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Bipolar fuzzy Fourier transform for bipolar fuzzy solution of the bipolar fuzzy heat equation

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Abstract

This article presents the exact solution of a bipolar fuzzy heat equation based on bipolar fuzzy Fourier transform under generalized Hukuhara partial (𝔤ℌ-p) differentiability. A bipolar fuzzy Fourier transform is defined, and the related key propositions and fundamental characteristics are discussed. Further, a bipolar fuzzy heat equation model is constructed using 𝔤ℌ-differentiability, and the analytical solution of a bipolar fuzzy heat equation with bipolar fuzzy Fourier transform approach is examined. Some illustrative examples are provided to check the suggested methodology's liability and efficiency. The type of differentiability and the solution of the bipolar fuzzy heat equation are shown graphically, demonstrating the versatility of the proposed methodology and elucidating the impact of differentiability types on the solution behavior of the bipolar fuzzy heat equation. Additionally, the impact of different parameters on the solution behavior is analyzed, revealing insights into the underlying dynamics.

Keywords: Bipolar fuzzy sets, generalized Hukuhara differentiability, heat equation, Fourier transform.

1 Introduction

After introducing fuzzy sets (FSs) by Zadeh [43], people began to realize how uncertainty originating from the human mind affects scientific problems. In 1978, Dubois and Prade [23] considered the term of fuzzy number (FN). Following [20, 24, 25], Dubois and Prade[26] provided the idea of fuzzy derivative. Later, Zhang [44] first proposed the notion of YinYang bipolar fuzzy sets (BFSs), which deal with dual-sided information. Positive information reveals what is accepted as feasible, whereas negative information reveals what is suggested to be impracticable to contemplate. A BFS is an extension of FS. In 2019, Akram and Arshad [4] introduced the concept of bipolar fuzzy numbers. A Bipolar fuzzy mathematical model is more appealing to researchers than the fuzzy model.

While modeling a real-world scenario using differential equations (DEs), we notice that the parameters and the variables involved in the equations are inexact, uncertain, or imprecise. This imprecision and uncertainty are expressed via FNs. The fuzzy differential equations (FDEs) can be used to avoid this ambiguity or uncertainty. Fuzzy set theory has many applications in various fields as provided by [22, 29, 35]. In 1987, Kaleva [28] investigated the solution of FDEs. In 1999, Buckley and Feuring [19] suggested the notion of fuzzy partial differential equations (FPDEs). Puri and Ralescu [33] proposed the definition of Hukuhara differentiability (H-differentiability) of a fuzzy valued function (FVF). The H-differentiability was the foundation for studying FDEs and, subsequently, FPDEs. Bede and Gal [14] demonstrated a strong idea of generalized Hukuhara differentiability (𝔤ℌ-differentiability) for FVF. Later, H-derivatives were considered to define the strongly generalized Hukuhara differentiability (SGH-differentiability). As a result, Stefanini and Bede [41] considered a generalization of H-difference. The concept of 𝔤ℌ-difference suggested by Stefanini [40], and the 𝔤ℌ-differentiability for FVF was considered by studying the concept of 𝔤ℌ-difference given by Bede and Stefanini [16]. In

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2021, Saqib et al. [37, 38] provided the notion of $\mathfrak{gH}$-differentiability for bipolar FVF's and suggested some techniques to investigate the numerical solution of bipolar fuzzy initial value problem. Several remarkable achievement regrading the FDEs of fractional order using $\mathfrak{gH}$-differentiability can be seen in [5, 7, 8]. Table 1 lists the contributions to solving ordinary differential equations (ODEs) and PDEs by using different approaches in fuzzy environments.

Table 1: List of relevant contributions in different environments

Reference	Year	Contribution
Oberguggenberger [31]	2004	Discussed fuzzy and weak solution of DEs
Bede et al. [15]	2005	Considered an application of fuzzy Fourier transform(FFT)
Allahviranloo [9]	2006	Solved FPDEs using difference methods
Allahviranloo and Taheri [13]	2009	Used Adomian decomposition approach to solved heat equation
Allahviranloo and Afshar [11]	2010	Provided a method to solve linear FPDEs
Bede and Stefanini [16]	2013	Developed the concept of $\mathfrak{gH}$-differentiability for FVFs
Allahviranloo et al. [12]	2015	Proposed fuzzy solution for fuzzy heat equation(FHE)
Gouyandeh et al.[27]	2017	Solved FHE by FFT
Akram et al. [6]	2019	Introduced bipolar fuzzy linear system of equations(BFLSEs)
Saqib et al.[36]	2020	Developed iterative methods for BFLSEs
Ahmad et al. [1]	2021	Discussed the solutions of 3rd-order fuzzy dispersive PDEs
Nagalakshmi and Kumar [30]	2021	Solved FHE under $\mathfrak{gH}$-differentiability
Akram et al. [2]	2022	Developed a method for solving bipolar fuzzy complex linear systems
Ullah et al. [42]	2023	Investigated the solution of fuzzy complex population dynamical model
Qayyum et al. [34]	2024	Provided Fuzzy-fractional modeling of Korteweg-de Vries equations

The main motivation behind this work is to extend the concept of FFT to a bipolar FFT and determine a bipolar fuzzy solution to heat equation problems. This concept results from establishing a few characteristics of the multivariate BFVF's $\mathfrak{gH}$-p differentiability.
The key desires that motivated the recommended formulation are as follows:

- Investigate the behavior of the heat problem's positive and negative membership function to understand better how bipolar fuzzy variables affect the solution.
- The $\mathfrak{gH}$-differentiability concept of FVF allows us to expand the work of Gouyandeh et al. [27] in a bipolar fuzzy environment. This addition simplifies the use of sophisticated mathematical approaches to bipolar fuzzy systems. Item Develop an algorithm to solve heat problems under $\mathfrak{gH}$-differentiability in a bipolar fuzzy environment. This algorithmic development seeks to provide an efficient and systematic approach to solving heat equation problems, including bipolar fuzzy variables.
- Define switching boundary for multivariate BFVFs.

The main objective of this article is to utilize bipolar FFT to solve a bipolar fuzzy heat equation (BFHE). For this, we define improper integral for multivariate BFVFs and demonstrate various aspects of $\mathfrak{gH}$-p differentiability. These hypotheses are utilized to implement the bipolar FFT.
This study has made the following contributions:

- The multivariate calculus concept of BFVF with useful features is studied.
- Improper integrals for multivariate BFVFs for solving BFHE are discussed.
- The concept of bipolar FFT for BFVFs is provided.
- The model for finding the bipolar solution of the BFHE using the bipolar FFT method under $\mathfrak{gH}$-differentiability is suggested.
- The efficiency and the liability of the proposed methodology and the concept of switching boundaries are illustrated with examples.
- The results are presented graphically, which not only enhances the precision and physical situations of the theoretical conclusion as well as an understanding of the proposed technique but also serves as visual validation of its efficiency in capturing the complex structure of bipolar fuzzy systems.

The rest of the article is organized as follows: Section 2 contains some basic ideas. Section 3 considers multivariate calculus for BFVFs and presents numerous characteristics. A bipolar FFT for BFVFs is presented in Section 4, along with some properties. In Section 5, $\mathfrak{gH}$-differentiability is used to construct a BFHE model, and a bipolar FFT technique is used to solve a BFHE. Various examples are provided in Section 6 to illustrate the findings. Section 7 contains the conclusions based on the current study.

2 Preliminary concepts

Definition 2.1. [44] *Given a universal set M, a bipolar fuzzy set $\widetilde{\mathfrak{S}}$ in M is an object of the form*

$$\widetilde{\mathfrak{S}} = \{< \mathfrak{z}, \mu^P_{\widetilde{\mathfrak{S}}}(\mathfrak{z}), \mu^N_{\widetilde{\mathfrak{S}}}(\mathfrak{z}) > \mid \mathfrak{z} \in M\},$$

where the mappings $\mu^P_{\widetilde{\mathfrak{S}}} : M \to [0,1]$ and $\mu^N_{\widetilde{\mathfrak{S}}} : M \to [-1,0]$ represent the degree of satisfaction and the degree of dissatisfaction, respectively.

Definition 2.2. [4] *Suppose $\widetilde{\Omega}$ is a bipolar fuzzy number(BFN). For $\tau \in [0,1]$, $\kappa \in [-1,0]$, the (τ,κ)-cut of $\widetilde{\Omega}$, denoted by $\widetilde{\Omega}^{(\tau,\kappa)}$, and defined as:*

$$\widetilde{\Omega}^{(\tau,\kappa)} = \{\mathfrak{z} \in \mathbb{R} \mid \mu^P_{\widetilde{\Omega}}(\mathfrak{z}) \geq \tau, \mu^N_{\widetilde{\Omega}}(\mathfrak{z}) \leq \kappa\}.$$

Set of all BFNs is denoted by $\mathbb{R}_{\mathcal{B}}$.

Definition 2.3. [6] *A BFN Ω in parametric form, is a quadrivial $\prec [\underline{\Omega}^P, \overline{\Omega}^P], [\underline{\Omega}^N, \overline{\Omega}^N] \succ$ of the mappings*

$$\underline{\Omega}^P(\imath), \overline{\Omega}^P(\imath), \underline{\Omega}^N(\mathfrak{j}), \overline{\Omega}^N(\mathfrak{j}); \quad 0 \leq \imath \leq 1, -1 \leq \mathfrak{j} \leq 0,$$

which satisfies the following conditions:

(i) $\underline{\Omega}^P(\imath)$ *is left-continuous, bounded, monotonically non-decreasing mapping on* $(0,1]$ *and right-continuous at* 0,

(ii) $\overline{\Omega}^P(\imath)$ *is left-continuous, bounded, monotonically non-increasing mapping on* $(0,1]$ *and right-continuous at* 0,

(iii) $\underline{\Omega}^N(\mathfrak{j})$ *is left-continuous, bounded, monotonically non-increasing mapping on a* $(-1,0]$ *and right-continuous at* -1,

(iv) $\overline{\Omega}^N(\mathfrak{j})$ *is left-continuous, bounded, monotonically non-decreasing mapping on a* $(-1,0]$ *and right-continuous at* -1,

(v) $\underline{\Omega}^P(\imath) \leq \overline{\Omega}^P(\imath)$,

(vi) $\underline{\Omega}^N(\mathfrak{j}) \leq \overline{\Omega}^N(\mathfrak{j})$.

Definition 2.4. [37] *For $\Omega = \prec [\underline{\Omega}^P(\imath), \overline{\Omega}^P(\imath)], [\underline{\Omega}^N(\mathfrak{j}), \overline{\Omega}^N(\mathfrak{j})] \succ$ and $\mathfrak{U} = \prec [\underline{\mathfrak{U}}^P(\imath), \overline{\mathfrak{U}}^P(\imath)], [\underline{\mathfrak{U}}^N(\mathfrak{j}), \overline{\mathfrak{U}}^N(\mathfrak{j})] \succ$ two BFNs, a metric $\mathfrak{d} : \mathbb{R}_{\mathcal{B}} \times \mathbb{R}_{\mathcal{B}} \to \mathbb{R}^+ \cup \{0\}$ is defined as:*

$$\begin{aligned} \mathfrak{d}(\Omega, \mathfrak{U}) \quad = \quad & \max[\max_{0 \leq \imath \leq 1}\{|\underline{\Omega}^P(\imath) - \underline{\mathfrak{U}}^P(\imath)|, |\overline{\Omega}^P(\imath) - \overline{\mathfrak{U}}^P(\imath)|\}, \\ & \max_{-1 \leq \mathfrak{j} \leq 0}\{|\underline{\Omega}^N(\mathfrak{j}) - \underline{\mathfrak{U}}^N(\mathfrak{j})|, |\overline{\Omega}^N(\mathfrak{j}) - \overline{\mathfrak{U}}^N(\mathfrak{j})|\}]. \end{aligned}$$

Metric space $(\mathbb{R}_{\mathcal{B}}, \mathfrak{d})$ is complete, locally compact, separable. For $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S} \in \mathbb{R}_{\mathcal{B}}$ and $\forall\ k \in \mathbb{R}$, the following axioms are valid for metric $\mathfrak{d}$:

(i) $\mathfrak{d}(k\mathcal{P}, k\mathcal{Q}) = |k|\mathfrak{d}(\mathcal{P}, \mathcal{Q})$,

(ii) $\mathfrak{d}(\mathcal{P} \oplus \mathcal{R}, \mathcal{Q} \oplus \mathcal{R}) = \mathfrak{d}(\mathcal{P}, \mathcal{Q})$,

(iii) $\mathfrak{d}(\mathcal{P} \ominus \mathcal{Q}, \mathcal{R} \ominus \mathcal{S}) \leq \mathfrak{d}(\mathcal{P}, \mathcal{R}) + \mathfrak{d}(\mathcal{Q}, \mathcal{S})$,

(iv) $\mathfrak{d}(\mathcal{P} \oplus \mathcal{Q}, \mathcal{R} \oplus \mathcal{S}) \leq \mathfrak{d}(\mathcal{P}, \mathcal{R}) + \mathfrak{d}(\mathcal{Q}, \mathcal{S})$.

Where, $\ominus$ is the H-difference and $\mathcal{P} \oplus \mathcal{Q} = \mathcal{R} \Leftrightarrow \mathcal{R} \ominus \mathcal{Q} = \mathcal{P}$.

