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Percentile-based $\bar{X}$ and R control charts for triangular fuzzy quality

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Abstract

Process monitoring using control charts is a common quality control method to plot the manufacturing process data and compare it to the control limits in the manufacturing process. Construction of the statistical control charts is recently suggested on the basis of the flexible triangular fuzzy quality rather than common interval-valued quality. Two new percentile-based approaches are investigated in this paper to construct mean and range control charts for the degree of belonging observations to the triangular fuzzy quality. A real-world case study about automobile engine piston rings is presented to show the performance of the proposed control charts.

Keywords: Quantile, kernel density estimation, quality control charts, process monitoring, fuzzy quality.

1 Introduction

In the realm of statistical quality control, we may confront imprecise concepts. Imagine a scenario where the quality is defined by a fuzzy set instead of the conventional set $[LSL, USL]$. This scenario is referred to as "fuzzy quality", a concept initially introduced by Yongting [18] in 1996. This idea involves substituting the membership function of fuzzy quality with the indicator function of specification limits. In such situations characterized by imprecision, products aren't simply classified as qualified or unqualified based on a Boolean perspective. Instead, their qualification is determined to varying degrees based on their quality levels. The motivations of fuzzy quality were mentioned in [13]. A new innovation for using a Monte Carlo technique was discussed in [11] to test the quality of a manufacturing process based on Yongting's index. Sadeghpour-Gildeh [15] compared capability indices C_p, C_{pk} and Yongting's index with respect to the measurement error occurrence. The structure of testing the fuzzy process capability indices was presented in [12] to measure the capability of a Normal process when the specification limits are imprecise. The third author, the first author, and Sadeghpour Gildeh [8] proposed the statistical fuzzy quality test for analyzing the manufacturing process based on fuzzy specification limits. More details about "simulation testing of fuzzy quality" and its applications were provided in [7, 10]. Also, the Monte Carlo simulation method was presented in [6] to test the quality of a production process based on the popular capability index C_{pk}. A few robust estimators for the yield index were defined in [5].

Broadly, employing control charts is essential to monitor and maintain the stability and consistency of processes over time in statistical quality control. Control charts are indispensable tools in quality management, offering a systematic approach to monitor and control processes. While they are commonly associated with symmetric quality characteristics, such as dimensions or weights, control charts are also applicable to asymmetric quality characteristics, like proportions or percentages. Zarandi, Turksen, and Kashan [4] designed the control charts for both variable ($\bar{X}$ chart) and attribute (u and c charts) quality characteristics, when there is uncertainty about the process parameters or sample data. Alizadeh and Fatemi Ghomi [1] develop mean and range control charts using defuzzification methods when the measurements are triangular fuzzy numbers. A fuzzy p-chart based on the mean degree of nonconformity was designed by the second author, Mashinchi, and the first author [2] in the precision of both input and output of the model which differs from the control charts introduced in [4]. Two new control charts and process capability ratios

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based on the Skew Normal distribution were presented by Tsai [17] to monitor the process average and evaluate the process capability of non-Normal data.

The motivation of this paper is presenting two new percentile-based approaches to design the mean and range control charts of the belonging degree based on the triangular fuzzy quality.

This paper is organized as follows. The triangular fuzzy quality and its importance in quality control are presented in Section 2. Also, the relation between specification limits and triangular fuzzy quality is discussed in Section 2. The mean and range control charts are developed based on triangular fuzzy quality in Section 3 by two different percentile-based approaches. A case study on the automobile engine piston ring has been provided based on triangular fuzzy quality in Section 4. All investigated percentile-based approaches have been compared by simulation in Section 5. The final section is conclusions and future works.

2 Triangular fuzzy quality

Consider the univariate quality characteristic X with support/domain χ. The traditional/crisp quality is determined by the crisp interval of specification limits $[LSL, USL]$ which can be indicated by the indicator function $I_{[LSL,USL]}(x)$ too. For a determined product with the certain specification limits, $x \in [LSL, USL]$ (or equivalently $I_{[LSL,USL]}(x) = 1$) means that the product has quality, and on the contrary, $I_{[LSL,USL]}(x) = 0$ means that the product has not quality. Like any fuzzy set that is defined by its membership function, "fuzzy quality" is also defined by its membership function as follows, which is a simple generalization of the indicator function $I_{[LSL,USL]}(x)$.

Definition 2.1. *The membership function of "fuzzy quality" is a function from χ into the interval $[0, 1]$ such that $\tilde{Q}(x)$ shows the quality degree of the measured/observed quality characteristic x for a product. In other words, the fuzzy set $\tilde{Q} \in F(\chi)$ is named "fuzzy quality" for the quality characteristic X, if and only if $\tilde{Q}(x)$ represents the degree of conformity product with the standard quality — or briefly the degree of quality — for any $x \in \chi$ where $F(\chi)$ is the set of all fuzzy sets on χ.*

