Iranian Journal of Fuzzy Systems

Volume 21, Number 2, (2024), pp. 129-140

Original paper

University of Sistan and Baluchestan

Existence and stability of equilibrium for the uncertainty abstract economy in fuzzy environment

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Abstract

For a class of uncertainty abstract economies in fuzzy environment (UASIFEs), we establish for each UASIFE, the existence of fuzzy strong Berge equilibrium by the Kakutani-Fan Glicksberg fixed point theorem. Moreover, we investigate the stability of equilibrium, and prove that most UASIFEs are essential and each UASIFE has at least an essential component by Fort theorem and reduction to absurdity, respectively.

Keywords: Fuzzy environment, strong Berge equilibrium, generic stability, abstract economy, essential components.

1 Introduction

Zadeh [40] provided a systematic theoretical basis for the study of fuzzy mathematics so that people have more analysis methods for uncertainty in mathematics. Nowadays, Many scholars have carried out extensive research on the game in the fuzzy environment. Kim and Lee [17] studied the generalized fuzzy games . Scalzo [25] gave a sufficient conditions for the existence of fuzzy equilibria of generalized fuzzy games. Yang [36] extended the $\alpha-$core to the fuzzy games and studied its stability. More researches on game theory can be found in the literature [14, 20, 29, 30, 31, 34, 45].

Nash [22] proposed the Nash equilibrium of the n-person noncooperative game. There are many Nash equilibria in a game, how to choose the best Nash equilibrium is always a problem worthy of study and analysis. Many scholars have put forward different equilibrium concepts. Schelling [27] proposed the ' gathering point ' equilibrium. Aumann [4] proposed ' correlation equilibrium '. Seten [28] proposed 'trembling hand equilibrium '. Ichiishi [15] proposed the social alliance equilibrium. The above equilibrium concepts are a refinement of Nash equilibrium. Nash equilibrium means that after each player in the game deviates from the Nash equilibrium strategy, the player 's payoff does not increase. However, if there exists a deviation of the game from a coalition, it is possible to bring better benefits to the members. Based on this situation, Berge [5] proposed the concept of Berge equilibrium. Zhukovskii [46] introduced the Berge equilibrium in the sense of Zhukovskii. Abalo et al. [1] defined simple Berge equilibrium. Larbani and Nessah [19] proposed the strong Berge equilibrium. Kim [16] studied the general concept of strong Berge equilibrium in generalized games. For more research on Berge equilibrium, see article [2, 23, 24]. All of the above are to study the equilibrium existence of non-cooperative game under the condition of certain environment. However, in practice, there are often some uncertain parameters. For example, Larbani and Lebbah [18] pointed out that uncertain parameters in the economy may be due to incomplete information or natural factors such as weather and temperature. The effects of uncertain parameters in industry may come from errors in measuring instruments. If the uncertainty of the experiment has little effect on the results, we can obtain an approximate solution. Thus, the uncertain parameters can be ignored. However, if the uncertainty of the experiment has huge effect on the results, an error solution may be obtained. At this moment, we cannot ignore the uncertain parameters. The equilibrium problem of game theory is actually a decision problem, so it is necessary to consider the uncertain parameters. Zhukovskii [46] introduced the NS equilibrium concept of uncertain game. Larbani and Lebbah [18] defined the ZS equilibrium of game under uncertainty. Zhang and Zhang

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Received: August 2023; Revised: February 2024; Accepted: April 2024.

https://doi.org/10.22111/IJFS.2024.46311.8154

[42, 43] proved the existence of NS equilibrium and simple Berge equilibrium of finite noncooperative game under uncertainty. Yang and Pu [35] defined the NS equilibrium point in generalized game under generalized uncertainty, and proved the existence theorem of NS equilibrium point by Fan-Glicksberg fixed point theorem. Yang [33] defined the Berge-NS equilibrium of noncooperative game under generalized uncertainty and proved its existence by maximum element theorem. Deng and Xiang [7, 8] studied the existence theorems of strong Berge equilibrium of generalized game under uncertainty and generalized multi-objective weak Pareto strong Berge equilibrium under uncertainty.

Fort [10, 11] introduced the concept of essential fixed point. Wu and Jiang [32] introduced the definition of essential Nash equilibrium point. Yu (1999) [37] extended Wu and Jiang 's results not only to games defined in infinite dimensional spaces, but also to other more general parameterizations of games. Yu [38] studied and summarized the existence and stability of Nash equilibrium. [26, 36, 45] also got some stability results about the game under certain conditions. [6, 12, 41, 44] studied the stability of game equilibrium under uncertainty. Obviously, many scholars have done many researchs on the strong Berge equilibrium of noncooperative games under uncertainty.

Inspired by the above works, we extend the generalized game under uncertainty to the fuzzy environment. we make every efforts to study the existence of UASIFE. Moreover, we study the stability of the fuzzy strong Berge equilibrium for a group of UASIFEs in the sense of the essential solutions method and verifying the stability results by giving an example. The sections of this article are as follows. We recall necessary definitions and theorems in section 2; We propose the game model for UASIFE in section 3; We study the stability of a group of UASIFEs in section 4.

2 Preliminaries

We review some definitions from [3, 6, 17, 21, 40].

Let E and H be two Hausdorff topological spaces, $R : E \to P_0(H)$ is said to be:

(1) upper semicontinuous at $e \in E$ if, for any open set Φ in H with $R(e) \subset \Phi$, there exists an open neighborhood $\mathrm{N}(e)$ of e such that $R(e') \subset \Phi$ for any $e' \in \mathrm{N}(e)$.

(2) lower semicontinuous at $e \in E$ if, for any open set Φ in H with $R(e) \cap \Phi \neq \emptyset$, there exists an open neighborhood $\mathrm{N}(e)$ of e such that $R(e') \cap \Phi \neq \emptyset$ for any $e' \in \mathrm{N}(e)$.

The subset M of E is a fuzzy set if there exists a membership function $A_M : E \to [0,1]$, which meets $A_M(e) = 1$ when $e \in M$, and $A_M(e) = 0$ with $e \notin M$. The $\alpha-$cut set of E is setted by

$$]A_M[^{\alpha} = \{e \in E : A_M(e) > \alpha\}.$$

We employ $\zeta(E)$ to express family of all fuzzy sets of E. A set-valued mapping $T : H \to \zeta(E)$ is said to be a fuzzy mapping from H to $\zeta(E)$. For each $h \in H$, the membership function $A_{T(h)}$ of the fuzzy set $\mathrm{T}(h)$ is denoted by $T(\cdot, h)$. We construct a fuzzy set-valued mapping $T : H \to \zeta(E)$ by the function $T : E \times H \to [0,1]$, i.e., $T(e,h) = A_{T(h)}(e)$. Therefor, for the fuzzy set-valued mapping $T : H \to \zeta(E)$ and a level function of satisfaction $t : F \to [0,1]$, we define

$$]T(\cdot,h)[^{t(h)} = \{h \in H : T(e,h) > t(h)\}, \forall h \in H.$$

Let $R : E \to 2^H$ is a set-valued mapping. R be a *usco* mapping if, R is upper semicontinuous in e and $R(e)$ is nonempty compact set for any $e \in E$.