Definition 2.5. [37] *For two BFNs $\Omega, \mathfrak{U} \in \mathbb{R}_{\mathcal{B}}$, the $\mathfrak{gH}$-difference is defined as:*

$$\Omega \ominus_{\mathfrak{gH}} \mathfrak{U} = \mathfrak{C} \Leftrightarrow \begin{cases} (\mathfrak{X}_0)\Omega = \mathfrak{U} \oplus \mathfrak{C}, \\ (\mathfrak{Z}_0)\mathfrak{U} = \Omega \oplus (-1)\mathfrak{C}. \end{cases}$$

$\Omega \ominus_{\mathfrak{gH}} \mathfrak{U}$ will exist if $\Omega^P \ominus \mathfrak{U}^P$ and $\Omega^N \ominus \mathfrak{U}^N$ exist, which implies

$$\Omega^P \ominus_{\mathfrak{gH}} \mathfrak{U}^P = \mathfrak{C}^P \Leftrightarrow \begin{cases} (\mathfrak{X}_1)\Omega^P = \mathfrak{U}^P \oplus \mathfrak{C}^P, \\ (\mathfrak{Z}_1)\mathfrak{U}^P = \Omega^P \oplus (-1)\mathfrak{C}^P, \end{cases}$$

and

$$\Omega^N \ominus_{\mathfrak{gH}} \mathfrak{U}^N = \mathfrak{C}^N \Leftrightarrow \begin{cases} (\mathfrak{X}_2)\Omega^N = \mathfrak{U}^N \oplus \mathfrak{C}^N, \\ (\mathfrak{Z}_2)\mathfrak{U}^N = \Omega^N \oplus (-1)\mathfrak{C}^N. \end{cases}$$

Proposition 2.6. *For $\dot{\mathfrak{a}}, \ddot{\mathfrak{a}} \in \mathbb{R}_{\mathcal{B}}$,*

$$\mathfrak{d}(\dot{\mathfrak{a}} \ominus \ddot{\mathfrak{a}}, 0) = \mathfrak{d}(\dot{\mathfrak{a}}, \ddot{\mathfrak{a}}).$$

Proposition 2.7. [37] *Let $\dot{\mathfrak{a}} \in \mathbb{R}_{\mathcal{B}}$ and $k_1, k_2 \in \mathbb{R}$ such that $k_1 k_2 \geq 0$, then*

$$\mathfrak{d}(k_1 \odot \dot{\mathfrak{a}} \ominus k_2 \odot \dot{\mathfrak{a}}, 0) = |k_1 - k_2| \mathfrak{d}(\dot{\mathfrak{a}}, 0).$$

Definition 2.8. [37] *Let $\pounds : \mathcal{I} \to \mathbb{R}_{\mathcal{B}}$ be BFVF, then (τ, κ)-cut of $\pounds$ is given by*

$$[\pounds]^{(\tau,\kappa)} = \prec [\underline{\pounds}^P(\xi;\tau), \overline{\pounds}^P(\xi;\tau)], [\underline{\pounds}^N(\xi;\kappa), \overline{\pounds}^N(\xi;\kappa)] \succ,$$

where

$$\begin{aligned} \underline{\pounds}^P(\xi;\tau) &= \min\{\pounds^P(\xi;\tau) : \xi \in \mathcal{I}, 0 \leq \tau \leq 1\}, \\ \overline{\pounds}^P(\xi;\tau) &= \max\{\pounds^P(\xi;\tau) : \xi \in \mathcal{I}, 0 \leq \tau \leq 1\}, \\ \underline{\pounds}^N(\xi;\kappa) &= \min\{\pounds^N(\xi;\kappa) : \xi \in \mathcal{I}, 0 \leq \kappa \leq 1\}, \\ \overline{\pounds}^N(\xi;\kappa) &= \max\{\pounds^N(\xi;\kappa) : \xi \in \mathcal{I}, 0 \leq \kappa \leq 1\}, \end{aligned}$$

and

$$\underline{\pounds}^P(\xi;\tau) \leq \overline{\pounds}^P(\xi;\tau) \leq \underline{\pounds}^N(\xi;\kappa) \leq \overline{\pounds}^N(\xi;\kappa).$$

Definition 2.9. [38] *Let $\pounds : \mathcal{I} \to \mathbb{R}_{\mathcal{B}}$ and $\xi_0 \in \mathcal{I}$, then $\pounds$ is called generalized Hukuhara differentiable (ghd) at $\xi_0 \in \mathcal{I}$ if there is $\pounds'(\xi_0) \in \mathbb{R}_{\mathcal{B}}$ such that*

$$\lim_{\mathfrak{k} \to 0^+} \frac{\pounds(\xi_0 + \mathfrak{k}) \ominus_{\mathfrak{gH}} \pounds(\xi_0)}{\mathfrak{k}}$$

exists and

$$\pounds'(\xi_0) = \lim_{\mathfrak{k} \to 0^+} \frac{\pounds(\xi_0 + \mathfrak{k}) \ominus_{\mathfrak{gH}} \pounds(\xi_0)}{\mathfrak{k}}.$$

Definition 2.10. [37] *A BFVF $\pounds : \mathcal{I} \to \mathbb{R}_{\mathcal{B}}$ ia called*

(i) *$(*)$-ghd if for $w > 0$, the differences $\pounds^P(\xi_0 + w) \ominus_{\mathfrak{gH}} \pounds^P(\xi_0)$ and $\pounds^N(\xi_0 + w) \ominus_{\mathfrak{gH}} \pounds^N(\xi_0)$ hold as case $(\mathfrak{X}_1)$ and $(\mathfrak{X}_2)$ of Definition 2.5, respectively, the limits*

$$\begin{cases} \lim\limits_{w \to 0^+} \frac{\pounds^P(\xi_0 + w) \ominus_{\mathfrak{gH}} \pounds^P(\xi_0)}{w}, \\ \lim\limits_{w \to 0^+} \frac{\pounds^N(\xi_0 + w) \ominus_{\mathfrak{gH}} \pounds^N(\xi_0)}{w}, \end{cases}$$

hold, and

$$\begin{cases} \pounds'^P_{(*)-\mathfrak{gH}}(\xi_0) = \lim\limits_{w \to 0^+} \frac{\pounds^P(\xi_0 + w) \ominus_{\mathfrak{gH}} \pounds^P(\xi_0)}{w}, \\ \pounds'^N_{(*)-\mathfrak{gH}}(\xi_0) = \lim\limits_{w \to 0^+} \frac{\pounds^N(\xi_0 + w) \ominus_{\mathfrak{gH}} \pounds^N(\xi_0)}{w}. \end{cases}$$

(ii) $(*,**)$-*ghd if for* $w>0$, *the differences* $\pounds^P(\xi_0+w)\ominus_{\mathfrak{gH}}\pounds^P(\xi_0)$ *and* $\pounds^N(\xi_0+w)\ominus_{\mathfrak{gH}}\pounds^N(\xi_0)$ *hold as case* $(\mathfrak{X}_1)$ *and* $(\mathfrak{Z}_2)$ *of Definition 2.5, respectively, the limits*

$$\begin{cases}\lim\limits_{w\to 0^+}\frac{\pounds^P(\xi_0+w)\ominus_{\mathfrak{gH}}\pounds^P(\xi_0)}{w},\\ \\ \lim\limits_{w\to 0^+}\frac{\pounds^N(\xi_0+w)\ominus_{\mathfrak{gH}}\pounds^N(\xi_0)}{w},\end{cases}$$

hold, and

$$\begin{cases}\pounds'^P_{(*)-\mathfrak{gH}}(\xi_0)=\lim\limits_{w\to 0^+}\frac{\pounds^P(\xi_0+w)\ominus_{\mathfrak{gH}}\pounds^P(\xi_0)}{w},\\ \\ \pounds'^N_{(**)-\mathfrak{gH}}(\xi_0)=\lim\limits_{w\to 0^+}\frac{\pounds^N(\xi_0+w)\ominus_{\mathfrak{gH}}\pounds^N(\xi_0)}{w}.\end{cases}$$

(iii) $(**,*)$-*ghd if for* $w>0$, *the differences* $\pounds^P(\xi_0+w)\ominus_{\mathfrak{gH}}\pounds^P(\xi_0)$ *and* $\pounds^N(\xi_0+w)\ominus_{\mathfrak{gH}}\pounds^N(\xi_0)$ *hold as case* $(\mathfrak{X}_2)$ *and* $(\mathfrak{Z}_1)$ *of Definition 2.5, respectively, the limits*

$$\begin{cases}\lim\limits_{w\to 0^+}\frac{\pounds^P(\xi_0+w)\ominus_{\mathfrak{gH}}\pounds^P(\xi_0)}{w},\\ \\ \lim\limits_{w\to 0^+}\frac{\pounds^N(\xi_0+w)\ominus_{\mathfrak{gH}}\pounds^N(\xi_0)}{w},\end{cases}$$

hold, and

$$\begin{cases}\pounds'^P_{(**)-\mathfrak{gH}}(\xi_0)=\lim\limits_{w\to 0^+}\frac{\pounds^P(\xi_0+w)\ominus_{\mathfrak{gH}}\pounds^P(\xi_0)}{w},\\ \\ \pounds'^N_{(*)-\mathfrak{gH}}(\xi_0)=\lim\limits_{w\to 0^+}\frac{\pounds^N(\xi_0+w)\ominus_{\mathfrak{gH}}\pounds^N(\xi_0)}{w}.\end{cases}$$

(iv) $(**)$-*ghd if for* $w>0$, *the differences* $\pounds^P(\xi_0+w)\ominus_{\mathfrak{gH}}\pounds^P(\xi_0)$ *and* $\pounds^N(\xi_0+w)\ominus_{\mathfrak{gH}}\pounds^N(\xi_0)$ *hold as case* $(\mathfrak{X}_2)$ *and* $(\mathfrak{Z}_2)$ *of Definition 2.5, respectively, and the limits*

$$\begin{cases}\lim\limits_{w\to 0^+}\frac{\pounds^P(\xi_0+w)\ominus_{\mathfrak{gH}}\pounds^P(\xi_0)}{w},\\ \\ \lim\limits_{w\to 0^+}\frac{\pounds^N(\xi_0+w)\ominus_{\mathfrak{gH}}\pounds^N(\xi_0)}{w},\end{cases}$$

hold, and

$$\begin{cases}\pounds'^P_{(**)-\mathfrak{gH}}(\xi_0)=\lim\limits_{w\to 0^+}\frac{\pounds^P(\xi_0+w)\ominus_{\mathfrak{gH}}\pounds^P(\xi_0)}{w},\\ \\ \pounds'^N_{(**)-\mathfrak{gH}}(\xi_0)=\lim\limits_{w\to 0^+}\frac{\pounds^N(\xi_0+w)\ominus_{\mathfrak{gH}}\pounds^N(\xi_0)}{w},\end{cases}$$

Definition 2.11. [37] *Let* $\pounds:\mathcal{I}\to\mathbb{R}_{\mathcal{B}}$ *and* $[\pounds(\xi)]^{(\tau,\kappa)}=\prec[\underline{\pounds}^P(\xi;\tau),\overline{\pounds}^P(\xi;\tau)][\underline{\pounds}^N(\xi;\kappa),\overline{\pounds}^N(\xi;\kappa)]\succ$, *for all* $\tau\in[0,1]$ *and* $\kappa\in[-1,0]$,

1. *If* $\pounds$ *is* $(*)$*-ghd then* $\underline{\pounds}^P_\tau,\overline{\pounds}^P_\tau,\underline{\pounds}^N_\kappa$ *and* $\overline{\pounds}^N_\kappa$ *are also differentiable, and*

$$[\pounds'(\xi)]^{(\tau,\kappa)}=\prec[\underline{\pounds}'^P(\xi;\tau),\overline{\pounds}'^P(\xi;\tau)],[\underline{\pounds}'^N(\xi;\kappa),\overline{\pounds}'^N(\xi;\kappa)]\succ.$$

2. *If* $\pounds$ *is* $(*,**)$*-ghd then* $\underline{\pounds}^P_\tau,\overline{\pounds}^P_\tau,\underline{\pounds}^N_\kappa$ *and* $\overline{\pounds}^N_\kappa$ *are also differentiable, and*

$$[\pounds'(\xi)]^{(\tau,\kappa)}=\prec[\underline{\pounds}'^P(\xi;\tau),\overline{\pounds}'^P(\xi;\tau)],[\overline{\pounds}'^N(\xi;\kappa),\underline{\pounds}'^N(\xi;\kappa)]\succ.$$

3. *If* $\pounds$ *is* $(**,*)$*-ghd then* $\underline{\pounds}^P_\tau,\overline{\pounds}^P_\tau,\underline{\pounds}^N_\kappa$ *and* $\overline{\pounds}^N_\kappa$ *are also differentiable, and*

$$[\pounds'(\xi)]^{(\tau,\kappa)}=\prec[\overline{\pounds}'^P(\xi;\tau),\underline{\pounds}'^P(\xi;\tau)],[\underline{\pounds}'^N(\xi;\kappa),\overline{\pounds}'^N(\xi;\kappa)]\succ.$$

4. *If $\pounds$ is $(**)$-ghd then $\underline{\pounds}_{\tau}^{P}, \overline{\pounds}_{\tau}^{P}, \underline{\pounds}_{\kappa}^{N}$ and $\overline{\pounds}_{\kappa}^{N}$ are also differentiable, and*

$$[\pounds'(\xi)]^{(\tau,\kappa)} =\prec [\overline{\pounds}'^{P}(\xi;\tau), \underline{\pounds}'^{P}(\xi;\tau)], [\overline{\pounds}'^{N}(\xi;\kappa), \underline{\pounds}'^{N}(\xi;\kappa)] \succ .$$