The interval of specification limits $[LSL, USL]$ provides an acceptable range for the output distribution which are derived from the process requirements. In other words, lower and upper specification limits LSL and USL specify the minimum and maximum acceptable limits for the process based on the requirements and standards of the quality characteristic X. Also, an optional target value T may be considered between specification limits $[LSL, USL]$ to specify the target of the quality characteristic X in the manufacturing process. Regarding to the above introduction about the target value and specification limits, let us to symbolically define the triangular fuzzy quality by a triangular fuzzy number on the basis of T, LSL and USL. The fuzzy number $\tilde{Q}_\Delta$ of the real line is said to be "triangular fuzzy quality", and denoted by $Tr(LSL, T, USL) \in F(\chi)$, if its membership function is

$$\tilde{Q}_\Delta(x) = \begin{cases} \frac{x-LSL}{T-LSL} & \text{if} \quad LSL \le x < T, \\ \frac{USL-x}{USL-T} & \text{if} \quad T \le x < USL, \\ 0 & \text{elsewhere} \end{cases}$$

in which the target value T is considered as the core and the bound of specification limits $[LSL, USL]$ is considered as the support of triangular fuzzy quality and also x is the quality characteristic of the considered product. Sometimes we write $\tilde{Q}$ instead of $\tilde{Q}_\Delta$ in this paper for simplicity in notations and formulas. The parameters of triangular fuzzy quality must be defined based on the customer requirements or expectations in a manufacturing quality process, since the target value and specification limits are determined according to the customer expectations. Triangular fuzzy quality is considered in this paper since only one point from χ is usually considered as the target value for the one-dimensional continuous quality characteristic X in manufacturing processes. It should also be mentioned that the membership function $\tilde{Q}_\Delta$ is basically subjective and it reveals a subjective relationship only between the triangular fuzzy quality and specification limits.

In the next section, we are going to investigate and develop $\bar{X}$ and R control charts for the degree of belonging observations to the triangular fuzzy quality $\tilde{Q}_\Delta$.

3 Construction of $\bar{X}$ and R control charts on the basis of triangular fuzzy quality

Suppose that f is the probability density function of one-dimensional continuous quality characteristic X, which is assumed to be unknown in this paper. Let m samples are available, each containing n observations for the quality

characteristic X. Usually, the sample sizes in quality control will be small since the sampling and inspection costs are often relatively large. It must be noted that the main goal of this paper is the construction of two mean and range control charts for the quality degree of x_{ij}'s, that is $\tilde{Q}(x_{ij})$'s (and not for x_{ij}'s). The distribution of $\tilde{Q}(x_{ij})$'s is unknown since it depends on the membership function of fuzzy quality and also the distribution of x_{ij}'s. That is cause of considering the percentile-based approach to construct control charts in this work. Two different percentile-based approaches are investigated in this section to construct mean and range control charts for the degree of belonging observations to the triangular fuzzy quality.

3.1 Data-based approach using the quantiles of quality data

Suppose that $x_j = (x_{1j}, ..., x_{nj})$ be the j-th sample / subgroup for $j = 1, ..., m$. Consequently, $\tilde{Q}_j = \tilde{Q}(x_j) = \left(\tilde{Q}(x_{1j}), ..., \tilde{Q}(x_{nj})\right)$ is the j-th sample for the quality degree where $\tilde{Q}(x_{ij}) \in [0, 1]$ for $i = 1, ..., n$ and $j = 1, ..., m$. Let

$$\overline{\tilde{Q}_j} = \frac{1}{n} \sum_{i=1}^{n} \tilde{Q}(x_{ij}), \tag{1}$$

and

$$R_j = \max_{i=1,...,n} \tilde{Q}(x_{ij}) - \min_{i=1,...,n} \tilde{Q}(x_{ij}), \tag{2}$$

are respectively the average/mean and the range of j-th sample of the quality degrees which have the ability to draw in $\bar{X}$ and R fuzzy quality control charts according to sample number for $j = 1, ..., m$. The center line (CL), lower control limit (LCL) and upper control limit (UCL) of the percentile-based $\bar{X}$ control chart for triangular fuzzy quality can be computed from the following equations:

$$\begin{cases} LCL_{\bar{X}}^{Q} = Quantile_{p/2}\left\{\overline{\tilde{Q}_1}, ..., \overline{\tilde{Q}_m}\right\}, \\ CL_{\bar{X}}^{Q} = Quantile_{1/2}\left\{\overline{\tilde{Q}_1}, ..., \overline{\tilde{Q}_m}\right\}, \\ UCL_{\bar{X}}^{Q} = Quantile_{1-p/2}\left\{\overline{\tilde{Q}_1}, ..., \overline{\tilde{Q}_m}\right\}, \end{cases} \tag{3}$$

in which $Quantile_p\{x_1, ..., x_m\} = x_{(k)} + \{(m+1)p - k\}\left(x_{(k+1)} - x_{(k)}\right)$ is the p-th quantile of data set $x_1, ..., x_m$ for $0 < p < 1$ and $k = [(m+1)p]$. In other words, the center line of the proposed $\bar{X}$ control chart is the median of the quality means. Meanwhile, the lower and upper control limits are $\frac{p}{2}$-th and $(1-\frac{p}{2})$-th quantiles of the quality means, respectively. Usually, we choose $p = 0.0027$ since $P(-3 \leq Z \leq 3) = 1 - p = 0.9973$ where Z is the standard Normal random variable [9]. In special case where the fuzzy quality is defined by the indicator function $\tilde{Q}(x) = \begin{cases} 1, & LSL \leq x \leq USL \\ 0, & otherwise \end{cases}$, the j-th sample for the quality degree becomes a set of zeros and ones, i.e. $\tilde{Q}(x_{ij}) \in \{0, 1\}$, for $i = 1, ..., n$ and $j = 1, ..., m$. Then in such crisp situation $\overline{\tilde{Q}_j}$ becomes the proportion of non-defective items in the j-th sample for $j = 1, ..., m$.

