau_L(\epsilon_i)) \times \frac{\tau_L(\epsilon_i) + \tau_P(\epsilon_i)}{2} \\
&\quad + \sum_{i=1}^{n} \psi_i \frac{k_1 - k_2}{2(k_1 + k_2)} \times (\tau_L(\epsilon_i) - \tau_N(\epsilon_i)) \times \frac{\tau_L(\epsilon_i) + \tau_N(\epsilon_i)}{2} \\
&= \sum_{i=1}^{n} \psi_i \frac{(\delta_N(\epsilon_i) - \delta_P(\epsilon_i)) - (\xi_N(\epsilon_i) - \xi_P(\epsilon_i))}{2} + \sum_{i=1}^{n} \psi_i \frac{k_1 - k_2}{4(k_1 + k_2)} \times (\tau_P^2(\epsilon_i) - \tau_L^2(\epsilon_i)) \\
&\quad + \sum_{i=1}^{n} \psi_i \frac{k_1 - k_2}{4(k_1 + k_2)} \times (\tau_L^2(\epsilon_i) - \tau_N^2(\epsilon_i)) \\
&= \sum_{i=1}^{n} \psi_i \left[\begin{array}{l} \frac{(\delta_N(\epsilon_i) - \delta_P(\epsilon_i)) - (\xi_N(\epsilon_i) - \xi_P(\epsilon_i))}{2} + \frac{k_1 - k_2}{4(k_1 + k_2)} \times (\tau_P^2(\epsilon_i) - \tau_L^2(\epsilon_i)) \\ + \frac{k_1 - k_2}{4(k_1 + k_2)} \times (\tau_L^2(\epsilon_i) - \tau_N^2(\epsilon_i)) \end{array} \right] \\
&= \sum_{i=1}^{n} \psi_i \left[\frac{(\delta_N(\epsilon_i) - \delta_P(\epsilon_i)) - (\xi_N(\epsilon_i) - \xi_P(\epsilon_i))}{2} + \frac{k_1 - k_2}{4(k_1 + k_2)} \times (\tau_P^2(\epsilon_i) - \tau_N^2(\epsilon_i)) \right] \\
&= \sum_{i=1}^{n} \psi_i \left[\begin{array}{l} \frac{(\delta_N(\epsilon_i) - \delta_P(\epsilon_i)) - (\xi_N(\epsilon_i) - \xi_P(\epsilon_i))}{2} + \frac{k_1 - k_2}{4(k_1 + k_2)} \times ((\tau_P(\epsilon_i) - \tau_N(\epsilon_i)) \\ + (\tau_P(\epsilon_i) + \tau_N(\epsilon_i)) \end{array} \right] \\
&= \sum_{i=1}^{n} \psi_i \left[\begin{array}{l} \frac{(\delta_N(\epsilon_i) - \delta_P(\epsilon_i)) - (\xi_N(\epsilon_i) - \xi_P(\epsilon_i))}{2} + \frac{k_1 - k_2}{4(k_1 + k_2)} ((\delta_N(\epsilon_i) - \delta_P(\epsilon_i)) \\ + (\xi_N(x_i) - \xi_P(x_i))) + (\tau_P(x_i) + \tau_N(\epsilon_i)) \end{array} \right] \\
&= \sum_{i=1}^{n} \psi_i \left[\begin{array}{l} \left(\frac{1}{2} + \frac{k_1 - k_2}{4(k_1 + k_2)} \times (\tau_P(\epsilon_i) + \tau_N(\epsilon_i)) \right) (\delta_N(\epsilon_i) - \delta_P(\epsilon_i)) \\ - \left(\frac{1}{2} - \frac{k_1 - k_2}{4(k_1 + k_2)} \times (\tau_P(\epsilon_i) + \tau_N(\epsilon_i)) \right) (\xi_N(\epsilon_i) - \xi_P(\epsilon_i)) \end{array} \right] \\
&= \sum_{i=1}^{n} \psi_i \left[\begin{array}{l} \left(\frac{1}{2} + \frac{k_1 - k_2}{4(k_1 + k_2)} \times (\tau_P(\epsilon_i) + \tau_N(\epsilon_i)) \right) (\delta_N(\epsilon_i) - \delta_P(\epsilon_i)) \\ + \left(\frac{1}{2} - \frac{k_1 - k_2}{4(k_1 + k_2)} \times (\tau_P(\epsilon_i) + \tau_N(\epsilon_i)) \right) (\xi_P(\epsilon_i) - \xi_N(\epsilon_i)) \end{array} \right] \\
&= \sum_{i=1}^{n} \psi_i \left(\frac{1}{2} + \frac{k_1 - k_2}{4(k_1 + k_2)} \times (\tau_P(\epsilon_i) + \tau_N(\epsilon_i)) \right) (\delta_N(\epsilon_i) - \delta_P(\epsilon_i)) \\
&\quad + \sum_{i=1}^{n} \psi_i \frac{k_1 \left(\xi_P(\epsilon_i) + \delta_P(\epsilon_i) + \xi_N(\epsilon_i) + \delta_N(\epsilon_i) \right) + 2 + k_2(\tau_P(\epsilon_i) + \tau_N(\epsilon_i))}{4(k_1 + k_2)} (\xi_P(\epsilon_i) - \xi_N(\epsilon_i)) \\
&= \sum_{i=1}^{n} \psi_i \left(\frac{1}{2} + \frac{k_1 - k_2}{4(k_1 + k_2)} \times (\tau_P(\epsilon_i) + \tau_N(\epsilon_i)) \right) (\delta_N(\epsilon_i) - \delta_P(\epsilon_i)) \\
&\quad + \sum_{i=1}^{n} \psi_i \frac{(k_1 - k_2) \left(\xi_P(\epsilon_i) + \delta_P(\epsilon_i) + \xi_N(\epsilon_i) + \delta_N(\epsilon_i) \right) + 2 + 2k_2}{4(k_1 + k_2)} \\
&\geq 0
\end{aligned}
$$

Therefore, $d_{GC}(L, P) \geq d_{GC}(L, N)$. Similarly, $d_{GC}(L, P) \geq d_{GC}(N, P)$. From the above analysis and proof, if $L \subseteq N \subseteq P$ then $d_{GC}(L, P) \geq d_{GC}(L, N)$ and $d_{GC}(L, P) \geq d_{GC}(N, P)$, the proof is completed.

