Iranian Journal of Fuzzy Systems

Volume 21, Number 3, (2024), pp. 1-17

Original paper

University of Sistan and Baluchestan

Robust nonfragile H_∞ filtering for fuzzy fractional order systems with uncertainties

B. X. Li [1], X. F. Zhang [2] and X. Zhao [3]

[1,3] *Institute of Robotics and Automatic Information System (IRAIS) and the Tianjin Key Laboratory of Intelligent Robotic (tjKLIR), Nankai University, Tianjin, 300350, China*

[2] *College of Sciences, Northeastern University, Shenyang, 110819, Liaoning, China*

[3] *Institute of Intelligence Technology and Robotic Systems, Shenzhen Research Institute of Nankai University, Shenzhen, 518083, China*

libingxin2017@163.com, zhangxuefeng@mail.neu.edu.cn, zhaoxin@nankai.edu.cn

Abstract

The problem of robust nonfragile H_∞ filtering for fuzzy fractional order (FFO) systems $0 < \alpha < 1$ with uncertainties is studied. First, a new sufficient condition of H_∞ control for fractional order systems is given to overcome the shortcoming of solving the complex matrix. Then, based on the condition and the linear matrix inequality (LMI) approach, the conditions of robust H_∞ control for FFO systems are proposed, which can guarantee the prescribed noise attenuation level in the H_∞ sense. Furthermore, the FFO filter is constructed, and sufficient conditions are proposed for FFO filter systems. Finally, two examples are given to verify the effectiveness of our conditions.

Keywords: Nonfragile H_∞ filtering, fuzzy fractional order (FFO) systems, linear matrix inequality (LMI).

1 Introduction

Recently, fractional calculus has received extensive attention and applications [20], for its theoretically challenging nature. When fractional calculus was introduced into the control field, it made great progress. On the one hand, many conditions have been published for fractional order systems based on the linear matrix inequality (LMI) approach, which is often used in the control field [6, 17]. Furthermore, a unified condition of stability for fractional order systems is derived in [38]. Based on the criterion in [38], observer-based control for fractional order switched systems has been studied in [30, 41]. On the other hand, [13] studied the stability of fractional order nonlinear systems by using the fractional order Lyapunov method and the Mittag-Leffler function. Moreover, [2] proposed more concise and easy conditions to analyze the problem for fractional order nonlinear systems with order $0 < \alpha < 1$.

The T-S fuzzy model consists of a set of IF-THEN rules that represent the local linear input-output relationship of nonlinear systems [7, 21, 31]. Fuzzy systems based on the T-S fuzzy model have attracted extensive attention. In particular, a lot of conditions have been published for integer order fuzzy systems, such as stability [9, 26, 28], performance analysis [25], H_∞ control [32, 34, 40], the non-singular terminal sliding mode control [12], and the networks [22, 29]. Recently, fuzzy fractional order (FFO) systems have attracted the attention of researchers, because they can more accurately describe the physical phenomena of the real world. Output feedback stabilization for FFO linear and nonlinear systems has been studied in [11] and [27]. Moreover, [8] proposed the conditions of fuzzy variable structure control and composite learning control for FFO nonlinear systems. [18] has done research work on practical engineering applications of FFO systems. Furthermore, [37] and [35] developed the conditions of the nonfragile controller and observer-based control for singular FFO systems.

H_∞ control can guarantee the attenuation level of the bounded disturbance. Recently, several results of H_∞ control for fractional order systems have appeared. In [15], Liang *et al.* derived the conditions of H_∞ control for fractional

Corresponding Author: B. X. Li

Received: April 2023; Revised: May 2024; Accepted: July 2024.

https://doi.org/10.22111/IJFS.2024.45384.8019

order systems based on the generalized KYP lemma. Furthermore, Farges *et al.* improved the conditions of H_∞ control in [5]. However, the problem of solving the complex matrix exists in these conditions. Output feedback H_∞ control for order $1 < \alpha < 2$ has been studied in [14]. Based on the conditions in [5, 15], observer-based H_∞ control for fractional order systems has been studied in [1]. However, dynamic output feedback H_∞ control for order $0 < \alpha < 1$ is still an open issue. H_∞ control of fractional order singular systems is studied in [4, 39]. [3, 10, 16] studied the guaranteed cost control and output feedback H_∞ control for fractional order nonlinear systems. For FFO systems, H_∞ output feedback control and robust H_∞ adaptive sliding mode control have been developed [19, 36]. However, based on the function $\frac{1}{2}D^\alpha x^2(t) \leq x(t)D^\alpha x(t)$ in [2], the conditions in [3, 19] are conservative, because the characteristic roots of the positive part are not considered. H_∞ filtering is often used to estimate the system state. Compared with the classical Kalman filtering approach, the H_∞ filtering technology is capable of providing a boundary for the estimation error without the knowledge of the noise statistics [24]. This technology has been widely used in integer order control systems. Up to now, few papers have reported about H_∞ filtering for FFO systems.

Motivated by the above observations, the problem of robust nonfragile H_∞ filtering for FFO systems is studied. For the FFO system, our goal is to propose a filter design method that makes the error system stable. The paper has the following advantages.

- Compared with the results in [5, 15], a new LMI-based sufficient condition of H_∞ control for order $0 < \alpha < 1$ is improved, which overcomes the shortcoming of solving the complex matrix. Based on the condition, robust H_∞ control and the state feedback controller design for FFO systems with uncertainties are studied.
- The nonfragile full-order fuzzy filter is designed. The LMI-based condition of nonfragile H_∞ filtering for FFO systems is derived. Then, the condition based on the state feedback controller is designed.

The rest is organized as follows. Preliminaries are presented in Section 2. The condition of H_∞ control for order $0 < \alpha < 1$ is given. In Section 3 and Section 4, LMI-based conditions of H_∞ control and nonfragile H_∞ filtering for FFO systems are given. Two illustrative examples are presented in Section 5.

Notation: In the paper, $\mathbb{R}$ is the Euclidean space. X^T and X^* are the transpose, and the conjugate transpose of X, respectively. $\|\cdot\|_\infty$ denotes the H_∞ norm. sym(X) and syms(X) denote $X + X^T$ and $X + X^*$. $X < (>)0$ denotes X is the negative (positive) definite. diag$\{\cdot\}$ is the diagonal matrix. $\begin{bmatrix} P & * \\ Q & P \end{bmatrix}$ denots $\begin{bmatrix} P & Q^T \\ Q & P \end{bmatrix}$.

2 Preliminaries

Caputo derivative definition [20] is given by

$$D^\alpha f(t) = \frac{1}{\Gamma(k-\alpha)} \int_0^t \frac{f^{(k)}(\tau)}{(t-\tau)^{\alpha-k+1}} d\tau,$$

where $\alpha \in (k-1, k)$, $\Gamma(\cdot)$ denotes the Euler Gamma function. The Caputo operator will be used throughout this paper.

Borrowed from the definition in [15], the fractional order system with disturbance is as follows

$$\begin{aligned} D^\alpha x(t) &= A_1 x(t) + B_1 w(t), \\ z(t) &= C_1 x(t) + D_1 w(t), \end{aligned} \tag{1}$$

where $\alpha \in (0, 1)$. $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^m$ and $z(t) \in \mathbb{R}^v$ are state vector, disturbance and output, respectively. A_1, B_1, C_1 and D_1 are known matrices. The initial condition of the system is zero.

From [5], the definition of the transfer matrix between $w(t)$ and $z(t)$ is as follows

$$T_{zw}(s) = \frac{Z(s)}{W(s)} = C_1(s^\alpha I - A_1)^{-1} B_1 + D_1. \tag{2}$$

Before proceeding, three lemmas and one definition are introduced.

Definition 2.1. [1] *The system (1) has the H_∞ norm less than a scalar $\gamma > 0$, $\|T_{zw}(s)\|_\infty < \gamma$ if it is stable and the following condition*

$$\sqrt{\frac{\int_0^\infty z^T(t) z(t) dt}{\int_0^\infty w^T(t) w(t) dt}} < \gamma, \tag{3}$$

is satisfied for any disturbances $w(t)$ belonging to L_2 and with zero initial conditions.

The condition of the H_∞ norm for fractional order systems with $0 < \alpha < 1$ based on the generalized KYP lemma has been derived in [15] and [5]. The condition is shown in Lemma 2.2.