Remark 3.1. *Points that are plotted above the upper control limit should be interpreted with special care (although they are satisfactory). In most cases, the existence of such points does not mean a real improvement in the quality of the process. They may be observed/occurred due to an error during the inspection. Or they may be from reporting fake quality information by operators, to show the high quality of the process. Therefore, not all upward changes in $\tilde{Q}$ indicate an improvement in process quality.*

Moreover, the process variability may be monitored by plotting values of the sample range of quality observations over the percentile-based R control chart. The center line and control limits of the percentile-based R control chart for triangular fuzzy quality are as follows:

$$\begin{cases} LCL_{R}^{Q} = Quantile_{p/2}\{R_1, ..., R_m\}, \\ CL_{R}^{Q} = Quantile_{1/2}\{R_1, ..., R_m\} = Median\{R_1, ..., R_m\}, \\ UCL_{R}^{Q} = Quantile_{1-p/2}\{R_1, ..., R_m\}. \end{cases} \tag{4}$$

3.2 Density-based approach by kernel density estimation

As a non-parametric method, the kernel density estimation (KDE) is useful applicable method to estimate the probability density function of a continuous random variable. Let $x_1, ..., x_n$ is the observed random sample from continuous univariate random variable X which its probability density function is unknown. The kernel density estimation based on observations $x_1, ..., x_n$ is defined by

$$\hat{f}\left[x_1, ..., x_n\right](x) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x - x_i}{h}\right), \tag{5}$$

in which $h > 0$ is the smoothing parameter and K is non-negative kernel function. Some of common kernel functions are: triangular $K(x) = \max\{1 - |x|, 0\}$, Gaussian $K(x) = \frac{1}{\sqrt{2\pi}} exp(\frac{-x^2}{2})$, and rectangular $K(x) = \frac{I(|x|\leq 1)}{2}$ where I is the indicator function (see [3] for more details about KDE and the methods of optimal h estimation).

According to this introduction, the kernel estimation of control limits in $\bar{X}$ and R percentile-based control charts can be computed from equations

$$\begin{cases} \int_0^{LCL_{\bar{X}}^{KDE}} \hat{f}\left[\overline{\tilde{Q}_1}, ..., \overline{\tilde{Q}_m}\right](x)\, dx = \frac{p}{2}, \\ \int_0^{CL_{\bar{X}}^{KDE}} \hat{f}\left[\overline{\tilde{Q}_1}, ..., \overline{\tilde{Q}_m}\right](x)\, dx = \frac{1}{2}, \\ \int_0^{UCL_{\bar{X}}^{KDE}} \hat{f}\left[\overline{\tilde{Q}_1}, ..., \overline{\tilde{Q}_m}\right](x)\, dx = 1 - \frac{p}{2}, \end{cases} \tag{6}$$

and

$$\begin{cases} \int_0^{LCL_{R}^{KDE}} \hat{f}\left[R_1, ..., R_m\right](x)\, dx = \frac{p}{2}, \\ \int_0^{CL_{R}^{KDE}} \hat{f}\left[R_1, ..., R_m\right](x)\, dx = \frac{1}{2}, \\ \int_0^{UCL_{R}^{KDE}} \hat{f}\left[R_1, ..., R_m\right](x)\, dx = 1 - \frac{p}{2}, \end{cases} \tag{7}$$

respectively. A real-world case study is presented in the next section to show the performance of the proposed control charts.

Remark 3.2. *Although $\tilde{Q}$ is a fuzzy set and its range is interval $[0, 1]$, but it is possible to obtain control limits out of reasonable range $[0, 1]$ in the density-based approach by KDE. That is because of regarding to the value of smoothing parameter and the kernel function, the range of the KDE for any data set is a little wider than the range of data set. In such situations, one can respectively consider $\max\left\{0, LCL^{KDE}\right\}$ and $\min\left\{1, UCL^{KDE}\right\}$ as the modified lower and upper control limits. Moreover, the user can decrease the smoothing parameter to avoid facing such a situation.*

Remark 3.3. *Amirzadeh et al.* [2] *have worked on control chart extension based on the fuzzy quality but their method differs from the investigated approaches outlined here in the following ways:*
(1) *Construction p control charts have been considered in* [2] *but here $\bar{X}$ and R charts are constructed,*
(2) *Normality assumption for the quality characteristic is considered in* [2] *whereas two robust approaches — with respect to the distribution — are proposed here, and*
(3) *Percentile-based approaches are considered here to estimate control limits but Amirzadeh et al.* [2] *have considered Shewhart-based control charts for control limits estimation.*

