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Original paper

Migrativity of uninorms not internal on the boundary over continuous t-(co)norms

D. X. Jiang [1] and H. W. Liu [2]

[1,2] *School of Mathematics, Shandong University, 250100 Jinan, PR China*

1477109118@qq.com, hw.liu@sdu.edu.cn

Abstract

Uninorms are a special type of associative aggregation functions, which have received widespread attention in the theoretical and practical fields since their introduction. Durante and Sarkoci introduced the migrativity property in 2008. Afterwards, this property was widely applied in numerous fields like image processing and decision analysis, which has sparked a series of studies. There have been a large number of research results on the migrativity involving uninorms, but the work has mainly focused on the uninorms internal on the boundary. In this paper, we will concentrate on the uninorms not internal on the boundary. First, we discuss the characterization of the α-migrativity of conjunctive uninorms over continuous t-norms according to the value of α. Then, the consequences of the α-migrativity of disjunctive uninorms over continuous t-conorms can be obtained dually.

Keywords: Migrativity, uninorm, triangular norm, triangular conorm.

1 Introduction

1.1 Migrativity

The concept of α-migrativity was introduced by Durante and Sarkoci [6] when they investigated the convex combinations of two triangular norms. In fact, the α-migrativity of t-norms has been raised as an open problem in [24]. Since it was proposed, the α-migrativity of aggregation functions has aroused the research interest of numerous scholars, mainly owing to its wide application. For instance, in image processing [2], a certain property of an image does not change with the proportional adjustment of some portion of the image. In decision-making problems [31], it is essential to aggregate some information from various sources into the global evaluation results. When the information parameters are repeated, it is crucial that the parameters have a certain proportion of interchangeability, that is, the selection of parameters does not affect the overall results.

Fodor et al. [8] presented the continuous t-norm solutions of α-migrativity equation, a similar result was also achieved by Ouyang et al. in [27]. After that, α-migrative semicopulas, quasi-copulas and copulas have been considered in [23]. Further, Fodor et al. [9] extended the notion of α-migrative t-norms by replacing the typical product operation with any given t-norm, i.e. (α, T_0)-migrative t-norms T and described (α, T_M)-migrative and (α, T_L)-migrative continuous t-norms. They also characterized continuous t-norms that are α-migrative over continuous ordinal sums in [10]. Since then, many colleagues have studied α-migrativity of other aggregation functions based on the same idea, such as t-subnorms [41], uninorms [34], nullnorms [48], overlap and grouping functions [46], uni-nullnorms [38, 39], 2-uninorms [19], Mayor's aggregation operators [40] and so on.

Corresponding Author: H. W. Liu

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Orcid address: https://orcid.org/0009-0006-0225-9990 (D. X. Jiang), https://orcid.org/0009-0000-2696-8413 (H. W. Liu)

1.2 Uninorms

Uninorms are a special type of aggregation functions which generalize the concepts of t-norms and t-conorms. They introduce a neutral element that can be positioned anywhere within the unit interval. Since they were put forward by Yager and Rybalov [45], many scholars have conducted extensive applied research on them taking into account that they can model bipolar behavior [25]. It has been shown that uninorms have a wide range of applications, such as decision making [44], fuzzy system modeling [43], expert systems [4], neural networks [18] and so on. Due to their wide application, many scholars studied uninorms from the theoretical perspective. Initially, Fodor et al. [11] gave the structural framework of uninorms, allowing them to play a t-norm on $[0,e]^2$ and a t-conorm on $[e,1]^2$, and on the remaining part of the unit square, any uninorm is limited by the minimum and the maximum of its arguments. Meanwhile, they introduced two classes of uninorms: $\mathcal{U}_{min}$ and $\mathcal{U}_{max}$, and almost continuous uninorms which have been shown to be equivalent to representable uninorms [32]. Gradually, researchers became interested in introducing new classes of uninorms and characterizing those that fulfill specific additional properties [3, 7, 17, 20, 26]. So far, there have been various usual classes of uninorms: $\mathcal{U}_{min}$ and $\mathcal{U}_{max}$, idempotent uninorms $\mathcal{U}_{ide}$, representable uninorms $\mathcal{U}_{rep}$, uninorms continuous in the open unit square $\mathcal{U}_{cos}$ and uninorms with continuous underlying operators $\mathcal{U}_{cts}$. Moreover, a new class of uninorms $\mathcal{U}_{A(e)}$ [5] was introduced with the help of the concept of locally interal: $U(x,y) \in \{x,y\}$ for any $(x,y) \in A(e) = [0,e) \times (e,1] \cup (e,1] \times [0,e)$, where e is the neutral element of the uninorm U. Along this line of thought, a more general kind of uninorms which are locally internal on the boundary of unit interval $\mathcal{U}_{locb}$ was raised in [22] by Mas et al., and then, Li et al. discussed the features and portrayals of this class of uninorms in [14]. As we all know, all of the most common types of uninorms are in $\mathcal{U}_{locb}$ [22]. Considering the existence of uninorms not internal on the boundary, i.e. $\mathcal{U}_{nlocb}$ [14], Xie [42] discussed the structure of uninorms in $\mathcal{U}_{nlocb}$ in cases where their boundary functions have only two discontinuities and one of the discontinuous points belongs to the set $\{0,1,e\}$. Recently, in order to further characterize the uninorms in $\mathcal{U}_{nlocb}$, Qin et al. [29] studied the structure of uninorms whose boundary functions have a finite number of discontinuities and extended the results in [42].

1.3 Migrativity of uninorms

Up to now, there have been a great deal of research findings on the α-migrativity of uninorms. Mas et al. [21] put forward the conception of (α, U_0)-migrative uninorms U, with U_0 being a uninorm that shares the same neutral element as U and explored the α-migrativity of different combinations of U and U_0, where U and U_0 belong to the most common types of uninorms: $\mathcal{U}_{min}$ and $\mathcal{U}_{max}$, $\mathcal{U}_{ide}$, $\mathcal{U}_{rep}$ and $\mathcal{U}_{cos}$. Qin and Su et al. [30, 35, 37] respectively characterized the α-migrativity of uninorms, which are among the most common types of uninorms possessing the different neutral elements, i.e., $\mathcal{U}_{min}$ and $\mathcal{U}_{max}$, $\mathcal{U}_{ide}$, $\mathcal{U}_{rep}$ and $\mathcal{U}_{cos}$. In view of the fact that there are few non-trivial solutions when assuming that both uninorms belong to the most usual classes, Su et al. [33] examined the α-migrativity of uninorms within a broader framework. In this context, the second uninorm can be one of any common classes, while the first uninorm can be any uninorm without extra assumptions. In 2021, Li et al. [16] analysed the α-migrativity of uninorms in cases where both uninorms have only continuous underlying operators.

In addition, the α-migrativity of uninorms and other operators has been studied extensively, including t-(co)norms [22], nullnorms [1], semi t-operators [36], overlap and grouping functions [15, 28, 46], 2-uninorms [12] and so on. Specifically, Mas et al. [22] put forward and explored the concepts of α-migrative uninorms and nullnorms over t-norms T and over t-conorms S, they characterized (α, T)-migrative uninorms U which are among $\mathcal{U}_{min}$ and $\mathcal{U}_{max}$, $\mathcal{U}_{ide}$, $\mathcal{U}_{rep}$ and $\mathcal{U}_{cos}$. In 2021, Zhu et al. [47] inspected the α-migrativity of conjunctive (resp. disjunctive) uninorms belonging to $\mathcal{U}_{locb}$ over continuous t-norms (resp. t-conorms). In this paper, we devote to characterize the α-migrativity of a uninorm which is not internal on the boundary over a continuous t-(co)norm.

The remaining sections of this paper are structured as follows. In Section 2, we recall key conceptions and characteristics of t-norms and uninorms relevant for this study. Section 3 addresses α-migrativity of conjunctive uninorms belonging to $\mathcal{U}_{nlocb}$ over continuous t-norms. Section 4 presents findings on the α-migrativity of disjunctive uninorms belonging to $\mathcal{U}_{nlocb}$ over continuous t-conorms directly through duality. Finally, we give the conclusion in Section 5.

2 Preliminaries

In this section, we review some essential definitions and results that will be utilized in this paper.

Definition 2.1. [13] *If a binary function $T : [0,1]^2 \to [0,1]$ meets commutativity, associativity, and is increasing in both variables such that $T(x,1) = x$ for any $x \in [0,1]$, it is called a triangular norm (t-norm for short).*

Definition 2.2. [13] *If a binary function $S : [0,1]^2 \to [0,1]$ meets commutativity, associativity, and is increasing in both variables such that $S(x,0) = x$ for any $x \in [0,1]$, it is called a triangular conorm (t-conorm for short).*

Remark 2.3. [13] *The associativity allows us to extend each t-norm T in a unique way to an n-ary operation in the usual way by induction, defining for each n-tuple $(x_1, x_2, ..., x_n) \in [0,1]^n$*

$$\top_{i=1}^{n} x_i = T(\top_{i=1}^{n-1} x_i, x_n) = T(x_1, x_2, ..., x_n).$$

In particular, if $x_1 = x_2 = \cdots = x_n = x$, we can briefly write $x_T^{(n)} = T(x, x, ..., x)$. For each t-conorm S, we can similarly extend it to an n-ary operation, we omit the definition here.