Lemma 2.2. [15] *$T_{zw}(s)$ in (2) satisfies $\|T_{zw}(s)\|_\infty < \gamma$ if there exist a scalar $\gamma > 0$ and a hermitian matrix $Z > 0$ such that the following condition holds*

$$\begin{bmatrix} syms(A_1(e^{j\theta}Z + e^{-j\theta}\bar{Z})) & * & * \\ C_1(e^{j\theta}Z + e^{-j\theta}\bar{Z}) & -\gamma I & * \\ B_1^T & D_1^T & -\gamma I \end{bmatrix} < 0, \tag{4}$$

where $\theta = \frac{\pi}{2}(1-\alpha)$, and $\bar{Z}$ is the conjugate of Z.

Remark 2.3. *Although Lemma 2.2 can be used to solve the H_∞ norm problem for fractional order systems, there exists the problem of solving complex matrix. This will lead to the calculation being more complicated than the real matrix solution. Hence, the new condition is derived to overcome the shortcoming in the following corollary.*

Corollary 2.4. *$T_{zw}(s)$ in (2) satisfies $\|T_{zw}(s)\|_\infty < \gamma$ if there exist a scalar γ and two matrices X, Y such that the following conditions hold*

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} > 0, \tag{5}$$

$$\begin{bmatrix} sym(A_1(aX + bY)) & * & * \\ C_1(aX + bY) & -\gamma I & * \\ B_1^T & D_1^T & -\gamma I \end{bmatrix} < 0, \tag{6}$$

where a and b denote $\sin(\alpha\frac{\pi}{2})$ and $\cos(\alpha\frac{\pi}{2})$.

Proof. Let P and Q be real matrices. Substituting $Z = P - jQ > 0$ into the inequality of Lemma 2.2, obtain

$$\begin{bmatrix} \text{syms}(A_1(e^{j\theta}(P - jQ) + e^{-j\theta}(P + jQ)) & * & * \\ C_1(e^{j\theta}(P - jQ) + e^{-j\theta}(P + jQ)) & -\gamma I & * \\ B_1^T & D_1^T & -\gamma I \end{bmatrix} < 0, \tag{7}$$

According to Euler's formula, we have

$$\begin{bmatrix} \text{sym}(A_1[2(aP + bQ)]) & * & * \\ C_1(2(aP + bQ)) & -\gamma I & * \\ B_1^T & D_1^T & -\gamma I \end{bmatrix} < 0, \tag{8}$$

where $a = \cos(\frac{\pi}{2}(1-\alpha)) = \sin(\alpha\frac{\pi}{2})$ and $b = \sin(\frac{\pi}{2}(1-\alpha)) = \cos(\alpha\frac{\pi}{2})$.

Let $X = 2P$ and $Y = 2Q$, the above inequality (8) is equivalent to (6). According to Fact 1 in [17] and $Z = \frac{1}{2}X - \frac{1}{2}jY > 0$, (5) holds. Hence, Corollary 2.4 is proved completely. □

Lemma 2.5. [33] *For given a scalar $\epsilon > 0$, $F^T(t)F(t) \le I$ and matrices H, E have*

$$HF(t)E + E^TF^T(t)H^T \le \epsilon HH^T + \epsilon^{-1}E^TE.$$

Lemma 2.6. [23] *Matrices $S_1 = S_1^T, S_2$ and $S_3 > 0$,*

$$S_1 + S_2S_3^{-1}S_2^T < 0,$$

if and only if

$$\begin{bmatrix} S_1 & S_2 \\ S_2^T & -S_3 \end{bmatrix} < 0.$$

3 Robust H_∞ control for fuzzy fractional order systems

In the paper, consider FFO systems based on the T-S fuzzy model with parametric uncertainties:

Plant Rule IF $\mu_1(t)$ is H_{i1}, ..., $\mu_2(t)$ is H_{i2}, $\cdots$, and $\mu_s(t)$ is H_{is}, THEN

$$\begin{aligned} D^\alpha x(t) =&(A_i + \Delta A_i(t))x(t) + B_i^u u(t) + (B_i + \Delta B_i(t))w(t), \\ y(t) =&C_i x(t) + D_i w(t), \\ z(t) =&(L_i + \Delta L_i(t))x(t), \end{aligned} \tag{9}$$

where $\alpha \in (0,1)$. $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^q$, $w(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^l$ and $z(t) \in \mathbb{R}^v$ are the state, the input, the disturbance, the measurement output, and the estimated output, respectively. A_i, B_i^u, B_i, C_i, D_i, and L_i are known matrices. μ_j and H_{ij}, $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, s$ are premise variables and fuzzy sets. $\Delta A_i(t)$, $\Delta B_i(t)$ and $\Delta L_i(t)$ satisfy

$$\Delta A_i(t) = M_{Ai}N(t)H_{Ai}, \Delta B_i(t) = M_{Bi}N(t)H_{Bi}, \Delta L_i(t) = M_{Li}N(t)H_{Li}, \tag{10}$$

where $N(t)$ satisfies $N^T(t)N(t) \le I$. M_{Ai}, M_{Bi}, M_{Li}, H_{Ai}, H_{Bi}, and H_{Li} are known matrices.

Then, The fuzzy basic function is defined as follows:

$$h_i(\mu(t)) = \frac{\eta_i(\mu(t))}{\sum\limits_{i=1}^{r} \eta_i(\mu(t))}, \eta_i(\mu(t)) = \Pi_{j=1}^{s} H_{ij}(\mu_j(t)), \tag{11}$$

where $\mu(t) = \begin{bmatrix}\mu_1(t) & \mu_2(t) & \ldots & \mu_s(t)\end{bmatrix}$, $H_{ij}(\mu_j(t))$ are the grade of membership of $\mu_j(t)$ in set H_{ij}. Hence, $h_i(\mu(t)) \ge 0$ and $\sum\limits_{i=1}^{r} h_i = 1$, for all $t > 0$.

For brevity, system (9) can be expressed as

$$\begin{aligned} D^\alpha x(t) =&\sum_{i=1}^{r} h_i(\mu(t))[(A_i + \Delta A_i(t))x(t) + B_i^u u(t) + (B_i + \Delta B_i(t))w(t)], \\ y(t) =&\sum_{i=1}^{r} h_i(\mu(t))[C_i x(t) + D_i w(t)], \\ z(t) =&\sum_{i=1}^{r} h_i(\mu(t))[(L_i + \Delta L_i(t))x(t)]. \end{aligned} \tag{12}$$

Based on Corollary 2.4, the following theorem of H_∞ control is derived for FFO systems $0 < \alpha < 1$ with uncertainties.

Theorem 3.1. *For a given $\gamma > 0$, the transfer function of the system (12) with $u(t) = 0$ has an H_∞ norm less than γ if there exist a scalar $\epsilon > 0$ and matrices X, Y such that the following conditions hold*

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} > 0, \tag{13}$$

$$\begin{bmatrix} \Pi_1 & * & * & * & * & * \\ L_i(aX + bY) & \Pi_2 & * & * & * & * \\ B_i^T & 0 & -\gamma I & * & * & * \\ H_{Ai}(aX + bY) & 0 & 0 & -\epsilon I & * & * \\ 0 & 0 & H_{Bi} & 0 & -\epsilon I & * \\ H_{Li}(aX + bY) & 0 & 0 & 0 & 0 & -\epsilon I \end{bmatrix} < 0, \tag{14}$$

where $a = \sin(\frac{\alpha\pi}{2})$, $b = \cos(\frac{\alpha\pi}{2})$, and

$$\Pi_1 = sym(A_i(aX + bY)) + \epsilon M_{Ai}M_{Ai}^T + \epsilon M_{Bi}M_{Bi}^T, \Pi_2 = -\gamma I + \epsilon M_{Li}M_{Li}^T.$$

Proof. Substituting $A_i + \Delta A_i(t)$, $B_i + \Delta B_i(t)$ and $L_i + \Delta L_i(t)$ into (7), we have

$$\begin{bmatrix} \mathrm{sym}((A_i + \Delta A_i(t))(aX + bY)) & * & * \\ (L_i + \Delta L_i(t))(aX + bY) & -\gamma I & * \\ (B_i + \Delta B_i(t))^T & 0 & -\gamma I \end{bmatrix} < 0. \tag{15}$$