Definition 2.12. [37] *Let $\pounds : \mathcal{I} \to \mathbb{R}_{\mathcal{B}}$, a point $\xi_0 \in \mathcal{I}$ is called a switching point for differentiability of $\pounds$ if for any neighborhood ϑ of ξ_0, there are points $\xi_1 < \xi_0 < \xi_2$ such that*

Type (1) *For $\xi = \xi_1$, $\pounds$ is $(*)$-ghd but not $(*,**)$-ghd, for $\xi = \xi_2$, $\pounds$ is $(*,**)$-ghd but not $(*)$-ghd, or*

Type (2) *For $\xi = \xi_1$, $\pounds$ is $(*,**)$-ghd but not $(**)$-ghd, for $\xi = \xi_2$, $\pounds$ is $(**)$-ghd but not $(*,**)$-ghd, or*

Type (3) *For $\xi = \xi_1$, $\pounds$ is $(**,*)$-ghd but not $(*)$-ghd, for $\xi = \xi_2$, $\pounds$ is $(*)$-ghd but not $(**,*)$-ghd, or*

Type (4) *For $\xi = \xi_1$, $\pounds$ is $(**)$-ghd but not $(**,*)$-ghd, for $\xi = \xi_2$, $\pounds$ is $(**,*)$-ghd but not $(**)$-ghd, or*

Type (5) *For $\xi = \xi_1$, $\pounds$ is $(*)$-ghd but not $(**,*)$-ghd, for $\xi = \xi_2$, $\pounds$ is $(**,*)$-ghd but not $(*)$-ghd, or*

Type (6) *For $\xi = \xi_1$, $\pounds$ is $(**,*)$-ghd but not $(**)$-ghd, for $\xi = \xi_2$, $\pounds$ is $(**)$-ghd but not $(**,*)$-ghd, or*

Type (7) *For $\xi = \xi_1$, $\pounds$ is $(*,**)$-ghd but not $(*)$-ghd, for $\xi = \xi_2$, $\pounds$ is $(*)$-ghd but not $(*,**)$-ghd, or*

Type (8) *For $\xi = \xi_1$, $\pounds$ is $(**)$-ghd but not $(*,**)$-ghd, for $\xi = \xi_2$, $\pounds$ is $(**)$-ghd but not $(*,**)$-ghd, or*

Type (9) *For $\xi = \xi_1$, $\pounds$ is $(*)$-ghd but not $(**)$-ghd, for $\xi = \xi_2$, $\pounds$ is $(**)$-ghd but not $(*)$-ghd, or*

Type (10) *For $\xi = \xi_1$, $\pounds$ is $(**)$-ghd but not $(*)$-ghd, for $\xi = \xi_2$, $\pounds$ is $(*)$-ghd but not $(**)$-ghd, or*

Type (11) *For $\xi = \xi_1$, $\pounds$ is $(**,*)$-ghd but not $(*,**)$-ghd, for $\xi = \xi_2$, $\pounds$ is $(*,**)$-ghd but not $(**,*)$-ghd, or*

Type (12) *For $\xi = \xi_1$, $\pounds$ is $(*,**)$-ghd but not $(**,*)$-ghd, for $\xi = \xi_2$, $\pounds$ is $(**,*)$-ghd but not $(*,**)$-ghd.*

Definition 2.13. *Suppose that $\pounds : \mathcal{I} \to \mathbb{R}_{\mathcal{B}}$ is ghd of order $j, j = 0, 1$, at ξ_0 and have no switching point in $[a, b]$, then $\pounds(\xi)$ is ghd of 2nd-order at ξ_0, if $\pounds''(\xi_0) \in \mathbb{R}_{\mathcal{B}}$,*

$$\pounds''(\xi_0) = \lim_{\mathfrak{k}\to 0^+} \frac{\pounds'_{\mathfrak{gH}}(\xi_0 + \mathfrak{k}) \ominus_{\mathfrak{gH}} \pounds'_{\mathfrak{gH}}(\xi_0)}{\mathfrak{k}}$$

exists and $\pounds'_{\mathfrak{gH}}(\xi_0 + \mathfrak{k}) \ominus_{\mathfrak{gH}} \pounds'_{\mathfrak{gH}}(\xi_0) \in \mathbb{R}_{\mathcal{B}}$.

Definition 2.14. [37] *A BFVF $\pounds : \mathcal{I} \to \mathbb{R}_{\mathcal{B}}$ is called Riemann integrable (RI) in $[\dot{c},\ \ddot{c}]$, if $\exists\ \mathbb{J} \in \mathbb{R}_{\mathcal{B}}$ and for any $\epsilon > 0$, $\exists$ a $\delta > 0$, such that for any $\chi : \dot{c} = \xi_0 < \cdots < \xi_m = \ddot{c}$ of $[\dot{c},\ \ddot{c}]$ with $||\chi|| < \delta$ and for all $\xi_{\mathfrak{i}} \in [\xi_{\mathfrak{i}}, \xi_{\mathfrak{i}+1}], \mathfrak{i} = 0, 1, \cdots, m-1$, we have*

$$\mathfrak{d}\left(\sum_{\mathfrak{i}=o}^{m-1} \pounds(\xi_i) \odot (\xi_{i+1} - \xi_i), \mathbb{J}\right) < \epsilon.$$

We write $\mathbb{J} = \int_{\dot{c}}^{\ddot{a}} \pounds(\xi)d\xi$. Thus if $\pounds$ is RI then the functions $\pounds^P$ and $\pounds^N$ are RI and $\mathbb{J}^P = \int_c^d \pounds^P(\xi)d\xi$ and $\mathbb{J}^N = \int_{\dot{c}}^{\ddot{c}} \pounds^N(\xi)d\xi$.

Definition 2.15. *A BFVF $\pounds : \mathbb{R} \to \mathbb{R}_{\mathcal{B}}$ is bipolar fuzzy absolutely integrable (BFAI) at $\xi \in \mathbb{R}$ if $\int_{-\infty}^{\infty} \mathfrak{d}(\pounds(\xi), 0)d\xi < \infty$.*

Definition 2.16. [37] *A BFVF $\pounds : \mathcal{I} \to \mathbb{R}_{\mathcal{B}}$ is said to be continuous at $\xi_0 \in \mathcal{I}$ if for any $\epsilon > 0$, $\exists$ a $\varrho > 0$ such that $\mathfrak{d}(\pounds(\xi), \pounds(\xi_0)) < \epsilon$, whenever $|\xi - \xi_0| < \varrho$.*

Proposition 2.17. *If $\pounds(\xi)$ is bipolar fuzzy continuous in $[c, d]$, it is bipolar fuzzy uniformly continuous in $[c, d]$.*

Proposition 2.18. *If $\pounds$ is ghd and have no switching point in $[c, d]$, then*

$$\int_c^d \pounds'_{\mathfrak{gH}}(\xi)d\xi = \pounds(d) \ominus_{\mathfrak{gH}} \pounds(c).$$

Proposition 2.19. *Let $\pounds : (-\infty, +\infty) \to \mathbb{R}_{\mathcal{B}}$ be a BFVF and it is represented by*

$$[\pounds(\xi)]^{(\tau,\kappa)} =\prec [\underline{\pounds}^P(\xi;\tau), \overline{\pounds}^P(\xi;\tau)], [\underline{\pounds}^N(\xi;\kappa), \overline{\pounds}^N(\xi;\kappa)] \succ,$$

for all $\tau \in [0,1]$ and $\kappa \in [-1,0]$. Suppose that $\underline{\pounds}^P(\xi;\tau), \overline{\pounds}^P(\xi;\tau), \underline{\pounds}^N(\xi;\kappa)$ and $\overline{\pounds}^N(\xi;\kappa)$ are Riemann integrable on $(-\infty,+\infty)$ for all $\tau \in [0,1]$ and $\kappa \in [-1,0]$. Then

$$\int_{-\infty}^{+\infty} \pounds(\xi)d\xi =\prec \left[\int_{-\infty}^{+\infty} \underline{\pounds}^P(\xi;\tau)d\xi, \int_{-\infty}^{+\infty} \overline{\pounds}^P(\xi;\tau)d\xi\right], \left[\int_{-\infty}^{+\infty} \underline{\pounds}^N(\xi;\kappa)d\xi, \int_{-\infty}^{+\infty} \overline{\pounds}^N(\xi;\kappa)d\xi\right] \succ .$$

Proposition 2.20. *If $\pounds : \mathcal{I} \to \mathbb{R}_{\mathcal{B}}$ and $\mu : \mathcal{I} \to \mathbb{R}$ are ghd and differentiable function at $\xi \in \mathcal{I}$, respectively, then*

$$(\pounds \odot \mu)'_{\mathfrak{gH}}(\xi) = \pounds'_{\mathfrak{gH}}(\xi) \odot \mu(\xi) \oplus \pounds(\xi) \odot \mu'(\xi).$$

Proof. Since $\mu(\xi)$ is continuous function, so by using the Definition 2.4 and Proposition 2.7, it follows that

$$\begin{aligned}
&\mathfrak{d}\left(\frac{\pounds(\xi+\mathfrak{k}) \odot \mu(\xi+\mathfrak{k}) \ominus_{\mathfrak{gH}} \pounds(\xi) \odot \mu(\xi)}{\mathfrak{k}}, \pounds'_{\mathfrak{gH}}(\xi) \odot \mu(\xi) \oplus \pounds(\xi) \odot \mu'(\xi)\right) \\
&= \mathfrak{d}\left(\frac{\pounds(\xi+\mathfrak{k}) \odot \mu(\xi+\mathfrak{k}) \ominus_{\mathfrak{gH}} \pounds \odot \mu(\xi+\mathfrak{k}) \oplus \pounds(\xi) \odot \mu(\xi+\mathfrak{k}) \ominus_{\mathfrak{gH}} \pounds(\xi) \odot \mu(\xi)}{\mathfrak{k}}, \pounds'_{\mathfrak{gH}}(\xi) \odot \mu(\xi) \oplus \pounds(\xi) \odot \mu'(\xi)\right) \\
&= \mathfrak{d}\left(\frac{(\pounds(\xi+\mathfrak{k}) \ominus_{\mathfrak{gH}} \pounds(\xi)) \odot \mu(\xi+\mathfrak{k}) \oplus \pounds(\xi) \odot (\mu(\xi+\mathfrak{k}) - \mu(\xi))}{\mathfrak{k}}, \pounds'_{\mathfrak{gH}}(\xi) \odot \mu(\xi) \oplus \pounds(\xi) \odot \mu'(\xi)\right) \\
&\leq \mathfrak{d}\left(\frac{(\pounds(\xi+\mathfrak{k}) \ominus_{\mathfrak{gH}} \pounds(\xi)) \odot \mu(\xi+\mathfrak{k})}{\mathfrak{k}}, \pounds'_{\mathfrak{gH}}(\xi) \odot \mu(\xi)\right) + \mathfrak{d}\left(\frac{\pounds(\xi) \odot (\mu(\xi+\mathfrak{k}) - \mu(\xi))}{\mathfrak{k}}, \pounds(\xi) \odot \mu'(\xi)\right) \\
&= \mathfrak{d}\left(\frac{(\pounds(\xi+\mathfrak{k}) \ominus_{\mathfrak{gH}} \pounds(\xi))}{\mathfrak{k}} \odot \mu(\xi+\mathfrak{k}), \pounds'_{\mathfrak{gH}}(\xi) \odot \mu(\xi)\right) + \mathfrak{d}\left(\pounds(\xi) \odot \frac{(\mu(\xi+\mathfrak{k}) - \mu(\xi))}{\mathfrak{k}}, \pounds(\xi) \odot \mu'(\xi)\right).
\end{aligned}$$

So $\mathfrak{k} \to 0$ implies the required result. □

Proposition 2.21. *Let $\pounds : \mathcal{I} \to \mathbb{R}_{\mathcal{B}}$ and $\mu : \mathcal{I} \to \mathbb{R}$ be differentiable functions , then*

$$\int_c^d \pounds'_{\mathfrak{gH}}(\xi) \odot \mu(\xi)d\xi = (\pounds(d) \odot \mu(d)) \ominus_{\mathfrak{gH}} (\pounds(c) \odot \mu(c)) \ominus_{\mathfrak{gH}} \int_c^d \pounds(\xi) \odot \mu'(\xi)d\xi.$$

3 Multivariate calculus

This section examines the primary characteristics of the multivariate BFFs. We will review some aspects of the multivariate BFF's $\mathfrak{gH}$-p differentiability. We'll also look at bipolar fuzzy improper integration, one of the multivariate calculus processes.

Definition 3.1. *The parametric representation of the two variable BFVF $\pounds : \mathcal{M} \subseteq \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}_{\mathcal{B}}$ is given by*

$$\pounds(\mathfrak{z},\xi) =\prec [\underline{\pounds}^P(\mathfrak{z},\xi;\tau), \overline{\pounds}^P(\mathfrak{z},\xi;\tau)], [\underline{\pounds}^N(\mathfrak{z},\xi;\kappa), \overline{\pounds}^N(\mathfrak{z},\xi;\kappa)] \succ,$$

for all $(\mathfrak{z},\xi) \in \mathcal{M}$ and $\tau \in [0,1], \kappa \in [-1,0]$.