Let B be a nonempty convex set of M and Z be a closed, convex and pointed cone of linear topological space H satisfies $\mathrm{int}Z \neq \emptyset$. $R : B \to H$ is a vector-valued function. Then

(1) R is $Z-$concave if, for each $z_1, z_2 \in B$ and $\lambda \in (0,1)$, $R(\lambda z_1 + (1-\lambda)z_2) - (\lambda R(z_1) + (1-\lambda)R(z_2)) \in Z$.

(2) R is $Z-$quasiconcave if, for each $z_1, z_2 \in B$ and $\lambda \in (0,1)$, $R(z_1) \in x + Z$, $R(z_2) \in x + Z$. Then, $R(\lambda z_1 + (1-\lambda)z_2) \in x + Z$.

Lemma 2.1. [39] *R must be a usco mapping, if $R : E \to H$ is a closed set-valued mapping and H is a compact set.*

Lemma 2.2. [39] *v is upper semicontinuous on M if and only if $\forall c \in R, \{m \in M : v(m) < c\}$ is an open set in E.*

Lemma 2.3. [39] *Let S and Q be two Hausdorsff topological spaces, $\{S_\alpha\}$ is a family of nonempty compact sets in S, $S_\alpha \to S$, where S is a nonempty compact set in S, $\{q_\alpha\}$ is a network in Q, $q_\alpha \to q \in Q, \{u^\alpha(s,y)\}$ is a family of continuous function networks on $S \times Q$, $\sup_{(s,q) \in S \times Q} |u^\alpha(s,q) \to u(s,q)| \to 0$, where $u(s,q)$ is a continuous function on $S \times Q$, then*

$$\max_{s \in S_\alpha} u^\alpha(s, q^\alpha) \to \max_{s \in S} u(s,q).$$

Theorem 2.4. [39] *Let N be a nonempty convex compact set in Hausdorff locally convex linear topological space Z. $V \subset N \times N$ satisfies :*
(1) $\forall e \in V, \{m \in N : (e,m) \in V\}$ *is open in* V *;*
(2) $\forall m \in V, \{e \in N : (e,m) \in V\}$ *is convex in* R*;*
(3) $\forall e \in V, (e,m) \notin V$,
then, there exists a $m \in V$ *such that for* $\forall e \in R$, $(e,m) \notin V$.

Theorem 2.5. [9, 13] *Let M be a nonempty convex compact set in Hausdorff locally convex linear topological space Z, the set-valued mapping $R : M \to P_0(M)$ satisfies that $R(m)$ is nonempty convex compact set for any $m \in M$ and R is upper semicontinuous on M. Then $\exists m^* \in M$ such that $m^* \in R(m^*)$.*

Theorem 2.6. [11] *Let M be a Hausdorff topological space and H be a metric space. If a set-valued mapping $R : M \to P_0(H)$ is upper semicontinuous on M with nonempty compact-valued (denoted briefly by usco), then there exists a residual subset Q in M such that R is lower semicontinuous on Q.*

3 Model and existence

The model for the UASIFE is depicted as follows:

$$\varphi = (S_k, u_k, G_k, g_k, B)_{k \in K},$$

where $K = \{1, \cdots, n\}$ is the set of all players. S_k is the strategy set of player k, $s \in S = \prod_{k=1}^{n} S_k$ is the game situation, B is an unknown parameter set. $u = (u_1, u_2, \cdots, u_n)$, $u_k : S \times B \to R$ is the payment function of the player k. $G_k : S_{-k} \to \zeta(S_k)$ is the fuzzy constraint mapping of the player k, $g_k : S_{-k} \to [0,1]$ is the corresponding level of feasibility. Define a set-valued mapping $W_k : S_{-k} \to P_0(S_k)$ by

$$W_k(s_{-k}) =]G_k(\cdot, s_{-k})[^{g_k(s_{-k})} = \{s_k \in S_k : G_{-k}(s_k, s_{-k}) \geq g_{-k}(s_k)\}.$$

Definition 3.1. *A strategy profile $(\bar{s}, \bar{b}) \in S \times B$ is called a fuzzy strong Berge equilibrium point of a $\varphi = (S_k, u_k, G_k, g_k, B)_{k \in K}$ if, the $(\bar{s}, \bar{b})$ satisfies the following conditions:*
(1) $\forall k \in K, \bar{s}_k \in W_k(\bar{s}_{-k})$*;*
(2) $\forall j \in K - \{k\}, \forall s_{-k} \in W_{-k}(\bar{s}_k), u_j(\bar{s}_k, s_{-k}, \bar{b}) \leq u_j(\bar{s}_k, \bar{s}_{-k}, \bar{b})$*;*
(3) $\forall b \in B, u(\bar{s}, b) - u(\bar{s}, \bar{b}) \notin \mathrm{int} R_+^n$,
where

$$R_+^m = \{s = (s_1, s_2, \cdots, s_m) \in R^m \,|\, s_k \geq 0, k = 1, \cdots, m\},$$
$$\mathrm{int} R_+^m = \{s = (s_1, s_2, \cdots, s_m) \in R^m \,|\, s_k > 0, k = 1, \cdots, m\}.$$

Remark 3.2. *The condition* (1) *and* (2) *show that $\bar{s}$ is a fuzzy strong Berge equilibrium of $\varphi = (S_k, u_k, G_k, g_k, B)_{k \in K}$. The condition* (3) *shows that $\bar{b}$ is a weak Pareto efficient solution of the multi-objective problem $(B, u(\bar{s}, b))$, that is, all players are optimistic about the uncertain parameter b.*

Remark 3.3. *If $B = \emptyset$ or $B = \{b\}$ and $\zeta(S_k)$ is no longer a fuzzy set, then φ becomes a classical generalized game.*

For any $s \in S, -k \in K - \{k\}$. We define the best response mapping $\Gamma_{-k} : S \times B \to P_0(S)$ of the alliance $K - \{k\}$ by

$$\Gamma_{-k}(s,b) = \{\mu \in S \,|\, \mu_{-k} \in W_{-k}(s_k) : u_{-k}(s_k, \mu_{-k}, b) \geq u_{-k}(s_k, q_{-k}, b), \forall q_{-k} \in S_{-k}, G_{-k s_k}(q_{-k}) \geq g_{-k}(s_k)\}.$$

Define the best response set of the alliance $K - \{k\}$ by $\forall (s,b) \in S \times B, \Gamma^1(s,b) = \bigcap_{k \in K} \Gamma_{-k}(s,b), \Gamma^1 : S \times B \to P_0(S \times B)$.