From (15), obtain

$$\begin{bmatrix} \mathrm{sym}(A_i(aX+bY)) & * & * \\ L_i(aX+bY) & -\gamma I & * \\ B_i^T & 0 & -\gamma I \end{bmatrix} + \begin{bmatrix} \mathrm{sym}(\Delta A_i(t)(aX+bY)) & * & * \\ \Delta L_i(t)(aX+bY) & 0 & * \\ \Delta B_i^T(t) & 0 & 0 \end{bmatrix} < 0. \tag{16}$$

According to (10) and (16), get

$$\begin{bmatrix} \mathrm{sym}(\Delta A_i(t)(aX+bY)) & * & * \\ \Delta L_i(t)(aX+bY) & 0 & * \\ \Delta B_i^T(t) & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathrm{sym}(M_{Ai}N(t)H_{Ai}(aX+bY)) & * & * \\ M_{Li}N(t)H_{Li}(aX+bY) & 0 & * \\ (M_{Bi}N(t)H_{Bi})^T & 0 & 0 \end{bmatrix} =$$
$$\begin{bmatrix} M_{Ai} & M_{Bi} & 0 \\ 0 & 0 & M_{Li} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} N(t) & 0 & 0 \\ 0 & N(t) & 0 \\ 0 & 0 & N(t) \end{bmatrix} \begin{bmatrix} H_{Ai}(aX+bY) & 0 & 0 \\ 0 & 0 & H_{Bi} \\ H_{Li}(aX+bY) & 0 & 0 \end{bmatrix} +$$
$$\left(\begin{bmatrix} M_{Ai} & M_{Bi} & 0 \\ 0 & 0 & M_{Li} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} N(t) & 0 & 0 \\ 0 & N(t) & 0 \\ 0 & 0 & N(t) \end{bmatrix} \begin{bmatrix} H_{Ai}(aX+bY) & 0 & 0 \\ 0 & 0 & H_{Bi} \\ H_{Li}(aX+bY) & 0 & 0 \end{bmatrix} \right)^T. \tag{17}$$

From (17) and Lemma 2.5, (16) can be expressed as

$$\begin{bmatrix} \mathrm{sym}(A_i(aX+bY)) & * & * \\ L_i(aX+bY) & -\gamma I & * \\ B_i^T & 0 & -\gamma I \end{bmatrix} + \begin{bmatrix} \epsilon(M_{Ai}M_{Ai}^T + M_{Bi}M_{Bi}^T) & * & * \\ 0 & \epsilon M_{Li}M_{Li}^T & * \\ 0 & 0 & 0 \end{bmatrix} +$$
$$\epsilon^{-1} \begin{bmatrix} H_{Ai}(aX+bY) & 0 & 0 \\ 0 & 0 & H_{Bi} \\ H_{Li}(aX+bY) & 0 & 0 \end{bmatrix}^T \begin{bmatrix} H_{Ai}(aX+bY) & 0 & 0 \\ 0 & 0 & H_{Bi} \\ H_{Li}(aX+bY) & 0 & 0 \end{bmatrix} < 0. \tag{18}$$

From (18) and Lemma 2.6, have

$$\begin{bmatrix} \Pi_1 & * & * & * & * & * \\ L_i(aX+bY) & \Pi_2 & * & * & * & * \\ B_i^T & 0 & -\gamma I & * & * & * \\ H_{Ai}(aX+bY) & 0 & 0 & -\epsilon I & * & * \\ 0 & 0 & H_{Bi} & 0 & -\epsilon I & * \\ H_{Li}(aX+bY) & 0 & 0 & 0 & 0 & -\epsilon I \end{bmatrix} < 0, \tag{19}$$

where $\Pi_1 = \mathrm{sym}(A_i(aX+bY)) + \epsilon M_{Ai}M_{Ai}^T + \epsilon M_{Bi}M_{Bi}^T$, and $\Pi_2 = -\gamma I + \epsilon M_{Li}M_{Li}^T$.

(19) is equivalent to (14). This completes the proof. □

Then, the state feedback control $u(t) = Kx(t)$ for the system (12) is given in the following Theorem 3.2.

Theorem 3.2. *For a given $\gamma > 0$, the transfer function of the system (12) with $u(t) = Kx(t)$ has an H_∞ norm less than γ if there exist a scalar $\epsilon > 0$ and matrices X, Y, Z such that the following conditions hold*

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} > 0, \tag{20}$$

$$\begin{bmatrix} \Pi_1 & * & * & * & * & * \\ L_i(aX+bY) & -\gamma I + \epsilon M_{Li}M_{Li}^T & * & * & * & * \\ B_i^T & 0 & -\gamma I & * & * & * \\ H_{Ai}(aX+bY) & 0 & 0 & -\epsilon I & * & * \\ 0 & 0 & H_{Bi} & 0 & -\epsilon I & * \\ H_{Li}(aX+bY) & 0 & 0 & 0 & 0 & -\epsilon I \end{bmatrix} < 0, \tag{21}$$

where $a = \sin(\frac{\alpha\pi}{2})$, $b = \cos(\frac{\alpha\pi}{2})$,

$$\Pi_1 = sym(A_i(aX+bY) + B_i^u Z) + \epsilon M_{Ai}M_{Ai}^T + \epsilon M_{Bi}M_{Bi}^T,$$

and the state feedback gain

$$K = Z(aX+bY)^{-1} \tag{22}$$

Proof. According to Theorem 3.1 and $u(t) = Kx(t)$, let $Z = K(aX+bY)$, Theorem 3.2 can be directly proved. □

4 Robust nonfragile H_∞ filtering for fuzzy fractional order systems

4.1 Nonfragile fuzzy filter design

Then, the nonfragile full-order filters are constructed for the system (9) in the following form.

Plant Rule IF $\mu_1(t)$ is H_{i1}, ..., $\mu_2(t)$ is H_{i2}, $\cdots$, and $\mu_s(t)$ is H_{is}, THEN

$$\begin{aligned} D^\alpha x_F(t) =&(A_{Fi} + N_{AFi}(t))x_F(t) + (B_{Fi} + N_{BFi}(t))y(t), \\ y(t) =&C_i x(t) + D_i w(t), \\ z_F(t) =&(L_{Fi} + N_{LFi}(t))x_F(t), \end{aligned} \tag{23}$$

where $i = 1, 2, \ldots, r$, $x_F(t) \in \mathbb{R}^n$ is the state variable of the filter, and $z_F(t) \in \mathbb{R}^v$ is the estimation of $z(t)$. A_{Fi}, B_{Fi}, and L_{Fi} are unknown gain matrices. $N_{AFi}(t)$, $N_{BFi}(t)$, and $N_{LFi}(t)$ are norm-bounded uncertainties and satisfy

$$N_{AFi}(t) = M_{AFi}N(t)H_{Ai}, N_{BFi}(t) = M_{BFi}N(t)H_{Bi}, N_{LFi}(t) = M_{LFi}N(t)H_{Li}, \tag{24}$$

where M_{AFi}, M_{BFi} and M_{LFi} are known matrices.

The nonfragile filter (23) can be obtained

$$\begin{aligned} D^\alpha x_F(t) =&\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\mu(t))h_j(\mu(t))[(A_{Fi} + N_{AFi}(t))x_F(t), \\ &+ (B_{Fi} + N_{BFi}(t))C_j x(t) + (B_{Fi} + N_{BFi}(t))D_j w(t)], \\ y(t) =&\sum_{i=1}^{r} h_i(\mu(t))[C_i x(t) + D_i w(t)], \\ z_F(t) =&\sum_{i=1}^{r} h_i(\mu(t))[(L_{Fi} + N_{LFi}(t))x_F(t)]. \end{aligned} \tag{25}$$

Denote $\bar{x}(t) = \begin{bmatrix} x(t) \\ x_F(t) \end{bmatrix}$ and $\bar{z}(t) = z(t) - z_F(t)$, from (12) and (25), the following filter error system can be shown

$$\begin{aligned} D^\alpha \bar{x}(t) =&\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\mu(t))h_j(\mu(t))[\bar{A}_{ij}\bar{x}(t) + \bar{B}_i^u u(t) + \bar{B}_{ij}w(t)], \\ \bar{z}(t) =&\sum_{i=1}^{r} h_i(\mu(t))\bar{L}_i\bar{x}(t), \end{aligned} \tag{26}$$

where

$$\begin{aligned} &\bar{A}_{ij} = \begin{bmatrix} A_i + \Delta A_i(t) & 0 \\ (B_{Fi} + N_{BFi}(t))C_j & A_{Fi} + N_{AFi}(t) \end{bmatrix}, \bar{B}_i^u = \begin{bmatrix} B_i^u \\ 0 \end{bmatrix}, \bar{B}_{ij} = \begin{bmatrix} B_i + \Delta B_i(t) \\ (B_{Fi} + N_{BFi}(t))D_j \end{bmatrix}, \\ &\bar{L}_i = \begin{bmatrix} L_i + \Delta L_i(t) & -L_{Fi} - N_{LFi}(t) \end{bmatrix}, \end{aligned}$$

and $\bar{T}_{zw}(s) = \frac{\bar{z}(t)}{w(t)} = \bar{L}_i(s^\alpha I - \bar{A}_{ij})\bar{B}_{ij}$.