Definition 2.4. [13] *Assume that T is a t-norm.*

(i) *If $T(a,a) = a$ for some $a \in [0,1]$, a is called an idempotent element of T. If $a \in (0,1)$, it is said to be a non-trivial idempotent element of T.*

(ii) *For an element $b \in (0,1)$, if $b_T^{(n)} = 0$ for some $n \in N$, it is called a nilpotent element of T .*

Lemma 2.5. [13] *Assume that T is a continuous t-norm, then $a \in [0,1]$ is an idempotent element of T if and only if $T(a,x) = min(a,x)$ for all $x \in [0,1]$.*

Definition 2.6. [13] *Consider a t-norm T, then:*

(i) *If $T(x,y) < T(x,z)$ when $x > 0$ and $y < z$, we say T is strictly monotone.*

(ii) *If $T(x,y) = T(x,z)$ implies $x = 0$ or $y = z$, we say T fulfills the cancellation law.*

(iii) *If $T(x,y) = T(x,z) > 0$ implies $y = z$, we say T fulfills the conditional cancellation law.*

(iv) *If for any $(x,y) \in (0,1)^2$, $x_T^{(n)} < y$ for some $n \in N$, we say T is Archimedean.*

Definition 2.7. [13] *Suppose that T is a t-norm.*

(i) *If T is continuous and strictly monotonic, we refer to it as strict.*

(ii) *If T is continuous and any $a \in (0,1)$ is a nilpotent element of it, we refer to it as nilpotent.*

Lemma 2.8. [13]

(i) *A t-norm T is strict if and only if it is continuous and fulfills the cancellation law.*

(ii) *Any nilpotent t-norm T fulfills the conditional cancellation law.*

Definition 2.9. [13] *Suppose $(T_m)_{m\in M}$ is a family of t-norms and $((a_m, e_m))_{m\in M}$ is a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. The t-norm T given by*

$$T(x,y) = \begin{cases} a_m + (e_m - a_m)T_m(\frac{x-a_m}{e_m-a_m}, \frac{y-a_m}{e_m-a_m}), & \text{if } x,y \in [a_m, e_m], \\ min(x,y), & \text{otherwise.} \end{cases}$$

is said to be the ordinal sum of $\langle a_m, e_m, T_m \rangle$, $m \in M$, and we shall write

$$T = (\langle a_m, e_m, T_m \rangle)_{m\in M}.$$

Lemma 2.10. [13] *Let $b \in (0,1)$ and T be a t-norm satisfying $T(b,x) = min(b,x)$ for any $x \in [0,1]$, then b is a non-trivial idempotent element of T if and only if there exist t-norms T_m and T_n such that $T = (\langle 0, b, T_m \rangle, \langle b, 1, T_n \rangle)$.*

Theorem 2.11. [13] *Assume $T : [0,1]^2 \to [0,1]$ is a binary function, then T is a continuous t-norm if and only if there is a unique countable index set M, a family of unique disjoint open subintervals $((a_m, e_m))_{m\in M}$ of $[0,1]$ and a family of unique continuous Archimedean t-norms $(T_m)_{m\in M}$ such that*

$$T = (\langle a_m, e_m, T_m \rangle)_{m\in M}.$$

Lemma 2.12. [13] *Assume that T is a continuous Archimedean t-norm. Then T is nilpotent if and only if T is not strict.*

For an arbitrary t-conorm, we can get corresponding properties and results by duality. We omit the results here and more details can be found in [13].

Definition 2.13. [45] *Let $0 \leq p < q \leq 1$. A uninorm is a binary function $U : [p,q]^2 \longrightarrow [p,q]$ which meets commutativity, associativity, and is increasing in both variables such that $U(x,e) = x$ for some element $e \in [p,q]$.*

Remark 2.14. *If $U(p,q) = p$ (resp. $U(p,q) = q$), the uninorm U is known as conjunctive (resp. disjunctive). The uninorm U, t-norm T and t-conorm S in this paper are binary functions on $[0,1]$ unless specifically emphasized.*

So far, many common types of uninorms have appeared in the literature, such as $\mathcal{U}_{min}, \mathcal{U}_{max}, \mathcal{U}_{rep}, \mathcal{U}_{cts}$ and so on. A more general type of uninorms in $\mathcal{U}_{locb}$ was put forward in [22] by Mas et al.

Definition 2.15. [22] *Assume U is a conjunctive uninorm. U is called locally internal on the boundary if it fulfills $U(1,y) \in \{1,y\}$ whenever $y \in [0,1]$. In the same way, if U is a disjunctive uninorm, it is said to be locally internal on the boundary if it fulfills $U(0,y) \in \{0,y\}$ whenever $y \in [0,1]$.*

In this paper, we only discuss the migrativity of a conjunctive uninorm over a continuous t-norm in detail. The results about the migrativity of disjunctive uninorms over continuous t-conorms can be obtained dually. Thus, we only recall some properties of conjunctive uninorms in $\mathcal{U}_{nlocb}$. For disjunctive uninorms in $\mathcal{U}_{nlocb}$, we will provide relevant notations for easier comprehension. For more details, please refer to [29].

As for arbitrary uninorm U, its boundary functions g_0, g_1: $[0,1] \to [0,1]$ which are determined by $g_0 = U(0,y)$, $g_1 = U(1,y)$ for any $y \in [0,1]$ can be defined.

For arbitrary conjunctive uninorm U, we denote $a = inf\{x \in [0,1] | U(x,1) = 1\}$. It has been shown in [29] that a must be a discontinuity of g_1, where $0 \leq a \leq e$ and $g_1(a) = 1$ or a. We list the result corresponding to $g_1(a) = 1$ here. As for $g_1(a) = a$, please refer to [29].

Theorem 2.16. [29] *Assume that U is a conjunctive uninorm possessing the neutral element $e \in (0,1)$ whose boundary function g_1 has a finite number of different discontinuities $a_1, \cdots, a_n$ with $0 \leq a_1 < \cdots < a_n = a \leq e$, then $U \in \mathcal{U}_{nlocb}$ if and only if*

$$g_1(y) = \begin{cases} l_i, & y \in (a_i, l_i], i \in \{1,2,\cdots,n\}, \\ y, & y \in \bigcup_{i=0}^{(n-1)} [l_i, a_{i+1}), \\ a_i \, or \, l_i, & y = a_i, i \in \{1,2,\cdots,n\}, \end{cases}$$

where $l_i \in (a_i, a_{i+1}]$, $i \in \{1,2,\cdots,n-1\}$, $l_0 = 0$, $l_n = 1$ and $l_i = lim_{y \to (a_i)^+} g_1(y)$.

From Theorem 2.16, for arbitrary conjunctive uninorm possessing the neutral element $e \in (0,1)$ whose boundary function g_1 has a finite number of different discontinuities $a_1, \cdots, a_n$ with $0 \leq a_1 < \cdots < a_n = a < e$, we have the following facts:

(1) $Ran(g_1) = \cup_{i=0}^{n-1} [l_i, a_{i+1}) \cup \{g_1(a_i) | i = 1,2,\cdots,n\} \cup \{1\}$.

(2) We denote

$$H_i = \begin{cases} [l_i, a_{i+1}), & g_1(a_{i+1}) = l_{i+1}, \\ [l_i, a_{i+1}], & g_1(a_{i+1}) = a_{i+1}, \end{cases}$$

where $i = 0, \cdots, n-1$. Clearly, $Ran(g_1) \backslash \{1\} = \cup_{i=0}^{n-1} H_i$, when $i,j \in \{0,\cdots,n-1\}$ with $i \neq j$, $H_i \cap H_j = \emptyset$, and $g_1(y) = y$ for any $y \in Ran(g_1)$.

(3) Denote

$$J_i = \begin{cases} (a_i, l_i), & g_1(a_i) = a_i, \\ [a_i, l_i), & g_1(a_i) = l_i, \end{cases}$$

where $i = 1, \cdots, n-1$. Clearly, $[0,a) \backslash Ran(g_1) = \cup_{i=1}^{n-1} J_i$, when $i,j \in \{0,\cdots,n-1\}$ with $i \neq j$, $J_i \cap J_j = \emptyset$, and $g_1(y) = l_i$ for any $y \in J_i$.

(4) We write $g_1^{-1}(x) = \{y \in [0,1] | g_1(y) = x\}$, then

$$g_1^{-1}(l_i) = \begin{cases} (a_i, l_i], & g_1(a_i) = a_i, \\ [a_i, l_i], & g_1(a_i) = l_i, \end{cases}$$

where $i = 1, \cdots, n-1$.

Theorem 2.17. [29] *A binary function U is a conjunctive uninorm $U \in \mathcal{U}_{nlocb}$ possessing the neutral element $e \in (0,1)$ whose boundary function g_1 has a finite number of different discontinuities $a_1, \cdots, a_n$ with $0 \leq a_1 < \cdots < a_n = a < e$, and $g_1(a) = 1$ if and only if $l_{n-1} < a_n$ and*

$$U(y,z) = \begin{cases} U_1(y,z), & (y,z) \in [a,1]^2, \\ F_1(y,z), & (y,z) \in [0,a]^2, \\ E_{1,i}(y,z), & (y,z) \in g_1^{-1}(l_i) \times [a,1], \\ E_{1,i}(z,y), & (y,z) \in [a,1] \times g_1^{-1}(l_i), \\ min(y,z), & otherwise, \end{cases}$$

where U_1 is a disjunctive uninorm possessing the neutral element e on $[a,1]$; $E_{1,i} : g_1^{-1}(l_i) \times [a,1] \to g_1^{-1}(l_i)$ is increasing, $l_i = lim_{y \to (a_i)^+} g_1(y)$, and for any $i \in \{1, 2, \cdots, n-1\}$ the following items are fulfilled :

- *$E_{1,i}(y,e) = y$, for any $y \in g_1^{-1}(l_i)$,*
- *$E_{1,i}(y,1) = l_i$, for any $y \in g_1^{-1}(l_i)$,*
- *$E_{1,i}(l_i, y) = l_i$, for any $y \in [a,1]$,*
- *$E_{1,i}(E_{1,i}(y,z), w) = E_{1,i}(y, U_1(z,w))$, for any $y \in g_1^{-1}(l_i)$ and $z, w \in [a,1]$.*

F_1 is a t-subnorm on $[0,a]$, and the following items are fulfilled:

- *$F_1(y,a) = y$ whenever $y \in [l_i, a_{i+1}]$ and $i \in \{0, 1, \cdots, n-1\}$; $E_{1,i}(y,a) = F_1(y,a)$ whenever $y \in (a_i, l_i)$ and $i \in \{1, \cdots, n-1\}$,*
- *$F_1(y,z) \in Ran(g_1) \backslash \{1\}$ for any $y, z \in Ran(g_1) \backslash \{1\}$,*
- *$a_j \geq F_1(y,z) = F_1(y, l_j) \in Ran(g_1) \backslash \{1\}$ for any $y \in Ran(g_1) \backslash \{1\}, z \in J_j$, $j \in \{1, \cdots, n-1\}$,*
- *If $F_1(y,z) \in Ran(g_1) \backslash \{1\}$ for any $y \in g_1^{-1}(l_i), z \in g_1^{-1}(l_j)$, $i, j \in \{1, \cdots, n-1\}$, then $F_1(y,z) = F_1(l_i, l_j)$. If there exist $y_0 \in g_1^{-1}(l_i), z_0 \in g_1^{-1}(l_j)$ satisfying $F_1(y_0, z_0) \notin Ran(g_1) \backslash \{1\}$, then there is $p \leq min(i,j)$ such that $F_1(y,z) \in g_1^{-1}(l_p)$ for any $y \in g_1^{-1}(l_i), z \in g_1^{-1}(l_j)$ and $E_{1,p}(F_1(y,z), w) = F_1(y, E_{1,j}(z,w)) = F_1(z, E_{1,i}(y,w))$ for any $w \in [a,1]$.*

For arbitrary disjunctive uninorm U, we can define $a' = sup\{x \in [0,1] | U(x,0) = 0\}$ through duality, then a' has to be a discontinuity of g_0 and $e \leq a' \leq 1$. Moreover, $g_0(a') = 0$ or $g_0(a') = a'$.