Theorem 3.4. *For each $k \in K$, let S_k and B be nonempty convex compact set of locally convex Hausdorff topological space E_k and F. For a $\varphi = (S_k, u_k, G_k, g_k, B)_{k \in K}$ if, it satisfies the following conditions:*
(1) $G_k : S_{-k} \to \zeta(S_k)$ *is upper semicontinuous with convex-valued and $g_k : S_{-k} \to [0,1]$ is lower semicontinuous for each $s_{-k} \in S_{-k}$;*
(2) $u_k : S \times B \to R$ *is continuous;*
(3) $\forall s_k \in S_k, \forall b \in B, x_{-k} \to u_{-k}(s_k, x_{-k}, b)$ *is quasi-concave on S_k;*
(4) $\forall (s,b) \in S \times B, \Gamma^1(s,b) \neq \emptyset$*;*
(5) $\forall s \in S, b \to u(s,b)$ *is R_+^n- quasi-concave.*
Then, the game φ possesses at least a fuzzy strong Berge equilibrium.

Proof. We define a set-valued mapping $\Gamma^2 : S \to P_0(B)$ by

$$\Gamma^2(s) = \left\{b \in B : u(s,v) - u(s,b) \notin \text{int}R_+^n, \forall v \in B\right\}.$$

(i) From the condition (4), we know easily that $\forall (s,b) \in S \times B, \Gamma_{-k}(s,b)$ is nonempty set. Let any $w^1, w^2 \in \Gamma_{-k}(s,b)$ and $\theta \in [0.1]$. By convexity of G_k and [[45], Lemma 2.2], $W_{-k}(s_k)$ is a convex set. That is, $\theta w_{-k}^1 + (1-\theta) w_{-k}^2 \in W_{-k}(s_k)$. From the condition (3), for any $t_{-k} \in W_{-k}(s_k)$, we have

$$u_{-k}\left(s_k, \theta w_{-k}^1 + (1-\theta) w_{-k}^2, b\right) \geq \min\left\{u_{-k}\left(s_k, w_{-k}^1, b\right), u_{-k}\left(s_k, w_{-k}^2, b\right)\right\} \geq u_{-k}(s_k, t_{-k}, b).$$

This implies $\theta w^1 + (1-\theta) w^2 \in \Gamma_{-k}(s,b)$. Thus, $\forall (s,b) \in S \times B, \Gamma_{-k}(s,b)$ is a nonempty convex set.
(ii) By the compactness of S and [[3], Theorem 17.11], we only need to prove that the $Graph(\Gamma_{-k}(s,b))$ is closed. That is, for $\forall \omega^\alpha \in \Gamma_{-k}(s^\alpha, b^\alpha), \forall (s^\alpha, b^\alpha) \to (s,b)$ and $\omega^\alpha \to \omega$, we need to prove

$$\omega \in \Gamma_{-k}(s,b).$$

Since $\omega_{-k}^\alpha \in W_{-k}(s_k)$ and $\omega_{-k}^\alpha \to \omega_{-k}$. By the condition (1) and [[45], Lemma 2.3], we know that $Graph(\Gamma_{-k}(s,b))$ is closed. Thus, $\omega_{-k} \in W_{-k}(s_k)$. Assume that $\omega_{-k} \notin \Gamma_{-k}(s,b)$. Then, there exists a $x_{-k} \in W_{-k}(s_k)$ satisfying

$$u_{-k}(s_k, x_{-k}, b) \geq u_{-k}(s_k, \omega_{-k}, b).$$

By the condition (2) and $\left(s_k^\alpha, x_{-k}^\alpha, b^\alpha\right) \to (s_k, x_{-k}, b)$, there exists α_0 such that for any $\alpha \succ \alpha_0$, we have

$$\lim_{\alpha\to\infty} u_{-k}\left(s_k^\alpha, x_{-k}^\alpha, b^\alpha\right) = u_{-k}(s_k, x_{-k}, b) \geq u_{-k}(s_k, \omega_{-k}, b).$$

Because any $\left(s_k^\alpha, \omega_{-k}^\alpha, b^\alpha\right) \to (s_k, \omega_{-k}, b)$, then there exists α_1 such that for any $\alpha_1 \succ \alpha_0$, we have

$$u_{-k}(s_k, s_{-k}, b) \geq u_{-k}(s_k, \omega_{-k}, b) = \lim_{\alpha\to\infty} u_{-k}\left(s_k^\alpha, \omega_{-k}^\alpha, b^\alpha\right).$$

Thus,

$$\lim_{\alpha\to\infty} u_{-k}\left(s_k^\alpha, x_{-k}^\alpha, b^\alpha\right) \geq \lim_{\alpha\to\infty_{-k}} \left(s_k^\alpha, \omega_{-k}^\alpha, b^\alpha\right).$$

This implies a contradiction with $\forall \omega^\alpha \in \Gamma_{-k}(s^\alpha, b^\alpha)$. Then, $\omega \in \Gamma_{-k}(s,b)$. By Lemma 2.1, $\Gamma_{-k} : S \times B \to P_0(S)$ is a *usco* mapping. Thus, the set-valued mapping $\Gamma_{-k}(s,b)$ is nonempty compact-valued and upper semicontinuous. Because the intersection of convex compact sets is convex compact set, the intersection of upper semicontinuous functions is upper semicontinuous. Therefore, $\forall (s,b) \in S \times B, \Gamma^1(s,b)$ must be a nonempty convex compact set and Γ^1 must be upper semicontinuous.
(iii) For any $s \in S$, define a set $C = \left\{(v,b) \in B \times B : u(s,v) - u(s,b) \in \text{int}R_+^n\right\}$. For each $v \in B$ and $k \in K$, we have

$$\{b \in B : (v,b) \in C\} = \left\{b \in B : u(s,v) - u(s,b) \in \text{int}R_+^n\right\} = \{b \in B : u_k(s,v) > u_k(s,b)\}.$$

By continuity of u_k and Lemma 2.2, we know that $\{b \in B : (v,b) \in C\}$ is an open set in C. Let any $v_1, v_2 \in \{v \in B : (v,b) \in C\}$ and $\lambda \in (0,1)$, $u(s, \lambda v_1 + (1-\lambda) v_2) \in u(s,v_1) + R_+^n$. From the condition (5), we have

$$u(s, \lambda v_1 + (1-\lambda) v_2) - u(s,b) \in u(s,v_1) - u(s,b) + R_+^n \subset \text{int}R_+^n + R_+^n \subset \text{int}R_+^n.$$