By Definition 3.2, it can be concluded that

(1) When $k_1 = 2, k_2 = 1$ the distance measure based on centroid of the isosceles TFNs,

$$
\begin{aligned}
d_C(L, N) = &\sum_{i=1}^{n} \psi_i \left(\frac{|(\xi_L(\epsilon_i) - \xi_N(\epsilon_i)) - (\delta_L(\epsilon_i) - \delta_N(\epsilon_i))|}{2} \right) \times \left(1 - \frac{\tau_L(\epsilon_i) + \tau_N(\epsilon_i)}{2}\right) \\
&+ \sum_{i=1}^{n} \psi_i \left(\frac{|2(\delta_L(\epsilon_i) - \delta_N(\epsilon_i)) - (\xi_L(\epsilon_i) - \xi_N(\epsilon_i))|}{3} \right) \times \frac{\tau_L(\epsilon_i) + \tau_N(\epsilon_i)}{2}.
\end{aligned}
$$

(2) When $k_1 = 3, k_2 = 1$ the distance measure based on orthocenter of the isosceles TFNs,

$$d_{OC}(L,N) = \sum_{\mathrm{i}=1}^{\mathrm{n}} \psi_{\mathrm{i}} \left(\frac{|(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))-(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))|}{2} \right) \times (1 - \frac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2}) + \sum_{\mathrm{i}=1}^{\mathrm{n}} \psi_{\mathrm{i}} \left(\frac{|3(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))-(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))|}{4} \right) \times \frac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2}.$$

(3) When $k_1 = 5, k_2 = 3$ the distance measure based on circumcenter of the isosceles TFNs,

$$d_{CC}(L,N) = \sum_{\mathrm{i}=1}^{\mathrm{n}} \psi_{\mathrm{i}} \left(\frac{|(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))-(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))|}{2} \right) \times (1 - \frac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2}) + \sum_{\mathrm{i}=1}^{\mathrm{n}} \psi_{\mathrm{i}} \left(\frac{|5(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))-3(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))|}{8} \right) \times \frac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2}.$$

(4) When $k_1 = \sqrt{5}, k_2 = 1$ the distance measure based on incenter of the isosceles TFNs,

$$d_{IC}(L,N) = \sum_{\mathrm{i}=1}^{\mathrm{n}} \psi_{\mathrm{i}} \left(\frac{|(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))-(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))|}{2} \right) \times (1 - \frac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2}) + \sum_{\mathrm{i}=1}^{\mathrm{n}} \psi_{\mathrm{i}} \left(\frac{\left|\sqrt{5}(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))-(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))\right|}{\sqrt{5}+1} \right) \times \frac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2}.$$

(5) When $k_1 = 11, k_2 = 5$ the distance measure based on the center of nine-point circle of the isosceles TFNs,

$$d_{9C}(L,N) = \sum_{\mathrm{i}=1}^{\mathrm{n}} \psi_{\mathrm{i}} \left(\frac{|(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))-(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))|}{2} \right) \times (1 - \frac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2}) + \sum_{\mathrm{i}=1}^{\mathrm{n}} \psi_{\mathrm{i}} \left(\frac{|11(\delta_L(\epsilon_i)-\delta_N(\epsilon_i))-5(\xi_L(\epsilon_i)-\xi_N(\epsilon_i))|}{16} \right) \times \frac{\tau_L(\epsilon_i)+\tau_N(\epsilon_i)}{2}.$$

□

4 Experiment and analysis the proposed measure

In this section, we compare the existing measures to discover the superiority of our proposed similarity measure.

Property 4.1. *For IFSs $L = \{\langle \epsilon_i, \xi_L(\epsilon_i), \delta_L(\epsilon_i)\rangle\, \epsilon_i \in \Lambda\}$ and $N = \{\langle \epsilon_i, \xi_N(\epsilon_i), \delta_N(\epsilon_i)\rangle\, \epsilon_i \in X\}$ in $\Lambda = \{\epsilon_1, \epsilon_2, \epsilon_3 ..., \epsilon_n\}$, if $s_{9C}(L,N) = s_{GC}(L,N) = 1$, then $L = N$.*

Proof. By Definition 2.4, $d_{GC}(L,N) = 0$ implies $L = N$. Because $s_{GC}(L,N) = 1 - d_{GC}(L,N)$. Therefore $s_{GC}(L,N) = 1$ implies $L = N$. □

Remark 4.2. *The above property is one of the basic properties of similarity measure. However, the similarity measures s_{KA} and s_{DC} do not satisfy this property. For instance, if $L = \{< \epsilon, 0.6, 0.4 >\}$ and $N = \{< \epsilon, 0.3, 0.1 >\}$, for all $\epsilon \in \Lambda$. we have $s_{KA}(L,N) = 1$, however $L \neq N$. Likewise, for p=1, we have $s_{DC}(L,N) = 1$ whereas $L \neq N$. Consequently, the proposed similarity measure is superior to the existing measures s_{KA} and s_{DC}.*

Property 4.3. *Let $L = \{\langle \epsilon, 0.0, 0.0\rangle\}$ and $N = \{\langle \epsilon, 0.0, 0.0\rangle\}$ be two IFSs, Hereafter $s_{9C}(L,N) = s_{GC}(L,N) = 1$.*

Proof. It can be proved easily. □

Example 4.4. (Zhang and Yu,2013[34]) *Consider three given patterns $L_1, L_2,, L_3$ denoted by IFSs in $\Lambda = \{\epsilon_1, \epsilon_2, \epsilon_3\}$ as follows:*

$$L_1 = \{< \epsilon_1, 0.2, 0.3 >, < \epsilon_2, 0.1, 0.4 >, < \epsilon_3, 0.2, 0.6 >\},$$
$$L_2 = \{< \epsilon_1, 0.3, 0.2 >, < \epsilon_2, 0.4, 0.1 >, < \epsilon_3, 0.5, 0.3 >\},$$
$$L_3 = \{< \epsilon_1, 0.2, 0.3 >, < \epsilon_2, 0.4, 0.1 >, < \epsilon_3, 0.5, 0.3 >\}.$$

Assume an unknown pattern N *whose ratings are specified as:*

$$N = \{< \epsilon_1, 0.1, 0.2 >, < \epsilon_2, 0.4, 0.5 >, < \epsilon_3, 0.0, 0.0 >\}.$$