Theorem 2.18. [29] *Assume that U is a disjunctive uninorm possessing the neutral element $e \in (0,1)$ whose boundary function g_0 has a finite number of different discontinuities $a'_1, \cdots, a'_n$, where $1 \geq a'_1 > \cdots > a'_n = a' \geq e$, then $U \in \mathcal{U}_{nlocb}$ if and only if*

$$g_0(y) = \begin{cases} l'_i, & y \in [l'_i, a'_i), i \in \{1, 2, \cdots, n\}, \\ y, & y \in \bigcup_{i=0}^{(n-1)} (a'_{i+1}, l'_i], \\ a'_i \; or \; l'_i, & y = a'_i, i \in \{1, 2, \cdots, n\}, \end{cases}$$

where $l'_i \in (a'_{i+1}, a'_i)$, $i \in \{1, 2, \cdots, n-1\}$, $l'_0 = 1$, $l'_n = 0$ and $l'_i = lim_{y \to (a_i)^-} g_0(y)$.

For arbitrary disjunctive uninorm possessing the neutral element $e \in (0,1)$ whose boundary function g_0 has a finite number of different discontinuities $a'_1, \cdots, a'_n$, where $1 \geq a'_1 > \cdots > a'_n = a' \geq e$, we have the following facts from Theorem 2.18:

(1) $Ran(g_0) = \cup_{i=0}^{n-1} (a'_{i+1}, l'_i] \cup \{g_0(a'_i) | i = 1, 2, \cdots, n\} \cup \{0\}$.

(2) We denote

$$H'_i = \begin{cases} (a'_{i+1}, l'_i], & g_0(a'_{i+1}) = l'_{i+1}, \\ [a'_{i+1}, l'_i], & g_0(a'_{i+1}) = a'_{i+1}, \end{cases}$$

where $i = 0, \cdots, n-1$. Clearly, $Ran(g_0) \backslash \{0\} = \cup_{i=0}^{n-1} H'_i$, when $i, j \in \{0, \cdots, n-1\}$ with $i \neq j$, $H'_i \cap H'_j = \emptyset$, and $g_0(y) = y$ for all $y \in Ran(g_0)$.

(3) Denote

$$J'_i = \begin{cases} (l'_i, a'_i), & g_0(a'_i) = a'_i, \\ (l'_i, a'_i], & g_0(a'_i) = l'_i, \end{cases}$$

where $i = 1, \cdots, n-1$. Clearly, $(a', 1] \backslash Ran(g_0) = \cup_{i=1}^{n-1} J'_i$, when $i, j \in \{1, \cdots, n-1\}$ with $i \neq j$, $J'_i \cap J'_j = \emptyset$, and $g_0(y) = l'_i$ for all $y \in J'_i$.

(4) We write $g_0^{-1}(x) = \{y \in [0,1] | g_0(y) = x\}$, then

$$g_0^{-1}(l'_i) = \begin{cases} [l'_i, a'_i), & g_0(a'_i) = a'_i, \\ [l'_i, a'_i], & g_0(a'_i) = l'_i, \end{cases}$$

for $i = 1, \cdots, n-1$.

Note that, if we respectively change uninorm for t-conorm, and t-subnorm for t-norm, these consequences still apply to the case $a = e$ although $a<e$ is assumed in Theorem 2.17. In Section 3, for convenience, we only analyze case $a<e$, the consequences also apply to the case $a = e$. Similarly, in Section 4, we only analyze case $a'>e$, the consequences also apply to the case $a' = e$.

Now, we recall some results about the migrativity of uninorms over t-norms. The consequences for the migrativity of uninorms over t-conorms can be obtained by duality, we omit them here and more details can be found in [47] and [22].

Definition 2.19. [22]

(i) *Suppose that $\alpha \in [0,1]$, T is a fixed t-norm, a uninorm U is called α-migrative over T or (α, T)-migrative if*

$$U(T(\alpha, x), y) = U(x, T(\alpha, y)), \tag{1}$$

for all $x, y \in [0,1]$.

(ii) *Suppose that $\alpha \in [0,1]$, S is a fixed t-conorm, a uninorm U is called α-migrative over S or (α, S)-migrative if*

$$U(S(\alpha, x), y) = U(x, S(\alpha, y)), \tag{2}$$

for all $x, y \in [0,1]$.

Proposition 2.20. [22] *Assume that T is a t-norm and U is a uninorm possessing the neutral element $e \in (0,1)$. Then*

(i) *U is $(1, T)$-migrative.*

(ii) *U is $(0, T)$-migrative if and only if U is conjunctive.*

Proposition 2.21. [22] *Assume that $\alpha \in (0,1)$, U is a uninorm possessing the neutral element $e \in (0,1)$ and T is a t-norm. If U is (α, T)-migrative, then:*

(i) *$U(0,1) = 0$.*

(ii) *$U(\alpha, x) = U(1, T(\alpha, x))$ for all $x \in [0,1]$.*

From Proposition 2.20 and Proposition 2.21, we only study the α-migrativity of a conjunctive uninorm possessing the neutral element $e \in (0,1)$ over a continuous t-norm when $\alpha \in (0,1)$.

Theorem 2.22. [47] *Let T be a given t-norm, U a conjunctive uninorm possessing the neutral element $e \in (0,1)$, and $\alpha \in (0,1)$. Then U is α-migrative over T if and only if $T(\alpha, y) = U(T(\alpha, e), y)$ for all $y \in [0,1]$.*

3 Migrativity of a conjunctive uninorm $U \in \mathcal{U}_{nlocb}$ over a continuous t-norm

Note that Qin et al. [29] discussed the structure of a conjunctive uninorm $U \in \mathcal{U}_{nlocb}$ possessing the neutral element $e \in (0,1)$ whose boundary function g_1 has a finite number of different discontinuities $a_1, \cdots, a_n$ with $0 \leq a_1 < \cdots < a_n = a < e$ according to the value of $g_1(a)$, i.e. $g_1(a) = 1$ and $g_1(a) = a$.

While in this section, we consider the α-migrativity of a conjunctive uninorm $U \in \mathcal{U}_{nlocb}$ over a continuous t-norm in the case of $g_1(a) = 1$ first, the results of the case $g_1(a) = a$ are the same as the case $g_1(a) = 1$, because in the proof, we only use the same structures of the two classes of uninorms. Therefore, this indicates that the result of α-migrativity of a conjunctive uninorm $U \in \mathcal{U}_{nlocb}$ whose boundary function g_1 has a finite number of different discontinuities over a continuous t-norm is independent of the value of $g_1(a)$.

For convenience, we denote all uninorms satisfying that their boundary function g_1 has a finite number of different discontinuities $a_1, \cdots, a_n$ with $0 \leq a_1 < \cdots < a_n = a < e$ and $g_1(a) = 1$ by $\mathcal{U}_{cdis}$ in the following.

3.1 Case for $g_1(a) = 1$

Lemma 3.1. *Assume U is a conjunctive uninorm $U \in \mathcal{U}_{nlocb} \cup \mathcal{U}_{cdis}$ possessing the neutral element $e \in (0,1)$, T is a continuous t-norm and $\alpha \in (0,1)$. If U is (α, T)-migrative, then $\alpha = U(1, T(\alpha, e))$ and $\alpha \in Ran(g_1) \backslash \{0,1\}$.*

Proof. Let $x = 1$ in the equation (1), from Theorem 2.22 we have

$$U(\alpha, y) = U(T(\alpha,1), y) = U(1, T(\alpha, y)) = U(1, U(T(\alpha, e), y)) = U(U(1,y), T(\alpha, e)) = T(\alpha, U(1,y)). \tag{3}$$

Let $y \in (a, 1]$ in (3), then we have $U(1, y) = 1$ and

$$U(1, T(\alpha, e)) = U(U(1,y), T(\alpha, e)) = T(\alpha, U(1,y)) = T(\alpha, 1) = \alpha.$$

Thus, $\alpha \in Ran(g_1)$, since $\alpha \in (0,1)$, then $\alpha \in Ran(g_1) \backslash \{0,1\}$. □

In fact, Lemma 3.1 applies to arbitrary t-norm.

According to Lemma 3.1, we only discuss the α-migrativity of a conjunctive uninorm $U \in \mathcal{U}_{nlocb} \cup \mathcal{U}_{cdis}$ over a continuous t-norm in the case of $\alpha \in Ran(g_1) \backslash \{0,1\}$. Furthermore, we study it depend on α is an idempotent element of the continuous t-norm T or not.

Remark 3.2. *It should be noticed that $Ran(g_1) \backslash \{0,1\} = \cup_{i=0}^{n-1} H_i \backslash \{0\} = (H_0 \backslash \{0\}) \cup (\cup_{i=1}^{n-1} H_i)$.*

3.1.1 Case for $T(\alpha, \alpha) = \alpha$

Lemma 3.3. *Assume U is a conjunctive uninorm $U \in \mathcal{U}_{nlocb} \cup \mathcal{U}_{cdis}$ possessing the neutral element $e \in (0,1)$, T is a continuous t-norm and $T(\alpha, \alpha) = \alpha$, $\alpha \in Ran(g_1) \backslash \{0,1\}$.*

(i) *If U is (α, T)-migrative, then $\alpha \in H_0 \backslash \{0\}$.*

(ii) *U is (α, T)-migrative if and only if $U(x, \alpha) = T(x, \alpha) = min(x, \alpha)$ for all $x \in [0,1]$.*

Proof. Since T is continuous and $T(\alpha, \alpha) = \alpha$, then $T(\alpha, y) = min(\alpha, y)$ for all $y \in [0,1]$ by Lemma 2.5.

(i) Suppose that $\alpha \in H_i$, $i \in \{1, \cdots, n-1\}$. Let $y \in J_i$, $i \in \{1, \cdots, n-1\}$ in (3), then we obtain

$$U(\alpha, y) = T(\alpha, U(1,y)) = T(\alpha, l_i) = min(\alpha, l_i) = l_i > y = U(e, y),$$

this contradicts the monotonicity of U. Thus, $\alpha \in Ran(g_1) \backslash \{0,1\} \backslash (\cup_{i=1}^{n-1} H_i) = H_0 \backslash \{0\}$.

(ii) By (i), we obtain $T(\alpha, e) = min(\alpha, e) = \alpha$. Then the result can be deduced from Theorem 2.22 immediately.

□

Theorem 3.4. *Assume U is a conjunctive uninorm $U \in \mathcal{U}_{nlocb} \cup \mathcal{U}_{cdis}$ possessing the neutral element $e \in (0,1)$, T is a continuous t-norm and $T(\alpha, \alpha) = \alpha$, $\alpha \in Ran(g_1) \backslash \{0,1\}$. Then U is (α, T)-migrative if and only if the following items are fulfilled:*

(i) $\alpha \in H_0 \backslash \{0\}$.

