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Extensions of $Z-$ideals in $MV-$algebras

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Abstract

In this paper, we introduce a special type of $Z-$ideals in $MV-$algebras and delve into their properties. These particular ideals are thoroughly examined within the context of $MV-$chains, semi-simple $MV-$algebras and Hyperarchimedean structures. Furthermore, we provide characterizations for these ideals, offering equivalent definitions that enhance our understanding of their nature. Our investigation also encompasses the relationships between these special $Z-$ideals and other types of ideals, such as the minimal prime ideals and the maximal ideals. Overall, this paper aims to provide a comprehensive analysis of special $Z-$ideals in $MV-$algebras.

Keywords: $MV-$algebra, Z_J-ideal, $Z_{NJ}-$ideal.

1 Introduction

MV-algebras were first introduced by C. C. Chang in 1958 as an algebraic counterpart to Łukasiewicz infinite-valued propositional logic [3]. $MV-$algebras are algebraic structures that generalize classical Boolean algebras. They are used in Mathematical logic and have applications in various fields, such as fuzzy logic, probability theory, and artificial intelligence. $MV-$algebras, also known as many-valued algebras, play a crucial role in fuzzy logic. Fuzzy logic is a Mathematical framework that enables reasoning and decision-making in conditions of uncertainty and ambiguity. Unlike classical binary logic, which is based on true or false values, fuzzy logic allows for intermediate truth values between true and false, representing degrees of truth or membership. $MV-$algebras provide a formal algebraic structure that can handle these intermediate truth values, serving as the foundation for fuzzy logic. They offer a rigorous Mathematical framework for the representation, manipulation, and reasoning about middle truth values, making them an essential tool for modeling and analyzing fuzzy systems. By applying the algebraic operations and rules defined within $MV-$algebras, fuzzy logic systems can perform various reasoning tasks, such as fuzzy inference, approximate reasoning, and fuzzy control. This formal algebraic structure enables the definition and manipulation of fuzzy rules and fuzzy concepts, providing a solid foundation for the implementation and analysis of fuzzy logic-based applications. In $MV-$algebras, the operations of conjunction and disjunction are replaced by a single operation called "the $MV-$operation," which captures the notion of truth degrees. While we aim to keep this paper concise, we encourage readers to refer to [3, 4, 11] for additional results on $MV-$algebras, particularly focusing on the ideal theory of $MV-$algebras [8]. Chang's work included the introduction of prime ideals and the definition of maximal ideals in $MV-$algebras [3]. It was proven that every proper ideal of an $MV-$algebra A is contained within a maximal ideal of A and every prime ideal of an $MV-$algebra A is contained within a unique maximal ideal of A [11]. M. Bedrood et al. extended the concept of ideals in $MV-$algebras by introducing $Z-$ideals and $Z^\circ-$ideals, exploring their relationships with other ideals of $MV-$algebras [2]. They also investigated $Z^\circ-$ideals and $Z-$ideals in $MV-$algebras of continuous functions [1].
Now, when we talk about extensions of $Z-$ideals, we are considering the process of enlarging or extending the $Z-$ideal to a larger subset while preserving its properties. This extension allows us to explore the structure of the $Z-$ideal in a more comprehensive manner and potentially uncover new insights or connections within the $MV-$algebra. By studying

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extensions of $Z-$ideals, one aims to understand the relationships between different subsets and their properties within $MV-$algebras. This exploration can lead to a deeper understanding of the algebraic structure and provide a basis for developing new Mathematical techniques or applications in related fields. Motivated by the significance of $Z-$ideals and maximal ideals, we delve deeper into the subject and explore new avenues. In this paper, we propose generalizations of $Z-$ideals based on an ideal J of $MV-$algebras, which we named $Z-$factors. We introduce these ideals and meticulously examine their properties, aiming to gain a deeper understanding of their characteristics and behaviors. Additionally, by exploring the relationship between these ideals and the radical of the $MV-$algebra, we obtain valuable results and findings. Through the study of semi-simple $MV-$algebras, we conclude that every ideal is either a $Z_{NJ}-$ideal or an essential ideal and the ideal generated by any idempotent element is a Z_J-ideal. We provide a proof that every proper ideal is a Z_J-ideal in the Hyperarchimedean and in $MV-$chains every $Z_{NJ}-$ideal is a $Z-$ideals. Finally, we introduce the lattices of Z_J-ideals and $Z-$factors, offering a comprehensive framework for understanding these concepts.

2 Preliminaries

We recollect some definitions and results which will be used in the sequel:

Definition 2.1. [3] *An $MV-$algebra is a structure $(A, \oplus, *, 0)$ of type $(2, 1, 0)$ such that the following axioms hold, for each $a, b \in A$:*
$(MV1)$ *$(A, \oplus, 0)$ is an abelian monoid;*
$(MV2)$ $(a^*)^* = a$;
$(MV3)$ $0^* \oplus a = 0^*$;
$(MV4)$ $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

Define $1 = 0^*$ and the auxiliary operations $\odot$, $\vee$ and $\wedge$ are defined on A:
$a \odot b = (a^* \oplus b^*)^*$, $\quad a \vee b = a \oplus (a^* \odot b) = b \oplus (a \odot b^*)$ and $\quad a \wedge b = a \odot (a^* \oplus b) = b \odot (b^* \oplus a)$. Also, for any two elements $a, b \in A$, $a \leq b$ if and only if $a^* \oplus b = 1$. Obviously, $\leq$ is a partial order on A which is called the natural order on A. We say that an $MV-$algebra A is an $MV-$chain if it is linearly ordered relative to natural order. A is called Hyperarchimedean if for every element $a \in A$ there is $n \in \mathbb{N}$ such that $na = (n+1)a$.
In this paper, A is an $MV-$algebra.