In the paper, the filtering problem is shown in Figure 1. In the following, our goal is to design the filter (23) for system (9) that simultaneously satisfies the following two performance metrics.

1. The system (26) is stable when $w(t) = 0$.
2. For given a disturbance attenuation level γ, $\bar{z}(t)$ should satisfy the H_∞ performance

$$\frac{\|\bar{z}(t)\|_\infty}{\|w(t)\|_\infty} \leq \gamma, \forall w(t) \neq 0,$$

and that is to say

$$\|\bar{T}_{zw}(s)\|_\infty < \gamma.$$

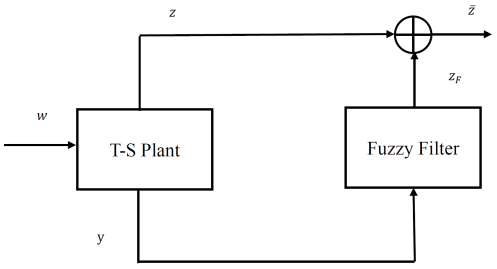


Figure 1: Filtering problem

4.2 Robust H_∞ filtering

In the section, the condition of H_∞ filtering for the system (26) is given.

Theorem 4.1. *The transfer function of the system (26) with $u(t) = 0$ has an H_∞ norm less than γ, if there exist matrices X_1, X_2, Y_1, Y_2, G, W_{1i}, W_{2i} and two scalars $\epsilon_1, \epsilon_2 > 0$ such that the following conditions hold:*

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}, Y = \begin{bmatrix} Y_1 & 0 \\ 0 & Y_2 \end{bmatrix}, \tag{27}$$

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} > 0, \tag{28}$$

$$\begin{bmatrix}
\Phi_{11} & * & * & * & * & * & * & * & * & * & * & * \\
G & \Phi_{22} & * & * & * & * & * & * & * & * & * & * \\
L_iP_1 & -W_{2i} & \Phi_{33} & * & * & * & * & * & * & * & * & * \\
B_i^T & D_j^TB_{Fi}^T & 0 & -\gamma I & * & * & * & * & * & * & * & * \\
H_{Ai}P_1 & 0 & 0 & 0 & -\epsilon_1 I & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & -\epsilon_1 I & * & * & * & * & * & * \\
0 & 0 & 0 & H_{Bi} & 0 & 0 & -\epsilon_1 I & * & * & * & * & * \\
H_{Li}P_1 & 0 & 0 & 0 & 0 & 0 & 0 & -\epsilon_1 I & * & * & * & * \\
H_{Bi}C_jP_1 & 0 & 0 & H_{Bi}D_j & 0 & 0 & 0 & 0 & -\epsilon_2 I & * & * & * \\
0 & H_{Ai}P_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\epsilon_2 I & * & * \\
0 & H_{Li}P_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\epsilon_2 I & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0-\epsilon_2 I
\end{bmatrix} < 0, \tag{29}$$

where $P_1 = aX_1 + bY_1$, $P_2 = aX_2 + bY_2$, $a = \sin(\frac{\alpha\pi}{2})$, $b = \cos(\frac{\alpha\pi}{2})$, and,

$$\begin{aligned}
&\Phi_{11} = sym(A_iP_1) + \epsilon_1(M_{Ai}M_{Ai}^T + M_{Bi}M_{Bi}^T), \\
&\Phi_{22} = sym(W_{1i}) + \epsilon_2(M_{AFi}M_{AFi}^T + M_{BFi}M_{BFi}^T), \\
&\Phi_{33} = -\gamma I + \epsilon_1 M_{Li}M_{Li}^T + \epsilon_2 M_{LFi}M_{LFi}^T, A_{Fi} = W_{1i}P_2^{-1}, L_{Fi} = W_{2i}P_2^{-1}.
\end{aligned}$$

Proof. Substituting $\bar{A}_{ij}$, $\bar{B}_{ij}$ and $\bar{L}_i$ into (7), we have

$$\begin{bmatrix} \mathrm{sym}(\bar{A}_{ij}(aX + bY)) & * & * \\ \bar{L}_i(aX + bY) & -\gamma I & * \\ \bar{B}_{ij}^T & 0 & -\gamma I \end{bmatrix} < 0, \tag{30}$$

From (27) and (30), get

$$\begin{bmatrix}
\mathrm{sym}(\begin{bmatrix} A_i + \Delta A_i(t) & 0 \\ (B_{Fi} + N_{BFi}(t))C_j & A_{Fi} + N_{AFi}(t) \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}) & * & * \\
\begin{bmatrix} L_i + \Delta L_i(t) & -L_{Fi} - N_{LFi}(t) \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} & -\gamma I & * \\
\begin{bmatrix} B_i + \Delta B_i(t) \\ (B_{Fi} + N_{BFi}(t))D_j \end{bmatrix}^T & 0 & -\gamma I
\end{bmatrix} < 0, \tag{31}$$

where $P_1 = aX_1 + bY_1$ and $P_2 = aX_2 + bY_2$.

Then, obtain

$$\begin{bmatrix} \mathrm{sym}((A_i+\Delta A_i(t))P_1) & * & * & * \\ (B_{Fi}+N_{BFi}(t))C_jP_1 & \mathrm{sym}((A_{Fi}+N_{AFi}(t))P_2) & * & * \\ (L_i+\Delta L_i(t))P_1 & (-L_{Fi}-N_{LFi}(t))P_2 & -\gamma I & * \\ (B_i+\Delta B_i(t))^T & ((B_{Fi}+N_{BFi}(t))D_j)^T & 0 & -\gamma I \end{bmatrix} =$$

$$\begin{bmatrix} \mathrm{sym}(A_iP_1) & * & * & * \\ B_{Fi}C_jP_1 & \mathrm{sym}(A_{Fi}P_2) & * & * \\ L_iP_1 & -L_{Fi}P_2 & -\gamma I & * \\ B_i^T & (B_{Fi}D_j)^T & 0 & -\gamma I \end{bmatrix} + \begin{bmatrix} \mathrm{sym}(\Delta A_i(t)P_1) & * & * & * \\ 0 & 0 & * & * \\ \Delta L_i(t)P_1 & 0 & 0 & * \\ \Delta B_i^T(t) & 0 & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & * & * & * \\ N_{BFi}(t)C_jP_1 & \mathrm{sym}(N_{AFi}(t)P_2) & * & * \\ 0 & -N_{LFi}(t)P_2 & 0 & * \\ 0 & (N_{BFi}(t)D_j)^T & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathrm{sym}(A_iP_1) & * & * & * \\ B_{Fi}C_jP_1 & \mathrm{sym}(A_{Fi}P_2) & * & * \\ L_iP_1 & -L_{Fi}P_2 & -\gamma I & * \\ B_i^T & (B_{Fi}D_j)^T & 0 & -\gamma I \end{bmatrix} +$$