(ii) $T = (\langle 0, \alpha, T_1\rangle, \langle \alpha, 1, T_2\rangle)$, *where* T_1, T_2 *are continuous t-norms.*

(iii) $U(y,z) = min(y,z)$ *for all* $(y,z) \in [\alpha, a] \times [0, \alpha] \cup [0, \alpha] \times [\alpha, a]$.

Proof. Necessity.

(i) It can be deduced from Lemma 3.3.

(ii) Since T is continuous and $T(\alpha, \alpha) = \alpha$, then we know from Lemma 2.10 that $T = (\langle 0, \alpha, T_1\rangle, \langle \alpha, 1, T_2\rangle)$, where T_1, T_2 are continuous t-norms.

(iii) Since $T(\alpha, \alpha) = \alpha$ and T is continuous, then $T(\alpha, x) = min(\alpha, x)$ for all $x \in [0,1]$ by Lemma 2.5. It holds from Lemma 3.3 that, $U(\alpha, x) = min(\alpha, x)$ for all $x \in [0, 1]$. Take $x \in (0, \alpha), y \in (\alpha, a]$ in (1), then $U(x, y) = U(min(\alpha, x), y) = U(T(\alpha, x), y) = U(x, T(\alpha, y)) = U(x, \alpha) = min(x, \alpha) = x = min(x, y)$. Thus, the item holds from the commutativity of U.

Sufficiency. According to the conditions, we have $U(\alpha, z) = T(\alpha, z) = min(\alpha, z)$ for all $z \in [0, 1]$. Then it is obvious from Lemma 3.3. □

3.1.2 Case for $T(\alpha, \alpha) < \alpha$

Now, we discuss the case $T(\alpha, \alpha) < \alpha$. Note that $Ran(g_1)\backslash\{0, 1\} = \cup_{i=0}^{n-1} H_i\backslash\{0\} = (H_0\backslash\{0\}) \cup (\cup_{i=1}^{n-1} H_i)$, we discuss the migrativity in two cases: $\alpha \in H_0\backslash\{0\}$ and $\alpha \in H_i$, $i \in \{1, \cdots, n-1\}$.

Remark 3.5. *Assume* T *is a continuous t-norm and* $T(\alpha, \alpha) < \alpha$ *for some* $\alpha \in [0, 1]$, *then we know that* $T = (\cdots, \langle c, d, T_*\rangle, \cdots)$, *where* T_* *is a continuous Archimedean t-norm and* $0 \leq c < \alpha < d \leq 1$ *by Theorem 2.11. A direct outcome of Lemma 2.8 and Lemma 2.12 is that* T_* *meets the conditional cancellation law. The symbols* c, d *and* T_* *are defined as the same unless specifically emphasized in the following.*

Lemma 3.6. *Assume* U *is a conjunctive uninorm* $U \in \mathcal{U}_{nlocb} \cup \mathcal{U}_{cdis}$ *possessing the neutral element* $e \in (0, 1)$, T *is a continuous t-norm and* $T(\alpha, \alpha) < \alpha$, $\alpha \in H_0\backslash\{0\}$.

(i) *If* U *is* (α, T)*-migrative, then* $\alpha < d \leq e$.

(ii) U *is* (α, T)*-migrative if and only if* $T(x, \alpha) = U(x, \alpha)$ *for any* $x \in [0, 1]$.

Proof. Since $T(\alpha, \alpha) < \alpha$, then we know that $T = (\cdots, \langle c, d, T_*\rangle, \cdots)$, where $0 \leq c < \alpha < d \leq 1$ and T_* is a continuous Archimedean t-norm by Theorem 2.11.

(i) Suppose that $d > e$. From Lemma 3.1, we have

$$U(1, T(\alpha, e)) = \alpha,$$

since $T(\alpha, e) \leq min(\alpha, e) = \alpha \in H_0\backslash\{0\}$, we have $T(\alpha, e) = \alpha = T(\alpha, 1)$ by the definition of g_1. This is a contradiction, because T_* is continuous Archimedean. Thus $\alpha < d \leq e$.

(ii) By (i), we have $d \leq e$, then $T(\alpha, e) = min(\alpha, e) = \alpha$ as a result of the ordinal sums structure of T. Therefore, the result holds from Theorem 2.22.

□

Theorem 3.7. *Assume that* U *is a conjunctive uninorm* $U \in \mathcal{U}_{nlocb} \cup \mathcal{U}_{cdis}$ *possessing the neutral element* $e \in (0, 1)$, T *is a continuous t-norm and* $T(\alpha, \alpha) < \alpha$, $\alpha \in H_0\backslash\{0\}$. *Then* U *is* (α, T)*-migrative if and only if one of the following items is fulfilled:*

(i) $T = (\cdots, \langle c, d, T_*\rangle, \cdots)$, *where* $0 < c < \alpha < d \leq a_1$, T_* *is a continuous Archimedean t-norm, and*

$$U(\alpha, y) = \begin{cases} y, & y \in [0, c], \\ \alpha, & y \in [d, 1], \\ c + (d-c)T_*(\frac{\alpha - c}{d-c}, \frac{y-c}{d-c}), & y \in (c, d). \end{cases} \quad (4)$$

Moreover, $U(y, z) = min(y, z)$ *for any* $(y, z) \in [0, c] \times [d, a] \cup [d, a] \times [0, c]$

(ii) $T = (\langle 0, d, T_* \rangle, \cdots)$, *where* $\alpha < d \leq a_1$, T_* *is a continuous Archimedean t-norm,* $T(\alpha, l_1) > 0$ *and*

$$U(\alpha, y) = \begin{cases} dT_*(\frac{\alpha}{d}, \frac{y}{d}), & y \in [0, d), \\ \alpha, & y \in [d, 1]. \end{cases}$$

(iii) $T = (\langle 0, d, T_* \rangle, \cdots)$, *where* $l_1 < d \leq a$, T_* *is a continuous Archimedean t-norm,* $T(\alpha, l_1) = 0$, *and*

$$U(\alpha, y) = \begin{cases} 0, & y \in [0, l_1], \\ dT_*(\frac{\alpha}{d}, \frac{y}{d}), & y \in (l_1, d), \\ \alpha, & y \in [d, 1]. \end{cases} \tag{5}$$

In this case, $U(y, z) = T(y, z) = 0$ *whenever* $(y, z) \in [0, \alpha]^2 \cup [0, \alpha] \times [\alpha, l_1] \cup [\alpha, l_1] \times [0, \alpha]$.

Proof. Since $T(\alpha, \alpha) < \alpha$, then it holds from Theorem 2.11 that $T = (\cdots, \langle c, d, T_* \rangle, \cdots)$, where $0 \leq c < \alpha < d \leq 1$ and T_* is a continuous Archimedean t-norm.

Necessity. If U is (α, T)-migrative, we have $U(x, \alpha) = T(x, \alpha)$ for all $x \in [0, 1]$ by Lemma 3.6.

(i) Suppose that $c > 0$, then we prove $\alpha < d \leq a_1$. Assume that $d > a_1$. Let $y \in J_1$ in (3), then $U(1, y) = l_1$ and

$$U(\alpha, y) = T(\alpha, U(1, y)) = T(\alpha, l_1),$$

furthermore, we have

$$T(\alpha, y) = U(\alpha, y) = T(\alpha, l_1) \geq T(\alpha, c) = c > 0,$$

for any $y \in J_1$ by the increasing property of T. As a result, $T(\alpha, y) = T(\alpha, l_1) > 0$ for all $y \in (a_1, d]$ (here $d < l_1$) or $y \in (a_1, l_1)$ (here $d \geq l_1$), which contradicts with the Archimedean property of T_*. Thus, $\alpha < d \leq a_1$. According to the ordinal sums structure of T, we can get an expression for $U(\alpha, y)$ like (4).

Let $x \in (0, c]$, $y \in [d, a]$ in (1), then $T(\alpha, x) = min(\alpha, x) = x$, $T(\alpha, y) = min(\alpha, y) = \alpha$ and $U(x, y) = U(T(\alpha, x), y) = U(x, T(\alpha, y)) = U(x, \alpha) = T(x, \alpha) = x$, Thus, we have $U(x, y) = min(x, y)$ for all $(x, y) \in [0, c] \times [d, a] \cup [d, a] \times [0, c]$ by the commutativity of U.

(ii) If $c = 0$, $T(\alpha, l_1) > 0$, we can prove it in an analogous manner to (i).

(iii) If $c = 0$, $T(\alpha, l_1) = 0$, then $d > l_1$, otherwise, we have $T(\alpha, d) = 0$ which contradicts with $T(\alpha, d) = \alpha$. Since $U(\alpha, y) = min(\alpha, y) = \alpha$ for any $y \in (a, 1]$ and $T(\alpha, l_1) = 0$, then $T(\alpha, y) = \alpha$ for any $y \in (a, 1]$ and $U(\alpha, l_1) = 0$. And we can obtain $d \leq a$, otherwise, we have $T(\alpha, y) = \alpha$ for all $y \in (a, d)$ which contradicts the fact that T_* is a continuous Archimedean t-norm. According to the ordinal sums structure of T, we can get an expression for $U(\alpha, y)$ as (5). And we also have $U(y, z) = T(y, z) = 0$ for any $(y, z) \in [0, \alpha]^2 \cup [0, \alpha] \times [\alpha, l_1] \cup [\alpha, l_1] \times [0, \alpha]$ by the monotonicity and commutativity of T and U.

Sufficiency. If any one of the statements is fulfilled, it is easy to check that $U(\alpha, z) = T(\alpha, z)$ for all $z \in [0, 1]$, then the result holds from Lemma 3.6.

□

Lemma 3.8. *Assume* U *is a conjunctive uninorm* $U \in \mathcal{U}_{nlocb} \cup \mathcal{U}_{cdis}$ *possessing the neutral element* $e \in (0, 1)$, T *is a continuous t-norm and* $T(\alpha, \alpha) < \alpha$, $\alpha \in H_i \backslash \{l_i\}$, $i \in \{1, \cdots, n-1\}$.

(i) *If* U *is* (α, T)*-migrative, then* $\alpha < d \leq e$.

(ii) U *is* (α, T)*-migrative if and only if* $T(x, \alpha) = U(x, \alpha)$ *for any* $x \in [0, 1]$.

Proof. We can prove it in an analogous manner to the proof of Lemma 3.6. □

Theorem 3.9. *Assume* U *is a conjunctive uninorm* $U \in \mathcal{U}_{nlocb} \cup \mathcal{U}_{cdis}$ *possessing the neutral element* $e \in (0, 1)$, T *is a continuous t-norm and* $T(\alpha, \alpha) < \alpha$, $\alpha \in H_i \backslash \{l_i\}$, $i \in \{1, \cdots, n-1\}$. *Then* U *is* (α, T)*-migrative if and only if the following statements are fulfilled:*

(i) $T = (\langle 0, d, T_* \rangle, \cdots)$, *where* $\alpha < d \leq a$, T_* *is a continuous Archimedean t-norm and* $T(\alpha, l_i) = 0$.