Table 1: Results of Example 4.4 (The classification results of the proposed measure s_{9C} and other measures (p = 1) in $s_{DC}, s_{LS1}, s_{LS2}, s_M, s_{Hy4}, s_{HY5}, s_{HY6}, s_{NL}$, $\psi_i = 1/3$ in $s_C, s_{OC}, s_{CC}, s_{IC}, s_{9C}$)

Measures	$s(L_1, N)$	$s(L_2, N)$	$s(L_3, N)$	Classification result
s_{KA}	**0.9000**	0.8667	**0.9000**	Cannot be recognized
s_{Chen}	**0.9000**	0.8667	**0.9000**	Cannot be recognized
s_{HK}	**0.7667**	**0.7667**	**0.7667**	Cannot be recognized
s_{DC}	**0.9000**	0.8667	**0.9000**	Cannot be recognized
s_{LS1}	**0.7667**	**0.7667**	**0.7667**	Cannot be recognized
s_{LS2}	**0.7333**	**0.7333**	**0.7333**	Cannot be recognized
s_M	**0.7667**	**0.7667**	**0.7667**	Cannot be recognized
s_{SK}	**1.0000**	**1.0000**	**1.0000**	Cannot be recognized
s_{HY1}	**0.6667**	**0.6333**	**0.6667**	Cannot be recognized
s_{HY2}	**0.5516**	0.5144	**0.5516**	Cannot be recognized
s_{HY3}	**0.5000**	0.4634	**0.5000**	Cannot be recognized
s_{WX}	**0.7167**	0.7000	**0.7167**	Cannot be recognized
s_L	**0.5239**	0.5204	**0.5239**	Cannot be recognized
s_{VS}	**NaN**	**NaN**	**NaN**	Cannot be determined
s_{HY4}	**0.7667**	**0.7667**	**0.7667**	Cannot be recognized
s_{HY5}	**0.5687**	**0.5687**	**0.5687**	Cannot be recognized
s_{HY6}	**0.5227**	**0.5227**	**0.5227**	Cannot be recognized
s_{HY7}	**0.3852**	**0.3852**	**0.3852**	Cannot be recognized
s_{HY8}	**0.7667**	**0.7667**	**0.7667**	Cannot be recognized
s_{HY9}	**0.3636**	**0.3636**	**0.3636**	Cannot be recognized
s_{HY10}	**0.5333**	**0.5333**	**0.5333**	Cannot be recognized
s_Y	**NaN**	**NaN**	**NaN**	Cannot be determined
s_{NL}	0.8889	0.8556	0.8778	L_1
s_S	0.8194	0.8317	0.8328	L_3
s_{CL}	0.9556	0.7956	0.8311	L_1
s_N	0.7778	**0.8000**	**0.8000**	Cannot be determined
s_{GK}	0.8178	0.8000	0.8089	L_1
s_H	0.7946	0.6874	0.6846	L_1
s_C	0.8733	0.8800	0.9000	L_3
s_{OC}	0.8600	0.8667	0.8800	L_3
s_{CC}	0.8800	0.8867	0.9100	L_3
s_{IC}	0.8664	0.8742	0.8924	L_3
s_{9C}	0.8700	0.8767	0.8950	L_3

NaN denotes that similarity cannot be computed owing to division by zero problem.

Bold denotes unreasonable results. ψ_i is determined based on the number of attributes.

Calculate Example 4.4 with existing similarity measures, The results are presented in Table 2 and some observations are as follows:

(1) *The measures $s_{KL}, s_{Chen}, s_{DC}, s_{HY1}, s_{HY2}, s_{HY3}, s_{WX}, s_L$ fails to recognize owing to $s(L_1, N) = s(L_3, N)$.*

(2) *The measure s_N fails to recognize on account of $s(L_2, N) = s(L_3, N)$.*

(3) *The measures $s_{HK}, s_{LS1}, s_{LS2}, s_M, s_{SK}, s_{HY4}, s_{HY5}, s_{HY6}, s_{HY7}, s_{HY8}, s_{HY9}, s_{HY10}$ unable to distinguish the results, because $s(L_1, N) = s(L_2, N) = s(L_3, N)$.*

(4) *The measures s_{VS}, s_Y unable to provide results owing to division by zero problem.*

(5) *The measures $s_{NL}, s_{CL}, s_{GK}, s_H$ offer identification of errors.*

(6) *The result calculated by s_{9C} is the same as that of $s_C, s_{OC}, s_{CC}, s_{IC}$. We provide the measure s_{9C} has the same advantage with $s_C, s_{OC}, s_{CC}, s_{IC}$.*

Example 4.5. (Iancu,2014[17]) *Let $L_1, L_{2,}, L_3$ be three known pattern given in the form of IFSs defined on $\Lambda = \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6\}$ as follows:*

$L_1 = \{< \epsilon_1, 0.94, 0.00 >, < \epsilon_2, 0.88, 0.00 >, < \epsilon_3, 0.82, 0.00 >, < \epsilon_4, 0.78, 0.02 >, < \epsilon_5, 0.75, 0.05 >, < \epsilon_6, 0.72, 0.08 >\}$,

$L_2 = \{< \epsilon_1, 0.86, 0.07 >, < \epsilon_2, 0.92, 0.04 >, < \epsilon_3, 0.98, 0.01 >, < \epsilon_4, 0.98, 0.00 >, < \epsilon_5, 0.95, 0.00 >, < \epsilon_6, 0.92, 0.00 >\}$,

$L_3 = \{< \epsilon_1, 0.66, 0.14 >, < \epsilon_2, 0.72, 0.08 >, < \epsilon_3, 0.78, 0.02 >, < \epsilon_4, 0.84, 0.00 >, < \epsilon_5, 0.90, 0.00 >, < \epsilon_6, 0.96, 0.00 >\}$.

Assume an unknown pattern N whose ratings are specified as

$N = \{< \epsilon_1, 0.53, 0.27 >, < \epsilon_2, 0.56, 0.24 >, < \epsilon_3, 0.59, 0.21 >, < \epsilon_4, 0.64, 0.18 >, < \epsilon_5, 0.70, 0.15 >, < \epsilon_6, 0.76, 0.12 >\}$.

To classify N with one of the class of L, we compute the degree of similarity between N and L_1, L_2, L_3 with existing measures and s_{9C}. The calculation results are shown in Table 2. The measure s_{VS} again fails to rank the object due to division by zero problem. The wrong resolution is provided by the measure s_{SK}, s_L and s_H. The measure s_{9C} provides the same calculation result with some other existing measures.