$$\begin{bmatrix} \mathrm{sym}(M_{Ai}N(t)H_{Ai}P_1) & * & * & * \\ 0 & 0 & * & * \\ M_{Li}N(t)H_{Li}P_1 & 0 & 0 & * \\ (M_{Bi}N(t)H_{Bi})^T & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & * & * & * \\ M_{BFi}N(t)H_{Bi}C_jP_1 & \mathrm{sym}(M_{AFi}N(t)H_{Ai}P_2) & * & * \\ 0 & -M_{LFi}N(t)H_{Li}P_2 & 0 & * \\ 0 & (M_{BFi}N(t)H_{Bi}D_j)^T & 0 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} \mathrm{sym}(A_iP_1) & * & * & * \\ B_{Fi}C_jP_1 & \mathrm{sym}(A_{Fi}P_2) & * & * \\ L_iP_1 & -L_{Fi}P_2 & -\gamma I & * \\ B_i^T & (B_{Fi}D_j)^T & 0 & -\gamma I \end{bmatrix} + \begin{bmatrix} M_{Ai} & 0 & M_{Bi} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{Li} \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbb{N}(t) \begin{bmatrix} H_{Ai}P_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & H_{Bi} \\ H_{Li}P_1 & 0 & 0 & 0 \end{bmatrix} +$$

$$\left(\begin{bmatrix} M_{Ai} & 0 & M_{Bi} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{Li} \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbb{N}(t) \begin{bmatrix} H_{Ai}P_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & H_{Bi} \\ H_{Li}P_1 & 0 & 0 & 0 \end{bmatrix} \right)^T +$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ M_{BFi} & M_{AFi} & 0 & 0 \\ 0 & 0 & -M_{LFi} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbb{N}(t) \begin{bmatrix} H_{Bi}C_jP_1 & 0 & 0 & H_{Bi}D_j \\ 0 & H_{Ai}P_2 & 0 & 0 \\ 0 & H_{Li}P_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ M_{BFi} & M_{AFi} & 0 & 0 \\ 0 & 0 & -M_{LFi} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbb{N}(t) \begin{bmatrix} H_{Bi}C_jP_1 & 0 & 0 & H_{Bi}D_j \\ 0 & H_{Ai}P_2 & 0 & 0 \\ 0 & H_{Li}P_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)^T < 0, \tag{32}$$

where $\mathbb{N}(t) = \mathrm{diag}\{N(t), N(t), N(t), N(t)\}$.

From Lemma 2.5 and (32), get

$$\begin{bmatrix} \mathrm{sym}(A_iP_1) & * & * & * \\ B_{Fi}C_jP_1 & \mathrm{sym}(A_{Fi}P_2) & * & * \\ L_iP_1 & -L_{Fi}P_2 & -\gamma I & * \\ B_i^T & (B_{Fi}D_j)^T & 0 & -\gamma I \end{bmatrix} + \epsilon_1 \begin{bmatrix} M_{Ai} & 0 & M_{Bi} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{Li} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} M_{Ai} & 0 & M_{Bi} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{Li} \\ 0 & 0 & 0 & 0 \end{bmatrix}^T +$$
$$\epsilon_1^{-1} \begin{bmatrix} H_{Ai}P_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & H_{Bi} \\ H_{Li}P_1 & 0 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} H_{Ai}P_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & H_{Bi} \\ H_{Li}P_1 & 0 & 0 & 0 \end{bmatrix} +$$
$$\epsilon_2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ M_{BFi} & M_{AFi} & 0 & 0 \\ 0 & 0 & -M_{LFi} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ M_{BFi} & M_{AFi} & 0 & 0 \\ 0 & 0 & -M_{LFi} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^T +$$
$$\epsilon_2^{-1} \begin{bmatrix} H_{Bi}C_jP_1 & 0 & 0 & H_{Bi}D_j \\ 0 & H_{Ai}P_2 & 0 & 0 \\ 0 & H_{Li}P_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} H_{Bi}C_jP_1 & 0 & 0 & H_{Bi}D_j \\ 0 & H_{Ai}P_2 & 0 & 0 \\ 0 & H_{Li}P_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} < 0. \tag{33}$$

Furthermore, from Lemma 2.6, the following inequality can be represented

$$\begin{bmatrix} \Phi_{11} & * & * & * & * & * & * & * & * & * & * & * \\ B_{Fi}C_jP_1 & \Phi_{22} & * & * & * & * & * & * & * & * & * & * \\ L_iP_1 & -L_{Fi}P_2 & \Phi_{33} & * & * & * & * & * & * & * & * & * \\ B_i^T & D_j^TB_{Fi}^T & 0 & -\gamma I & * & * & * & * & * & * & * & * \\ H_{Ai}P_1 & 0 & 0 & 0 & -\epsilon_1 I & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & -\epsilon_1 I & * & * & * & * & * & * \\ 0 & 0 & 0 & H_{Bi} & 0 & 0 & -\epsilon_1 I & * & * & * & * & * \\ H_{Li}P_1 & 0 & 0 & 0 & 0 & 0 & 0 & -\epsilon_1 I & * & * & * & * \\ H_{Bi}C_jP_1 & 0 & 0 & H_{Bi}D_j & 0 & 0 & 0 & 0 & -\epsilon_2 I & * & * & * \\ 0 & H_{Ai}P_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\epsilon_2 I & * & * \\ 0 & H_{Li}P_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\epsilon_2 I & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\epsilon_2 I \end{bmatrix} < 0, \tag{34}$$

where

$$\Phi_{11} = \mathrm{sym}(A_iP_1) + \epsilon_1(M_{Ai}M_{Ai}^T + M_{Bi}M_{Bi}^T),$$
$$\Phi_{22} = \mathrm{sym}(A_{Fi}P_2) + \epsilon_2(M_{AFi}M_{AFi}^T + M_{BFi}M_{BFi}^T),$$
$$\Phi_{33} = -\gamma I + \epsilon_1 M_{Li}M_{Li}^T + \epsilon_2 M_{LFi}M_{LFi}^T.$$

Then, let $G = B_{Fi}C_jP_1$, $W_{1i} = A_{Fi}P_2$ and $W_{2i} = L_{Fi}P_2$, (34) is equivalent to (29). Hence, this completes the proof. □

Remark 4.2. *Robust H_∞ control for the system (23) can be solved by using Theorem 3.2 which is a strict LMI condition. Although the condition of dynamic output feedback H_∞ control for order $1 < \alpha < 2$ has been proposed in* [1]. *The study for order $0 < \alpha < 1$ is more difficult and interesting, let alone FFO systems with order $0 < \alpha < 1$. Moreover, Theorem 3.2 avoids the singular value decomposition approach which is inherently conservative, particularly for such a large number of decision variables* [11, 30]. *Hence, Theorem 3.2 overcomes the nonlinear terms and the shortcomings of the singular value decomposition approach.*

In the following, the condition for the system (26) with state feedback controller $u(t) = Kx(t)$ is given.

Theorem 4.3. *The transfer function of the system (26) with $u(t) = Kx(t)$ has an H_∞ norm less than γ, if there exist matrices X_1, X_2, Y_1, Y_2, G, W_{1i}, W_{2i}, Z and a scalar $\epsilon > 0$ such that the following conditions hold:*

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}, Y = \begin{bmatrix} Y_1 & 0 \\ 0 & Y_2 \end{bmatrix}, \tag{35}$$

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} > 0, \tag{36}$$

$$\begin{bmatrix}
\Phi_{11} & * & * & * & * & * & * & * & * & * & * & * \\
G & \Phi_{22} & * & * & * & * & * & * & * & * & * & * \\
L_iP_1 & -W_{2i} & \Phi_{33} & * & * & * & * & * & * & * & * & * \\
B_i^T & D_j^TB_{Fi}^T & 0 & -\gamma I & * & * & * & * & * & * & * & * \\
H_{Ai}P_1 & 0 & 0 & 0 & -\epsilon_1 I & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & -\epsilon_1 I & * & * & * & * & * & * \\
0 & 0 & 0 & H_{Bi} & 0 & 0 & -\epsilon_1 I & * & * & * & * & * \\
H_{Li}P_1 & 0 & 0 & 0 & 0 & 0 & 0 & -\epsilon_1 I & * & * & * & * \\
H_{Bi}C_jP_1 & 0 & 0 & H_{Bi}D_j & 0 & 0 & 0 & 0 & -\epsilon_2 I & * & * & * \\
0 & H_{Ai}P_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\epsilon_2 I & * & * \\
0 & H_{Li}P_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\epsilon_2 I & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0-\epsilon_2 I
\end{bmatrix} < 0, \tag{37}$$

where $P_1 = aX_1 + bY_1$, $P_2 = aX_2 + bY_2$, $a = \sin(\frac{\alpha\pi}{2})$, $b = \cos(\frac{\alpha\pi}{2})$, *and*

$$\begin{aligned}
&\Phi_{11} = sym(A_iP_1 + B_i^uZ) + \epsilon_1(M_{Ai}M_{Ai}^T + M_{Bi}M_{Bi}^T),\\
&\Phi_{22} = sym(W_{1i}) + \epsilon_2(M_{AFi}M_{AFi}^T + M_{BFi}M_{BFi}^T),\\
&\Phi_{33} = -\gamma I + \epsilon_1 M_{Li}M_{Li}^T + \epsilon_2 M_{LFi}M_{LFi}^T, A_{Fi} = W_{1i}P_2^{-1}, L_{Fi} = W_{2i}P_2^{-1}.
\end{aligned}$$

and the state feedback gain

$$K = ZP_1^{-1} \tag{38}$$

Proof. From Theorem 4.1 and $u(t) = Kx(t)$, let $Z = KP_1$, Theorem 4.3 can be directly proved. □