Definition 3.2. [38] *Let $\pounds(\mathfrak{z},\xi)$ be BFVF, then the first-order $\mathfrak{gH}$-p derivatives of $\pounds$ at $(\mathfrak{z}_0,\xi_0) \in \mathcal{M}$ w.r.t. $\mathfrak{z}$ and ξ is denoted by $\partial_{\mathfrak{z}_{\mathfrak{gH}}} \pounds(\mathfrak{z}_0,\xi_0)$ and $\partial_{\xi_{\mathfrak{gH}}} \pounds(\mathfrak{z}_0,\xi_0)$ and defined as*

$$\begin{aligned}
\partial_{\mathfrak{z}_{\mathfrak{gH}}} \pounds(\mathfrak{z}_0,\xi_0) &= \lim_{k\to 0} \frac{\pounds(\mathfrak{z}_0+\mathfrak{k},\xi_0) \ominus_{\mathfrak{gH}} \pounds(\mathfrak{z}_0,\xi_0)}{\mathfrak{k}}, \\
\partial_{\xi_{\mathfrak{gH}}} \pounds(\mathfrak{z}_0,\xi_0) &= \lim_{h\to 0} \frac{\pounds(\mathfrak{z}_0,\xi_0+h) \ominus_{\mathfrak{gH}} \pounds(\mathfrak{z}_0,\xi_0)}{h},
\end{aligned}$$

provided that limit exist and $\partial_{\mathfrak{z}_{\mathfrak{gH}}} \pounds(\mathfrak{z}_0,\xi_0)$ and $\partial_{\xi_{\mathfrak{gH}}} \pounds(\mathfrak{z}_0,\xi_0) \in \mathbb{R}_{\mathcal{B}}$.

Definition 3.3. *Let* $\pounds(\mathfrak{z},\xi):\mathcal{M}\to\mathbb{R}_{\mathcal{B}}$ *and* $\pounds(\mathfrak{z},\xi)=\prec[\underline{\pounds}^P(\mathfrak{z},\xi;\tau),\overline{\pounds}^P(\mathfrak{z},\xi;\tau)],[\underline{\pounds}^N(\mathfrak{z},\xi;\kappa),\overline{\pounds}^N(\mathfrak{z},\xi;\kappa)]\succ,\ \forall\ \tau\in[0,1]$ *and* $\kappa\in[-1,0]$,

1. *If* $\pounds(\mathfrak{z},\xi)$ *is* $(*)-\mathfrak{gH}$ *partial differentiable (p-differentiable) w.r.t.* ξ *at* $(\mathfrak{z}_0,\xi_0)\in\mathcal{M}$ *then* $\underline{\pounds}^P_\tau,\overline{\pounds}^P_\tau,\underline{\pounds}^N_\kappa$ *and* $\overline{\pounds}^N_\kappa$ *all are p-differentiable and*
$$[\partial_{\xi_{\mathfrak{gH}}}\pounds(\mathfrak{z}_0,\xi_0)]^{(p,\kappa)}=\prec[\partial_\xi\underline{\pounds}^P(\mathfrak{z}_0,\xi_0;\tau),\partial_\xi\overline{\pounds}^P(\mathfrak{z}_0,\xi_0;\tau)],[\partial_\xi\underline{\pounds}^N(\mathfrak{z}_0,\xi_0;\kappa),\partial_\xi\overline{\pounds}^N(\mathfrak{z}_0,\xi_0;\kappa)]\succ.$$

2. *If* $\pounds(\mathfrak{z},\xi)$ *is* $(**,*)-\mathfrak{gH}$ *p-differentiable w.r.t.* ξ *at* $(\mathfrak{z}_0,\xi_0)\in\mathcal{M}$ *then* $\underline{\pounds}^P_\tau,\overline{\pounds}^P_\tau,\underline{\pounds}^N_\kappa$ *and* $\overline{\pounds}^N_\kappa$ *all are p-differentiable and*
$$[\partial_{\xi_{\mathfrak{gH}}}\pounds(\mathfrak{z}_0,\xi_0)]^{(\tau,\kappa)}=\prec[\partial_\xi\overline{\pounds}^P(\mathfrak{z}_0,\xi_0;\tau),\partial_\xi\underline{\pounds}^P(\mathfrak{z}_0,\xi_0;\tau)],[\partial_\xi\underline{\pounds}^N(\mathfrak{z}_0,\xi_0;\kappa),\partial_\xi\overline{\pounds}^N(\mathfrak{z}_0,\xi_0;\kappa)]\succ.$$

3. *If* $\pounds(\mathfrak{z},\xi)$ *is* $(*,**)-\mathfrak{gH}$ *p-differentiable w.r.t.* ξ *at* $(\mathfrak{z}_0,\xi_0)\in\mathcal{M}$ *then* $\underline{\pounds}^P_\tau,\overline{\pounds}^P_\tau,\underline{\pounds}^N_\kappa$ *and* $\overline{\pounds}^N_\kappa$ *all are p-differentiable and*
$$[\partial_{\xi_{\mathfrak{gH}}}\pounds(\mathfrak{z}_0,\xi_0)]^{(\tau,\kappa)}=\prec[\partial_\xi\underline{\pounds}^P(\mathfrak{z}_0,\xi_0;\tau),\partial_\xi\overline{\pounds}^P(\mathfrak{z}_0,\xi_0;\tau)],[\partial_\xi\overline{\pounds}^N(\mathfrak{z}_0,\xi_0;\kappa),\partial_\xi\underline{\pounds}^N(\mathfrak{z}_0,\xi_0;\kappa)]\succ.$$

4. *If* $\pounds(\mathfrak{z},\xi)$ *is* $(**)-\mathfrak{gH}$ *p-differentiable w.r.t.* ξ *at* $(\mathfrak{z}_0,\xi_0)\in\mathcal{M}$ *then* $\underline{\pounds}^P_\tau,\overline{\pounds}^P_\tau,\underline{\pounds}^N_\kappa$ *and* $\overline{\pounds}^N_\kappa$ *all are p-differentiable and*
$$[\partial_{\xi_{\mathfrak{gH}}}\pounds(\mathfrak{z}_0,\xi_0)]^{(\tau,\kappa)}=\prec[\partial_\xi\overline{\pounds}^P(\mathfrak{z}_0,\xi_0;\tau),\partial_\xi\underline{\pounds}^P(\mathfrak{z}_0,\xi_0;\tau)],[\partial_\xi\overline{\pounds}^N(\mathfrak{z}_0,\xi_0;\kappa),\partial_\xi\underline{\pounds}^N(\mathfrak{z}_0,\xi_0;\kappa)]\succ.$$

Definition 3.4. *Let* $\pounds(\mathfrak{z},\xi):\mathcal{M}\to\mathbb{R}_{\mathcal{B}}$. *The function* $\xi=k(\mathfrak{z})$, *is called switching boundary for the differentiability of the function* $\pounds(\mathfrak{z},\xi)$ *w.r.t.* ξ, *if* $\forall\ \mathfrak{z}$ *belongs to domain of* $k(\mathfrak{z})$ *and* $\forall\ \xi\in\mathcal{M},\ \exists\ (\mathfrak{z},\xi_1)<(\mathfrak{z},\xi)<(\mathfrak{z},\xi_2)$ *such that*

$$\textbf{\textit{Type (1):}}\begin{cases}\pounds\ is\ (*)-\mathfrak{gH}\ p-differentiable\ \ but\ \ not\ (**,*)-\mathfrak{gH}\ p-differentiable & at\ (\mathfrak{z},\xi_1),\\ \pounds\ is\ (**,*)-\mathfrak{gH}\ p-differentiable\ \ but\ \ not\ (*)-\mathfrak{gH}\ p-differentiable & at\ (\mathfrak{z},\xi_2),\end{cases}$$

or,

$$\textbf{\textit{Type (2):}}\begin{cases}\pounds\ is\ (*,**)-\mathfrak{gH}\ p-differentiable\ \ but\ \ not\ (**)-\mathfrak{gH}\ p-differentiable & at\ (\mathfrak{z},\xi_1),\\ \pounds\ is\ (**)-\mathfrak{gH}\ p-differentiable\ \ but\ \ not\ (*,**)-\mathfrak{gH}\ p-differentiable & at\ (\mathfrak{z},\xi_2),\end{cases}$$

or,

$$\textbf{\textit{Type (3):}}\begin{cases}\pounds\ is\ (**,*)-\mathfrak{gH}\ p-differentiable\ \ but\ \ not\ (*)-\mathfrak{gH}\ p-differentiable & at\ (\mathfrak{z},\xi_1),\\ \pounds\ is\ (*)-\mathfrak{gH}\ p-differentiable\ \ but\ \ not\ (**,*)-\mathfrak{gH}\ p-differentiable & at\ (\mathfrak{z},\xi_2),\end{cases}$$

or,

$$\textbf{\textit{Type (4):}}\begin{cases}\pounds\ is\ (**)-\mathfrak{gH}\ p-differentiable\ \ but\ \ not\ (**,*)-\mathfrak{gH}\ p-differentiable & at\ (\mathfrak{z},\xi_1),\\ \pounds\ is\ (**,*)-\mathfrak{gH}\ p-differentiable\ \ but\ \ not\ (**)-\mathfrak{gH}\ p-differentiable & at\ (\mathfrak{z},\xi_2),\end{cases}$$

or,

$$\textbf{\textit{Type (5):}}\begin{cases}\pounds\ is\ (*)-\mathfrak{gH}\ p-differentiable\ \ but\ \ not\ (**,*)-\mathfrak{gH}\ p-differentiable & at\ (\mathfrak{z},\xi_1),\\ \pounds\ is\ (**,*)-\mathfrak{gH}\ p-differentiable\ \ but\ \ not\ (*)-\mathfrak{gH}\ p-differentiable & at\ (\mathfrak{z},\xi_2),\end{cases}$$

or,

$$\textbf{\textit{Type (6):}}\begin{cases}\pounds\ is\ (**,*)-\mathfrak{gH}\ p-differentiable\ \ but\ \ not\ (**)-\mathfrak{gH}\ p-differentiable & at\ (\mathfrak{z},\xi_1),\\ \pounds\ is\ (**)-\mathfrak{gH}\ p-differentiable\ \ but\ \ not\ (**,*)-\mathfrak{gH}\ p-differentiable & at\ (\mathfrak{z},\xi_2),\end{cases}$$

or,

$$\textbf{\textit{Type (7):}}\begin{cases}\pounds\ is\ (*,**)-\mathfrak{gH}\ p-differentiable\ \ but\ \ not\ (*)-\mathfrak{gH}\ p-differentiable & at\ (\mathfrak{z},\xi_1),\\ \pounds\ is\ (*)-\mathfrak{gH}\ p-differentiable\ \ but\ \ not\ (*,**)-\mathfrak{gH}\ p-differentiable & at\ (\mathfrak{z},\xi_2),\end{cases}$$

or,

$$\textbf{\textit{Type (8):}}\begin{cases} \pounds \text{ is } (**)-\mathfrak{gH} \text{ } p-\text{differentiable but not } (*,**)-\mathfrak{gH} \text{ } p-\text{differentiable} & \text{at } (\mathfrak{z},\xi_1), \\ \pounds \text{ is } (*,**)-\mathfrak{gH} \text{ } p-\text{differentiable but not } (**)-\mathfrak{gH} \text{ } p-\text{differentiable} & \text{at } (\mathfrak{z},\xi_2), \end{cases}$$

or,

$$\textbf{\textit{Type (9):}}\begin{cases} \pounds \text{ is } (*)-\mathfrak{gH} \text{ } p-\text{differentiable but not } (**)-\mathfrak{gH} \text{ } p-\text{differentiable} & \text{at } (\mathfrak{z},\xi_1), \\ \pounds \text{ is } (**)-\mathfrak{gH} \text{ } p-\text{differentiable but not } (*)-\mathfrak{gH} \text{ } p-\text{differentiable} & \text{at } (\mathfrak{z},\xi_2), \end{cases}$$

or,

$$\textbf{\textit{Type (10):}}\begin{cases} \pounds \text{ is } (**)-\mathfrak{gH} \text{ } p-\text{differentiable but not } (*)-\mathfrak{gH} \text{ } p-\text{differentiable} & \text{at } (\mathfrak{z},\xi_1), \\ \pounds \text{ is } (*)-\mathfrak{gH} \text{ } p-\text{differentiable but not } (**)-\mathfrak{gH} \text{ } p-\text{differentiable} & \text{at } (\mathfrak{z},\xi_2), \end{cases}$$

or,

$$\textbf{\textit{Type (11):}}\begin{cases} \pounds \text{ is } (**,*)-\mathfrak{gH} \text{ } p-\text{differentiable but not } (*,**)-\mathfrak{gH} \text{ } p-\text{differentiable} & \text{at } (\mathfrak{z},\xi_1), \\ \pounds \text{ is } (*,**)-\mathfrak{gH} \text{ } p-\text{differentiable but not } (**,*)-\mathfrak{gH} \text{ } p-\text{differentiable} & \text{at } (\mathfrak{z},\xi_2), \end{cases}$$

or,

$$\textbf{\textit{Type (12):}}\begin{cases} \pounds \text{ is } (*,**)-\mathfrak{gH} \text{ } p-\text{differentiable but not } (**,*)-\mathfrak{gH} \text{ } p-\text{differentiable} & \text{at } (\mathfrak{z},\xi_1), \\ \pounds \text{ is } (**,*)-\mathfrak{gH} \text{ } p-\text{differentiable but not } (*,**)-\mathfrak{gH} \text{ } p-\text{differentiable} & \text{at } (\mathfrak{z},\xi_2). \end{cases}$$