Thus, for any $v \in B$, $\{b \in B : (v,b) \in C\}$ must be a convex set. By Theorem 2.1, there exists $b \in B$ and $\forall v \in B$, $(v,b) \in C$.
That is,

$$u(s,b) - u(s,v) \notin \text{int}R_+^n, \forall v \in B.$$

Thus,

$$\forall s \in S, \Gamma^2(s) \neq \emptyset.$$

Let $b_1, b_2 \in \Gamma^2(s)$ and any $\lambda \in (0,1)$, $\lambda b_1 + (1-\lambda) b_2 \notin \Gamma^2(s)$. Then, there exists a $v \in B$ such that $u(s, \lambda b_1 + (1-\lambda) b_2) - u(s,v) \in \text{int}R_+^n$. By the condition (5) and $u(s, \lambda b_1 + (1-\lambda) b_2) \in u(s,b_1) + R_+^n$, we have

$$u(s,v) - u(s,b_1) = u(s,v) - u(s, \lambda b_1 + (1-\lambda) b_2) + u(s, \lambda b_1 + (1-\lambda) b_2) - u(s,b_1) \subset \text{int}R_+^n + R_+^n \subset \text{int}R_+^n.$$

This implies a contradiction with $b_1 \in \Gamma^2(s)$. Thus, $\lambda b_1 + (1-\lambda) b_2 \in \Gamma^2(s)$. Then, $\Gamma^2(s)$ is a convex set.
(iv) By compactness of B and [[3], Theorem 17.11], we only need to prove that the $Graph\left(\Gamma^2(s)\right)$ is closed. That is,

for s^α in S, $s^\alpha \to s \in S$ and $b^\alpha \in \Gamma^2(s)$, $b^\alpha \to b$ with $b \in \Gamma^2(s)$. Assume that $b \notin \Gamma^2(s)$, that is, there exists a $v \in B$ such that

$$u(s,v) - u(s,b) \in \text{int}R_+^n.$$

Thus, there exists an open neighborhood Ψ of a zero element in R_+^n such that

$$u(s,v) - u(s,b) + \Psi \in \text{int}R_+^n.$$

Since $(s^\alpha, b^\alpha) \to (s,b)$, then there exists an α_2 such that for any $\alpha > \alpha_2$, we have

$$(s^\alpha, b^\alpha) \in U(s,b), u(s^\alpha, b) \to u(s^\alpha, b^\alpha) \in \text{int}R_+^n.$$

This implies a contradiction with $b^\alpha \in \Gamma^2(s)$. Thus, $b \in \Gamma^2(s)$ and $\Gamma^2 : S \to P_0(B)$ is a *usco* mapping. Then, Γ^2 is nonempty convex compact-valued and upper semicontinuous.
(v) Construct a that set-valued mapping $\Gamma : S \times B \to P_0(S \times B), \Gamma(s,b) = \Gamma^1(s,b) \times \Gamma^2(s)$. It can be seen from the proof of (i)-(iv), the set-valued mapping Γ is a nonempty convex compact-valued and upper semicontinuous. By Theorem 2.2, there exists $(\bar{s}, \bar{b}) \in \Gamma(\bar{s}, \bar{b})$. That is,

$$(\bar{s}, \bar{b}) \in \Gamma^1(\bar{s}, \bar{b}) \times \Gamma^2(\bar{s}).$$

Thus,

$$\bar{s} \in \Gamma^1(\bar{s}, \bar{b}), \bar{b} \in \Gamma^2(\bar{s}).$$

By $\bar{s} \in \Gamma^1(\bar{s}, \bar{b})$, we have

$$\forall k \in K, \bar{s} \in \Gamma_{-k}(\bar{s}, \bar{b}).$$

That is,

$$\forall k \in K, j \in K - \{k\}, \forall \bar{s}_{-k} \in W_{-k}(\bar{s}_k), u_j(\bar{s}_k, \bar{s}_{-k}, \bar{b}) \geq u_j(\bar{s}_k, s_{-k}, \bar{b}).$$

By $\bar{b} \in \Gamma^2(\bar{s})$, we have

$$\forall b \in B, u(\bar{s}, b) - u(\bar{s}, \bar{b}) \notin \text{int}R_+^n.$$

□

4 Stability

Let E_k be a Hausdorff normed linear space, $S_k \subset E_k$ and B be a nonempty convex compact set. Ω is the problem space of UASIFE: $\varphi = (S_k, u_k, G_k, g_k, B)_{k \in K}$. Ω satisfies the following conditions:
(1) $G_k : S_{-k} \to \zeta(S_k)$ is upper semicontinuous with convex-valued and $g_k : S_{-k} \to [0,1]$ is lower semicontinuous for each $s_{-k} \in S_{-k}$;
(2) $u_k : S \times B \to R$ is continuous;
(3) $\forall s_k \in S_k, \forall b \in B, x_{-k} \to u_{-k}(s_k, x_{-k}, b)$ is quasi-concave on S_k;
(4) $\forall (s,b) \in S \times B, \Gamma^1(s,b) \neq \emptyset$;
(5) $\forall s \in S, b \to u(s,b)$ is R_+^n- quasi-concave;
(6) $\sum\limits_{k \in K, s \in S, b \in B} |u_k(s,b)| < \infty$.
For any $\varphi_1 = \left(S_k, u_k^1, G_k^1, g_k^1, B\right)_{k \in K}$ and $\varphi_2 = \left(S_k, u_k^2, G_k^2, g_k^2, B\right)_{k \in K}$, we define

$$\rho(\varphi_1, \varphi_2) = \sup_{s \in S, b \in B} \sum_{k \in K} \left|u_k^1(s,b) - u_k^2(s,b)\right| + \sup_{s \in S, b \in B} \sum_{k \in K} \left|G_{ks_{-k}}^1(s_k) - G_{ks_{-k}}^2(s_k)\right| + \sup_{s \in S, b \in B} \sum_{k \in K} \left|g_k^1(s_{-k}) - g_k^2(s_{-k})\right|.$$

Theorem 4.1. (Ω, ρ) *is a complete metric space.*

Proof. We know easily that (Ω, ρ) is a metric space and claim that (Ω, ρ) is complete. Indeed, assume that $\varphi_m = (S_k, u_k^m, G_k^m, g_k^m, B)_{k \in K}$ is any Cauchy sequence in Ω. Then, for any $\vartheta > 0$, there exists an integer $Y = Y(\vartheta)$ such that

for any $n, m > Y$, we have

$$\begin{aligned}\rho(\varphi_n, \varphi_m) = & \sup_{s\in S, b\in B} \sum_{k\in K} |u_k^n(s,b) - u_k^m(s,b)| \\ & + \sup_{s\in S, b\in B} \sum_{k\in K} \left|G_{ks_{-k}}^n(s_k) - G_{ks_{-k}}^m(s_k)\right| \\ & + \sup_{s\in S, b\in B} \sum_{k\in K} |g_k^n(s_{-k}) - g_k^m(s_{-k})| \\ < & \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon\end{aligned}$$