5 Examples

5.1 Example 1

Consider a fractional order system

$$\begin{aligned}
D^{0.4}x(t) &= \sum_{i=1}^{2} h_i(\mu(t))[(A_i + \Delta A_i(t))x(t) + (B_i + \Delta B_i(t))w(t)],\\
z(t) &= \sum_{i=1}^{2} h_i(\mu(t))[(L_i + \Delta L_i(t))x(t)],
\end{aligned} \tag{39}$$

where

Subsystem 1:

$$A_1 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, L_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, M_{A1} = M_{B1} = \begin{bmatrix} 0.01 & 0.02 & 0.02 \\ 0.01 & 0.01 & 0.03 \\ 0.03 & 0.04 & 0.03 \end{bmatrix},$$

$$M_{L1} = \begin{bmatrix} 0.01 & 0.02 & 0.02 \end{bmatrix}, H_{A1} = H_{L1} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}, H_{B1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Subsystem 2:

$$A_2 = \begin{bmatrix} 2 & 3 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, L_2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}, M_{A2} = M_{B2} = \begin{bmatrix} 0.01 & 0.02 & 0.02 \\ 0.01 & 0.01 & 0.05 \\ 0.05 & 0.03 & 0.02 \end{bmatrix},$$

$$M_{L2} = \begin{bmatrix} 0.03 & 0.04 & 0.02 \end{bmatrix}, H_{A2} = H_{L2} = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 4 \\ 2 & 2 & 3 \end{bmatrix}, H_{B2} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

Then, $N(t)$ and the membership functions are as follows

$$N(t) = \text{diag}\{\frac{1}{4}\sin(t), \frac{1}{4}\cos(t), \frac{1}{4}\sin(t^2)\}, h_1(t) = \frac{1-\cos(x_1)}{2}, h_2(t) = \frac{1+\cos(x_1)}{2}.$$

From Theorem 3.1, the feasibility of (13) and (14) can be checked. Then, a feasible solution can be found as follows:

$$X = \begin{bmatrix} 0.0979 & 0.0133 & -0.0962 \\ 0.0133 & 0.1943 & -0.0273 \\ -0.0962 & -0.0273 & 0.1062 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0.0962 & 0.0273 \\ -0.0962 & 0 & -0.1062 \\ -0.0273 & 0.1062 & 0 \end{bmatrix},$$
$$\gamma = 5.9926, \epsilon = 9.1414.$$

Remark 5.1. *The eigenvalues of subsystem 1 and subsystem 2 without uncertainty in (39) are* $\{-3, 1+2i, 1-2i\}$ *and* $\{-1, 2+\sqrt{6}i, 2-\sqrt{6}i\}$. *According to Matignon's stability theorem* (Th 3, [6]), *we can get* $|\ arg(1 \pm 2i)\ | = 1.1071 > \frac{0.4\pi}{2}$ *and* $|\ arg(2 \pm \sqrt{6}i)\ | = 0.8861 > \frac{0.4\pi}{2}$. *Although these exist eigenvalues of the real part, subsystem 1 and subsystem 2 are still stable. Referring to Table 1, compared to existing methods in* [3, 5, 15, 19], *our conditions can easily solve the decision variables, and allow their roots with positive real parts. The conditions in* [3, 19] *can be obtained from the following information, which also means these conditions invalid in this case.*

Marginal infeasibility: *These LMI constraints may be feasible but are not strictly feasible.*

For Example 1, this means that these conditions in [3, 19] *do not yield a set of feasible solutions.*

Figures 2-4 represent the system states, the estimated outputs, and the disturbances. Figures 2 and 3 show the influence of the disturbance $w(t)$ in time $t \in [10, 20]s$ and the performances of the proposed methods in the paper.

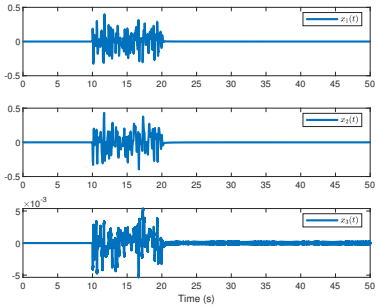


Figure 2: State response $x(t)$ in Example 1

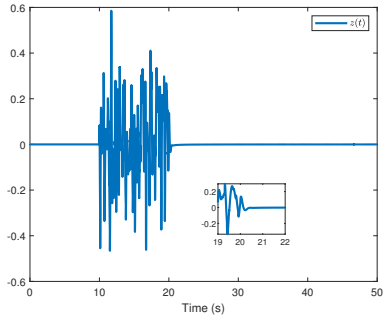


Figure 3: Estimated output $z(t)$ in Example 1

Table 1: The comparison of existing methods and ours

Ref.	Easy solve	Less conservative
[5, 15]	×	✓
[3, 19]	✓	×
ours	✓	✓

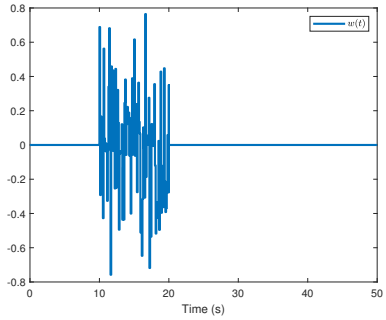


Figure 4: Disturbance input $w(t)$ in Example 1

5.2 Example 2

Inspired by [1], construct the following fractional order system

$$\begin{aligned} D^{0.5}x(t) &= \sum_{i=1}^{2} h_i(\mu(t))[(A_i + \Delta A_i(t))x(t) + B_i^u u(t) + (B_i + \Delta B_i(t))w(t)], \\ y(t) &= \sum_{i=1}^{2} h_i(\mu(t))[C_i x(t) + D_i w(t)], \\ z(t) &= \sum_{i=1}^{2} h_i(\mu(t))[(L_i + \Delta L_i(t))x(t)], \end{aligned} \tag{40}$$

where

Subsystem 1:

$$A_1 = \begin{bmatrix} -1 & 2 \\ 0.5 & 0 \end{bmatrix}, B_1^u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0.3 \end{bmatrix}, C_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}, D_1 = 1, L_1 = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$M_{A1} = M_{B1} = M_{AF1} = M_{BF1} = \begin{bmatrix} 0.01 & 0.02 \\ 0.01 & 0.01 \end{bmatrix}, M_{L1} = M_{LF1} = \begin{bmatrix} 0.01 & 0.02 \end{bmatrix},$$

$$H_{A1} = H_{L1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, H_{B1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Subsystem 2:

$$A_2 = \begin{bmatrix} -0.5 & -1 \\ 1 & 0 \end{bmatrix}, B_2^u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, D_2 = 2, L_2 = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

$$M_{A2} = M_{B2} = M_{AF2} = M_{BF2} = \begin{bmatrix} 0.02 & 0.01 \\ 0.01 & 0.02 \end{bmatrix}, M_{L2} = M_{LF2} = \begin{bmatrix} 0.02 & 0.01 \end{bmatrix},$$

$$H_{A2} = H_{L2} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, H_{B2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Then, $N(t)$ and the membership function are as follows

$$N(t) = \text{diag}\{\frac{1}{3}\sin(t^2), \frac{1}{3}\cos(t^2)\}, h_1(t) = \frac{1-\sin(x_1)}{2}, h_2(t) = \frac{1+\sin(x_1)}{2}.$$