(ii)

$$U(\alpha,y)=\begin{cases}0, & y\in[0,l_i],\\ dT_*(\frac{\alpha}{d},\frac{y}{d}), & y\in(l_i,d),\\ \alpha, & y\in[d,1].\end{cases}$$

Additionally, $U(y,z)=T(y,z)=0$ *whenever* $(y,z)\in[0,l_i]^2\cup[0,l_i]\times[l_i,\alpha]\cup[l_i,\alpha]\times[0,l_i]$.

Proof. If U is α-migrative over T, then from Lemma 3.8 we get $T(z,\alpha)=U(z,\alpha)$ for all $z\in[0,1]$.

Since $T(\alpha,\alpha)<\alpha$, then it holds from Theorem 2.11 that $T=(\cdots,\langle c,d,T_*\rangle,\cdots)$, where $0\le c<\alpha<d\le 1$ and T_* is a continuous Archimedean t-norm. Firstly, we prove that $c=0$. Suppose that $c>0$. Then we prove $l_i\le c<\alpha$. If $c<l_i$, Take $y\in(a_i,l_i)$ in (3), we have

$$U(\alpha,y)=T(\alpha,U(1,y))=T(\alpha,l_i)\ge T(\alpha,c)=c>0.$$

Therefore, we obtain $T(\alpha,y)=T(\alpha,l_i)>0$ for all $y\in(a_i,l_i)$ by Lemma 3.8 which contradicts with the continuous Archimedean property of T_*. Thus, $l_i\le c<\alpha$. It is clearly that $T(\alpha,l_i)=min(\alpha,l_i)=l_i$, and then $y=min(\alpha,y)=T(\alpha,y)=T(\alpha,l_i)=l_i$ for all $y\in(a_i,l_i)$, it is a contradiction. Hence, $c=0$.

If $T(\alpha,l_i)>0$, then we obtain $T(\alpha,y)=U(\alpha,y)=T(\alpha,l_i)>0$ for all $y\in(a_i,l_i)$, which is contradictory. Thus, $T(\alpha,l_i)=0$. Further, $U(\alpha,l_i)=T(\alpha,l_i)=0$, as a result of the commutativity and monotonicity of T and U, $U(x,y)=T(x,y)=0$ for any $(x,y)\in[0,l_i]^2\cup[l_i,\alpha]\times[0,l_i]\cup[0,l_i]\times[l_i,\alpha]$.

Now, we prove that $\alpha<d\le a$. Assume that $d>a$. Take $y\in(a,d)$ in (3), we have

$$T(\alpha,y)=U(\alpha,y)=T(\alpha,U(1,y))=T(\alpha,1)=\alpha>0,$$

Which is a contradiction because T_* is a continuous Archimedean t-norm. Hence, $\alpha<d\le a$.

Conversely, it is apparent that $T(z,\alpha)=U(z,\alpha)$ for any $z\in[0,1]$, then the result holds from Lemma 3.8. □

Lemma 3.10. *Assume* U *is a conjunctive uninorm* $U\in\mathcal{U}_{nlocb}\cup\mathcal{U}_{cdis}$ *possessing the neutral element* $e\in(0,1)$*,* T *is a continuous t-norm and* $\alpha=l_i$*,* $T(l_i,l_i)<l_i$*,* $i\in\{1,\cdots,n-1\}$*. If* $d\le e$*, then* U *is* (α,T)*-migrative if and only if* $T(x,l_i)=U(x,l_i)$ *for all* $x\in[0,1]$*.*

Proof. It is in the same way as the proof of Lemma 3.6. □

Theorem 3.11. *Assume* U *is a conjunctive uninorm* $U\in\mathcal{U}_{nlocb}\cup\mathcal{U}_{cdis}$ *possessing the neutral element* $e\in(0,1)$*,* T *is a continuous t-norm and* $\alpha=l_i$*,* $T(l_i,l_i)<l_i$*,* $i\in\{1,\cdots,n-1\}$*. If* $d\le e$*, then* U *is* (α,T)*-migrative if and only if the following items are satisfied:*

(i) $T=(\langle 0,d,T_*\rangle,\cdots)$*, where* $l_i<d\le a$*,* T_* *is a continuous Archimedean t-norm and* $T(l_i,l_i)=0$*.*

(ii)

$$U(\alpha,y)=\begin{cases}0, & y\in[0,l_i],\\ dT_*(\frac{\alpha}{d},\frac{y}{d}), & y\in(l_i,d),\\ l_i, & y\in[d,1].\end{cases}$$

Additionally, $U(y,z)=T(y,z)=0$ *whenever* $(y,z)\in[0,l_i]^2$*.*

Proof. It is in the same way as the proof of Theorem 3.9. □

Theorem 3.12. *Assume* U *is a conjunctive uninorm* $U\in\mathcal{U}_{nlocb}\cup\mathcal{U}_{cdis}$ *possessing the neutral element* $e\in(0,1)$*,* T *is a continuous t-norm and* $\alpha=l_i$*,* $T(l_i,l_i)<l_i$*,* $i\in\{1,\cdots,n-1\}$*. If* $d>e$*, then* U *is* (α,T)*-migrative if and only if one of the following items is satisfied:*

(i) *If* $a_i>0$*,* $U(x,y)\in Ran(g_1)\backslash\{1\}$ *for all* $x,y\in g_1^{-1}(l_i)$*, then* $T=(\langle 0,d,T_*\rangle,\cdots)$*,* $U(l_i,l_i)=T(l_i,l_i)=0$ *and*

$$T(l_i,y)=\begin{cases}0, & y\in[0,l_i],\\ U(T(l_i,e),y), & y\in(l_i,d),\\ l_i, & y\in[d,1].\end{cases}$$

and $U(T(l_i,y),z)=U(y,l_i)$ *for* $y\in(l_i,d)$*,* $z\in[d,1]$*. Specially,* $U(T(l_i,y),z)=l_i$ *for all* $y\in[a,d)$*,* $z\in[d,1]$*.*

(ii) *If* $a_i>0$, $U(x_0,y_0)\notin Ran(g_1)\backslash\{1\}$ *for some* $x_0,y_0\in g_1^{-1}(l_i)$, *then* $T=(\cdots,\langle c,d,T_*\rangle,\cdots)$, *where* $c<a_i$ *and*

$$T(l_i,y)=\begin{cases}y, & y\in[0,c],\\ U(T(l_i,e),y), & y\in(c,d),\\ l_i, & y\in[d,1].\end{cases}$$

and $U(T(l_i,y),z)=U(y,l_i)$ *for* $y\in(c,d)$, $z\in[d,1]$, $U(y,z)=U(y,T(l_i,z))$ *for* $y\in[0,c]$, $z\in(c,d)$, $U(y,z)=U(y,l_i)$ *for* $y\in[0,c]$, $z\in[d,1]$. *Specially,* $U(T(l_i,y),z)=l_i$ *for all* $y\in[a,d)$, $z\in[d,1]$.

(iii) *If* $i=1$ *and* $a_1=0$, *then* $T=(\langle 0,d,T_*\rangle,\cdots)$, $U(l_1,l_1)=T(l_1,l_1)=0$ *and*

$$T(l_1,y)=\begin{cases}0, & y\in[0,l_1],\\ U(T(l_1,e),y), & y\in(l_1,d),\\ l_1, & y\in[d,1].\end{cases}$$

and $U(T(l_1,y),z)=U(y,l_1)$ *for* $y\in(l_1,d)$, $z\in[d,1]$. *Specially,* $U(T(l_1,y),z)=l_1$ *for all* $y\in[a,d)$, $z\in[d,1]$.

Proof. For necessity. Take $y\in(a,d)$ in (3), we have

$$U(1,T(l_i,y))=T(l_i,U(1,y))=T(l_i,1)=l_i,$$

then $T(l_i,y)\in g_1^{-1}(l_i)$. Since T_* is continuous Archimedean, $T(l_i,y)<l_i$, hence, $T(l_i,y)\in J_i$ for any $y\in(a,d)$. Specially, $T(l_i,e)\in J_i$.

Take $y\in J_i$ in (3), we have

$$U(l_i,y)=T(l_i,U(1,y))=T(l_i,l_i),$$

Since $l_i\in Ran(g_1)\backslash\{1\}$, $y\in L_i$, then we get $U(l_i,y)=U(l_i,l_i)\le a_i$ by Theorem 2.17. Hence, $T(l_i,l_i)=U(l_i,l_i)\le a_i$, $i\in\{1,\cdots,n-1\}$.

Next, we prove in two cases: $a_i>0$ and $a_i=0$. Note that $a_i>0$ for $i\in\{2,\cdots,n-1\}$ and $a_1>0$ or $a_1=0$.

Case1 : $a_i>0, i\in\{1,2,\cdots,n-1\}$.

Firstly, we prove that $c<a_i$. Assume that $a_i<c<l_i$. From the ordinal sums structure of T, it is easy to obtain $T(l_i,c)=c>a_i\ge T(l_i,l_i)$ which contradicts the monotonicity of T. Suppose that $c=a_i$. From the discussion above, we have $T(l_i,l_i)\le a_i$. If $T(l_i,l_i)<a_i$, obviously, it leads to the monotonicity of T, which is a contradiction. If $T(l_i,l_i)=a_i$, we have $0<a_i=c=T(l_i,c)=T(l_i,l_i)$, then $T(l_i,y)=a_i$ for all $y\in[c,l_i]$ which contradicts the continuous Archimedean property of T_*. Thus, $c<a_i$.

Case1.1:

If $U(x,y)\in Ran(g_1)\backslash\{1\}$ for all $x,y\in g_1^{-1}(l_i)$, since $T(l_i,e)\in J_i\subset g_1^{-1}(l_i)$, then

$$T(l_i,x)=U(T(l_i,x),e)=U(x,T(l_i,e))=U(l_i,l_i)=T(l_i,l_i), \tag{6}$$

for all $x\in g_1^{-1}(l_i)$ by Theorem 2.17. If $0<c<a_i$, then $T(l_i,l_i)\ge T(l_i,c)=c>0$ and we can obtain from (6) that it contradicts the Archimedean property of T_*. If $c=0$ and $T(l_i,l_i)>0$, we can similarly obtain that (6) leads to a contradiction. Therefore, $c=0$ and $T(l_i,l_i)=0$, further, we can get $U(l_i,l_i)=0$ and $U(y,z)=T(y,z)=0$ for all $(y,z)\in[0,l_i]^2$ by the monotonicity of T and U. From the ordinal sums structure of T, we have $T(l_i,y)=0$ for any $y\in[0,l_i]$ and $T(l_i,y)=l_i$ for any $y\in[d,1]$. For any $y\in(l_i,d)$, we obtain $T(l_i,y)=U(T(l_i,e),y)$ by Theorem 2.22.