Example 4.6. (Boran and Akay,2014; Chen and Lan,2016[5, 9]) *Let $L_1, L_{2,}, L_3$ be three known pattern given in the form of IFSs defined on $\Lambda = \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$ as follows:*

$$L_1 = \{< \epsilon_1, 0.50, 0.30 >, < \epsilon_2, 0.70, 0.00 >, < \epsilon_3, 0.40, 0.50 >, < \epsilon_4, 0.70, 0.30 >\}$$
$$L_2 = \{< \epsilon_1, 0.50, 0.20 >, < \epsilon_2, 0.60, 0.10 >, < \epsilon_3, 0.20, 0.70 >, < \epsilon_4, 0.70, 0.30 >\}$$
$$L_3 = \{< \epsilon_1, 0.50, 0.40 >, < \epsilon_2, 0.70, 0.10 >, < \epsilon_3, 0.40, 0.60 >, < \epsilon_4, 0.70, 0.20 >\}$$

Table 2: Results of Example 4.5 (The classification results of the proposed measure(s_{9c}) and other measures ($p = 1$) in $s_{DC}, s_{LS1}, s_{LS2}, s_M, s_{HY4}, s_{HY5}, s_{HY6}, s_{BA}$, $\psi_i = 1/6$ in $s_C, s_{OC}, s_{CC}, s_{IC}, s_{9C}$)

Measures	$s(L_1, N)$	$s(L_2, N)$	$s(L_3, N)$	Classification result
s_{KA}	0.8225	0.7600	0.8325	L_3
s_{Chen}	0.8225	0.7600	0.8325	L_3
s_{HK}	0.8158	0.7600	0.8625	L_3
s_{DC}	0.8225	0.7600	0.8325	L_3
s_{LS1}	0.8158	0.7600	0.8325	L_3
s_{LS2}	0.8100	0.7600	0.8325	L_3
s_M	0.8158	0.7600	0.8325	L_3
s_{SK}	0.3364	0.4221	0.3013	L_2
s_{HY1}	0.7900	0.6950	0.8200	L_3
s_{HY2}	0.7003	0.5841	0.7394	L_3
s_{HY3}	0.6529	0.5326	0.6949	L_3
s_{WX}	0.8029	0.7275	0.8263	L_3
s_L	0.7764	0.7262	0.8278	L_1
s_{VS}	**NaN**	**NaN**	**NaN**	Cannot be determined
s_{HY4}	0.8158	0.7600	0.8325	L_3
s_{HY5}	0.6437	0.5591	0.6708	L_3
s_{HY6}	0.5962	0.5135	0.6236	L_3
s_{HY7}	0.6568	0.5785	0.6668	L_3
s_{HY8}	0.8158	0.7600	0.8325	L_3
s_{HY9}	0.6377	0.5752	0.6667	L_3
s_{HY10}	0.7788	0.7303	0.8000	L_3
s_Y	0.9517	0.9557	0.9669	L_3
s_{BL}	0.8203	0.7600	0.8325	L_3
s_S	0.9204	0.9154	0.9431	L_3
s_{CL}	0.8147	0.7351	0.8281	L_3
s_N	0.8111	0.7383	0.8283	L_3
s_{GK}	0.8076	0.7293	0.8192	L_3
s_H	0.4384	0.1414	0.3935	L_1
s_C	0.8228	0.7624	0.8329	L_3
s_{OC}	0.8229	0.7636	0.8332	L_3
s_{CC}	0.8227	0.7618	0.8328	L_3
s_{IC}	0.8217	0.7619	0.8320	L_3
s_{9C}	0.8228	0.7627	0.8330	L_3

NaN denotes that similarity cannot be computed owing to division by zero problem.

Bold denotes unreasonable results. ψ_i is determined based on the number of attributes.

Assume an unknown pattern N whose ratings are specified as:

$$N = \{< \epsilon_1, 0.40, 0.30 >, < \epsilon_2, 0.70, 0.10 >, < \epsilon_3, 0.30, 0.60 >, < \epsilon_4, 0.70, 0.30 >\}$$

Calculate Example 4.6 with existing similarity measures, the outcome are shown in Table 3 and some observations as, the measures $s_{HK}, s_{LS1}, s_M, s_{WX}, s_{HY4}, s_{HY5}, s_{HY6}, s_{HY8}$ is not sure classification result, as a result of $s(L_1, N) = s(L_3, N)$. Using the measures $s_{LS2}, s_{HY1}, s_{HY2}, s_{HY3}$ calculate Example 4.6 is $s(L_1, N) = s(L_2, N) = s(L_3, N)$, so the classification result is uncertain. The measure s_{VS} cannot be computed due to division by zero problem. The result calculated by s_{9C} is the same as that calculated by other measures ($s_{KA}, s_{Chen}, s_{DC}, s_{SK}, s_{HY7}, s_{HY9}, s_{HY10}, s_Y, s_{BA}, s_{CL}, s_N, s_{GK}, s_H, s_C, s_{OC}, s_{CC}, s_{IC}, s_{9C}$).

5 Illustrative examples of the proposed measure

In this section, More examples related to pattern recognition, medical diagnosis and clustering analysis by using proposed measure are debated.

Example 5.1. (Liang and Shi,2003[20]) *Consider a universal set $\Lambda = \{\epsilon_1, \epsilon_2, \epsilon_3\}$ over which three known patterns L_1, L_2 and L_3 are defined as :*

$$L_1 = \{< \epsilon_1, 0.1, 0.1 >, < \epsilon_2, 0.5, 0.1 >, < \epsilon_3, 0.1, 0.9 >\},$$
$$L_2 = \{< \epsilon_1, 0.5, 0.5 >, < \epsilon_2, 0.7, 0.3 >, < \epsilon_3, 0.0, 0.8 >\},$$
$$L_3 = \{< \epsilon_1, 0.7, 0.2 >, < \epsilon_2, 0.1, 0.8 >, < \epsilon_3, 0.4, 0.4 >\}.$$

Assume an unknown pattern N has represented as,

$$B = \{< \epsilon_1, 0.4, 0.4 >, < \epsilon_2, 0.6, 0.2 >, < \epsilon_3, 0.0, 0.8 >\}.$$