Proposition 3.5. *Suppose that* $\pounds : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}_{\mathcal{B}}$ *and* $\upsilon : \mathbb{R} \to \mathbb{R}_{\mathcal{B}}$ *are BFVFs such that*

$$[\upsilon(\mathfrak{z})]^{(\tau,\kappa)} =< [\underline{\upsilon}^P(\mathfrak{z};\tau), \overline{\upsilon}^P(\mathfrak{z};\tau)], [\underline{\upsilon}^N(\mathfrak{z};\kappa), \overline{\upsilon}^N(\mathfrak{z};\kappa)] > .$$

Let $\mu : \mathbb{R} \to \mathbb{R}$ *and* $\varphi : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}_+$ *be differentiable functions w.r.t.* ξ *and*

$$\partial_\xi \varphi(\mathfrak{z},\xi) = \begin{cases} \partial_\xi\varphi(\mathfrak{z},\xi) \geq 0, & \text{if } \mu_1(\xi) < \mathfrak{z} < \mu_2(\xi); \\ \partial_\xi\varphi(\mathfrak{z},\xi) < 0, & \text{if } \mu_2(\xi) \leq \mathfrak{z} < \mu_3(\xi), \end{cases} \tag{1}$$

and $\pounds(\mathfrak{z},\xi) = \varphi(\mathfrak{z},\xi)\odot\upsilon(\mathfrak{z})$. *Then* $\partial_{\xi_{\mathfrak{gH}}}\pounds(\mathfrak{z},\xi)$ *exists and*

$$\partial_{\xi_{\mathfrak{gH}}}\pounds(\mathfrak{z},\xi) = \begin{cases} \partial_{\xi_{(*)\mathfrak{gH}}}\pounds(\mathfrak{z},\xi) = \partial_\xi\varphi(\mathfrak{z},\xi)\odot\upsilon(\mathfrak{z}), & \text{if } \mu_1(\xi) < \mathfrak{z} < \mu_2(\xi); \\ \partial_{\xi_{(**)\mathfrak{gH}}}\pounds(\mathfrak{z},\xi) = \partial_\xi\varphi(\mathfrak{z},\xi)\odot\upsilon(\mathfrak{z}), & \text{if } \mu_2(\xi) \leq \mathfrak{z} < \mu_3(\xi). \end{cases} \tag{2}$$

Clearly, $\mu_2(\xi)$ *is the switching boundary of* ***Type (9)*** *for the differentiability of* $\pounds(\mathfrak{z},\xi)$ *w.r.t.* ξ.

Proof. Let $\varphi(\mathfrak{z},\xi)$ be a differentiable function and

$$\pounds(\mathfrak{z},\xi) = \varphi(\mathfrak{z},\xi)\odot\upsilon(\mathfrak{z}),$$

which implies

$$\partial_{\xi_{\mathfrak{gH}}}\pounds(\mathfrak{z},\xi) = \partial_\xi\varphi(\mathfrak{z},\xi)\odot < [\underline{\upsilon}^P(\mathfrak{z};\tau), \overline{\upsilon}^P(\mathfrak{z};\tau)], [\underline{\upsilon}^N(\mathfrak{z};\kappa), \overline{\upsilon}^N(\mathfrak{z};\kappa)] > .$$

If $\partial_\xi\varphi(\mathfrak{z},\xi) \geq 0$ for all $\mu_1(\xi) < \mathfrak{z} < \mu_2(\xi)$, then

$$\partial_{\xi_{\mathfrak{gH}}}\pounds(\mathfrak{z},\xi) =< \left[\partial_\xi\varphi(\mathfrak{z},\xi)\underline{\upsilon}^P(\mathfrak{z};\tau), \partial_\xi\varphi(\mathfrak{z},\xi)\overline{\upsilon}^P(\mathfrak{z};\tau)\right], \left[\partial_\xi\varphi(\mathfrak{z},\xi)\underline{\upsilon}^N(\mathfrak{z};\kappa), \partial_\xi\varphi(\mathfrak{z},\xi)\overline{\upsilon}^N(\mathfrak{z};\kappa)\right] > .$$

Using Definition 3.3, $\pounds(\mathfrak{z},\xi)$ is $(*)$-*ghd w.r.t.* ξ.
If $\partial_\xi\varphi(\mathfrak{z},\xi) < 0$ for all $\mu_2(\xi) \leq z < \mu_3(\xi)$, then

$$\partial_{\xi_{\mathfrak{gH}}}\pounds(\mathfrak{z},\xi) =< \left[\partial_\xi\varphi(\mathfrak{z},\xi)\overline{\upsilon}^P(\mathfrak{z};\tau), \partial_\xi\varphi(\mathfrak{z},\xi)\underline{\upsilon}^P(\mathfrak{z};\tau)\right], \left[\partial_\xi\varphi(\mathfrak{z},\xi)\overline{\upsilon}^N(\mathfrak{z};\kappa), \partial_\xi\varphi(\mathfrak{z},\xi)\underline{\upsilon}^N(\mathfrak{z};\kappa)\right] > .$$

Using Definition 3.3, $\pounds(\mathfrak{z},\xi)$ is $(**)$-*ghd w.r.t.* ξ. □

Proposition 3.6. *Assume that* $\pounds : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}_{\mathcal{B}}$ *is a BFVF and*

$$[\pounds(\mathfrak{z},\xi)]^{(\tau,\kappa)} =\prec [\underline{\pounds}^P(\mathfrak{z},\xi;\tau), \overline{\pounds}^P(\mathfrak{z},\xi;\tau)], [\underline{\pounds}^N(\mathfrak{z},\xi;\kappa), \overline{\pounds}^N(\mathfrak{z},\xi;\kappa)] \succ .$$

If the bipolar fuzzy integral $\int_{\mathfrak{c}}^{\infty} \pounds(\mathfrak{z},\xi)d\xi$ *convergent* $\forall\ \mathfrak{z} \in [\mathfrak{b},\infty)$ *and* $\int_{\mathfrak{b}}^{\infty} \pounds(\mathfrak{z},\xi)dz$ *is convergent on* $[\mathfrak{c},\infty)$. *Then*

$$\int_{\mathfrak{b}}^{\infty}\int_{\mathfrak{c}}^{\infty} \pounds(\mathfrak{z},\xi)d\xi dz = \int_{\mathfrak{c}}^{\infty}\int_{\mathfrak{b}}^{\infty} \pounds(\mathfrak{z},\xi)dzd\xi, \qquad \forall\ \mathfrak{b},\mathfrak{c} \in \mathbb{R}.$$

Proposition 3.7. *Let* $\pounds(\mathfrak{z},\xi)$ *and* $\partial_{\mathfrak{z}_{\mathfrak{g}\mathfrak{H}}}\pounds(\mathfrak{z},\xi)$, *are bipolar fuzzy continuous in* $[c,d]\times[e,\infty)$. *Assume that*

$$\mathfrak{F}(\mathfrak{z}) = \int_e^{+\infty} \pounds(\mathfrak{z},\xi)d\xi, \tag{3}$$

converges $\forall\ \mathfrak{z} \in \mathbb{R}$, *and* $\int_e^{\infty} \partial_{\mathfrak{z}_{\mathfrak{g}\mathfrak{H}}}\pounds(\mathfrak{z},\xi)d\xi$ *converges in* $[c,d]$. *Then* $\mathfrak{F}$ *is ghd on* $[c,d]$ *and*

$$\mathfrak{F}'_{\mathfrak{g}\mathfrak{H}}(\mathfrak{z}) = \int_e^{\infty} \partial_{\mathfrak{z}_{\mathfrak{g}\mathfrak{H}}}\pounds(\mathfrak{z},\xi)d\xi.$$

4 Bipolar fuzzy Fourier transform

In this section, we present the bipolar FFT and demonstrate how it may help to solve the BFPDEs.

Definition 4.1. *Let* $\pounds(\xi) : \mathbb{R} \to \mathbb{R}_{\mathcal{B}}$ *be a BFVF. The bipolar FFT of* $\pounds(\xi)$ *is a mapping* $\mathcal{F}\{\pounds(\xi)\} : \mathbb{R} \to \mathbb{C}_{\mathcal{B}}$ *given by*

$$\mathcal{F}\{\pounds(\xi)\} = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{+\infty} \pounds(\xi)e^{-\iota\omega\xi}d\xi = \mathfrak{F}(\omega).$$

Definition 4.2. *Let* $\mathfrak{F}(\omega)$ *be a bipolar FFT of* $\pounds(\xi)$, *then the bipolar fuzzy inverse Fourier transform (FIFT) of* $\mathfrak{F}(\omega)$ *is*

$$\mathcal{F}^{-1}\{\mathfrak{F}(\omega)\} = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} \mathfrak{F}(\omega)e^{\iota\omega\xi}d\omega = \pounds(\xi).$$

Remark 4.3. *Consider a BFVF* $[\pounds(\xi)]^{(\tau,\kappa)} =\prec [\underline{\pounds}^P(\xi;\tau), \overline{\pounds}^P(\xi;\tau)], [\underline{\pounds}^N(\xi;\kappa), \overline{\pounds}^N(\xi;\kappa)] \succ$, *for all* $\tau \in [0,1]$ *and* $\kappa \in [-1,0]$, *Proposition 2.19 implies*

$\mathcal{F}\{[\pounds(\xi)]^{(\tau,\kappa)}\}$

$$=\prec \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underline{\pounds}^P(\xi;\tau)e^{-\iota\omega\xi}d\xi, \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{\pounds}^P(\xi;\tau)e^{-\iota\omega\xi}d\xi\right], \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underline{\pounds}^N(\xi;\kappa)e^{-\iota\omega\xi}d\xi, \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{\pounds}^N(\xi;\kappa)e^{-\iota\omega\xi}d\xi\right] \succ,$$

by using the Definition 4.1, we get

$$\mathcal{F}\{[\pounds(\xi)]^{(\tau,\kappa)}\} =\prec \left[\mathcal{F}\{\underline{\pounds}^P(\xi;\tau)\}, \mathcal{F}\{\overline{\pounds}^P(\xi;\tau)\}\right], \left[\mathcal{F}\{\underline{\pounds}^N(\xi;\kappa)\}, \mathcal{F}\{\overline{\pounds}^N(\xi;\kappa)\}\right] \succ .$$

Proposition 4.4. *Let* $\pounds(\xi)$ *be a BFAI on* $(-\infty,\infty)$, *then the bipolar FFT* $\mathcal{F}\{\pounds(\xi)\} : \mathbb{R} \to \mathbb{C}_{\mathcal{B}}$ *exists.*

Proof. From Definition 4.1, it follows that

$$\begin{aligned}
\mathfrak{d}(\mathcal{F}\{\pounds(\xi)\},0) &= \mathfrak{d}\left(\int_{-\infty}^{\infty} \pounds(\xi)e^{-\iota\omega\xi}d\xi, 0\right)\\
&\leq \int_{-\infty}^{\infty} \mathfrak{d}\left(\pounds(\xi)e^{-\iota\omega\xi}, 0\right)d\xi\\
&\leq \int_{-\infty}^{\infty} \left|e^{-\iota\omega\xi}\right| \mathfrak{d}\left(\pounds(\xi), 0\right)d\xi
\end{aligned}$$

since $|e^{-\iota\omega\xi}| = 1$, which implies

$$\mathfrak{d}\left(\mathcal{F}\{\pounds(\xi)\}, 0\right) \leq \int_{-\infty}^{\infty} \mathfrak{d}\left(\pounds(\xi), 0\right)d\xi < \infty.$$

Which means $\mathcal{F}\{\pounds(\xi)\}$ is finite. Thus $\mathcal{F}\{\pounds(\xi)\}$ exists, when $\pounds(\xi)$ is BFAI. □

Proposition 4.5. *Let $\mathfrak{F}(\omega)$ be a BFAI, then the bipolar FIFT $\mathcal{F}^{-1}\{\mathfrak{F}(\omega)\}$ exists.*

Proof. It is easy to prove it using similar arguments in Proposition 4.4. □

Proposition 4.6. *Let $\pounds(\xi)$ and $\mu(\xi)$ be BFVFs, and the bipolar FFT of both functions exists. Then for any $k_1, k_2 \in \mathbb{R}$*

- $\mathcal{F}\{k_1\pounds(\xi) \oplus k_2\mu(\xi)\} = k_1\mathcal{F}\{\pounds(\xi)\} \oplus k_2\mathcal{F}\{\mu(\xi)\}$,
- $\mathcal{F}\{k_1\pounds(\xi) \ominus_{\mathfrak{gH}} k_2\mu(\xi)\} = k_1\mathcal{F}\{\pounds(\xi)\} \ominus_{\mathfrak{gH}} k_2\mathcal{F}\{\mu(\xi)\}$,

where $k_1 \geq k_2$.