Thus, for each fixed k and $\forall n, m > Y$, we have

$$\begin{aligned}|u_k^n(s,b) - u_k^m(s,b)| &< \frac{\varepsilon}{3}, \\ \left|G_{ks_{-k}}^n(s_k) - G_{ks_{-k}}^m(s_k)\right| &< \frac{\varepsilon}{3}, \\ |g_k^n(s_{-k}) - g_k^m(s_{-k})| &< \frac{\varepsilon}{3}.\end{aligned}$$

That is, the sequence $u_k^r, r = 1, 2, \cdots$ is a Cauchy sequence. Thus, there exists u_k such that $u_k^m \to u_k\,(m \to \infty)$. For $|u_k^n(s,b) - u_k^m(s,b)| < \frac{\varepsilon}{3}$, let $m \to \infty$. Then for each $m > Y$, we have

$$|u_k^n(s,b) - u_k(s,b)| < \frac{\varepsilon}{3} \qquad (i)$$

Because $u_k^n \in \Omega$, then there exists a real number Y_n such that $|u_k^n| < Y_n$. Thus,

$$|u_k| < |u_k - u_k^n| + |u_k^n| \le \varepsilon + Y_n < \infty.$$

Fix any $m > Y$ and $\forall (s,b) \in S \times B$. Since $u_k^n(s,b)$ is continuous on $S \times B$. Thus, for any $\varepsilon > 0, \exists \delta > 0$, when $|(s,b) - (s',b')| < \delta$ we have

$$|u_k^n(s,b) - u_k^n(s',b')| < \frac{\varepsilon}{3} \qquad (ii).$$

Then, by (i) and (ii) we can get

$$|u_k(s,b) - u_k(s',b')| < \varepsilon.$$

Thus, u_k is continuous on $S \times B$. Furthermore, there exists $G_k : S_{-k} \times S_k \to [0,1]$ and $g_k : S_{-k} \to [0,1]$ such that $\lim_{n\to\infty} G_{ks_{-k}}^n(s_k) = G_{ks_{-k}}(s_k)$ and $\lim_{n\to\infty} g_k^n(s_{-k}) = g_k(s_{-k})$. Note that G_k is an upper semicontinuous mapping with convex-valued on S_{-k} and that g_k is a lower semicontinuous mapping on S_{-k}. Since $x_{-k} \to u_{-k}^n(s_k, x_{-k}, b)$ is quasi-concave on S_k. Then, for any $x_{-k}^1, x_{-k}^2 \in S_{-k}$ and $\theta \in (0,1)$, we have

$$u_{-k}^n\left(s_k, \theta x_{-k}^1 + (1-\theta)x_{-k}^2, b\right) \ge \min\left\{u_{-k}^n\left(s_k, x_{-k}^1, b\right), u_{-k}^n\left(s_k, x_{-k}^2, b\right)\right\}.$$

Thus,

$$\begin{aligned}u_{-k}\left(s_k, \theta x_{-k}^1 + (1-\theta)x_{-k}^2, b\right) &> u_{-k}^n\left(s_k, \theta x_{-k}^1 + (1-\theta)x_{-k}^2, b\right) - \varepsilon \\ &\ge \min\left\{u_{-k}^n\left(s_k, x_{-k}^1, b\right), u_{-k}^n\left(s_k, x_{-k}^2, b\right)\right\} - \varepsilon \\ &\ge \min\left\{u_{-k}\left(s_k, x_{-k}^1, b\right), u_{-k}^n\left(s_k, x_{-k}^2, b\right)\right\} - 2\varepsilon \\ &\ge \min\left\{u_{-k}\left(s_k, x_{-k}^1, b\right), u_{-k}^n\left(s_k, x_{-k}^2, b\right)\right\}.\end{aligned}$$

Then, $x_{-k} \to u_{-k}(s_k, x_{-k}, b)$ is quasi-concave on S_k. Finally, we obtain $\varphi = (S_k, u_k, G_k, g_k, B)_{k\in K} \in \Omega$. Thus, (Ω, ρ) is complete. □

For any (Ω, ρ), we construct a set-valued mapping $\mathrm{T} : \Omega \to P_0(S \times B)$. $\mathrm{T}(\varphi)$ is the set of the fuzzy strong Berge equilibrium for φ. By Theorem 3.1, $\mathrm{T}(\varphi) \neq \emptyset$.

Theorem 4.2. *$\mathrm{T} : \Omega \to P_0(S \times B)$ is a usco mapping.*

Proof. By the compactness of $S \times B$ and [[3], Theorem 17.11], we claim that T is closed. That is,

$$\varphi^n \in \Omega, \varphi^n \to \varphi, \forall\, (s^n, b^n) \in \mathrm{T}\,(\varphi^n)\,, (s^n, b^n) \to (s, b)\,,$$

where

$$(s, b) \in \mathrm{T}\,(\varphi)\,.$$

Since $\varphi^n \to \varphi$, for any $k \in K$, we have $G^n_{-k} \to G_{-k}, g^n_{-k} \to g_{-k}$ and $u^n_{-k} \to u_{-k}\,(n \to \infty)$. By contradiction, assume that for a given $k \in K$, $G_{-ks_k}(s_{-k}) < g_{-k}\,(s_k)$. Let $\xi > 0$, then $g_{-k}\,(s_k) - G_{-ks_k}(s_{-k}) > \xi > 0$. By properties of G_k and g_k, for sufficiently large n, we have $g_{-k}\,(s^n_k) - G_{-ks^n_k}(s^n_{-k}) > \xi > 0$. Again, by $G^n_{-k} \to G_{-k}$ and $g^n_{-k} \to g_{-k}$, we have

$$g^n_{-k}\,(s^n_k) - G^n_{-ks^n_k}(s^n_{-k}) > g_{-k}\,(s^n_k) - G^n_{-ks^n_k}(s^n_{-k}) > 0.$$

Thus,

$$g^n_{-k}\,(s^n_k) > G^n_{-ks^n_k}(s^n_{-k}).$$

But $s^n_{-k} \in W_{-k}\,(s^n_k)$, which constitutes a contradiction.