4 Case study

Piston rings for a car engine are produced in a forging process. Twenty-five samples, each of size 5, are taken from the inside diameter length (in terms of one millimeter) which are quoted in six left columns of Table 1 from [2, 9]. It must be mentioned that all the programs and plots in this case study have been carried out with R software and also the piston rings data are accessible by "qcc" package [16]. Regarding to the Section 2 and also considering specification limits $[73.96, 74.03]$ and the target value $T = 74.005$ to specify the target of the quality characteristic X in the manufacturing process, we decide to consider the following triangular fuzzy quality (see Figure 1)

$$\tilde{Q}_{\Delta} = Tr(73.96, 74.005, 74.03).$$

Table 1: Forty samples of piston rings measurements with their quality degrees in the case study.

j	x_{ij}					$\tilde{Q}(x_{ij})$					$\bar{\tilde{Q}}_j$	R_j
1	74.030	74.002	74.019	73.992	74.008	0.000	0.933	0.440	0.711	0.880	0.5928	0.933
2	73.995	73.992	74.001	74.011	74.004	0.778	0.711	0.911	0.760	0.978	0.8276	0.267
3	73.988	74.024	74.021	74.005	74.002	0.622	0.240	0.360	1.000	0.933	0.6310	0.760
4	74.002	73.996	73.993	74.015	74.009	0.933	0.800	0.733	0.600	0.840	0.7812	0.333
5	73.992	74.007	74.015	73.989	74.014	0.711	0.920	0.600	0.644	0.640	0.7030	0.320
6	74.009	73.994	73.997	73.985	73.993	0.840	0.756	0.822	0.556	0.733	0.7414	0.284
7	73.995	74.006	73.994	74.000	74.005	0.778	0.960	0.756	0.889	1.000	0.8766	0.244
8	73.985	74.003	73.993	74.015	73.988	0.556	0.956	0.733	0.600	0.622	0.6934	0.400
9	74.008	73.995	74.009	74.005	74.004	0.880	0.778	0.840	1.000	0.978	0.8952	0.222
10	73.998	74.000	73.990	74.007	73.995	0.844	0.889	0.667	0.920	0.778	0.8196	0.253
11	73.994	73.998	73.994	73.995	73.990	0.756	0.844	0.756	0.778	0.667	0.7602	0.177
12	74.004	74.000	74.007	74.000	73.996	0.978	0.889	0.920	0.889	0.800	0.8952	0.178
13	73.983	74.002	73.998	73.997	74.012	0.511	0.933	0.844	0.822	0.720	0.7660	0.422
14	74.006	73.967	73.994	74.000	73.984	0.960	0.156	0.756	0.889	0.533	0.6588	0.804
15	74.012	74.014	73.998	73.999	74.007	0.720	0.640	0.844	0.867	0.920	0.7982	0.280
16	74.000	73.984	74.005	73.998	73.996	0.889	0.533	1.000	0.844	0.800	0.8132	0.467
17	73.994	74.012	73.986	74.005	74.007	0.756	0.720	0.578	1.000	0.920	0.7948	0.422
18	74.006	74.010	74.018	74.003	74.000	0.960	0.800	0.480	0.956	0.889	0.8170	0.480
19	73.984	74.002	74.003	74.005	73.997	0.533	0.933	0.956	1.000	0.822	0.8488	0.467
20	74.000	74.010	74.013	74.020	74.003	0.889	0.800	0.680	0.400	0.956	0.7450	0.556
21	73.988	74.001	74.009	74.005	73.996	0.622	0.911	0.840	1.000	0.800	0.8346	0.378
22	74.004	73.999	73.990	74.006	74.009	0.978	0.867	0.667	0.960	0.840	0.8624	0.311
23	74.010	73.989	73.990	74.009	74.014	0.800	0.644	0.667	0.840	0.640	0.7182	0.200
24	74.015	74.008	73.993	74.000	74.010	0.600	0.880	0.733	0.889	0.800	0.7804	0.289
25	73.982	73.984	73.995	74.017	74.013	0.489	0.533	0.778	0.520	0.680	0.6000	0.289
26	74.012	74.015	74.030	73.986	74.000	0.720	0.600	0.000	0.578	0.889	0.5574	0.889
27	73.995	74.010	73.990	74.015	74.001	0.778	0.800	0.667	0.600	0.911	0.7512	0.311
28	73.987	73.999	73.985	74.000	73.990	0.600	0.867	0.556	0.889	0.667	0.7158	0.333
29	74.008	74.010	74.003	73.991	74.006	0.880	0.800	0.956	0.689	0.960	0.8570	0.271
30	74.003	74.000	74.001	73.986	73.997	0.956	0.889	0.911	0.578	0.822	0.8312	0.378
31	73.994	74.003	74.015	74.020	74.004	0.756	0.956	0.600	0.400	0.978	0.7380	0.578
32	74.008	74.002	74.018	73.995	74.005	0.880	0.933	0.480	0.778	1.000	0.8142	0.520
33	74.001	74.004	73.990	73.996	73.998	0.911	0.978	0.667	0.800	0.844	0.8400	0.311
34	74.015	74.000	74.016	74.025	74.000	0.600	0.889	0.560	0.200	0.889	0.6276	0.689
35	74.030	74.005	74.000	74.016	74.012	0.000	1.000	0.889	0.560	0.720	0.6338	1.000
36	74.001	73.990	73.995	74.010	74.024	0.911	0.667	0.778	0.800	0.240	0.6792	0.671
37	74.015	74.020	74.024	74.005	74.019	0.600	0.400	0.240	1.000	0.440	0.5360	0.760
38	74.035	74.010	74.012	74.015	74.026	0.000	0.800	0.720	0.600	0.160	0.4560	0.800
39	74.017	74.013	74.036	74.025	74.026	0.520	0.680	0.000	0.200	0.160	0.3120	0.680
40	74.010	74.005	74.029	74.000	74.020	0.800	1.000	0.040	0.889	0.400	0.6258	0.960