Table 3: Results of Example 4.6 (The classification results of the proposed measure(s_{9C}) and other measures ($p = 1$) in $s_{DC}, s_{LS1}, s_{LS2}, s_M, s_{HY4}, s_{HY5}, s_{HY6}, s_{BA}$, $\psi_i = 1/4$ in $s_C, s_{OC}, s_{CC}, s_{IC}, s_{9C}$)

Measures	$s(L_1, N)$	$s(L_2, N)$	$s(L_3, N)$	Classification result
s_{KA}	0.9500	0.9375	0.9750	L_3
s_{Chen}	0.9500	0.9375	0.9750	L_3
s_{HK}	**0.9500**	0.9375	**0.9500**	Cannot be recognized
s_{DC}	0.9500	0.9375	0.9750	L_3
s_{LS1}	**0.9500**	0.9375	**0.9500**	Cannot be recognized
s_{LS2}	**0.9375**	**0.9375**	**0.9375**	Cannot be recognized
s_M	**0.9500**	0.9375	**0.9500**	Cannot be recognized
s_{SK}	0.2857	0.2292	0.3833	L_3
s_{HY1}	**0.9250**	**0.9250**	**0.9250**	Cannot be recognized
s_{HY2}	**0.8857**	**0.8857**	**0.8857**	Cannot be recognized
s_{HY3}	**0.8605**	**0.8605**	**0.8605**	Cannot be recognized
s_{WX}	**0.9375**	0.9313	**0.9375**	Cannot be recognized
s_L	**0.9134**	**0.9134**	0.8882	Cannot be recognized
s_{VS}	**NaN**	0.9666	0.9700	Cannot be recognized
s_{HY4}	**0.9500**	0.9375	**0.9500**	Cannot be recognized
s_{HY5}	**0.8899**	0.8641	**0.8899**	Cannot be recognized
s_{HY6}	**0.8636**	0.8333	**0.8636**	Cannot be recognized
s_{HY7}	0.8875	0.8562	0.8944	L_3
s_{HY8}	**0.9500**	0.9375	**0.9500**	Cannot be recognized
s_{HY9}	0.8889	0.8611	0.8919	L_3
s_{HY10}	0.9412	0.9254	0.9429	L_3
s_Y	0.9906	0.9871	0.9959	L_3
s_{BA}	0.9500	0.9375	0.9667	L_3
s_S	0.9844	0.9957	0.9807	L_2
s_{CL}	0.9492	0.9338	0.9658	L_3
s_N	0.9500	0.9333	0.9583	L_3
s_{GK}	0.9567	0.9417	0.9717	L_3
s_H	0.8911	0.9506	0.9642	L_3
s_C	0.9500	0.9385	0.9700	L_3
s_{OC}	0.9500	0.9391	0.9725	L_3
s_{CC}	0.9500	0.9383	0.9738	L_3
s_{IC}	0.9497	0.9382	0.9730	L_3
s_{9C}	0.9500	0.9387	0.9731	L_3

NaN denotes that similarity cannot be computed owing to division by zero problem.

Bold denotes unreasonable results. ψ_i is determined based on the number of attributes.

Table 4: Results of Example 5.1 (Post-classification of the proposed measure(s_{9C}) and other measures, ($p = 1$) in $s_{DC}, s_{LS1}, s_{LS2}, s_M, s_{HY4}, s_{HY5}, s_{HY6}, s_{BA}$, $\psi_i = 1/4$ in $s_C, s_{OC}, s_{CC}, s_{IC}, s_{9C}$)

Measures	$s(L_1, N)$	$s(L_2, N)$	$s(L_3, N)$	Classification result
s_{KA}	**1.0000**	**1.0000**	0.6000	Cannot be recognized
s_{Chen}	**1.0000**	**1.0000**	0.8000	Cannot be recognized
s_{HK}	0.8333	0.9333	0.6000	L_2
s_{DC}	**1.0000**	**1.0000**	0.6000	Cannot be recognized
s_{LS1}	0.8333	0.9333	0.6000	L_2
s_{LS2}	0.8333	0.9333	0.5833	L_2
s_M	0.8333	0.9333	0.6000	L_2
s_{SK}	0.5407	0.4667	0.7778	L_3
s_{HY1}	0.8333	0.9333	0.5667	L_2
s_{HY2}	0.7571	0.8980	0.4437	L_2
s_{HY3}	0.7143	0.8750	0.3953	L_2
s_{WX}	0.8333	0.9333	0.5833	L_2
s_L	0.6683	0.8586	0.5757	L_2
s_{VS}	**NaN**	**NaN**	**NaN**	Cannot be determined
s_{HY4}	0.8333	0.9333	0.6000	Cannot be recognized
s_{HY5}	0.6722	0.8556	0.3631	L_2
s_{HY6}	0.6250	0.8235	0.3333	L_2
s_{HY7}	0.6000	0.8667	0.3644	L_2
s_{HY8}	0.8333	0.9333	0.6000	L_2
s_{HY9}	0.6154	0.8571	0.3514	L_2
s_{HY10}	0.7619	0.9231	0.5200	L_2
s_Y	0.9954	0.9988	0.6709	L_2
s_{BA}	0.9444	0.9778	0.6000	L_2
s_S	0.9225	0.9626	0.8655	L_2
s_{CL}	0.9444	0.9778	0.6000	L_2
s_N	0.8667	0.9444	0.5889	L_2
s_{GK}	0.7337	0.8222	0.7200	L_2
s_H	**NaN**	0.5687	**NaN**	Cannot be recognized
s_C	0.9789	0.9978	0.6000	L_2
s_{OC}	0.9683	0.9967	0.6000	L_2
s_{CC}	0.9842	0.9983	0.6000	L_2
s_{IC}	0.9749	0.9947	0.5975	L_2
s_{9C}	0.9763	0.9975	0.6000	L_2

NaN denotes that similarity cannot be computed owing to division by zero problem.

Bold denotes unreasonable results. ψ_i is determined based on the number of attributes.

The measures s_{KA}, s_{Chen} and s_{DC} can't match the pattern owing to $s(L_1, N) = s(L_2, N)$.

The measures s_{VS} and s_H unable to present the result, owing to the division by zero problem. The result calculated by s_{9C} is coincide the same output L_2 with the $s_{HK}, s_{LS1}, s_{LS2}, s_M, s_{HY1}, s_{HY2}, s_{HY3}, s_{WX}, s_L, s_{HY4}, s_{HY5}, s_{HY6}, s_{HY7}, s_{HY8}, s_{HY9}, s_{HY10}, s_Y, s_{BA}, s_S, s_{CL}, s_N, s_{GK}, s_C, s_{OC}, s_{CC}, s_{IC}$.