Let $x\in(l_i,d)$, $y\in[d,1]$ in (1), we get

$$U(T(l_i,x),y)=U(x,T(l_i,y))=U(x,l_i).$$

Specially, $U(T(l_i,x),y)=l_i$ for all $x\in[a,d)$, $y\in[d,1]$.

Case1.2:

If $U(x_0,y_0)\notin Ran(g_1)\backslash\{1\}$ for some $x_0,y_0\in g_1^{-1}(l_i)$, analogously to the proof of Case 1.1, we obtain that $T(l_i,y)=y$ for any $y\in[0,c]$, $T(l_i,y)=l_i$ for all $y\in[d,1]$, $T(l_i,y)=U(T(l_i,e),y)$ for any $y\in(c,d)$ and $U(T(l_i,x),y)=U(x,l_i)$ for any $(x,y)\in(c,d)\times[d,1]$.

Moreover, let $x\in[0,c]$, $y\in(c,d)$ in (1), we get

$$U(x,y)=U(x,T(l_i,y)),$$

and take $x\in[0,c]$, $y\in[d,1]$ in (1), we obtain

$$U(x,y)=U(x,l_i).$$

Specially, $U(T(l_i,x),y) = l_i$ for all $x \in [a,d)$, $y \in [d,1]$.

Case2 : $i = 1, a_1 = 0$

If $i = 1$ and $a_1 = 0$, we can obtain $T(l_1,l_1) = 0$ from the fact that $0 \le T(l_1,l_1) \le a_1 = 0$, then $U(l_1,l_1) = T(l_1,l_1) = 0$ and $U(y,z) = T(y,z) = 0$ for all $(y,z) \in [0,l_1]^2$ by the monotonicity of T and U. Thus, $U(y,z) = 0 \in Ran(g_1)\backslash\{1\}$ for all $y,z \in g_1^{-1}(l_1)$. In addition, we have $c = T(l_1,c) = 0$. Analogously to the proof of Case 1.1, we get

$$T(l_1,y) = \begin{cases} 0, & y \in [0,l_1], \\ U(T(l_1,e),y), & y \in (l_1,d), \\ l_1, & y \in [d,1]. \end{cases}$$

and $U(T(l_1,y),z) = U(y,l_1)$ for $y \in (l_1,d)$, $z \in [d,1]$. Specially, $U(T(l_1,y),z) = l_1$ for all $y \in [a,d)$, $z \in [d,1]$.

For sufficiency, it is easy to check. □

Remark that, when $\alpha = l_i$, $T(l_i,l_i)<l_i$, if $n = 2$, we can get the following theorem because we get $U(y,z) = U(y,l_1) \le a_1$ for any $y \in [0,a)$, $z \in J_1$ by Theorems 3.15 and 3.17 in [29]:

Theorem 3.13. *Assume U is a conjunctive uninorm $U \in \mathcal{U}_{nlocb}$ possessing the neutral element $e \in (0,1)$ whose boundary function g_1 has only two discontinuities a_1,a_2 with $0 \le a_1<a_2 = a<e$, and $g_1(a) = 1$, T is a continuous t-norm and $\alpha = l_1$, $T(l_1,l_1)<l_1$. If $d>e$, then U is (α,T)-migrative if and only if $T = (\langle 0,d,T_*\rangle,\cdots)$, $U(l_1,l_1) = T(l_1,l_1) = 0$ and*

$$T(l_1,y) = \begin{cases} 0, & y \in [0,l_1], \\ U(T(l_1,e),y), & y \in (l_1,d), \\ l_1, & y \in [d,1]. \end{cases}$$

and $U(T(l_1,y),z) = U(y,l_1)$ for $y \in (l_1,d)$, $z \in [d,1]$. Specially, $U(T(l_1,y),z) = l_1$ for any $y \in [a,d)$, $z \in [d,1]$.

Proof. For necessity. It is like proving Theorem 3.12, we also obtain $T(l_1,y) \in J_1$ for any $y \in (a,d)$.

Take $y \in J_1$ in (3), we have

$$a_1 \ge U(l_1,l_1) = U(l_1,y) = T(l_1,U(1,y)) = T(l_1,l_1)$$

by Theorem 3.15 in [29].

Let $x \in J_1, y \in (a,d)$ in (1), we obtain $T(l_1,x) = min(T(l_1,x),y) = U(T(l_1,x),y) = U(x,T(l_1,y)) = U(T(l_1,y),l_1) = U(l_1,l_1) = T(l_1,l_1)$ by Theorem 3.15 in [29].

If $a_1>0$, the proof is similar to the Case 1.1 of Theorem 3.12. If $a_1 = 0$, the proof is similar to the Case 2 of Theorem 3.12.

For sufficiency, it is easy to check.

□

Remark 3.14. *From Lemmas 3.3, 3.6 and 3.8, it can be seen that the migrativity of conjunctive uninorms $U \in \mathcal{U}_{nlocb} \cup \mathcal{U}_{cdis}$ over continuous t-norms is due to whether the α-sections of T and U are equal, i.e. $T(\alpha,z) = U(\alpha,z)$ whenever $z \in [0,1]$, except when $\alpha = l_i$, $i \in \{1,\cdots,n-1\}$, $T(l_i,l_i)<l_i$.*

3.2 Case for $g_1(a) = a$

The results in this section are the same as those in Section 3.1, but it should be noted that when $\alpha = l_i$, $T(l_i,l_i)<l_i$, $i \in \{1,\cdots,n-1\}$ and $d>e$, the corresponding results are slightly different from Theorem 3.12 and Theorem 3.13. Thus, we list the results for this case and omit the other relevant results.

For convenience, we denote all uninorms satisfying that their boundary function g_1 has a finite number of different discontinuities $a_1,\cdots,a_n$ with $0 \le a_1<\cdots<a_n = a<e$ and $g_1(a) = a$ by $\mathcal{U}_{cdia}$ in the following.

Theorem 3.15. *Assume U is a conjunctive uninorm $U \in \mathcal{U}_{nlocb} \cup \mathcal{U}_{cdia}$ possessing the neutral element $e \in (0,1)$, T is a continuous t-norm and $\alpha = l_i$, $T(l_i,l_i)<l_i$, $i \in \{1,\cdots,n-1\}$. If $d>e$, then U is (α,T)-migrative if and only if one of the following items is satisfied:*

(i) *If $a_i>0$, $U(x,y) \in Ran(g_1)\backslash\{1\}$ for all $x,y \in g_1^{-1}(l_i)$, then $T = (\langle 0,d,T_*\rangle,\cdots)$, $U(l_i,l_i) = T(l_i,l_i) = 0$ and*

$$T(l_i,y) = \begin{cases} 0, & y \in [0,l_i], \\ U(T(l_i,e),y), & y \in (l_i,d), \\ l_i, & y \in [d,1]. \end{cases}$$

and $U(T(l_i,y),z) = U(y,l_i)$ for $y \in (l_i,d)$, $z \in [d,1]$. Specially, $U(T(l_i,y),z) = l_i$ for all $y \in (a,d)$, $z \in [d,1]$.

(ii) *If* $a_i>0$, $U(x_0, y_0) \notin Ran(g_1)\backslash\{1\}$ *for some* $x_0, y_0 \in g_1^{-1}(l_i)$, *then* $T = (\cdots, \langle c, d, T_* \rangle, \cdots)$, *where* $c<a_i$ *and*

$$T(l_i, y) = \begin{cases} y, & y \in [0, c], \\ U(T(l_i, e), y), & y \in (c, d), \\ l_i, & y \in [d, 1]. \end{cases}$$

and $U(T(l_i, y), z) = U(y, l_i)$ *for* $y \in (c, d)$, $z \in [d, 1]$, $U(y, z) = U(y, T(l_i, z))$ *for* $y \in [0, c]$, $z \in (c, d)$, $U(y, z) = U(y, l_i)$ *for* $y \in [0, c]$, $z \in [d, 1]$. *Specially,* $U(T(l_i, y), z) = l_i$ *for all* $y \in (a, d)$, $z \in [d, 1]$.

(iii) *If* $i = 1$ *and* $a_1 = 0$, *then* $T = (\langle 0, d, T_* \rangle, \cdots)$, $U(l_1, l_1) = T(l_1, l_1) = 0$ *and*

$$T(l_1, y) = \begin{cases} 0, & y \in [0, l_1], \\ U(T(l_1, e), y), & y \in (l_1, d), \\ l_1, & y \in [d, 1]. \end{cases}$$

and $U(T(l_1, y), z) = U(y, l_1)$ *for* $y \in (l_1, d)$, $z \in [d, 1]$. *Specially,* $U(T(l_1, y), z) = l_1$ *for all* $y \in (a, d)$, $z \in [d, 1]$.

Remark that, when $\alpha = l_i$, $T(l_i, l_i)<l_i$, if $n = 2$, we can get the following theorem because we get $U(y, z) = U(y, l_1) \leq a_1$ for any $y \in [0, a]$, $z \in J_1$ by Theorems 3.18 and 3.19 in [29]:

Theorem 3.16. *Assume* U *is a conjunctive uninorm* $U \in \mathcal{U}_{nlocb}$ *possessing the neutral element* $e \in (0, 1)$ *whose boundary function* g_1 *has only two discontinuities* a_1, a_2 *with* $0 \leq a_1<a_2 = a<e$, *and* $g_1(a) = a$, T *is a continuous t-norm and* $\alpha = l_1$, $T(l_1, l_1)<l_1$. *If* $d>e$, *then* U *is* (α, T)*-migrative if and only if* $T = (\langle 0, d, T_* \rangle, \cdots)$, $U(l_1, l_1) = T(l_1, l_1) = 0$ *and*

$$T(l_1, y) = \begin{cases} 0, & y \in [0, l_1], \\ U(T(l_1, e), y), & y \in (l_1, d), \\ l_1, & y \in [d, 1]. \end{cases}$$

and $U(T(l_1, y), z) = U(y, l_1)$ *for* $y \in (l_1, d)$, $z \in [d, 1]$. *Specially,* $U(T(l_1, y), z) = l_1$ *for any* $y \in (a, d)$, $z \in [d, 1]$.

4 Migrativity of a disjunctive uninorm $U \in \mathcal{U}_{nlocb}$ over a continuous t-conorm

Note that Qin et al. [29] discussed the structure of a disjunctive uninorm $U \in \mathcal{U}_{nlocb}$ possessing the neutral element $e \in (0, 1)$ whose boundary function g_0 has a finite number of different discontinuities $a'_1, \cdots, a'_n$ and $1 \geq a'_1 > \cdots > a'_n = a' > e$ according to the value of $g_0(a')$, i.e. $g_0(a') = 0$ and $g_0(a') = a'$.