Proposition 4.7. *Let $\pounds(\xi)$ be bipolar fuzzy continuous, BFAI and $\pounds(\xi) \to 0$ as $|\xi| \to \infty$. Further, suppose that $\pounds'_{\mathfrak{gH}}(\xi)$ is BFAI in $(-\infty, \infty)$. Then*

$$\mathcal{F}\{\pounds'_{\mathfrak{gH}}(\xi)\} = \iota\omega\mathcal{F}\{\pounds(\xi)\}.$$

Proof. From Definition 4.1, it follows that

$$\mathcal{F}\{\pounds'_{\mathfrak{gH}}(\xi)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \pounds'_{\mathfrak{gH}}(\xi)e^{-\iota\omega\xi}d\xi,$$

Proposition 2.21 implies that

$$\mathcal{F}\{\pounds'_{\mathfrak{gH}}(\xi)\} = \frac{1}{\sqrt{2\pi}}\left(\pounds(\xi)e^{-\iota\omega\xi}\mid_{-\infty}^{\infty} \ominus_{\mathfrak{gH}}(-\iota\omega)\int_{-\infty}^{\infty}\pounds(\xi)e^{-\iota\omega\xi}d\xi\right).$$

Since $\pounds(\xi) \to 0$ as $|\xi| \to \infty$, we obtain

$$\mathcal{F}\{\pounds'_{\mathfrak{gH}}(\xi)\} = \iota\omega\mathcal{F}\{\pounds(\xi)\}.$$

□

Corollary 4.8. *Suppose that $\pounds(\xi)$ and $\pounds'_{\mathfrak{gH}}(\xi)$ are bipolar fuzzy continuous on $(-\infty, +\infty)$, $\pounds^m_{\mathfrak{gH}} \to 0$ as $|\xi| \to \infty, m = 0, 1$, and $\pounds^n_{\mathfrak{gH}}$ be BFAI in $(-\infty, +\infty), n = 0, 1, 2$. Then*

$$\mathcal{F}\{\pounds''_{\mathfrak{gH}}(\xi)\} = -\omega^2\mathcal{F}\{\pounds(\xi)\}.$$

5 Method for solving bipolar fuzzy heat equation

Let

$$\begin{cases} \partial_{\xi_{\mathfrak{gH}}}\nu(\mathfrak{z},\xi) \ominus_{\mathfrak{gH}} c\ \partial_{\mathfrak{z}\mathfrak{z}_{\mathfrak{gH}}}\nu(\mathfrak{z},\xi) = 0, & -\infty < \mathfrak{z} < \infty, \xi \geq 0; \\ \nu(\mathfrak{z},0) = \pounds(\mathfrak{z}) & -\infty < \mathfrak{z} < \infty, \end{cases} \tag{4}$$

be a BFHE, where $c \in \mathbb{R}^+$ is a constant of heat conductivity, and $\pounds(\mathfrak{z}) \in \mathbb{R}_{\mathcal{B}}$ is a bipolar fuzzy continuous function. Consider $\nu(\mathfrak{z},\xi)$ is a BFAI, we define the bipolar FFT and the bipolar FIFT of $\nu(\mathfrak{z},\xi)$ by

$$\begin{aligned} \mathcal{F}\{\nu(\mathfrak{z},\xi)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\nu(\mathfrak{z},\xi)e^{-\iota\omega z}d\mathfrak{z} = V(\omega,\xi), \\ \mathcal{F}^{-1}\{V(\omega,\xi)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}V(\omega,\xi)e^{\iota\omega z}d\omega = \nu(\mathfrak{z},\xi). \end{aligned}$$

If $\partial_{\xi_{\mathfrak{gH}}}\nu(\mathfrak{z},\xi), \partial_{\mathfrak{z}_{\mathfrak{gH}}}\nu(\mathfrak{z},\xi)$ and $\partial_{\mathfrak{z}\mathfrak{z}_{\mathfrak{gH}}}\nu(\mathfrak{z},\xi)$ are BFAI in $(-\infty,\infty)$, then Proposition 4.6 and Equation (4) implies:

$$\mathcal{F}\{\partial_{\xi_{\mathfrak{gH}}}\nu(\mathfrak{z},\xi)\} \ominus_{\mathfrak{gH}} \mathcal{F}\{c\ \partial_{\mathfrak{z}\mathfrak{z}_{\mathfrak{gH}}}\nu(\mathfrak{z},\xi)\} = 0.$$

Corollary 4.8 implies that

$$\mathcal{F}\{c\ \partial_{\mathfrak{z}\mathfrak{z}_{\mathfrak{gH}}}\nu(\mathfrak{z},\xi)\} = \ominus_{\mathfrak{gH}} c\ \omega^2 V(\omega,\xi).$$

Moreover, bipolar fuzzy absolutely integrability of $\partial_{\xi_{\mathfrak{gH}}}\nu(\mathfrak{z},\xi)$ implies

$$\mathfrak{d}\left(\int_{-\infty}^{\infty}\partial_{\xi_{\mathfrak{gH}}}\nu(\mathfrak{z},\xi)d\mathfrak{z}, 0\right) \leq \int_{-\infty}^{\infty}\mathfrak{d}\left(\partial_{\xi_{\mathfrak{gH}}}\nu(\mathfrak{z},\xi), 0\right)d\mathfrak{z} < \infty,$$

then $\int_{-\infty}^{\infty} \partial_{\xi_{\mathfrak{g}\mathfrak{H}}} \nu(\mathfrak{z},\xi) d\mathfrak{z}$ converges uniformly. From Proposition 3.7, it follows that

$$\mathcal{F}\{\partial_{\xi_{\mathfrak{g}\mathfrak{H}}} \nu(\mathfrak{z},\xi)\} = \int_{-\infty}^{\infty} \partial_{\xi_{\mathfrak{g}\mathfrak{H}}} \nu(\mathfrak{z},\xi) e^{-\iota\omega\xi} d\mathfrak{z} = \frac{dV(\omega,\xi)}{d\xi}.$$

So

$$\frac{dV(\omega,\xi)}{d\xi} \oplus c\ \omega^2 V(\omega,\xi) = 0. \tag{5}$$

Multiply $e^{\int c\ \omega^2 d\xi}$, on both sides of Equation (5) implies

$$e^{\int c\ \omega^2 d\xi} \frac{dV(\omega,\xi)}{d\xi} \oplus e^{\int c\ \omega^2 d\xi} c\ \omega^2 V(\omega,\xi) = 0.$$

Proposition 2.18 and Proposition 2.20 implies

$$\left(e^{c\ \omega^2 \xi} V(\omega,\xi)\right)'_{\mathfrak{g}\mathfrak{H}} = 0.$$

Therefore

$$e^{c\ \omega^2 \xi} V(\omega,\xi) \ominus_{\mathfrak{g}\mathfrak{H}} V(\omega,0) = 0.$$

Thus

$$V(\omega,\xi) = V(\omega,0) e^{-c\ \omega^2 \xi},$$

where

$$V(\omega,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \pounds(\mathfrak{z}) e^{-\iota\omega z} d\mathfrak{z}.$$

Bipolar FIFT of $V(\omega,\xi)$ follows that

$$\begin{aligned}
\nu(\mathfrak{z},\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} V(\omega,0) e^{-c\ \omega^2 \xi} e^{\iota\omega z} d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \pounds(\varsigma) e^{-\iota\omega\varsigma} d\varsigma \right) e^{-c\ \omega^2 \xi} e^{\iota\omega z} d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \pounds(\varsigma) \int_{-\infty}^{\infty} e^{-c\ \omega^2 \xi} e^{\iota\omega(z-\varsigma)} d\omega d\varsigma,
\end{aligned}$$

as

$$-c\ \omega^2 \xi + \iota\omega(\mathfrak{z}-\varsigma) = -\xi c \left(\omega^2 - \iota\omega \frac{\mathfrak{z}-\varsigma}{\xi c}\right) = -\xi c \left[\left(\omega - \iota \frac{\mathfrak{z}-\varsigma}{2\xi c}\right)^2 - \frac{(\mathfrak{z}-\varsigma)^2}{4\xi^2 c^2}\right].$$

Consider

$$\mathcal{J}(\mathfrak{z},\xi,\varsigma) = \int_{-\infty}^{\infty} e^{-c\ \omega^2 \xi} e^{\iota\omega(\mathfrak{z}-\varsigma)} d\omega = \int_{-\infty}^{\infty} e^{\left[-\xi c\left(\omega - \iota\frac{\mathfrak{z}-\varsigma}{2\xi c}\right)^2 - \frac{(\mathfrak{z}-\varsigma)^2}{4\xi c}\right]} d\omega,$$

and setting $\omega - \iota\frac{(\mathfrak{z}-\varsigma)}{2\xi} = \frac{s}{\sqrt{c\xi}}$, implies $d\omega = \frac{ds}{\sqrt{c\xi}}$. Since $\int_{-\infty}^{\infty} e^{-s^2} ds = \sqrt{\pi}$, we have

$$\mathcal{J}(\mathfrak{z},\xi,\varsigma) = \int_{-\infty}^{\infty} e^{-s^2} e^{\left[-\frac{(\mathfrak{z}-\varsigma)^2}{tc}\right]} \frac{ds}{\sqrt{c\xi}} = \sqrt{\frac{\pi}{c\xi}} e^{\left(-\frac{(\mathfrak{z}-\varsigma)^2}{4c\xi}\right)}.$$

It follows that

$$\nu(\mathfrak{z},\xi) = \frac{1}{\sqrt{4\pi c\xi}} \int_{-\infty}^{\infty} \pounds(\varsigma) e^{\left(-\frac{(\mathfrak{z}-\varsigma)^2}{4c\xi}\right)} d\varsigma, \tag{6}$$

is the solution of Equation (4).

Remark 5.1. *Now, we justify the differentiability type of the solution. Let* $\psi(\mathfrak{z}-\varsigma,\xi) = \frac{1}{\sqrt{4c\pi\xi}} e^{-\frac{(\mathfrak{z}-\varsigma)^2}{4c\xi}}$. *Then*

$$\partial_\xi \psi(\mathfrak{z}-\varsigma,\xi) = c\partial_{\mathfrak{z}\mathfrak{z}} \psi(\mathfrak{z}-\varsigma,\xi) = -\frac{e^{-\frac{(\mathfrak{z}-\varsigma)^2}{4c\xi}}}{4(\xi c)^{\frac{3}{2}}\sqrt{\pi}} + \frac{e^{-\frac{(\mathfrak{z}-\varsigma)^2}{4c\xi}}(\mathfrak{z}-\varsigma)^2}{8\xi^2 c\sqrt{\xi\pi c}},$$

therefore

$$\begin{cases} \partial_{\xi}\psi(\mathfrak{z}-\varsigma,\xi)\geq 0, & \text{for } |\varsigma-z|\geq\sqrt{2c\xi}; \\ \\ \partial_{\xi}\psi(\mathfrak{z}-\varsigma,\xi)<0, & \text{for } |\varsigma-z|<\sqrt{2c\xi}. \end{cases} \tag{7}$$

Now consider $\mathcal{H}(\mathfrak{z}-\varsigma,\xi)=\psi(\mathfrak{z}-\varsigma,\xi)\odot\pounds(\varsigma)$. *Since* $\psi(\mathfrak{z}-\varsigma,\xi)>0$, *so Proposition 3.5 implies*

$$\partial_{\xi_{\mathfrak{g}\mathfrak{H}}}\mathcal{H}(\mathfrak{z}-\varsigma,\xi)=\partial_{\xi}\psi(\mathfrak{z}-\varsigma,\xi)\odot\pounds(\varsigma).$$

From Equation (7) it follows that

$$\partial_{\xi_{\mathfrak{g}\mathfrak{H}}}\mathcal{H}(\mathfrak{z}-\varsigma,\xi)=\begin{cases} \partial_{\xi_{(*)\mathfrak{g}\mathfrak{H}}}\mathcal{H}(\mathfrak{z}-\varsigma,\xi), & \text{for } |\varsigma-\mathfrak{z}|\geq\sqrt{2c\xi}; \\ \\ \partial_{\xi_{(**)\mathfrak{g}\mathfrak{H}}}\mathcal{H}(\mathfrak{z}-\varsigma,\xi), & \text{for } |\varsigma-\mathfrak{z}|<\sqrt{2c\xi}. \end{cases} \tag{8}$$

Accordingly,

$$\nu(\mathfrak{z},\xi)=\int_{-\infty}^{\infty}\partial_{\xi}\psi(\mathfrak{z}-\varsigma,\xi)\odot\pounds(\varsigma)d\varsigma,$$

Proposition 3.7 implies that

$$\partial_{\xi_{\mathfrak{g}\mathfrak{H}}}\nu(\mathfrak{z},\xi)=\int_{-\infty}^{\infty}\partial_{\xi}\psi(\mathfrak{z}-\varsigma,\xi)\odot\pounds(\varsigma)d\varsigma=\int_{-\infty}^{\infty}\partial_{\xi_{\mathfrak{g}\mathfrak{H}}}\mathcal{H}(\mathfrak{z}-\varsigma,\xi)d\varsigma.$$

Equation (8) implies $\nu(\mathfrak{z},\xi)$ *is* $(*)$-$\mathfrak{g}\mathfrak{H}$ *p-differentiable w.r.t.* ξ *in* $\left(-\infty,\mathfrak{z}-\sqrt{2c\xi}\right]$ *and* $\left[\mathfrak{z}+\sqrt{2c\xi},\infty\right)$. *Moreover* $\nu(\mathfrak{z},\xi)$ *is* $(**)$-$\mathfrak{g}\mathfrak{H}$ *p-differentiable w.r.t.* ξ *in* $\left(\mathfrak{z}-\sqrt{2c\xi},\mathfrak{z}+\sqrt{2c\xi}\right)$.