$\forall k \in K, u^n_{-k}\left(s^n_k, s^n_{-k}, b^n\right) = \max\limits_{h_{-k} \in S_{-k}} u^n_{-k}\,(s^n_k, h_{-k}, b^n)$. Obviously, u_{-k} is continuous on (s, b). By $u^n_{-k} \to u_k, (s^n, b^n) \to (s, b)$, we have

$$\begin{aligned}
&\left|u^n_{-k}\left(s^n_k, s^n_{-k}, b^n\right) - u_{-k}\,(s_k, s_{-k}, b)\right| \\
&\leq \left|u^n_{-k}\left(s^n_k, s^n_{-k}, b^n\right) - u_{-k}\left(s^n_k, s^n_{-k}, b^n\right)\right| + \left|u_{-k}\left(s^n_k, s^n_{-k}, b^n\right) - u_{-k}\,(s_k, s_{-k}, b)\right| \\
&\leq \rho_1\left(u^n_{-k}, u_{-k}\right) + \left|u_{-k}\left(s^n_k, s^n_{-k}, b^n\right) - u_{-k}\,(s_k, s_{-k}, b)\right| \to 0.
\end{aligned}$$

Thus,

$$u^n_{-k}\left(s^n_k, s^n_{-k}, b^n\right) \to u_{-k}\,(s_k, s_{-k}, b)\,.$$

For any $k \in K$, because u^n_{-k}, u_{-k} is continuous, and $u^n_{-k} \to u_{-k}, s^n_{-k} \to s_{-k}$. By Lemma 2.3, we have

$$\max_{s_{-k} \in S_{-k}} u^n_{-k}\,(s^n_k, s_{-k}, b^n) \to \max_{s_{-k} \in S_{-k}} u_{-k}\,(s_k, s_{-k}, b)\,.$$

Then,

$$u_{-k}\,(x_k, h_{-k}, b) \leq u_{-k}\,(s_k, s_{-k}, b)\,.$$

Assume that $(s, b) \notin \mathrm{T}\,(\varphi)$, then (s, b) does not satisfy

$$u\,(s, b') - u\,(s, b) \notin \mathrm{int}R^n_+, \forall b' \in B.$$

That is, there exists b^* such that $u\,(s, b^*) - u\,(s, b) \in \mathrm{int}R^n_+$. Thus, there exists an open neighborhood O of a zero element in R^n_+ such that $u\,(s, b^*) - u\,(s, b) + O \in \mathrm{int}R^n_+$. Because $u^n \to u$ and u is continuous. Then for sufficiently large n, we have

$$u\,(s^n, b^*) - u\,(s^n, b^n) \in u\,(s, b^*) - u\,(s, b) + \frac{O}{2} + R^n_+ \subset \mathrm{int}R^n_+ + R^n_+ \subset \mathrm{int}R^n_+.$$

This implies a contradiction with $b^n \in \mathrm{T}\,(\varphi^n)$. Therefore, $(s, b) \in \mathrm{T}\,(\varphi)$. That is, the set-valued mapping T is *usco* on φ. □

To study the essential stability results of UASIFE, we introduce the definitions of essential points, essential sets and essential components for UASIFE:

Definition 4.3. (1) *Assume that $\varphi \in \Omega$. A strategy $(s, b) \in \mathrm{T}\,(\varphi)$ is essential if for any open neighborhood $V((s, b))$ of (s, b), there exists an open neighborhood $U(\varphi)$ of φ such that for any $\varphi' \in U(\varphi)$, $\mathrm{T}\,(\varphi')$ contains a point (s', b') with $(s', b') \in V\,(s, b)$. UASIFE is called essential if every $(s, b) \in \mathrm{T}\,(\varphi)$ is essential.*
(2) *Assume that $\varphi \in \Omega$. A nonempty closed subset $\phi\,(\varphi) \in \mathrm{T}\,(\varphi)$ is essential if for any open set $U \supset \phi\,(\varphi)$, there exists $\mu > 0$ such that $\mathrm{T}\,(\lambda) \cap U \neq \emptyset$ for any $\lambda \in \Lambda$ with $\rho\,(\lambda, \varphi) < \mu$. $\phi\,(\varphi)$ is called a minimal essential set of $\mathrm{T}\,(\varphi)$ if it is a minimal element ordered by set inclusion.*
(3) *Assume that $\varphi \in \Omega$. The component of $\mathrm{T}\,(\varphi)$ is the union of all connected subsets of $\mathrm{T}\,(\varphi)$. A component $C\,(\varphi) \subset \mathrm{T}\,(\varphi)$ is essential if $C(\varphi)$ is essential.*

Theorem 4.4. *φ is essential if and only if the set-valued mapping T is lower semicontinuous in φ.*

Proof. We prove the sufficiency of the theorem. Let $\varphi \in \Omega$, for any $(s,b) \in \mathrm{T}(\varphi)$ and the open field U of (s,b) in $S \times B$, we have $U \cap \mathrm{T}(\varphi) \neq \emptyset$. Since T is lower semicontinuous in φ, then there exists an open field O of φ such that for any $\varphi' \in O$ satisfying $U \cap \mathrm{T}(\varphi') \neq \emptyset$. We choose a strategy $(s',b') \in U \cap \mathrm{T}(\varphi')$, then $(s',b') \in \mathrm{T}(\varphi')$ and $(s',b') \in U$. Therefore, (s,b) is essential. This implies that φ is essential.

Secondly, we prove the necessity of the theorem. For any open set G in $S \times B$, $G \cap \mathrm{T}(\varphi) \neq \emptyset$. Let $(s,b) \in G \cap \mathrm{T}(\varphi)$, then G is the open field of (s,b) in $S \times B$. Since φ is essential, then $(s,b) \in G$ is the essential equilibrium point. There exists an open field O of φ in Ω such that for any $\varphi' \in O$, we have $(s',b') \in \mathrm{T}(\varphi')$. Because $(s',b') \in G$, then $G \cap \mathrm{T}(\varphi') \neq \emptyset$ as $\varphi' \in O$. Thus, the set-valued mapping T is lower semicontinuous in φ. □

Theorem 4.5. *There exists a dense residual subset Q of Ω such that for $\forall \varphi \in Q$, the game φ is essential.*

Proof. By Theorem 4.1, (Ω, ρ) is a complete metric space. By Theorem 4.2, the set-valued mapping $\mathrm{T}: \Omega \to P_0(S \times B)$ is a *usco* mapping. Thus, by Theorem 2.3, there exists a dense residual subset Q of Ω such that for any $\varphi \in Q$, the set-valued mapping T is lower semicontinuous in φ. By Theorem 4.3, every φ in Q is essential. □

We prove by the following example that not all UASIFEs in Ω have the essential points; thus, $Q \neq \Omega$.