While in this paper, we study the case $g_0(a') = 0$ first, the results of the case $g_0(a') = a'$ are the same as the case $g_0(a') = 0$, because in the proof, we only use the same structures of the two classes of uninorms. Therefore, this shows that the result of migrativity of a disjunctive uninorm $U \in \mathcal{U}_{nlocb}$ whose boundary function g_0 has a finite number of different discontinuities over a t-conorm is independent of the value of $g_0(a')$.

The results in this section can be obtained directly from the results in Section 3 by duality, thus, we only give the results and omit the proof.

For convenience, we denote all uninorms satisfying that their boundary function g_0 has a finite number of different discontinuities $a'_1, \cdots, a'_n$ with $1 \geq a'_1 > \cdots > a'_n = a' > e$ and $g_0(a') = 0$ by $\mathcal{U}_{ddis}$ in the following.

4.1 Case for $g_0(a') = 0$

Lemma 4.1. *Suppose* U *is a disjunctive uninorm* $U \in \mathcal{U}_{nlocb} \cup \mathcal{U}_{ddis}$ *possessing the neutral element* $e \in (0, 1)$, S *is a continuous t-conorm and* $\alpha \in (0, 1)$. *If* U *is* (α, S)*-migrative, then* $\alpha = U(0, S(\alpha, e))$ *and* $\alpha \in Ran(g_0)\backslash\{0, 1\}$.

In fact, Lemma 4.1 holds for any t-conorm.

According to Lemma 4.1, we only discuss the migrativity of a disjunctive uninorm $U \in \mathcal{U}_{nlocb} \cup \mathcal{U}_{ddis}$ over a continuous t-conorm in the case $\alpha \in Ran(g_0)\backslash\{0, 1\}$. Furthermore, we study it in terms of whether α is an idempotent element of the continuous t-conorm S.

Remark 4.2. *It should be noticed that* $Ran(g_0)\backslash\{0, 1\} = \cup_{i=0}^{n-1} H'_i \backslash\{1\} = (H'_0\backslash\{1\}) \cup (\cup_{i=1}^{n-1} H'_i)$.

4.1.1 Case for $S(\alpha,\alpha)=\alpha$

Lemma 4.3. *Suppose U is a disjunctive uninorm $U\in\mathcal{U}_{nlocb}\cup\mathcal{U}_{ddis}$ possessing the neutral element $e\in(0,1)$, S is a continuous t-conorm and $S(\alpha,\alpha)=\alpha$, $\alpha\in Ran(g_0)\backslash\{0,1\}$.*

(i) *If U is (α,S)-migrative, then $\alpha\in H_0'\backslash\{1\}$.*

(ii) *U is (α,S)-migrative if and only if $U(x,\alpha)=S(x,\alpha)=max(x,\alpha)$ for all $x\in[0,1]$.*

Theorem 4.4. *Assume U is a disjunctive uninorm $U\in\mathcal{U}_{nlocb}\cup\mathcal{U}_{ddis}$ possessing the neutral element $e\in(0,1)$, S is a continuous t-conorm and $S(\alpha,\alpha)=\alpha$, $\alpha\in Ran(g_0)\backslash\{0,1\}$. Then U is (α,S)-migrative if and only if the following items are fulfilled:*

(i) *$\alpha\in H_0'\backslash\{1\}$.*

(ii) *$S=(\langle 0,\alpha,S_1\rangle,\langle\alpha,1,S_2\rangle)$, where S_1,S_2 are continuous t-conorms.*

(iii) *$U(y,z)=max(y,z)$ for all $(y,z)\in[a',\alpha]\times[\alpha,1]\cup[\alpha,1]\times[a',\alpha]$.*

4.1.2 Case for $S(\alpha,\alpha)>\alpha$

Now, we discuss the case $S(\alpha,\alpha)>\alpha$. Note that $Ran(g_0)\backslash\{0,1\}=\cup_{i=0}^{n-1}H_i'\backslash\{1\}=(H_0'\backslash\{1\})\cup(\cup_{i=1}^{n-1}H_i')$, we discuss the migrativity in two cases: $\alpha\in H_0'\backslash\{1\}$ and $\alpha\in H_i'$, $i\in\{1,\cdots,n-1\}$.

Remark 4.5. *Assume S is a continuous t-conorm and $S(\alpha,\alpha)>\alpha$ for some $\alpha\in[0,1]$, then we know that $S=(\cdots,\langle c',d',S_*\rangle,\cdots)$, where S_* is a continuous Archimedean t-norm and $0\le c'<\alpha<d'\le 1$ by Theorem 2.11. A direct outcome of Lemma 2.8 and Lemma 2.12 is that S_* meets the conditional cancellation law. The symbols c',d' and S_* are defined as the same unless specifically emphasized in the following.*

Lemma 4.6. *Suppose U is a disjunctive uninorm $U\in\mathcal{U}_{nlocb}\cup\mathcal{U}_{ddis}$ possessing the neutral element $e\in(0,1)$, S is a continuous t-conorm and $S(\alpha,\alpha)>\alpha$, $\alpha\in H_0'\backslash\{1\}$.*

(i) *If U is (α,S)-migrative, then $e\le c'<\alpha$.*

(ii) *U is (α,S)-migrative if and only if $S(x,\alpha)=U(x,\alpha)$ for any $x\in[0,1]$.*

Theorem 4.7. *Assume U is a disjunctive uninorm $U\in\mathcal{U}_{nlocb}\cup\mathcal{U}_{ddis}$ possessing the neutral element $e\in(0,1)$, S is a continuous t-conorm and $S(\alpha,\alpha)>\alpha$, $\alpha\in H_0'\backslash\{1\}$. Then U is (α,S)-migrative if and only if one of the following items is satisfied:*

(i) *$S=(\cdots,\langle c',d',S_*\rangle,\cdots)$, where $a_1'\le c'<\alpha<d'<1$, S_* is a continuous Archimedean t-conorm, and*

$$U(\alpha,y)=\begin{cases}\alpha, & y\in[0,c'],\\ y, & y\in[d',1],\\ c'+(d'-c')S_*(\frac{\alpha-c'}{d'-c'},\frac{y-c'}{d'-c'}), & y\in(c',d').\end{cases}$$

Moreover, $U(y,z)=max(y,z)$ whenever $(y,z)\in[d',1]\times[a',c']\cup[a',c']\times[d',1]$

(ii) *$S=(\cdots,\langle c',1,S_*\rangle)$, where $a_1'\le c'<\alpha$, S_* is a continuous Archimedean t-conorm, $S(\alpha,l_1')<1$ and*

$$U(\alpha,y)=\begin{cases}c'+(1-c')T_*(\frac{\alpha-c'}{1-c'},\frac{y-c'}{1-c'}), & y\in(c',1],\\ \alpha, & y\in[0,c'].\end{cases}$$

(iii) *$S=(\cdots,\langle c',1,S_*\rangle)$, where $a'\le c'<l_1'$, S_* is a continuous Archimedean t-conorm, $S(\alpha,l_1')=1$, and*

$$U(\alpha,y)=\begin{cases}\alpha, & y\in[0,c'],\\ c'+(1-c')T_*(\frac{\alpha-c'}{1-c'},\frac{y-c'}{1-c'}), & y\in(c',l_1'),\\ 1, & y\in[l_1',1].\end{cases}$$

In this case, $U(y,z)=S(y,z)=1$ whenever $(y,z)\in[\alpha,1]^2\cup[\alpha,1]\times[l_1',\alpha]\cup[l_1',\alpha]\times[\alpha,1]$.

Lemma 4.8. *Assume U is a disjunctive uninorm $U \in \mathcal{U}_{nlocb} \cup \mathcal{U}_{ddis}$ possessing the neutral element $e \in (0,1)$, S is a continuous t-conorm and $S(\alpha,\alpha)>\alpha$, $\alpha \in H_i'\backslash\{l_i'\}$, $i \in \{1,\cdots,n-1\}$.*

(i) *If U is (α,S)-migrative, then $e \le c'<\alpha$.*

(ii) *U is (α,S)-migrative if and only if $S(x,\alpha)=U(x,\alpha)$ for all $x \in [0,1]$.*

Theorem 4.9. *Assume U is a disjunctive uninorm $U \in \mathcal{U}_{nlocb} \cup \mathcal{U}_{ddis}$ possessing the neutral element $e \in (0,1)$, S is a continuous t-conorm and $S(\alpha,\alpha)>\alpha$, $\alpha \in H_i'\backslash\{l_i'\}$, $i \in \{1,\cdots,n-1\}$. Then U is (α,S)-migrative if and only if the following items are satisfied:*

(i) *$S=(\cdots,\langle c',1,S_*\rangle)$, where $a' \le c'<\alpha$, S_* is a continuous Archimedean t-conorm and $S(\alpha,l_i')=1$.*

(ii)
$$U(\alpha,y)=\begin{cases}\alpha, & y\in[0,c'],\\ c'+(1-c')S_*(\frac{\alpha-c'}{1-c'},\frac{y-c'}{1-c'}), & y\in(c',l_i'),\\ 1, & y\in[l_i',1].\end{cases}$$

Additionally, $U(y,z)=S(y,z)=1$ whenever $(y,z)\in[l_i',1]^2\cup[l_i',1]\times[\alpha,l_i']\cup[\alpha,l_i']\times[l_i',1]$.

Lemma 4.10. *Suppose U is a disjunctive uninorm $U \in \mathcal{U}_{nlocb} \cup \mathcal{U}_{ddis}$ possessing the neutral element $e \in (0,1)$, S is a continuous t-conorm and $\alpha=l_i'$, $S(l_i',l_i')>l_i'$, $i \in \{1,\cdots,n-1\}$. If $c' \ge e$, then U is (α,S)-migrative if and only if $S(l_i',x)=U(l_i',x)$ for all $x\in[0,1]$.*

Theorem 4.11. *Assume U is a disjunctive uninorm $U \in \mathcal{U}_{nlocb} \cup \mathcal{U}_{ddis}$ possessing the neutral element $e \in (0,1)$, S is a continuous t-conorm and $\alpha=l_i'$, $S(l_i',l_i')>l_i'$, $i \in \{1,\cdots,n-1\}$. If $c' \ge e$, then U is (α,S)-migrative if and only if the following items are satisfied:*

(i) *$S=(\cdots,\langle c',1,S_*\rangle)$, where $a' \le c'<l_i'$, S_* is a continuous Archimedean t-conorm and $S(l_i',l_i')=1$.*

(ii)
$$U(\alpha,y)=\begin{cases}l_i', & y\in[0,c'],\\ c'+(1-c')S_*(\frac{\alpha-c'}{1-c'},\frac{y-c'}{1-c'}), & y\in(c',l_i'),\\ 1, & y\in[l_i',1].\end{cases}$$

Additionally, $U(y,z)=S(y,z)=1$ whenever $(y,z)\in[l_i',1]^2$.