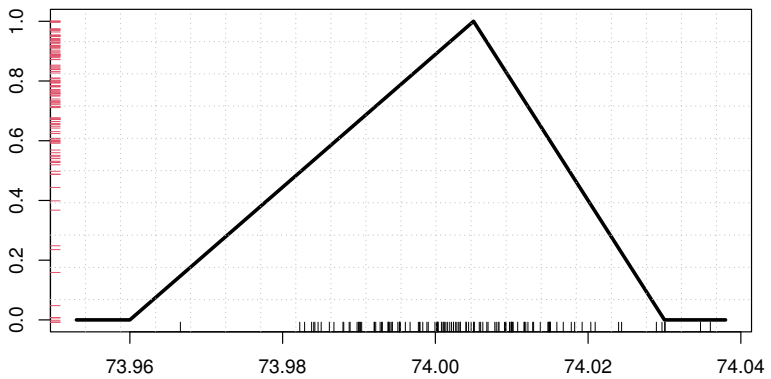


Figure 1: Membership function of fuzzy quality $\tilde{Q}_\Delta$ in the case study.

The quality degree of each inside diameter for forty samples are computed based on the considered fuzzy quality $\tilde{Q}_\Delta$ in Table 1. Moreover, the mean and range of quality degrees are calculated in Table 1 for each samples. For instance, the first sample / subgroup is $x_1 = (74.030, 74.002, 74.019, 73.992, 74.008)$. Therefore, by considering the triangular fuzzy quality $\tilde{Q}_\Delta$, the first sample of the quality degree is obtained as $\tilde{Q}_1 = (0.000, 0.933, 0.440, 0.711, 0.880)$. Hence, the mean and range for the quality degree of the first sample can be easily computed by

$$\overline{\tilde{Q}_1} = \frac{0.000 + 0.933 + 0.440 + 0.711 + 0.880}{5} = 0.5928,$$

and

$$R_1 = \max\{0.000, 0.933, 0.440, 0.711, 0.880\} - \min\{0.000, 0.933, 0.440, 0.711, 0.880\} = 0.933,$$

respectively. Table 2 contains the center line and control limits for the proposed $\bar{X}$ and R control charts in Subsection 3.1 with $p = 0.0027$ in the case study which are computed by Eq. (3) and Eq. (4). The treat of data-based mean and range control charts for the degree of belonging observations to the triangular fuzzy quality $\tilde{Q}_\Delta$ are plotted in Figure 2 by `qcc` package [16] based on the proposed approach in Subsection 3.1.

Table 2: Mean and range control limits in the case study based on the investigated approach in Subsection 3.1.