Example 4.6. *A two people game $\varphi = (S_k, u_k, G_k, g_k, B)_{k\in\{1.2\}}$, where $S_1 = [-1,1]$,$S_2 = [0,2]$. The fuzzy constraint mappings $G_1, G_2 : [-1,2] \to \mathcal{F}([0,1])$ and the degree functions $g_k : S_{-k} \to [0,1]$ are defined by*

$$G_{1s_2}(s_1) = \begin{cases} q, s_2 \in [0,2], s_1 \in \{-1,1\} \\ 0, s_2 \in [0,2], s_1 \in (-1,1) \end{cases} \qquad G_{2s_1}(s_2) = \begin{cases} q, s_1 \in [-1,1], s_2 \in [0,1] \\ 0, s_1 \in [-1,1], s_2 \in (1,2] \end{cases}$$

and $g_1(s_2) = g_2(s_1) = p$, $q > p > 1$, then we have $W_1(s_2) = \{-1,1\}, W_2(s_1) = [0,1]$. For any $(s_1, s_2, b_1, b_2) \in [-1,1] \times [0,2] \times [0,1] \times [0,1]$, the peyment functions of the player 1 and 2 are defined by

$$u_1(s_1, s_2, b_1, b_2) = |s_1| + b_1,$$

$$u_2(s_1, s_2, b_1, b_2) = s_2 + b_2.$$

It is easily proved that the fuzzy strong Berge equilibriums are $\{(-1,1),(1,1)\}$ and $\{(1,1),(1,1)\}$. Consider a game $\varphi^n = (S_k, u_k^n, G_k^n, g_k^n, B)_{k\in\{1.2\}}$, where

$$G^n{}_{2s_1}(s_2) = \begin{cases} q + \frac{s_2}{n}, s_1 \in [-1,1], s_2 \in [0,1] \\ 0, s_1 \in [-1,1], s_2 \in (1,2] \end{cases}, \qquad G_{1s_2}^n(s_1) = \begin{cases} q + \frac{s_1}{n}, s_2 \in [0,2], s_1 \in \{-1,1\} \\ 0, s_2 \in [0,2], s_1 \in (-1,1) \end{cases},$$

and $g_1^n(s_2) = p + \frac{s_2}{n}$, $g_2^n(s_1) = p + \frac{s_1}{n}$. We have $W_1^n(s_2) = \{-1,1\}, W_2^n(s_1) = [0,1]$. The peyment functions of the players 1 and 2 are defined by respectively,

$$u_1^n(s_1, s_2, b_1, b_2) = \begin{cases} \left(1 - \frac{1}{n}\right) s_1 + b_1, (s_1, s_2, b_1) \in [0,1] \times [0,1] \times [0,1] \times [0,1] \\ l - s_1 + b_1, (s_1, s_2, b_1, b_2) \in [-1,0] \times [0,1] \times [0,1] \times [0,1] \end{cases}$$

and

$$u_2^n(s_1, s_2, b_1, s_2) = \left(1 - \frac{1}{n}\right) s_2 + b_2.$$

It can be identified that $\varphi^n = (S_k, u_k^n, G_k^n, g_k^n, B)_{k\in\{1.2\}} \in \Omega$, $\varphi^n \to \varphi$ and the fuzzy strong Berge equilibrium of (u_1^n, u_2^n) is $\{(-1,1),(1,1)\}$. Thus, $\{(1,1),(1,1)\}$ is not essential.
Consider another sequence $\varphi^{n'} = \left(S_k, u_k^{n'}, G_k^{n'}, g_k^{n'}, B\right)_{k\in\{1.2\}}$. Similar to the above derivation, we take $W_1^{n'}(s_2) = \{-1,1\}$, $W_2^{n'}(s_1) = [0,1]$ and let the payment function be

$$u_1^{n'}(s_1, s_2, b_1, b_2) = \begin{cases} \left(1 + \frac{1}{n}\right) s_1 + b_1, (s_1, s_2, b_1, b_2) \in [0,1] \times [0,1] \times [0,1] \times [0,1] \\ l - s_1 + b_1, (s_1, s_2, b_1, b_2) \in [-1,0] \times [0,1] \times [0,1] \times [0,1] \end{cases}$$

$$u_2^{n'}(s_1, s_2, b_1, b_2) = \left(1 + \frac{1}{n}\right) s_2 + b_2.$$

It is easily proved that $\varphi^{n'} = \left(S_k, u_k^{n'}, G_k^{n'}, g_k^{n'}, B\right)_{k\in\{1.2\}} \in \Omega$,$\varphi^{n'} \to \varphi$ and the strong Berge equilibrium is $\{(1,1),(1,1)\}$. Thus, $\{(-1,1),(1,1)\}$ is not essential.

Theorem 4.7. *φ is essential if $\mathrm{T}(\varphi) = \{(s,b)\}$ (single point set).*

Proof. For any open set G in $S \times B$, $G \cap \mathrm{T}(\varphi) \neq \emptyset$. Because $\mathrm{T}(\varphi) = \{(s,b)\}$, then $\mathrm{T}(\varphi) \subset G$. By Theorem 4.2, the set-valued mapping T is upper semicontinuous. Then, there exists $\delta > 0$ such that $\mathrm{T}(\varphi') \subset G$ as $\rho(\varphi, \varphi') < \delta$. Thus, $\mathrm{T}(\varphi') \cap G \neq \emptyset$, the set-valued mapping T is lower semicontinuous. By Theorem 4.3, φ is essential. □

Theorem 4.8. *For any $\varphi \in \Omega$, there exists at least one minimal essential set of $\mathrm{T}(\varphi)$.*

Proof. By Theorem 4.2, $\mathrm{T}: \Omega \to P_0(S \times B)$ is an *usco* mapping, there exists an open neighborhood V of φ such that $\mathrm{T}(\varphi') \subset O$ for any $\varphi' \in \Omega$. This states that $\mathrm{T}(\varphi)$ is its own essential set.

Suppose that Θ is the family of all essential sets of $\mathrm{T}(\varphi)$ ordered by set inclusion. It can be seen that every decreasing chain of Θ possesses a lower bound. By the Zorn's lemma, there exists a minimal member of Θ, which is obviously a minimal essential set of $\mathrm{T}(\varphi)$. □

Theorem 4.9. *For any $\varphi \in \Omega$, all minimal essential subsets of $\mathrm{T}(\varphi)$ are connected.*

Proof. Assume that $m(\varphi)$ is a minimal essential set of $\mathrm{T}(\varphi)$, we claim that $m(\varphi)$ is connected. If $m(\varphi)$ is not connected, then there exists two nonempty closed subsets $E_1(\varphi)$ and $E_2(\varphi)$ such that $m(\varphi) = E_1(\varphi) \cup E_2(\varphi)$ and two disjoint open sets U_1 and U_2 with $E_1(\varphi) \subset U_1$ and $E_2(\varphi) \subset U_2$. Because $E_1(\varphi)$ and $E_2(\varphi)$ are not the essential set of $\mathrm{T}(\varphi)$, then there exists two open sets O_1 and O_2 with $O_1 \supset E_1(\varphi)$ and $O_2 \supset E_2(\varphi)$ and $\varphi_1, \varphi_2 \in \Omega$, where $\mathrm{T}(\varphi_1) \cap O_1 = \emptyset$ with $\rho(\varphi, \varphi_1) < \delta$ and $\mathrm{T}(\varphi_2) \cap O_2 = \emptyset$ with $\rho(\varphi, \varphi_2) < \delta$ for each $\delta > 0$. Because $m(\varphi)$ is not connected, assume that there exists two open sets Φ_1, Φ_2 such that $E_1(\varphi) \subset \Phi_1 \subset \bar{\Phi}_1 \subset U_1 \cup O_1$ and $E_2(\varphi) \subset \Phi_2 \subset \bar{\Phi}_2 \subset U_2 \cup O_2$.