6 Examples

This section studies some of BFHE's examples using the suggested methodology. Further, we take

$$erf(\gamma)=\frac{2}{(\pi)^{\frac{1}{2}}}\int_{0}^{\gamma}e^{-\xi^{2}}d\xi.$$

Example 6.1. *Let*

$$\begin{cases} \partial_{\xi_{\mathfrak{g}\mathfrak{H}}}\nu(\mathfrak{z},\xi)\ominus_{\mathfrak{g}\mathfrak{H}}0.25\partial_{\mathfrak{z}\mathfrak{z}_{\mathfrak{g}\mathfrak{H}}}\nu(\mathfrak{z},\xi)=0, & -\infty<\mathfrak{z}<\infty,\xi\geq 0; \\ \\ \nu(\mathfrak{z},0)=<[-5+2\tau,-1-2\tau],[-6-\kappa,-4+\kappa]>e^{-\mathfrak{z}^{2}}. \end{cases} \tag{9}$$

be a BFHE. By using Equation (6), it follows that

$$\begin{aligned} &\nu(\mathfrak{z},\xi) \\ &= \frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{+\infty}e^{-\frac{(\mathfrak{z}-\varsigma)^{2}}{\xi}}\odot<\left[(-5+2\tau)e^{-\varsigma^{2}},(-2-2\tau)e^{-\varsigma^{2}}\right],\left[(-6-\kappa)e^{-\varsigma^{2}},(-4+\kappa)e^{-\varsigma^{2}}\right]>d\varsigma \\ &= <\left[(-5+2\tau)\int_{-\infty}^{+\infty}\frac{e^{-\varsigma^{2}}}{\sqrt{\pi\xi}}e^{-\frac{(\mathfrak{z}-\varsigma)^{2}}{\xi}}d\varsigma,(-2-2\tau)\int_{-\infty}^{+\infty}\frac{e^{-\varsigma^{2}}}{\sqrt{\pi\xi}}e^{-\frac{(\mathfrak{z}-\varsigma)^{2}}{\xi}}d\varsigma\right], \\ &\quad \left[(-6-\kappa)\int_{-\infty}^{+\infty}\frac{e^{-\varsigma^{2}}}{\sqrt{\pi\xi}}e^{-\frac{(\mathfrak{z}-\varsigma)^{2}}{\xi}}d\varsigma,(-4+\kappa)\int_{-\infty}^{+\infty}\frac{e^{-\varsigma^{2}}}{\sqrt{\pi\xi}}e^{-\frac{(\mathfrak{z}-\varsigma)^{2}}{\xi}}d\varsigma\right]> \\ &= <[-5+2\tau,-1-2\tau],[-6-\kappa,-4+\kappa]>\frac{1}{\sqrt{1+\xi}}e^{-\frac{\mathfrak{z}^{2}}{1+\xi}}. \end{aligned}$$

Figure 1 shows the graphical depiction of the bipolar fuzzy solution of Equation (9) for $\tau=\frac{1}{2}$ *and* $\kappa=-\frac{1}{2}$. *The graphical depiction demonstrates the solution's behavior and properties inside the bipolar fuzzy framework. The graph illustrates the impact of varying* τ *and* κ *values on the system's dynamics by establishing the solution against key variables or parameters like time or space.*

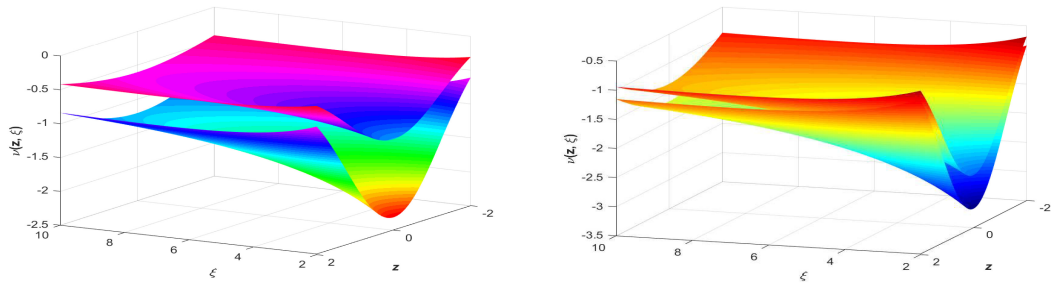


Figure 1: Animation of the positive and the negative membership function of $\nu(\mathfrak{z},\xi)$, respectively (from left to right).

Remark 5.1 implies

$$\partial_{\xi_{\mathfrak{g}\mathfrak{H}}}\nu(\mathfrak{z},\xi) =< [-5+2\tau,-1-2\tau],[-6-\kappa,-4+\kappa] > \int_{-\infty}^{+\infty}\partial_{\xi}\left(\frac{e^{-\frac{(\mathfrak{z}-\varsigma)^2}{\xi}}}{\sqrt{\pi\xi}}\right)e^{-\varsigma^2}d\varsigma. \tag{10}$$

Figure 2 shows the graphical depiction of $\partial_{\xi_{\mathfrak{g}\mathfrak{H}}}\nu(\mathfrak{z},\xi)$ *at* $\tau=\frac{1}{2}$ *and* $\kappa=-\frac{1}{2}$.

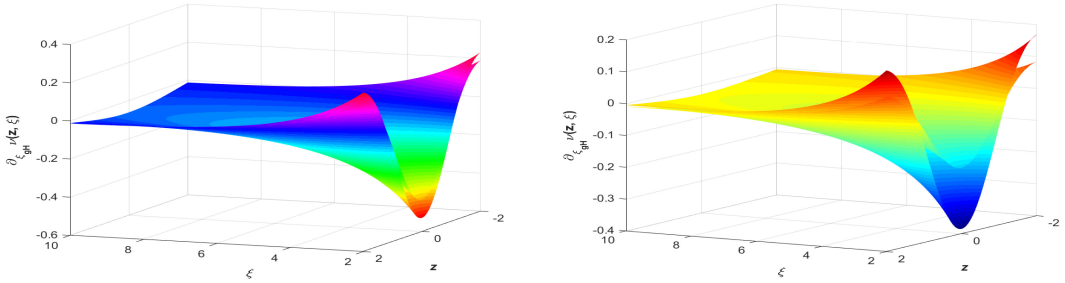


Figure 2: Animation of the positive and the negative membership function of $\partial_{\xi_{\mathfrak{g}\mathfrak{H}}}\nu(\mathfrak{z},\xi)$, respectively (from left to right).

Equation (10) follows that:

$$\partial_{\xi_{(*)\mathfrak{g}\mathfrak{H}}}\nu(\mathfrak{z},\xi) \quad =$$

$$< \left[(-5+2\tau)\int_{-\infty}^{+\infty}\frac{e^{-\frac{(\mathfrak{z}-\varsigma)^2}{\xi}-\varsigma^2}(2(\mathfrak{z}-\varsigma)^2-\xi)}{2\sqrt{\pi}\xi^{\frac{5}{2}}}d\varsigma,(-1-2\tau)\int_{-\infty}^{+\infty}\frac{e^{-\frac{(\mathfrak{z}-\varsigma)^2}{\xi}-\varsigma^2}(2(\mathfrak{z}-\varsigma)^2-\xi)}{2\sqrt{\pi}\xi^{\frac{5}{2}}}d\varsigma\right],$$
$$\left[(-6-\kappa)\int_{-\infty}^{+\infty}\frac{e^{-\frac{(\mathfrak{z}-\varsigma)^2}{\xi}-\varsigma^2}(2(\mathfrak{z}-\varsigma)^2-\xi)}{2\sqrt{\pi}\xi^{\frac{5}{2}}}d\varsigma,(-4+\kappa)\int_{-\infty}^{+\infty}\frac{e^{-\frac{(\mathfrak{z}-\varsigma)^2}{\xi}-\varsigma^2}(2(\mathfrak{z}-\varsigma)^2-\xi)}{2\sqrt{\pi}\xi^{\frac{5}{2}}}d\varsigma\right] >, \quad |\varsigma-\mathfrak{z}|\geq\sqrt{0.5\xi};$$

and

$$\partial_{\xi_{(**)\mathfrak{g}\mathfrak{H}}}\nu(\mathfrak{z},\xi) \quad =$$

$$< \left[(-1-2\tau)\int_{-\infty}^{+\infty}\frac{e^{-\frac{(\mathfrak{z}-\varsigma)^2}{\xi}-\varsigma^2}(2(\mathfrak{z}-\varsigma)^2-\xi)}{2\sqrt{\pi}\xi^{\frac{5}{2}}}d\varsigma,(-5+2\tau)\int_{-\infty}^{+\infty}\frac{e^{-\frac{(\mathfrak{z}-\varsigma)^2}{\xi}-\varsigma^2}(2(\mathfrak{z}-\varsigma)^2-\xi)}{2\sqrt{\pi}\xi^{\frac{5}{2}}}d\varsigma\right],$$
$$\left[(-4+\kappa)\int_{-\infty}^{+\infty}\frac{e^{-\frac{(\mathfrak{z}-\varsigma)^2}{\xi}-\varsigma^2}(2(\mathfrak{z}-\varsigma)^2-\xi)}{2\sqrt{\pi}\xi^{\frac{5}{2}}}d\varsigma,(-6-\kappa)\int_{-\infty}^{+\infty}\frac{e^{-\frac{(\mathfrak{z}-\varsigma)^2}{\xi}-\varsigma^2}(2(\mathfrak{z}-\varsigma)^2-\xi)}{2\sqrt{\pi}\xi^{\frac{5}{2}}}d\varsigma\right] >, \quad |\varsigma-\mathfrak{z}|<\sqrt{0.5\xi}.$$

Clearly, the solution $\nu(\mathfrak{z},\xi)$ *has switching boundary of* ***Type (9)*** *w.r.t.* ξ *for* $\mathfrak{z}=-\frac{\sqrt{1+\xi}}{\sqrt{2}}$ *and switching boundary of* ***Type (10)*** *w.r.t.* ξ *for* $\mathfrak{z}=\frac{\sqrt{1+\xi}}{\sqrt{2}}$. *In fact*

$$\partial_{\xi_{\mathfrak{gH}}}\nu(\mathfrak{z},\xi)=\begin{cases} <\left[\frac{(-5+2\tau)(1+\xi-2\mathfrak{z}^2)e^{\frac{-\mathfrak{z}^2}{1+\xi}}}{2(1+\xi)^{\frac{5}{2}}},\frac{(-1-2\tau)(1+\xi-2\mathfrak{z}^2)e^{\frac{-\mathfrak{z}^2}{1+\xi}}}{2(1+\xi)^{\frac{5}{2}}}\right],\left[\frac{(-6-\kappa)(1+\xi-2\mathfrak{z}^2)e^{\frac{-\mathfrak{z}^2}{1+\xi}}}{2(1+\xi)^{\frac{5}{2}}},\frac{(-4+\kappa)(1+\xi-2\mathfrak{z}^2)e^{\frac{-\mathfrak{z}^2}{1+\xi}}}{2(1+\xi)^{\frac{5}{2}}}\right]>, & |\mathfrak{z}|\geq\frac{\sqrt{1+\xi}}{\sqrt{2}};\\ <\left[\frac{(-1-2\tau)(1+\xi-2\mathfrak{z}^2)e^{\frac{-\mathfrak{z}^2}{1+\xi}}}{2(1+\xi)^{\frac{5}{2}}},\frac{(-5+2\tau)(1+\xi-2\mathfrak{z}^2)e^{\frac{-\mathfrak{z}^2}{1+\xi}}}{2(1+\xi)^{\frac{5}{2}}}\right],\left[\frac{(-4+\kappa)(1+\xi-2\mathfrak{z}^2)e^{\frac{-\mathfrak{z}^2}{1+\xi}}}{2(1+\xi)^{\frac{5}{2}}},\frac{(-6-\kappa)(1+\xi-2\mathfrak{z}^2)e^{\frac{-\mathfrak{z}^2}{1+\xi}}}{2(1+\xi)^{\frac{5}{2}}}\right]>, & |\mathfrak{z}|<\frac{\sqrt{1+\xi}}{\sqrt{2}}. \end{cases}$$

Example 6.2. *Let*

$$\begin{cases} \partial_{\xi_{\mathfrak{gH}}}\nu(\mathfrak{z},\xi)\ominus_{\mathfrak{gH}}2\partial_{\mathfrak{z}\mathfrak{z}_{\mathfrak{gH}}}\nu(\mathfrak{z},\xi)=0, & -\infty<\mathfrak{z}<\infty,\xi\geq 0;\\ \nu(\mathfrak{z},0)=<[2+5\tau,12-5\tau],[7-5\kappa,17+5\kappa]>|\mathfrak{z}|e^{-\mathfrak{z}^2}. \end{cases} \tag{11}$$

be a BFHE. From Equation (6), it follows:

$$\begin{aligned} \nu(\mathfrak{z},\xi) &= <[2+5\tau,12-5\tau],[7-5\kappa,17+5\kappa]>\int_{-\infty}^{+\infty}\frac{1}{\sqrt{8\pi\xi}}e^{-\frac{(\mathfrak{z}-\varsigma)^2}{8\xi}}\odot|\varsigma|e^{-\varsigma^2}d\varsigma\\ &= <[2+5\tau,12-5\tau],[7-5\kappa,17+5\kappa]>\left(\frac{2e^{\frac{-\mathfrak{z}^2}{4\xi}}\xi\left(4e^{\frac{\mathfrak{z}^2}{8\xi}}+e^{\frac{\mathfrak{z}^2+4\xi\mathfrak{z}^2}{4\xi+32\xi^2}}\sqrt{\frac{2\pi\mathfrak{z}^2}{\xi+8\xi^2}}erf\left[\sqrt{\frac{\mathfrak{z}^2}{8\xi+64\xi^2}}\right]\right)}{2\sqrt{2\pi\xi}(1+8\xi)}\right). \end{aligned}$$

Figure 3 shows the graphical depiction of the bipolar fuzzy solution of Equation (11) for $\tau=\frac{1}{2}$ *and* $\kappa=-\frac{1}{2}$.