According to Theorem 4.3, we can get

$$X_1 = \begin{bmatrix} 0.1311 & -0.0950 \\ -0.0950 & 0.0810 \end{bmatrix}, X_2 = \begin{bmatrix} 0.1165 & -0.1340 \\ -0.1340 & 0.2384 \end{bmatrix},$$
$$Y_1 = Y_2 = \begin{bmatrix} 0 & -0.0810 \\ 0.0810 & 0 \end{bmatrix}, Z = \begin{bmatrix} -0.1779 & -0.1922 \end{bmatrix},$$
$$\gamma = 1.0085, \epsilon_1 = 1.0073, \epsilon_2 = 1.0074,$$
$$A_{F1} = \begin{bmatrix} -10.3805 & -9.3628 \\ -2.3157 & -5.0753 \end{bmatrix}, B_{F1} = \begin{bmatrix} 1.7143 \\ 1.8351 \end{bmatrix}, L_{F1} = \begin{bmatrix} 1.4121 \\ 0.5423 \end{bmatrix}^T,$$
$$A_{F2} = \begin{bmatrix} -0.0208 & -0.0208 \\ -0.0108 & -0.0144 \end{bmatrix}, B_{F2} = \begin{bmatrix} 1.2133 \\ 1.0283 \end{bmatrix}, L_{F2} = \begin{bmatrix} 1.1121 \\ 1.0125 \end{bmatrix}^T.$$

From (38), the gain matrix K is obtained as

$$K = \begin{bmatrix} -2.9704 & -9.8118 \end{bmatrix}.$$

Based on the fractional order filter, Figures 5 and 6 show the state responses of the states $x(t)$ and $x_F(t)$. Figure 7 is the simulation results of signals $z(t)$, $z_F(t)$, and the filtering error $\bar{z}(t)$. The disturbance signal is shown in Figure 8. From Figures 5-7, it is easy to see that the designed nonfragile filter satisfies the system requirements.

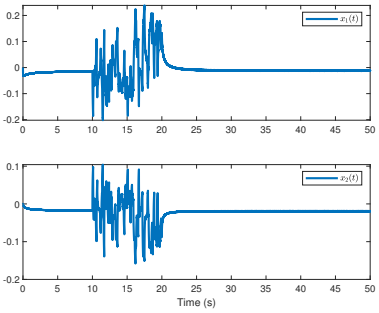


Figure 5: State response $x(t)$ in Example 2

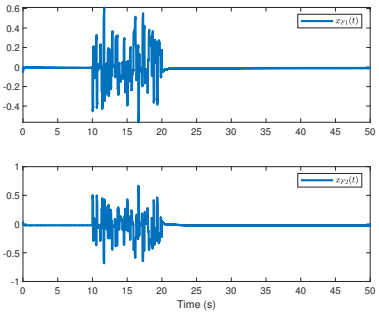


Figure 6: State response $x_F(t)$ in Example 2

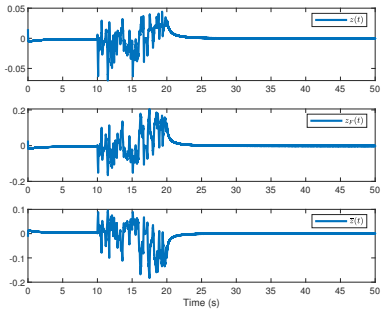


Figure 7: Response $z(t)$, $z_F(t)$ and $\bar{z}(t)$ in Example 2

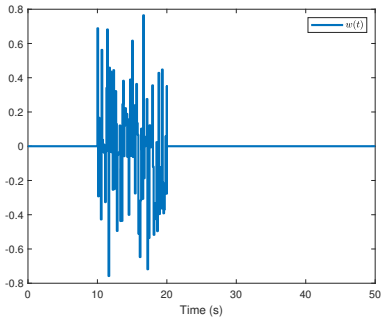


Figure 8: Disturbance input $w(t)$ in Example 2

Remark 5.2. *From Example 1 and Example 2, when the disturbance input $w(t)$ has been added, the estimated output $z(t)$ is in a bounded range. In addition, the estimation error converges to zero in the region where the disturbance input is equal to zero. The size of the disturbance input has little effect on the estimation error.*

6 Conclusions

In the paper, robust nonfragile H_∞ filtering for fuzzy fractional order (FFO) systems $0 < \alpha < 1$ with uncertainties is studied. First, a new sufficient condition of H_∞ control for fractional order systems is proved based on the linear matrix inequality (LMI) approach, which overcomes the shortcoming of solving the complex matrix. Then, based on the condition, the LMI-based condition of H_∞ control for FFO uncertain systems is proposed and the state feedback controller is designed. Furthermore, a nonfragile filter is designed for FFO systems. Then, the conditions of nonfragile H_∞ filtering for FFO systems are derived. Finally, two examples are given to verify the effectiveness of the proposed conditions.

Acknowledgement

This study was supported by the National Natural Science Foundation of China (Grant No.62027812, 62273185, 62273186).

The authors wish to express their appreciation for several excellent suggestions for improvements in this paper made by the referees.

References

[1] Y. Boukal, M. Darouach, M. Zasadzinski, N. E. Radhy, *Robust H_∞ observer based control of fractional-order systems with gain parametrization*, IEEE Transactions on Automatic Control, **62**(11) (2017), 5710-5723. https://ieeexplore.ieee.org/abstract/document/7890403

[2] N. A. Camacho, M. D. Mermoud, J. A. Gallegos, *Lyapunov functions for fractional order systems*, Communications in Nonlinear Science and Numerical Simulation, **19** (2014), 2951-2957. https://doi.org/10.1016/j.cnsns.2014.01.022

[3] L. P. Chen, R. C. Wu, L. G. Yuan, L. S. Yin, Y. Q. Chen, S. Q. Xu, *Guaranteed cost control of fractional-order linear uncertain systems with time-varying delay*, Optimal Control Applications and Methods, **42**(4) (2021), 1102-1118. https://doi.org/10.1002/oca.2718

[4] R. J. Cui, J. G. Lu, *H_-/H_∞ fault detection observer design for fractional-order singular systems in finite frequency domains*, ISA Transactions, **129**(8) (2022), 100-109. https://doi.org/10.1016/j.isatra.2022.02.04

[5] C. Farges, L. Fadiga, J. Sabatier, *H_∞ analysis and control of commensurate fractional order systems*, Mechatronics, **23**(7) (2013), 772-780. https://doi.org/10.1016/j.mechatronics.2013.06.005

[6] C. Farges, M. Moze, J. Sabatier, *Pseudo-state feedback stabilization of commensurate fractional order systems*, Automatica, **46** (2010), 1730-1734. https://ieeexplore.ieee.org/abstract/document/4982688

[7] Z. G. Feng, P. Shi, *Admissibilization of singular interval-valued fuzzy systems*, IEEE Transactions on Fuzzy Systems, **25**(6) (2017), 1765-1776. https://ieeexplore.ieee.org/abstract/document/7762091

[8] S. M. Hu, L. Y. Chen, H. Liu, *Adaptive fuzzy variable structure control of fractional-order nonlinear systems with input nonlinearities*, International Journal of Fuzzy Systems, **23**(7) (2021), 2309-2323. https://doi.org/10.1007/s40815-020-00814-z

[9] C. Hua, S. Wu, X. Guan, *Stabilization of T-S fuzzy system with time delay under sampled-data control using a new looped-functional*, IEEE Transactions on Fuzzy Systems, **28**(2) (2020), 400-407. https://ieeexplore.ieee.org/abstract/document/8668823

[10] D. C. Huong, M. V. Thuan, *Mixed H_∞ and passive control for fractional-order nonlinear systems via LMI approach*, Acta Applicandae Mathematicae, **170** (2020), 37-52. https://doi.org/10.1007/s10440-020-00323-z

[11] Y. D. Ji, L. Q. Su, J. Q. Qiu, *Design of fuzzy output feedback stabilization for uncertain fractional-order systems*, Neurocomputing, **173**(3) (2016), 1683-1693. https://ieeexplore.ieee.org/abstract/document/8788632

[12] M. A. L. Khaniki, M. Manthouri, M. A. Khanesar, *Adaptive non-singular fast terminal sliding mode control and synchronization of a chaotic system via interval type-2 fuzzy inference system with proportionate controller*, Iranian Journal of Fuzzy Systems, **20**(6) (2023), 171-185. https://doi.org/10.22111/IJFS.2023.39658.6889