Theorem 4.12. *Assume U is a disjunctive uninorm $U \in \mathcal{U}_{nlocb} \cup \mathcal{U}_{ddis}$ possessing the neutral element $e \in (0,1)$, S is a continuous t-conorm and $\alpha=l_i'$, $S(l_i',l_i')>l_i'$, $i \in \{1,\cdots,n-1\}$. If $c'<e$, then U is (α,S)-migrative if and only if one of the following items is satisfied:*

(i) *If $a_i'<1$, $U(x,y)\in Ran(g_0)\backslash\{0\}$ for all $x,y\in g_0^{-1}(l_i')$, then $S=(\cdots,\langle c',1,S_*\rangle)$, $U(l_i',l_i')=S(l_i',l_i')=1$ and*

$$S(l_i',y)=\begin{cases}l_i', & y\in[0,c'],\\ U(S(l_i',e),y), & y\in(c',l_i'),\\ 1, & y\in[l_i',1].\end{cases}$$

and $U(S(l_i',y),z)=U(y,l_i')$ when $y\in(c',l_i')$, $z\in[0,c']$. Specially, $U(S(l_i',y),z)=l_i'$ whenever $y\in(c',a']$, $z\in[0,c']$.

(ii) *If $a_i'<1$, $U(x_0,y_0)\notin Ran(g_0)\backslash\{0\}$ for some $x_0,y_0\in g_0^{-1}(l_i')$, then $S=(\cdots,\langle c',d',S_*\rangle,\cdots)$, where $d'>a_i'$ and*

$$S(l_i',y)=\begin{cases}l_i', & y\in[0,c'],\\ U(S(l_i',e),y), & y\in(c',d'),\\ y, & y\in[d',1].\end{cases}$$

and $U(S(l_i',y),z)=U(y,l_i')$ when $y\in(c',d')$, $z\in[0,c']$, $U(y,z)=U(y,S(l_i',z))$ whenever $y\in[d',1]$,$z\in(c',d')$, $U(y,z)=U(y,l_i')$ when $y\in[d',1]$,$z\in[0,c']$. Specially, $U(S(l_i',y),z)=l_i'$ whenever $y\in(c',a']$, $z\in[0,c']$.

(iii) *If $i=1$ and $a'_1=1$, then $S=(\cdots,\langle c',1,S_*\rangle)$, $U(l'_1,l'_1)=S(l'_1,l'_1)=1$ and*

$$S(l'_1,y)=\begin{cases}1, & y\in[l'_1,1],\\ U(S(l'_1,e),y), & y\in(c',l'_1),\\ l'_1, & y\in[0,c'].\end{cases}$$

and $U(S(l'_1,y),z)=U(y,l'_1)$ when $y\in(c',l'_1)$, $z\in[0,c']$. Specially, $U(S(l'_1,y),z)=l'_1$ whenever $y\in(c',a']$, $z\in[0,c']$.

Notice that, when $\alpha=l'_i$, $S(l'_i,l'_i)>l'_i$, if $n=2$, we can get the following theorem because we have $U(x,y)=U(x,l'_1)\geq a'_1$ for any $x\in(a',1]$, $y\in J'_1$ by Theorems 4.12 and 4.13 in [29]:

Theorem 4.13. *Assume U is a disjunctive uninorm $U\in\mathcal{U}_{nlocb}$ possessing the neutral element $e\in(0,1)$ whose boundary function g_0 has only two discontinuities a'_1,a'_2 with $e<a'=a'_2<a'_1\leq 1$, and $g_0(a')=0$, S is a continuous t-conorm and $\alpha=l'_1$, $S(l'_1,l'_1)>l'_1$. If $c'<e$, then U is (α,S)-migrative if and only if $S=(\cdots,\langle c',1,S_*\rangle)$, $U(l'_1,l'_1)=S(l'_1,l'_1)=1$ and*

$$S(l'_1,y)=\begin{cases}1, & y\in[l'_1,1],\\ U(S(l'_1,e),y), & y\in(c',l'_1),\\ l'_1, & y\in[0,c'].\end{cases}$$

and $U(S(l'_1,y),z)=U(y,l'_1)$ for $y\in(c',l'_1)$, $z\in[0,c']$. Specially, $U(S(l'_1,y),z)=l'_1$ for any $y\in(c',a']$, $z\in[0,c']$.

Remark 4.14. *From Lemmas 4.3, 4.6 and 4.8, it can be seen that the migrativity of disjunctive uninorms $U\in\mathcal{U}_{nlocb}\cup\mathcal{U}_{ddis}$ over continuous t-conorms is due to whether the α-sections of S and U are equal, i.e. $S(z,\alpha)=U(z,\alpha)$ whenever $z\in[0,1]$, except when $\alpha=l'_i$, $i\in\{1,\cdots,n-1\}$, $S(l'_i,l'_i)>l'_i$.*

4.2 Case for $g_0(a')=a'$

The results in this section are the same as those in Section 4.1, but it should be noted that when $\alpha=l'_i$, $S(l'_i,l'_i)>l'_i$, $i\in\{1,\cdots,n-1\}$ and $c'<e$, the corresponding results are slightly different from Theorem 4.12 and Theorem 4.13. Thus, we list the results for this case and omit the other relevant results.

For convenience, we denote all uninorms satisfying that their boundary function g_0 has a finite number of different discontinuities $a'_1,\cdots,a'_n$ with $1\geq a'_1>\cdots>a'_n=a'>e$ and $g_0(a')=a'$ by $\mathcal{U}_{ddia}$ in the following.

Theorem 4.15. *Assume U is a disjunctive uninorm $U\in\mathcal{U}_{nlocb}\cup\,\mathcal{U}_{ddia}$ possessing the neutral element $e\in(0,1)$, S is a continuous t-conorm and $\alpha=l'_i$, $S(l'_i,l'_i)>l'_i$, $i\in\{1,\cdots,n-1\}$. If $c'<e$, then U is (α,S)-migrative if and only if one of the following items is satisfied:*

(i) *If $a'_i<1$, $U(x,y)\in Ran(g_0)\backslash\{0\}$ for all $x,y\in g_0^{-1}(l'_i)$, then $S=(\cdots,\langle c',1,S_*\rangle)$, $U(l'_i,l'_i)=S(l'_i,l'_i)=1$ and*

$$S(l'_i,y)=\begin{cases}l'_i, & y\in[0,c'],\\ U(S(l'_i,e),y), & y\in(c',l'_i),\\ 1, & y\in[l'_i,1].\end{cases}$$

and $U(S(l'_i,y),z)=U(y,l'_i)$ when $y\in(c',l'_i)$, $z\in[0,c']$. Specially, $U(S(l'_i,y),z)=l'_i$ whenever $y\in(c',a')$, $z\in[0,c']$.

(ii) *If $a'_i<1$, $U(x_0,y_0)\notin Ran(g_0)\backslash\{0\}$ for some $x_0,y_0\in g_0^{-1}(l'_i)$, then $S=(\cdots,\langle c',d',S_*\rangle,\cdots)$, where $d'>a'_i$ and*

$$S(l'_i,y)=\begin{cases}l'_i, & y\in[0,c'],\\ U(S(l'_i,e),y), & y\in(c',d'),\\ y, & y\in[d',1].\end{cases}$$

and $U(S(l'_i,y),z)=U(y,l'_i)$ when $y\in(c',d')$, $z\in[0,c']$, $U(y,z)=U(y,S(l'_i,z))$ whenever $y\in[d',1]$,$z\in(c',d')$, $U(y,z)=U(y,l'_i)$ when $y\in[d',1]$,$z\in[0,c']$. Specially, $U(S(l'_i,y),z)=l'_i$ whenever $y\in(c',a')$, $z\in[0,c']$.

(iii) *If* $i = 1$ *and* $a'_1 = 1$*, then* $S = (\cdots, \langle c', 1, S_* \rangle)$*,* $U(l'_1, l'_1) = S(l'_1, l'_1) = 1$ *and*

$$S(l'_1, y) = \begin{cases} 1, & y \in [l'_1, 1], \\ U(S(l'_1, e), y), & y \in (c', l'_1), \\ l'_1, & y \in [0, c']. \end{cases}$$

and $U(S(l'_1, y), z) = U(y, l'_1)$ *when* $y \in (c', l'_1)$*,* $z \in [0, c']$*. Specially,* $U(S(l'_1, y), z) = l'_1$ *whenever* $y \in (c', a')$*,* $z \in [0, c']$*.*

Notice that, when $\alpha = l'_i$, $S(l'_i, l'_i) > l'_i$, if $n = 2$, we can get the following theorem because we have $U(x, y) = U(x, l'_1) \geq a'_1$ for any $x \in [a', 1]$, $y \in J'_1$ by Theorems 4.14 and 4.15 in [29]:

Theorem 4.16. *Assume* U *is a disjunctive uninorm* $U \in \mathcal{U}_{nlocb}$ *possessing the neutral element* $e \in (0, 1)$ *whose boundary function* g_0 *has only two discontinuities* a'_1, a'_2 *with* $e < a' = a'_2 < a'_1 \leq 1$*, and* $g_0(a') = a'$*,* S *is a continuous t-conorm and* $\alpha = l'_1$*,* $S(l'_1, l'_1) > l'_1$*. If* $c' < e$*, then* U *is* (α, S)*-migrative if and only if* $S = (\cdots, \langle c', 1, S_* \rangle)$*,* $U(l'_1, l'_1) = S(l'_1, l'_1) = 1$ *and*

$$S(l'_1, y) = \begin{cases} 1, & y \in [l'_1, 1], \\ U(S(l'_1, e), y), & y \in (c', l'_1), \\ l'_1, & y \in [0, c']. \end{cases}$$

and $U(S(l'_1, y), z) = U(y, l'_1)$ *for* $y \in (c', l'_1)$*,* $z \in [0, c']$*. Specially,* $U(S(l'_1, y), z) = l'_1$ *for any* $y \in (c', a')$*,* $z \in [0, c']$*.*

5 Conclusions

In this paper, we characterize the α-migrativity of uninorms belonging to $\mathcal{U}_{nlocb}$ over continuous t-norms and continuous t-conorms. The results show that the α-sections of uninorm U and t-norm T (resp. t-conorm S) play a crucial role in whether α-migrativity of uninorms belonging to $\mathcal{U}_{nlocb}$ over continuous t-norms (resp. t-conorms) is established. Additionally, we give the equivalent conditions of the α-migrativity of uninorms in $\mathcal{U}_{nlocb}$ over continuous t-norms (resp. t-conorms) based on α is an idempotent element of T (resp. t-conorm S) or not and the value of α. As a supplement to the α-migrativity of uninorms over continuous t-norms and t-conorms, this paper will make the work of α-migrativity more complete.

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