Example 5.2. (Ye,2011[31]) *Assume that there are four patients,* P=$\{Al, Bob, Joe, Ted\}$. *Their symptoms are* S = { *Temperature, Headache, Stomach pain, Cough, Chest pain*}. *The set of diagnoses is defined as* D = { *Viral fever, Malaria, Typhoid, Stomach problem, Chest problem*}.

Table 5: Results of Example 5.2 symptoms characteristic of the patients.

	Temperature	Headache	Stomach pain	Cough	Chest pain
AI	(0.8,0.1)	(0.6,0.1)	(0.2,0.8)	(0.6,0.1)	(0.1,0.6)
Bob	(0.0,0.8)	(0.4,0.4)	(0.6,0.1)	(0.1,0.7)	(0.1,0.8)
Joe	(0.8,0.1)	(0.8,0.1)	(0.0,0.6)	(0.2,0.7)	(0.0,0.5)
Ted	(0.6,0.1)	(0.5,0.4)	(0.3,0.4)	(0.7,0.2)	(0.3,0.4)

Table 6: Results of Example 5.2 symptoms characteristic of the diagnoses.

	Viral fever	Malaria	Typhoid	Stomach problem	Chest pain problem
Temperature	(0.40,0.00)	(0.70,0.00)	(0.30,0.30)	(0.10,0.70)	(0.10,0.80)
Headache	(0.30,0.50)	(0.20,0.60)	(0.60,0.10)	(0.20,0.40)	(0.00,0.80)
Stomach pain	(0.10,0.70)	(0.00,0.90)	(0.20,0.70)	(0.80,0.00)	(0.20,0.80)
Cough	(0.40,0.30)	(0.70,0.00)	(0.20,0.60)	(0.20,0.70)	(0.20,0.80)
Chest pain	(0.10,0.70)	(0.10,0.80)	(0.10,0.90)	(0.20,0.70)	(0.80,0.10)

Table 7: Results of Example 5.2 the proposed similarity measure s_{9C} between each patient's symptoms and the considered set of possible diagnoses, $\psi_i = 1/5$.

	Viral fever	Malaria	Typhoid	Stomach problem	Chest pain problem
AI	0.8539	0.8363	0.8000	0.5581	0.4989
Bob	0.6616	0.5366	0.7351	0.9038	0.6581
Joe	0.8094	0.7102	0.8613	0.6411	0.6411
Ted	0.8241	0.7660	0.7072	0.6313	0.5487

ψ_i is determined based on the number of attributes

As presented in Table 7, it can be concluded that Al afflicted with Viral Fever, Bob afflicted with Stomach problem, Joe afflicted with Typhoid, and Ted afflicted with Viral Fever.

Example 5.3. (Xu,2008[30]) *Consider the data set of ten patterns $i = 1, 2, ...10$, $\psi_i = 1/6$. Implemented with s_{9C} measure and the flowchart of Figure 2 to cluster these 10 patterns.*

$$G = \begin{pmatrix}
1.0000 & 0.6639 & 0.6384 & 0.7503 & 0.6707 & 0.8921 & 0.6727 & 0.6088 & 0.7806 & 0.6273 \\
0.6639 & 1.0000 & 0.9261 & 0.7022 & 0.7152 & 0.6893 & 0.9418 & 0.8444 & 0.7512 & 0.7264 \\
0.6384 & 0.9261 & 1.0000 & 0.7572 & 0.7702 & 0.6639 & 0.9324 & 0.8528 & 0.7918 & 0.7343 \\
0.7503 & 0.7022 & 0.7572 & 1.0000 & 0.7368 & 0.7091 & 0.7104 & 0.6591 & 0.8886 & 0.6777 \\
0.6707 & 0.7152 & 0.7702 & 0.7368 & 1.0000 & 0.7295 & 0.7208 & 0.8035 & 0.7753 & 0.8874 \\
0.8921 & 0.6893 & 0.6639 & 0.7091 & 0.7295 & 1.0000 & 0.6978 & 0.6343 & 0.7545 & 0.6861 \\
0.6727 & 0.9418 & 0.9324 & 0.7104 & 0.7208 & 0.6978 & 1.0000 & 0.8525 & 0.7832 & 0.7157 \\
0.6088 & 0.8444 & 0.8528 & 0.6591 & 0.8035 & 0.6343 & 0.8525 & 1.0000 & 0.6961 & 0.7972 \\
0.7806 & 0.7512 & 0.7918 & 0.8886 & 0.7753 & 0.7545 & 0.7832 & 0.6961 & 1.0000 & 0.6638 \\
0.6273 & 0.7264 & 0.7343 & 0.6777 & 0.8874 & 0.6861 & 0.7157 & 0.7972 & 0.6638 & 1.0000
\end{pmatrix},$$

$$G^2 = \begin{pmatrix}
1.0000 & 0.7512 & 0.7806 & 0.7806 & 0.7753 & 0.8921 & 0.7806 & 0.6961 & 0.7806 & 0.6861 \\
0.7512 & 1.0000 & 0.9324 & 0.7572 & 0.8035 & 0.7512 & 0.9418 & 0.8528 & 0.7918 & 0.7972 \\
0.7806 & 0.9324 & 1.0000 & 0.7918 & 0.8035 & 0.7545 & 0.9324 & 0.8528 & 0.7918 & 0.7972 \\
0.7806 & 0.7572 & 0.7918 & 1.0000 & 0.7753 & 0.7545 & 0.7832 & 0.7572 & 0.8886 & 0.7368 \\
0.7753 & 0.8035 & 0.8035 & 0.7753 & 1.0000 & 0.7545 & 0.8035 & 0.8035 & 0.7753 & 0.8874 \\
0.8921 & 0.7512 & 0.7545 & 0.7545 & 0.7545 & 1.0000 & 0.7545 & 0.7295 & 0.7806 & 0.7295 \\
0.7806 & 0.9418 & 0.9324 & 0.7832 & 0.8035 & 0.7545 & 1.0000 & 0.8528 & 0.7918 & 0.7972 \\
0.6961 & 0.8528 & 0.8528 & 0.7572 & 0.8035 & 0.7295 & 0.8528 & 1.0000 & 0.7918 & 0.8035 \\
0.7806 & 0.7918 & 0.7918 & 0.8886 & 0.7753 & 0.7806 & 0.7918 & 0.7918 & 1.0000 & 0.7753 \\
0.6861 & 0.7972 & 0.7972 & 0.7368 & 0.8874 & 0.7295 & 0.7972 & 0.8035 & 0.7753 & 1.0000
\end{pmatrix},$$