Because $m(\varphi)$ is the essential set of $\mathrm{T}(\varphi)$ and $m(\varphi) \subset \Phi_1 \cup \Phi_2$, then there exists a $\sigma^* > 0$ such that for any $\lambda \in \Omega$, $\mathrm{T}(\lambda) \cap (\Phi_1 \cup \Phi_2) = \emptyset$ with $\rho(\varphi, \lambda) < \sigma^*$. Moreover, since $E_1(\varphi)$ and $E_2(\varphi)$ are not essential set of $\mathrm{T}(\varphi)$, then there exists $\lambda_1, \lambda_2 \in \Omega$ such that $\mathrm{T}(\varphi_1) \cap \Phi_1 = \emptyset$ with $\rho(\varphi, \lambda_1) < \frac{\sigma^*}{2}$ and $\mathrm{T}(\lambda_2) \cap \Phi_2 = \emptyset$ with $\rho(\varphi, \lambda_2) < \frac{\sigma^*}{2}$.

Let $\lambda_1 = \left(S_k, u_k^1, G_k^1, g_k^1, B\right)$ and $\lambda_2 = \left(S_k, u_k^2, G_k^2, g_k^2, B\right)$. Construct a generalized fuzzy game under uncertainty $\lambda_3 = \left(S_k, u_k^3, G_k^3, g_k^3, B\right)$ denoted by

$$\begin{aligned} u_k^3(s,b) &= \alpha(s)\, u_k^1(s,b) + \beta(s)\, u_k^2(s,b), \\ G_{ks_{-k}}^3(s_k) &= \alpha(s)\, G_{ks_{-k}}^1(s_k) + \beta(s)\, G_{ks_{-k}}^2(s_k), \\ g_k^3(s_{-k}) &= \alpha(s)\, g_k^1(s_{-k}) + \beta(s)\, g_k^2(s_{-k}), \end{aligned}$$

where $\alpha(s) = \frac{d\left(s,\bar{\Phi}_2\right)}{d\left(s,\bar{\Phi}_1\right)+d\left(s,\bar{\Phi}_2\right)}$, $\beta(s) = \frac{d\left(s,\bar{\Phi}_1\right)}{d\left(s,\bar{\Phi}_1\right)+d\left(s,\bar{\Phi}_2\right)}$. It is easily verified that $\lambda_3 \in \Omega$. Furthermore, we can have

$$\begin{aligned} \rho(\varphi, \lambda_3) &= \sup_{s\in S, b\in B} \sum_{k\in K} \left|u_k(s,b) - u_k^3(s,b)\right| + \sup_{s\in S, b\in B} \sum_{k\in K} \left|G_{ks_{-k}}(s_k) - G_{ks_{-k}}^3(s_k)\right| + \sup_{s\in S, b\in B} \sum_{k\in K} \left|g_k(s_{-k}) - g_k^3(s_{-k})\right| \\ &= \sup_{s\in S, b\in B} \sum_{k\in K} \left|\alpha(s)\, u_k(s,b) + \beta(s)\, u_k(s,b) - \alpha(s)\, u_k^1(s,b) - \beta(s)\, u_k^2(s,b)\right| \\ &\quad + \sup_{s\in S, b\in B} \sum_{k\in K} \left|\alpha(s)\, G_{ks_{-k}}(s_k) + \beta(s)\, G_{ks_{-k}}(s_k) - \alpha(s)\, G_{ks_{-k}}^1(s_k) - \beta(s)\, G_{ks_{-k}}^2(s_k)\right| \\ &\quad + \sup_{s\in S, b\in B} \sum_{k\in K} \left|\alpha(x)\, g_k(s_{-k}) + \beta(s)\, g_k(s_{-k}) - \alpha(s)\, g_k^1(s_{-k}) - \beta(s)\, g_k^2(s_{-k})\right| \\ &\leq \rho(\varphi, \lambda_1) + \rho(\varphi, \lambda_2) < \delta^*. \end{aligned}$$

Thus, $\mathrm{T}(\lambda_3) \cap (\Phi_1 \cup \Phi_2) \neq \emptyset$. Without losing generality, it can be assumed that $\mathrm{T}(\lambda_3) \cap \Phi_1 \neq \emptyset$. Then, there exists $\left(\bar{s}, \bar{b}\right) \in \mathrm{T}(\lambda_3) \cap \Phi_1$ satisfying that $u_k^3\left(\bar{s}, \bar{b}\right) = u_k^1\left(\bar{s}, \bar{b}\right)$, $G_{k\bar{s}_{-k}}^3(\bar{s}_k) = G_{k\bar{s}_{-k}}^1(\bar{s}_k)$, $g_k^3(\bar{s}_{-k}) = g_k^1(\bar{s}_{-k})$. Thus, $\left(\bar{s}_k, \bar{b}\right) \in \mathrm{T}(\varphi)$ satisfies

$$\forall j \in K - \{k\}, \forall s_{-k} \in W_{-k}(s_k), u_j\left(\bar{s}_k, s_{-k}, b\right) \leq u_j\left(\bar{s}_k, \bar{s}_{-k}, \bar{b}\right).$$

$$\forall b \in B, u\left(\bar{s}, b\right) - u\left(\bar{s}, \bar{b}\right) \notin \mathrm{int}R_+^n.$$

Then $\left(\bar{s}, \bar{b}\right) \in \mathrm{T}(\lambda_1) \cap \Phi_1$. This implies a contradiction with $\mathrm{T}(\lambda_1) \cap \Phi_1 = \emptyset$. Then, $m(\varphi)$ is connected. □

5 Conclusions

We extended the generalized game model under uncertainty and introduced the concept of the fuzzy strong Berge equilibrium for UASIFE. We employed Theorem 2.1 and Theorem 2.2 to prove the existence of the fuzzy strong Berge equilibrium for UASIFE. Next, we constructed a UASIFE problem space Ω. By the existence result, we obtained that some stability results of the fuzzy strong Berge equilibrium for UASIFE. It was proved that there exists a dense residual set Q in the space Ω. For $\forall\varphi \in Q$, the game φ is essential. We proved that φ is essential when $\mathrm{T}(\varphi)$ is a single point set. Moreover, we obtained that there exists at least one essential component in $\mathrm{T}(\varphi)$ for any $\varphi \in \Omega$.

Acknowledgement

The authors wish to express their appreciation for several excellent suggestions for improvements in this paper made by the referees.

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