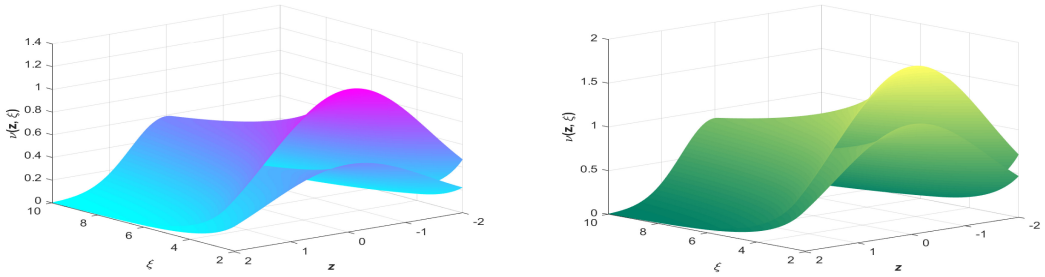


Figure 3: Animation of the positive and the negative membership function of $\nu(\mathfrak{z},\xi)$, respectively (from left to right).

7 Conclusion

The heat equation is a particular PDE type in mathematics and physics. These PDEs are also used to define a wide range of time-dependent problems. Under $\mathfrak{gH}$-p differentiability, a bipolar fuzzy solution to the BFHE is given in this study. To solve BFHE, a bipolar fuzzy improper integral for multivariate BFVF is presented. Using this notion, we have defined the bipolar FFT and demonstrated some characteristics. These characteristics have been utilized to solve the BFHE. Furthermore, in Remark 5.1, we have investigated the solution's $\mathfrak{gH}$-p differentiability type. A few examples are solved to examine the suggested methodology and a bipolar fuzzy solution for the BFHE is obtained. The BFHE's solution and differentiability have been graphically examined to illustrate and support theoretical conclusions. This work contributes to a better understanding and use of bipolar fuzzy mathematics for complex time-dependent problems, paving the way for future study and innovation in this multidisciplinary domain.

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References

[1] S. Ahmad, A. Ullah, A. Akgül, T. Abdeljawad, *Semi-analytical solutions of the 3rd order fuzzy dispersive partial differential equations under fractional operators*, Alexandria Engineering Journal, **60**(6) (2021), 5861-5878. https://doi.org/10.1016/j.aej.2021.04.065

[2] M. Akram, M. Ali, T. Allahviranloo, *A method for solving bipolar fuzzy complex linear systems with real and complex coefficients*, Soft Computing, **26**(5) (2022), 2157-2178. https://doi.org/10.1007/s00500-021-06672-7

[3] M. Akram, T. Allahviranloo, W. Pedrycz, M. Ali, *Methods for solving LR-bipolar fuzzy linear systems*, Soft Computing, **25**(1) (2021), 85-108. https://doi.org/10.1007/s00500-020-05460-z

[4] M. Akram, M. Arshad, *A novel trapezoidal bipolar fuzzy TOPSIS method for group decision-making*, Group Decision and Negotiation, **28** (2019), 565-584. https://doi.org/10.1007/s10726-018-9606-6

[5] M. Akram, G. Muhammad, *Analysis of incommensurate multi-order fuzzy fractional differential equations under strongly generalized fuzzy Caputo's differentiability*, Granular Computing, **8**(4) (2023), 809-825. https://doi.org/10.1007/s41066-022-00353-y

[6] M. Akram, G. Muhammad, T. Allahviranloo, *Bipolar fuzzy linear system of equations*, Computational and Applied Mathematics, **38**(69) (2019), 69. https://doi.org/10.1007/s40314-019-0814-8

[7] M. Akram, G. Muhammad, T. Allahviranloo, *Explicit analytical solutions of an incommensurate system of fractional differential equations in a fuzzy environment*, Information Sciences, **645** (2023), 119372. https://doi.org/10.1016/j.ins.2023.119372

[8] M. Akram, G. Muhammad, T. Allahviranloo, W. Pedrycz, *Incommensurate non-homogeneous system of fuzzy linear fractional differential equations using the fuzzy bunch of real functions*, Fuzzy Sets and Systems, **473** (2023), 108725. https://doi.org/10.1016/j.fss.2023.108725

[9] T. Allahviranloo, *Difference methods for fuzzy partial differential equtions*, Computational Methods in Applied Mathematics, **2**(3) (2006), 233-242. https://doi.org/10.2478/cmam-2002-0014

[10] T. Allahviranloo, S. Abbasbandy, H. Rouhparvar, *The exact solutions of fuzzy wave like equations with variable coefficients by a variational iteration method*, Applied Soft Computing, **11**(2) (2011), 2186-2192. https://doi.org/10.1016/j.asoc.2010.07.018

[11] T. Allahviranloo, K. M. Afshar, *Numerical methods for fuzzy linear partial differential equations under new definition for derivative*, Iranian Journal of Fuzzy Systems, **7**(3) (2010), 33-50. https://doi.org/10.22111/IJFS.2010.187

[12] T. Allahviranloo, Z. Gouyandeh, A. Armand, A. Hasanoglu, *On fuzzy solution for heat equation based on generalized hukuhara differentiability*, Fuzzy Sets and Systems, **265** (2015), 1-23. https://doi.org/10.1016/j.fss.2014.11.009

[13] T. Allahviranloo, N. Taheri, *An analytic approximation to the solution of fuzzy heat equation by adomian decompostion method*, International Journal of Contemporary Mathematical Sciences, **4**(3) (2009), 105-114.

[14] B. Bede, S. G. Gal, *Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy ordinary differential equations*, Fuzzy Sets and Systems, **151**(3) (2005), 581-599. https://doi.org/10.1016/j.fss.2004.08.001

[15] B. Bede, G. Lovane, I. Esposito, *Fuzzy Fourier transforms and their application to fingerprint identification*, Journal of Discrete Mathematical Sciences and Cryptography, **8**(1) (2005), 59-79. https://doi.org/10.1080/09720529.2005.10698021

[16] B. Bede, L. Stefanini, *Generalized differentiability of fuzzy-valued functions*, Fuzzy Sets and Systems, **230** (2013), 119-141. ttps://doi.org/10.1016/j.fss.2012.10.003

[17] A. M. Bertone, R. M. Jafelice, L. C. de Barros, R. C. Bassanezi, *On fuzzy solution for partial differential equations*, Fuzzy Sets and Systems, **219** (2013), 68–80. https://doi.org/10.1016/j.fss.2012.12.002

[18] J. J. Buckley, *Fuzzy complex number*, Fuzzy Sets and Systems, **33**(3) (1989), 333-345. https://doi.org/10.1016/0165-0114(89)90122-X

[19] J. J. Buckley, T. Feuring, *Introduction to fuzzy partial differential equtions*, Fuzzy Sets and Systems, **105**(2) (1999), 241-248. https://doi.org/10.1016/S0165-0114(98)00323-6

[20] S. S. Chang, L. A. Zadeh, *On fuzzy mapping and control*, IEEE Transaction on Systems, **2**(1) (1972), 30-34. https://doi.org/10.1109/TSMC.1972.5408553

[21] Y. Y. Chen, Y. T. Chang, B. S. Chen, *Fuzzy solution to partial differential equations: Adoptive approach*, IEEE Transactions on Fuzzy Systems, **17**(1) (2009), 116-127. https://doi.org/10.1109/TFUZZ.2008.2005010

[22] F. Dayan, N. Ahmed, M. Rafiq, A. Akgül, A. Raza, M. O. Ahmad, F. Jarad, *Construction and numerical analysis of a fuzzy non-standard computational method for the solution of an SEIQR model of COVID-19 dynamics*, AIMS Mathematics, **7**(5) (2022), 8449-8470. http://hdl.handle.net/20.500.12416/7409

[23] D. Dubois, H. Prade, *Operations on fuzzy numbers*, International Journal of Systems Sciences, **9**(6) (1978), 613-626. https://doi.org/10.1080/00207727808941724

[24] D. Dubois, H. Prade, *Towards fuzzy differential calculus part* 1*: Integration of fuzzy mappings*, Fuzzy Sets and Systems, **8**(1) (1982), 1-17. https://doi.org/10.1016/0165-0114(82)90025-2

[25] D. Dubois, H. Prade, *Towards fuzzy differential calculus part* 2*: Integration on fuzzy intervals*, Fuzzy Sets and Systems, **8**(2) (1982), 105-116. https://doi.org/10.1016/0165-0114(82)90001-X

[26] D. Dubois, H. Prade, *Towards fuzzy differential calculus part* 3*: Differentiation*, Fuzzy Sets and Systems, **8**(3) (1982), 225-233. https://doi.org/10.1016/S0165-0114(82)80001-8

[27] Z. Gouyandeh, T. Allahviranloo, S. Abbasbandy, A. Armand, *A fuzzy solution of heat equation under generalized Hukuhara differentiability by fuzzy Fourier transform*, In Fuzzy Sets and Systems, **309** (2017), 81-97. https://doi.org/10.1016/j.fss.2016.04.010

[28] O. Kaleva, *Fuzzy differential equations*, Fuzzy Sets and Systems, **24**(3) (1987), 301-317. https://doi.org/10.1016/0165-0114(87)90029-7

[29] A. Khan, M. Farman, A. Akgül, *Decision making under Pythagorean fuzzy soft environment*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, **31**(5) (2023), 773-793. https://doi.org/10.1142/S0218488523500368

[30] S. Nagalakshmi, G. S. Kumar, *Solution for one dimensional fuzzy heat equation under generalized Hukuhara differentiability*, AIP Conference Proceedings, **2375**(1) (2021), 020031. https://doi.org/10.1063/5.0066523

[31] M. Oberguggenbberger, *Fuzzy and weak solution to differential equations*, in: Proceedings of the 10th International IPMU Conference, (2004), 517-524.

[32] U. M. Pirzada, D. C. Vakaskar, *Fuzzy solution of homogeneous heat equation having solution in Fourier series form*, SeMA, **76** (2019), 181-194. https://doi.org/10.1007/s40324-018-0169-x

[33] M. L. Puri, D. A. Ralescu, *Differentials of fuzzy functions*, Journal of Mathematical Analysis and Application, **91**(12) (1983), 552-558. https://doi.org/10.1016/0022-247X(83)90169-5

[34] M. Qayyum, E. Ahmad, A. Akgül, S. M. El Din, *Fuzzy-fractional modeling of Korteweg-de Vries equations in Gaussian-Caputo sense: New solutions via extended He-Mahgoub algorithm*, Ain Shams Engineering Journal, **15**(4) (2024), 102623. https://doi.org/10.1016/j.asej.2023.102623

[35] M. Qayyum, A. Tahir, A. Bariq, A. Akgül, S. T. Saeed, *Modeling and analysis of thin film flow of fuzzified Johnson Segalman nanofluid using fuzzy extension of He-Laplace scheme*, Mathematical and Computer Modelling of Dynamical Systems, **29**(1) (2023), 286-314. https://doi.org/10.1080/13873954.2023.2276440

[36] M. Saqib, M. Akram, S. Bashir, *Certain efficient iterative methods for bipolar fuzzy system of linear equations*, Journal of Intelligent and Fuzzy Systems, **39**(3) (2020), 3971-3985. https://doi.org/10.3233/JIFS-200084

[37] M. Saqib, M. Akram, S. Bashir, T. Allahviranloo, *Numerical solution of bipolar fuzzy initial value problem*, Journal of Intelligent and Fuzzy Systems, **40**(1) (2021), 1309-1341. https://doi.org/10.3233/JIFS-201619

[38] M. Saqib, M. Akram, S. Bashir, T. Allahviranloo, *A Runge-Kutta numerical method to approximate the solution of bipolar fuzzy initial value problems*, Computational and Applied Mathematics, **40**(4) (2021), 151. https://doi.org/10.1007/s40314-021-01535-1

[39] S. Seikkala, *On the fuzzy initial value problm*, Fuzzy Sets and Systems, **24**(3) (1987), 319-330. https://doi.org/10.1016/0165-0114(87)90030-3

[40] L. Stefanini, *A generalization of hukuhara difference and division for interval and fuzzy arithmetic*, Fuzzy Sets and Systems, **161**(11) (2010), 1564-1584. https://doi.org/10.1016/j.fss.2009.06.009

[41] L. Stefanini, B. Bede, *Generalized hukuhara differentiability of fuzzy-number-valued functions with interval differential equations*, Nonlinear Analysis, **71** (2009), 1311-1328. https://doi.org/10.1016/j.na.2008.12.005

[42] A. Ullah, A. Ullah, S. Ahmad, I. Ahmad, A. Akgül,*On solutions of fuzzy fractional order complex population dynamical model*, Numerical Methods for Partial Differential Equations, **39**(6) (2023), 4595-4615. https://doi.org/10.1002/num.22654

[43] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8**(3) (1965), 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X

[44] W. R. Zhang, *Bipolar fuzzy sets and relations: A computational framework for cognitive modeling and multiagent decision analysis*, In Proceedings of IEEE Conference, (1994), 305-309. https://doi.org/10.1109/IJCF.1994.375115