$$G^4 = \begin{pmatrix}
1.0000 & 0.7806 & 0.7806 & 0.7806 & 0.7806 & 0.8921 & 0.7806 & 0.7806 & 0.7806 & 0.7806 \\
0.7806 & 1.0000 & 0.9324 & 0.7918 & 0.8035 & 0.7806 & 0.9418 & 0.8528 & 0.7918 & 0.8035 \\
0.7806 & 0.9324 & 1.0000 & 0.7918 & 0.8035 & 0.7806 & 0.9324 & 0.8528 & 0.7918 & 0.8035 \\
0.7806 & 0.7918 & 0.7918 & 1.0000 & 0.7918 & 0.7806 & 0.7918 & 0.7918 & 0.8886 & 0.7918 \\
0.7806 & 0.8035 & 0.8035 & 0.7918 & 1.0000 & 0.7753 & 0.8035 & 0.8035 & 0.7918 & 0.8874 \\
0.8921 & 0.7806 & 0.7806 & 0.7806 & 0.7753 & 1.0000 & 0.7806 & 0.7806 & 0.7806 & 0.7753 \\
0.7806 & 0.9418 & 0.9324 & 0.7918 & 0.8035 & 0.7806 & 1.0000 & 0.8528 & 0.7918 & 0.8035 \\
0.7806 & 0.8528 & 0.8528 & 0.7918 & 0.8035 & 0.7806 & 0.8528 & 1.0000 & 0.7918 & 0.8035 \\
0.7806 & 0.7918 & 0.7918 & 0.8886 & 0.7918 & 0.7806 & 0.7918 & 0.7918 & 1.0000 & 0.7918 \\
0.7806 & 0.8035 & 0.8035 & 0.7918 & 0.8874 & 0.7753 & 0.8035 & 0.8035 & 0.7918 & 1.0000
\end{pmatrix},$$

Table 8: Example 5.3 consider the data set of ten patterns, $\psi_i = 1/6$.

	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	ϵ_6
L_1	(0.30, 0.40)	(0.20, 0.70)	(0.40, 0.50)	(0.80, 0.10)	(0.40, 0.50)	(0.20, 0.70)
L_2	(0.40, 0.30)	(0.50, 0.10)	(0.60, 0.20)	(0.20, 0.70)	(0.30, 0.60)	(0.70, 0.20)
L_3	(0.40, 0.20)	(0.60, 0.10)	(0.80, 0.10)	(0.20, 0.60)	(0.30, 0.70)	(0.50, 0.20)
L_4	(0.30, 0.40)	(0.90, 0.00)	(0.80, 0.10)	(0.70, 0.10)	(0.10, 0.80)	(0.20, 0.80)
L_5	(0.80, 0.10)	(0.70, 0.20)	(0.70, 0.00)	(0.40, 0.10)	(0.80, 0.20)	(0.40, 0.60)
L_6	(0.40, 0.30)	(0.30, 0.50)	(0.20, 0.60)	(0.70, 0.10)	(0.50, 0.40)	(0.30, 0.60)
L_7	(0.60, 0.40)	(0.40, 0.20)	(0.70, 0.20)	(0.30, 0.60)	(0.30, 0.70)	(0.60, 0.10)
L_8	(0.90, 0.10)	(0.70, 0.20)	(0.70, 0.10)	(0.40, 0.50)	(0.40, 0.50)	(0.80, 0.00)
L_9	(0.40, 0.40)	(1.00, 0.00)	(0.90, 0.10)	(0.60, 0.20)	(0.20, 0.70)	(0.10, 0.80)
L_{10}	(0.90, 0.10)	(0.80, 0.00)	(0.60, 0.30)	(0.50, 0.20)	(0.80, 0.10)	(0.60, 0.40)

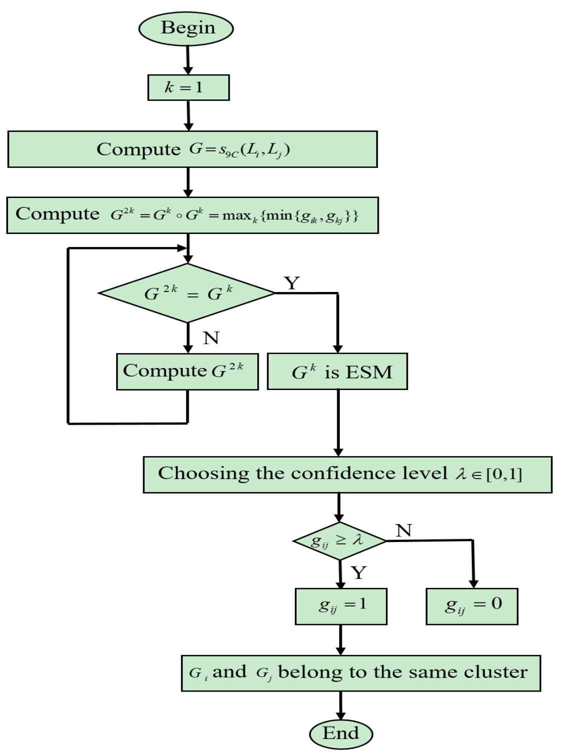

Figure 2: The flowchart of Example 5.3

$$G^8 = \begin{pmatrix} 1.0000 & 0.7806 & 0.7806 & 0.7806 & 0.7806 & 0.8921 & 0.7806 & 0.7806 & 0.7806 & 0.7806 \\ 0.7806 & 1.0000 & 0.9324 & 0.7918 & 0.8035 & 0.7806 & 0.9418 & 0.8528 & 0.7918 & 0.8035 \\ 0.7806 & 0.9324 & 1.0000 & 0.7918 & 0.8035 & 0.7806 & 0.9324 & 0.8528 & 0.7918 & 0.8035 \\ 0.7806 & 0.7918 & 0.7918 & 1.0000 & 0.7918 & 0.7806 & 0.7918 & 0.7918 & 0.8886 & 0.7918 \\ 0.7806 & 0.8035 & 0.8035 & 0.7918 & 1.0000 & 0.7806 & 0.8035 & 0.8035 & 0.7918 & 0.8874 \\ 0.8921 & 0.7806 & 0.7806 & 0.7806 & 0.7806 & 1.0000 & 0.7806 & 0.7806 & 0.7806 & 0.7806 \\ 0.7806 & 0.9418 & 0.9324 & 0.7918 & 0.8035 & 0.7806 & 1.0000 & 0.8528 & 0.7918 & 0.8035 \\ 0.7806 & 0.8528 & 0.8528 & 0.7918 & 0.8035 & 0.7806 & 0.8528 & 1.0000 & 0.7918 & 0.8035 \\ 0.7806 & 0.7918 & 0.7918 & 0.8886 & 0.7918 & 0.7806 & 0.7918 & 0.7918 & 1.0000 & 0.7918 \\ 0.7806 & 0.8035 & 0.8035 & 0.7918 & 0.8874 & 0.7806 & 0.8035 & 0.8035 & 0.7918 & 1.0000 \end{pmatrix},$$