[13] Y. Li, Y. Q. Chen, I. Podlubny, *Mittag-Leffler stability of fractional order nonlinear dynamic systems*, Automatica, **45**(8) (2009), 1965-1969. https://doi.org/10.1016/j.automatica.2009.04.003

[14] H. Li, G. H. Yang, *Dynamic output feedback H_∞ control for fractional-order linear uncertain systems with actuator faults*, Journal of the Franklin Institute, **356**(8) (2019), 442-446. https://doi.org/10.1016/j.jfranklin.2019.04.004

[15] S. Liang, Y. H. Wei, J. W. Pan, Q. Gao, Y. Wang, *Bounded real lemmas for fractional order systems*, International Journal of Automation and Computing, **12**(2) (2015), 192-198. https://doi.org/10.1007/s11633-014-0868-4

[16] L. P. Liu, X. Y. Cao, Z. M. Fu, S. Z. Song, H. Xing, *Guaranteed cost finite-time control of fractional-order nonlinear positive switched systems with D-perturbations via MDT*, Journal of Systems Science and Complexity, **32** (2019), 857-874. https://doi.org/10.1007/s11424-018-7347-x

[17] J. G. Lu, Y. Q. Chen, *Robust stability and stabilization of fractionalorder interval systems with the fractional order α: The $0 < \alpha < 1$ case*, IEEE Transactions on Automatic Control, **55**(1) (2010), 152-158. https://ieeexplore.ieee.org/abstract/document/4982688

[18] P. Mani, R. Rajan, L. Shanmugam, H. Y. Joo, *Adaptive fractional fuzzy integral sliding mode control for PMSM model*, IEEE Transactions on Fuzzy Systems, **27**(8) (2019), 1674-1686. https://ieeexplore.ieee.org/abstract/document/8571254

[19] J. H. Ning, C. C. Hua, *H_∞ output feedback control for fractional-order T-S fuzzy model with time-delay*, Applied Mathematics and Computation, **416** (2022), 126736. https://doi.org/10.1016/j.amc.2021.126736

[20] I. Podlubny, *Geometric and physical interpretation of fractional integration and fractional differentiation*, Fractional Calculus and Applied Analysis, **5** (2002), 367-386. https://doi.org/10.48550/arXiv.math/0110241

[21] R. E. Precup, A. T. Nguyen, S. Blažič, *A survey on fuzzy control for mechatronics applications*, International Journal of Systems Science, **55**(4) (2023), 771-813. https://doi.org/10.1080/00207721.2023.2293486

[22] R. E. Precup, S. Preitl, E. Petriu, C. A. Bojan-Dragos, A. I. Szedlak-Stinean, R. C. Roman, E. L. Hedrea, *Model-based fuzzy control results for networked control systems*, Reports in Mechanical Engineering, **1**(1) (2020), 10-25. https://doi.org/10.31181/rme200101010p

[23] C. Rao, S. Mitra, *Generalized inverse of matrices and its applications*, Wiley, 1971. https://doi.org/10.1525/9780520 325883-032

[24] W. J. Ren, N. Hou, Q. Wang, Y. Lu, X. H. Liu, *Non-fragile H_∞ filtering for nonlinear systems with randomly occurring gain variations and channel fadings*, Neurocomputing, **156**(25) (2015), 176-185. https://doi.org/10.1016/j.neucom.2014.12.065

[25] A. Sala, C. Arino, *Relaxed stability and performance LMI conditions for Takagi-Sugeno fuzzy systems with polynomial constraints on membership function shapes*, IEEE Transactions on Fuzzy Systems, **16**(5) (2008), 1328-1336. https://ieeexplore.ieee.org/abstract/document/4534857

[26] K. Tanaka, T. Hori, H. O. Wang, *A multiple Lyapunov function approach to stabilization of fuzzy control systems*, IEEE Transactions on Fuzzy Systems, **11**(4) (2003), 582-589. https://ieeexplore.ieee.org/abstract/document/1220303

[27] X. N. Tang, D. Zhai, Z. M. Fu, H. M. Wang, *Output feedback adaptive fuzzy control for uncertain fractional-order nonlinear switched system with output quantization*, International Journal of Fuzzy Systems, **22**(3) (2020), 943-955. https://doi.org/10.1007/s40815-020-00814-z

[28] M. L. Tomescu, S. Preitl, R. E. Precup, J. K. Tar, *Stability analysis method for fuzzy control systems dedicated controlling nonlinear processes*, Acta Polytechnica Hungarica, **4**(3) (2007), 127-141. https://acta.uni-obuda.hu//Issue11.htm

[29] L. X. Wang, *Hierarchical fuzzy opinion networks: Top-down for social organizations and bottom-up for election*, IEEE Transactions on Fuzzy Systems, **28**(7) (2020), 1265-1275. https://ieeexplore.ieee.org/abstract/document/8957283

[30] Z. Wang, D. Y. Xue, F. Pan, *Observer-based robust control for singular switched fractional order systems subject to actuator saturation*, Applied Mathematics and Computation, **411** (2021), 126538. https://doi.org/10.1016/j.amc.2021.126538

[31] H. N. Wu, H. X. Li, *H_∞ fuzzy observer-based control for a class of nonlinear distributed parameter systems with control constraints*, IEEE Transactions on Fuzzy Systems, **16**(2) (2008), 502-516. https://ieeexplore.ieee.org/abstract/document/4469891

[32] H. N. Wu, J. W. Wang, H. X. Li, *Design of distributed H_∞ fuzzy controllers with constraint for nonlinear hyperbolic PDE systems*, Automatica, **48**(10) (2012), 2535-2543. https://doi.org/10.1016/j.automatica.2012.06.043

[33] L. H. Xie, *Output feedback H_∞ control of systems with parameter uncertainty*, International Journal of Control, **63**(4) (1996), 741-750. https://doi.org/10.1080/00207179608921866

[34] W. Xie, Y. Wang, J. Zhang, M. Fu, *Novel separation principle based H_∞ observer controller design for a class of T-S fuzzy systems*, IEEE Transactions on Fuzzy Systems, **26**(6) (2018), 3206-3221. https://ieeexplore.ieee.org/abstract/document/8314705

[35] Y. Q. Yan, H. G. Zhang, Z. Y. Ming, Y. C. Wang, *Observer-based adaptive control and faults estimation for T-S fuzzy singular fractional order systems*, Neural Computing and Applications, **34** (2022), 4265-4275. https://doi.org/10.1007/s00521-021-06527-0

[36] X. F. Zhang, W. K. Huang, Q. G. Wang, *Robust H_∞ adaptive sliding mode fault tolerant control for T-S fuzzy fractional order systems with mismatched disturbances*, IEEE Transactions on Circuits and Systems I: Regular Papers, **68**(3) (2021), 1297-1307. https://ieeexplore.ieee.org/abstract/document/9281103

[37] X. F. Zhang, K. J. Jin, *Design of non-fragile controller for singular fractional order Takagi-Sugeno fuzzy systems*, International Journal of Fuzzy Systems, **22**(4) (2020), 1289-1298. https://doi.org/10.1007/s40815-020-00822-z

[38] X. F. Zhang, C. Lin, Y. Q. Chen, D. Boutat, *A unified framework of stability theorems for LTI fractional order systems with* $0 < \alpha < 2$, IEEE Transactions on Circuits and Systems II: Express Briefs, **67**(12) (2020), 3237-3241. https://ieeexplore.ieee.org/abstract/document/9026796

[39] Q. H. Zhang, J. G. Lu, *H_∞ control for singular fractional-order interval systems: The $0 < \alpha < 1$ case*, ISA Transactions, **110** (2021), 105-116. https://doi.org/10.1016/j.isatra.2020.10.003

[40] Y. Zhang, Y. Ma, L. Fu, W. Zhao, X. Huang, *Finite-time non-fragile H_∞ sampled-data control for uncertain T-S fuzzy system with time-varying delay and nonlinear perturbation subject to Markovian jump*, ISA Transactions, **99** (2020), 59-73. https://doi.org/10.1016/j.isatra.2019.09.007

[41] X. F. Zhang, Z. Wang, *Stability and robust stabilization of uncertain switched fractional order systems*, ISA Transactions, **103** (2020), 1-9. https://doi.org/10.1016/j.isatra.2020.03.019