Type of control chart	LCL	CL	UCL
$\bar{X}$	0.539	0.781	0.895
R	0.178	0.320	0.933

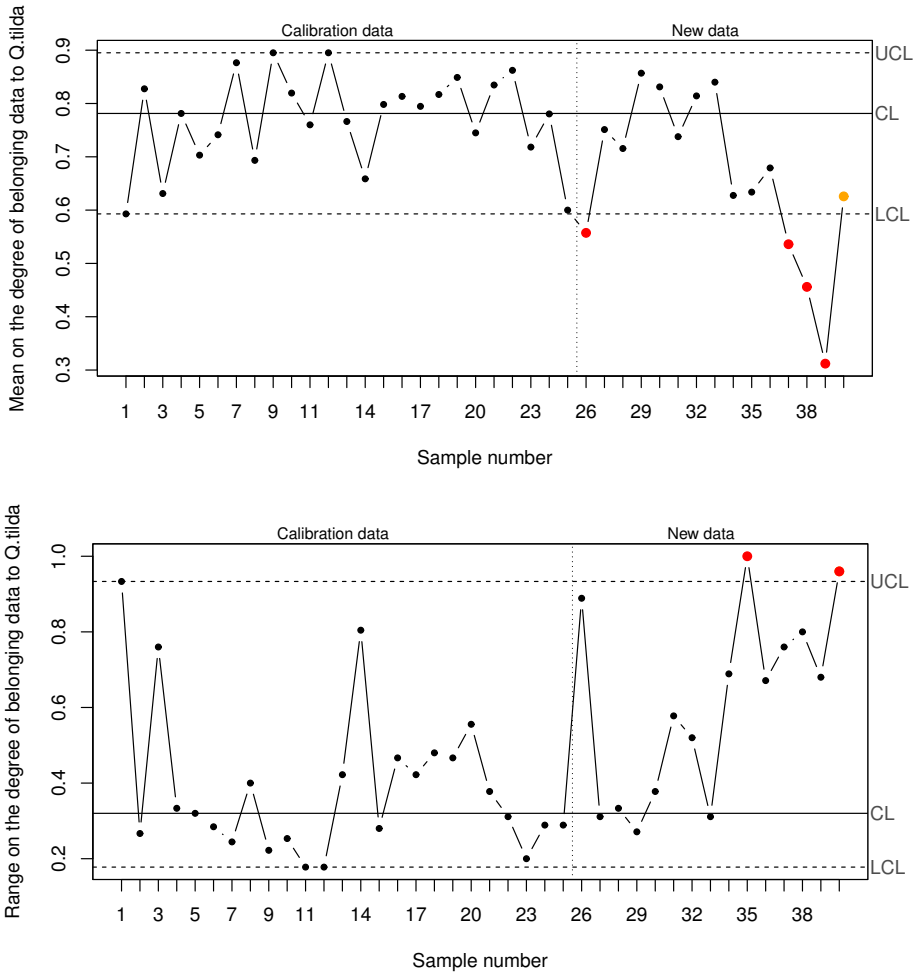


Figure 2: Data-based $\bar{X}$ and R control charts (phase I and phase II) for the degree of belonging observations to the triangular fuzzy quality $\tilde{Q}_\Delta$ in the case study.

The 25 initial subgroup points which were used to estimate control limits were in control at phase *I*. If some of these points were out of control, we would eliminate them, and then plot a control chart based on the remaining points. If all the points on this new chart were in control, they would be used to estimate control limits once again. If not, the out of control points would again be eliminated, and the procedure repeated until, as in our case study, all sample points fell within the control limits. Now phase *II* start with 15 additional samples (each with size 5). If any new subgroups are taken, a sample point falling outside these control limits will be considered out of control [2, 9]. Even if all the points plot inside the control limits, but if they behave in a systematic/nonrandom behavior, then this is an indication that the process is out of control. Figure 3 shows the kernel density estimations for the mean of belonging degrees to the triangular fuzzy quality $\tilde{Q}_\Delta$ with considering optimal smoothing parameter $h = 0.03857$. Also, Figure 4 shows the kernel density estimations for the range of belonging degrees to the triangular fuzzy quality $\tilde{Q}_\Delta$ with optimal smoothing parameter $h = 0.07056$. Table 3 contains the center line and control limits for the investigated $\bar{X}$ and R control charts in Subsection 3.2 by considering triangular, uniform and Gaussian kernel functions in the case study which are computed by Eq. (6) and Eq. (7). Furthermore, kernel-based mean and range control charts — based on triangular kernel function — are plotted in Figure 5 for the degree of belonging observations to the triangular fuzzy quality $\tilde{Q}_\Delta$ by the investigated approach in Subsection 3.2. The red and yellow points in $\bar{X}$ and R control charts of Figure 2 and Figure 5 indicate out of control points and systematic/nonrandom manner, respectively. There is usually a reason for the certain non-random pattern appears in control charts, and the performance of the process can be improved if the reason can be found and eliminated.

In monitoring the inside diameter of the piston rings data by conventional $\bar{X}$ and R control charts, samples 37, 38, 39 and 40 are determined as out-of-control samples [2, 9]. But samples 26, 35, 37, 38, 39 and 40 are determined as out-of-control in both presented percentile-based control charts which shows the higher sensitivity of the proposed charts on the basis of fuzzy quality. Of course, these results are not general and depend on the membership function of the considered fuzzy quality.

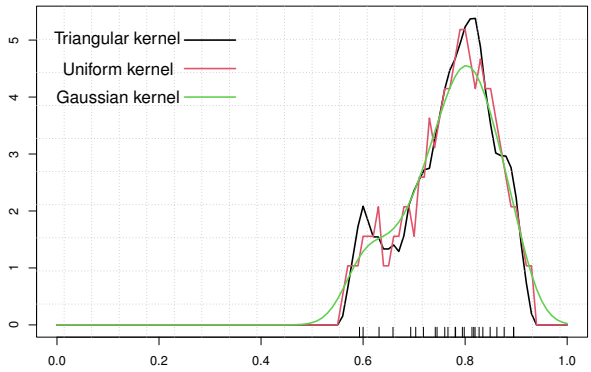


Figure 3: Triangular, uniform and Gaussian kernel density functions for the mean of belonging degrees to the triangular fuzzy quality $\tilde{Q}_\Delta$ in the case study.

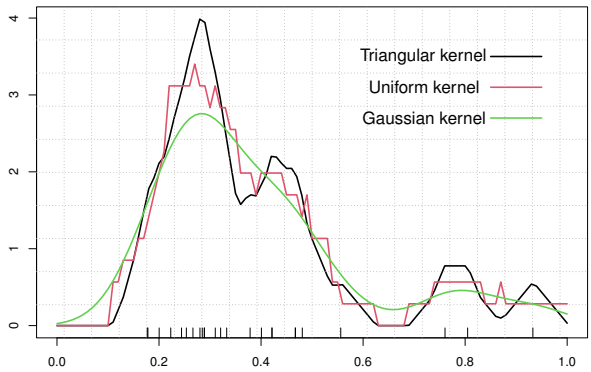


Figure 4: Triangular, uniform and Gaussian kernel density functions for the range of belonging degrees to the triangular fuzzy quality $\tilde{Q}_\Delta$ in the case study.