$$G^{16} = \begin{pmatrix} 1.0000 & 0.7806 & 0.7806 & 0.7806 & 0.7806 & 0.8921 & 0.7806 & 0.7806 & 0.7806 & 0.7806 \\ 0.7806 & 1.0000 & 0.9324 & 0.7918 & 0.8035 & 0.7806 & 0.9418 & 0.8528 & 0.7918 & 0.8035 \\ 0.7806 & 0.9324 & 1.0000 & 0.7918 & 0.8035 & 0.7806 & 0.9324 & 0.8528 & 0.7918 & 0.8035 \\ 0.7806 & 0.7918 & 0.7918 & 1.0000 & 0.7918 & 0.7806 & 0.7918 & 0.7918 & 0.8886 & 0.7918 \\ 0.7806 & 0.8035 & 0.8035 & 0.7918 & 1.0000 & 0.7806 & 0.8035 & 0.8035 & 0.7918 & 0.8874 \\ 0.8921 & 0.7806 & 0.7806 & 0.7806 & 0.7806 & 1.0000 & 0.7806 & 0.7806 & 0.7806 & 0.7806 \\ 0.7806 & 0.9418 & 0.9324 & 0.7918 & 0.8035 & 0.7806 & 1.0000 & 0.8528 & 0.7918 & 0.8035 \\ 0.7806 & 0.8528 & 0.8528 & 0.7918 & 0.8035 & 0.7806 & 0.8528 & 1.0000 & 0.7918 & 0.8035 \\ 0.7806 & 0.7918 & 0.7918 & 0.8886 & 0.7918 & 0.7806 & 0.7918 & 0.7918 & 1.0000 & 0.7918 \\ 0.7806 & 0.8035 & 0.8035 & 0.7918 & 0.8874 & 0.7806 & 0.8035 & 0.8035 & 0.7918 & 1.0000 \end{pmatrix}.$$

As $G^{16} = G^8$, Therefore, G^8 is an equivalent similarity matrix.

Table 9: The clustering results of ten IFS patterns for different confidence level λ.

Class	Confidence level	Clustering results
9	$0.9420 < \lambda \leq 1.0000$	$\{L_1\}, \{L_2\}, \{L_3\}, \{L_4\}, \{L_5\}, \{L_6\}, \{L_7\}, \{L_8\}, \{L_9\}, \{L_{10}\}$
8	$0.9400 < \lambda \leq 0.9420$	$\{L_1\}, \{L_2, L_7\}, \{L_3\}, \{L_4\}, \{L_5\}, \{L_6\}, \{L_8\}, \{L_9\}, \{L_{10}\}$
7	$0.9300 < \lambda \leq 0.9400$	$\{L_1\}, \{L_2, L_3, L_7\}, \{L_4\}, \{L_5\}, \{L_6\}, \{L_8\}, \{L_9\}, \{L_{10}\}$
6	$0.8900 < \lambda \leq 0.9300$	$\{L_1, L_6\}, \{L_2, L_3, L_7\}, \{L_4\}, \{L_5\}, , \{L_8\}, \{L_9\}, \{L_{10}\}$
5	$0.8800 < \lambda \leq 0.8900$	$\{L_1, L_6\}, \{L_2, L_3, L_7\}, \{L_4, L_9\}, \{L_5, L_{10}\}, \{L_8\}$
4	$0.8520 < \lambda \leq 0.8800$	$\{L_1, L_6\}, \{L_2, L_3, L_7, L_8\}, \{L_4, L_9\}, \{L_5, L_{10}\}$
3	$0.8000 < \lambda \leq 0.8520$	$\{L_1, L_6\}, \{L_2, L_3, L_5, L_7, L_8, L_{10}\}, \{L_4, L_9\}$
2	$0.7810 < \lambda \leq 0.8520$	$\{L_1, L_6\}, \{L_2, L_3, L_4, L_5, L_7, L_8, L_9, L_{10}\}$
1	$0.7800 < \lambda \leq 0.8520$	$\{L_1, L_2, L_3, L_4, L_5, L_6, L_7, L_8, L_9, L_{10}\}$

From Table 9, it can be observed that different values of λ yield different classification results. A detailed analysis reveals that as lambda increases, the number of classification patterns also increases, and they become more distinct. From the correlation matrix studied by Xu et al. It can be observed that the correlation coefficients in the second row are highly similar. Furthermore, the clustering results of this measure have been supported by other studies. For example, as shown by the clustering result with four clusters $\{L_1, L_6\}, \{L_4, L_9\}, \{L_2, L_3, L_7, L_8\}, \{L_5, L_{10}\}$ are supported by Hwang and Yang (2012) researches demonstrates the effectiveness of the proposed measure.

6 Conclusions

In this paper, a novel similarity/distance measure based on the center of nine-point circle is presented. It has been demonstrated that it satisfies the theories and properties of distance measure. It generalizes the distance measure based on centroid, orthocenter, circumcenter, incenter and nine-point circle center of the isosceles triangles. Comparison with other similarity/distance measures , as well as Examples 4.4, 4.5, and 4.6, highlight the effectiveness and superiority of this similarity/distance measure. Furthermore, Examples 5.1, 5.2, and 5.3 provide concrete applications of this similarity/distance measure in classification, medical diagnosis, and pattern recognition, respectively. In the forthcoming period, we will strive to extend the proposed measure to interval-valued intuitionistic fuzzy sets and type-2 fuzzy sets, to handle real-world problems that involve uncertainty and fuzziness, such as medical diagnosis and image processing.

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