Table 3: Mean and range control limits for triangular, uniform and Gaussian kernel functions based on the investigated approach in Subsection 3.2 in the case study.

Type of kernel function	Type of control chart	LCL	CL	UCL
triangular	X	0.564	0.785	0.926
triangular	R	0.120	0.329	0.985
uniform	$\bar{X}$	0.557	0.784	0.933
uniform	R	0.110	0.336	0.995
Gaussian	$\bar{X}$	0.514	0.782	0.979
Gaussian	R	0.029	0.343	1.000

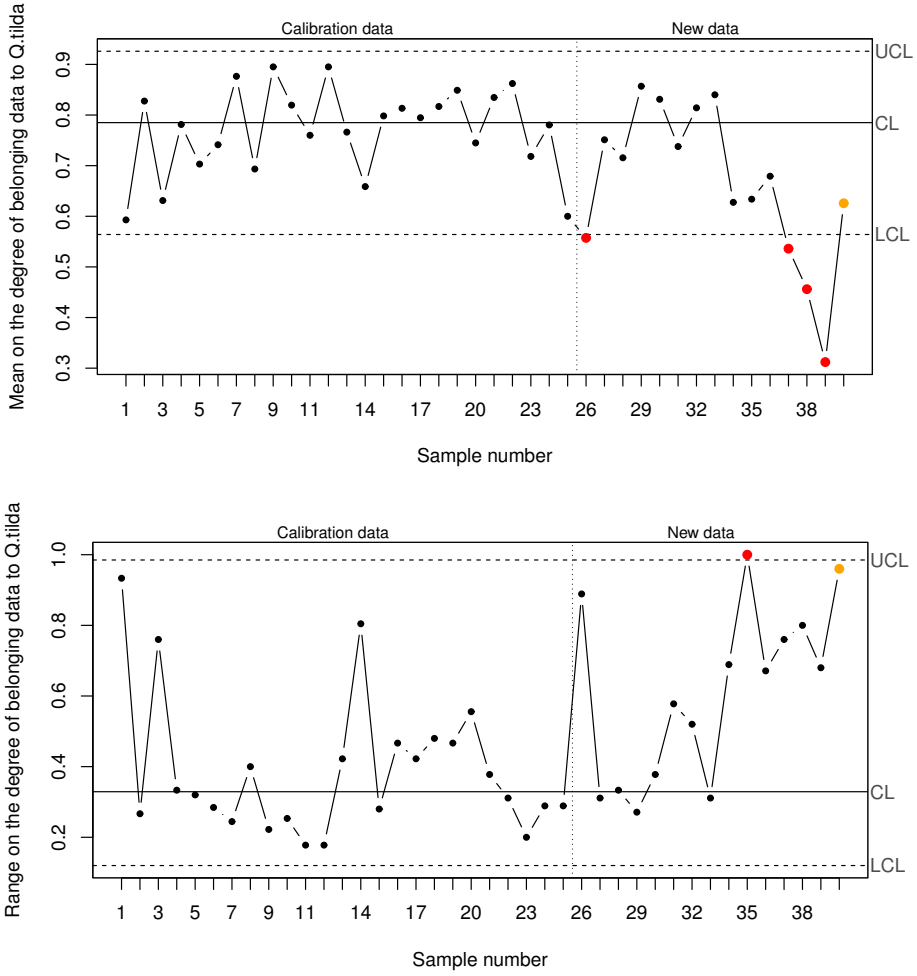


Figure 5: Kernel-based $\bar{X}$ and R control charts (phase I and phase II) for the degree of belonging observations to the triangular fuzzy quality $\tilde{Q}_{\Delta}$ in the case study.

5 Modeling control charts by simulation

In this section, we compare four proposed control charts in Subsection 3.1, i.e. data-based and density-based approaches, by simulation on the basis of piston rings data. Considering automobile engine piston rings observations in Table 1, the process mean and process standard deviation can be estimated by $\hat{\mu} = \bar{\bar{x}} = 74$ and $\hat{\sigma} = \dfrac{\bar{s}}{c_4} = 0.0096$, respectively. A simulation is done in two following phases, to have a comparison between the proposed quantile-based control charts:
Simulation of phase *I*: The main goal in this phase is the simulation of control limits by two proposed quantile-based control charts based on the introduced triangular fuzzy quality $\tilde{Q}_{\Delta}$. One thousand samples ($m = 10^3$) — each of

size five — have been simulated independently from the Normal distribution with mean $\hat{\mu}$ and $\hat{\sigma}$ from the in-control process. Then, control limits have been simulated in Table 4 for both data-based and density-based proposed $\bar{X}$-R control charts in Subsection 3.1. In kernel density-base approach, the triangular kernel is considered for the simulation of control limits.

Simulation of phase *II*: The probability of a point between the control limits for an in-control process must be equal to $1-p=0.9973$. In phase II of the simulation, we are going to simulate this probability for all investigated quantile-based charts in Subsection 3.1 by Monte Carlo method when the process is in-control. Therefore, one million samples — each of size five — have been simulated independently from the Normal distribution with mean $\hat{\mu}$ and standard deviation $\hat{\sigma}$. The simulated samples in phase I must be independent of the simulated samples in phase II. Monte Carlo simulation for the following probabilities is considered in Table 4 for phase II where the process is in-control:

- the probability of a random point being bellow the simulated LCL,
- the probability of a random point being between the simulated control limits, and
- the probability of a random point being above the simulated UCL.

It must be mentioned that symbol $\bullet$ in Table 4 denotes the random point/statistic on control chart in phase II of simulation. For example, the probability of a random point $\overline{\tilde{Q}}$ being between the simulated control limits of the data-based $\bar{X}$ control chart is estimated/simulated for in-control process in Table 4 by the proportion formula

$$\hat{P}\left(LCL_{\bar{X}}^{Q}<\overline{\tilde{Q}}<UCL_{\bar{X}}^{Q}\right)=\frac{1}{10^6}\sum_{j=1}^{10^6}I\left(LCL_{\bar{X}}^{Q}<\overline{\tilde{Q}_j}<UCL_{\bar{X}}^{Q}\right)=0.996836,$$

in which the control limits $LCL_{\bar{X}}^{Q}$ and $UCL_{\bar{X}}^{Q}$ are simulated in phase I and the simulated data $\overline{\tilde{Q}_1},...,\overline{\tilde{Q}_{10^6}}$ are independently simulated in phase II.

Table 4: Simulation results of two investigated quantile-based charts in Subsection 3.1 based on the introduced triangular fuzzy quality $\tilde{Q}_{\Delta}$ in the case study.

Type of approach and chart	LCL	UCL	$\hat{P}(\bullet\leq LCL)$	$\hat{P}(LCL<\bullet<UCL)$	$\hat{P}(UCL\leq\bullet)$	$\hat{p}$
Data-based $\bar{X}$ control chart	0.531	0.955	0.002548	0.996836	0.000616	0.003164
Data-based R control chart	0.049	0.993	0.006350	0.999015	0.000350	0.000985
Density-based $\bar{X}$ control chart	0.535	0.954	0.002853	0.996444	0.000703	0.003556
Density-based R control chart	0.051	0.988	0.001303	0.997445	0.001252	0.002555

It seems that the simulated $\hat{p}$ in Table 4 for density-based control charts is more closed to $p=0.0027$ with respect to data-based control charts and therefore density-based charts are a little better than data-based charts in this simulation.

Table 5: Simulation results of two investigated quantile-based charts in Subsection 3.1 based on the introduced triangular fuzzy quality $\tilde{Q}_{\Delta}$ with $m=10^6$.

Type of approach and chart	LCL	UCL	$\hat{P}(\bullet\leq LCL)$	$\hat{P}(LCL<\bullet<UCL)$	$\hat{P}(UCL\leq\bullet)$	$\hat{p}$
Data-based $\bar{X}$ control chart	0.511	0.9464	0.001377	0.997197	0.001426	0.002803
Data-based R control chart	0.058	0.972	0.001326	0.997332	0.001342	0.002668
Density-based $\bar{X}$ control chart	0.511	0.946	0.001377	0.997143	0.00148	0.002857
Density-based R control chart	0.058	0.974	0.001303	0.997445	0.001252	0.002555

A similar simulation is considered with $m=10^6$ in Table 5. The simulation results in this table show that the probability of a random point being out of control limits when the process is in-control (i.e., the probability of type I error occurrence) is estimated/simulated near to $p=0.0027$ for all proposed percentile-based $\bar{X}$ and R control charts. Moreover, "the probability of random point being less than the simulated LCL" is approximately equal to "the probability of random point being bigger than the simulated UCL" for large sample sizes. Besides, the simulated control limits for both kinds data-based and density-based $\bar{X}$-R charts are approximately the same in Table 5 and it shows the tendency of both charts for large sample sizes. Also in simulation with large sample size, simulated $\hat{p}$ are very closed to 0.0027 which indicates the appropriate performance of data-based and density-based $\bar{X}$-R control charts.

6 Conclusions and future works

The $\bar{X}$ chart indicates the central tendency for a manufacturing process, whereas the R chart indicates its dispersion/variance. Because of both variables are important, it makes sense to simultaneously monitor a process using both $\bar{X}$ and R control charts. Construction both of $\bar{X}$ and R control charts was investigated in this article on the basis of "fuzzy quality" rather than "crisp quality". In the other words, two new percentile-based approaches are proposed in this paper to simultaneously control both process mean and process range of the belonging degree to the triangular fuzzy quality. The first approach is based on the quantiles of quality degrees and the second approach is on the basis of kernel density estimation. A case study on automobile engine piston rings is presented to show the performance of two proposed approaches. The simulated probability of a random point being out of control limits when the process is in-control is approximately equal to the nominal value 0.0027 for all investigated percentile-based $\bar{X}$ and R control charts. Although the investigated control charts have several differences with Amirzadeh et al. [2] chart which are presented in Remark 3, but the most advantage of proposed quantile-based charts is their robustness with respect to the distribution of quality characteristics. Meanwhile with respect to the traditional/crisp quality, the users can have more flexibility in design of the fuzzy quality membership function when work with the proposed quantile-based charts in this paper. Construction other types of control charts by a similar percentile-based approach is a potential field for future works.

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