Lemma 2.2. [11] *If $a, a_1, a_2, ..., a_n$ are elements of A, then the following hold:*

$$a \wedge (a_1 \oplus a_2 \oplus ... \oplus a_n) \leq (a \wedge a_1) \oplus (a \wedge a_2) \oplus ... \oplus (a \wedge a_n).$$

Definition 2.3. [11] *An element a of A is called an idempotent element, if $a \oplus a = a$ and the collection of all idempotent elements is displayed with $B(A)$. Also, if every member of A is an idempotent element, then $MV-$algebra A is called a Boolean algebra.*

Definition 2.4. [4, 11] *An ideal of A is a non-empty subset I of A which is closed under $\oplus$ and such that if $b \in I$, $a \in A$ and $a \leq b$, then $a \in I$. $Id(A)$ is the set of all ideals of A.*
A proper ideal I of A is called:
• *A prime ideal, if $a \wedge b \in I$ implies that $a \in I$ or $b \in I$, for each $a, b \in A$. $Spec(A)$ is the set of all prime ideals of A.*
• *A minimal prime ideal of A, if there exists $J \in Spec(A)$ such that $J \subseteq I$, then $I = J$. $Min(A)$ is the set of all minimal prime ideals of A.*
• *A maximal ideal if and only if whenever J is an ideal such that $I \subseteq J \subseteq A$, then either $J = I$ or $J = A$. $Max(A)$ is the set of all maximal ideals of A and $Rad(A)$ is the intersection of all maximal ideals of A.*
• *Local $MV-$algebras are $MV-$algebras with only one maximal ideal.*
• *A is a semi-simple $MV-$algebra if and only if it is non-trivial and $Rad(A) = \{0\}$.*

Note: Prime ideal P of A is called a minimal prime ideal over ideal I, if
1) $I \subseteq P$;
2) If there exists $Q \in Spec(A)$ such that $I \subseteq Q \subseteq P$, then $P = Q$.
$Min(I)$ is the set of all minimal prime ideals over ideal I.

Corollary 2.5. [11] *Let $I \in Id(A)$. Then $I = \cap \{P \in Spec(A) \mid I \subseteq P\}$.*

Lemma 2.6. [11] *The proper ideal P is a prime ideal if and only if $I \cap J \subseteq P$, then $I \subseteq P$ or $J \subseteq P$, for all $I, J \in Id(A)$.*

Notation 2.7. [4] *Every finite $MV-$algebra is semi-simple.*

Notation 2.8. [11] *(i) Let $X \subseteq A$. Denote by $(X]$ the ideal generated by X. Then we have* $(X] = \{a \in A \mid a \leqslant x_1 \oplus x_2 \oplus ... \oplus x_n, \ for\ some\ n \in \mathbb{N}\ and\ x_1, ..., x_n \in X\}$.
In particular, $(a] = \{x \in A \mid x \leqslant na, \ for\ some\ n \in \mathbb{N}\}$.
(ii) For $I, J \in Id(A)$, we put

$$I \wedge J = I \cap J, \qquad I \vee J = (I \cup J], \qquad (I \cup J] = \{x \in A \mid x \leq a \oplus b, \text{ for some } a \in I \text{ and } b \in J\}.$$

(iii) If $a, b \in A$, then $(a] \cap (b] = (a \wedge b]$ and $(a] \vee (b] = (a \oplus b]$.

Theorem 2.9. [10] *Let I be a proper ideal of A and P be a prime ideal of A such that $I \subseteq P$. Then $P \in Min(I)$ if and only if for each $a \in P$ there exists $r \in A \setminus P$ such that $r \wedge a \in I$.*

Definition 2.10. [6] *Let X be a non-empty subset of A. $Ann(X)$ is the annihilator of X defined by:*

$$Ann(X) = \{a \in A : a \wedge x = 0, \forall x \in X\}.$$

Lemma 2.11. [2] *Let X and Y be two subsets of A. Then the following sentences hold:*
(i) $Ann(Ann(Ann(X))) = Ann(X)$.
(ii) If $X \subseteq Y$, then $Ann(Y) \subseteq Ann(X)$.

Definition 2.12. [2] *Let X be a non-empty subset of A. For an ideal I of A, the set*

$$(I : X) = \{a \in A \mid a \wedge x \in I, \text{ for all } x \in X\}.$$

• *$(I : X)$ is clearly an ideal of A and if $I = 0$, then $(0 : X) = Ann(X)$.*

Definition 2.13. [8] *A non-zero ideal I of A is called an essential ideal if $Ann(I) = 0$.*

Theorem 2.14. [11] *Let I be an ideal of A and $a \in A \setminus I$. Then there exists $P \in Spec(A)$ such that $I \subseteq P$ and $a \notin P$.*

Let $a \in A$ and I be an ideal of A. Define:

$$M_a = \bigcap\{M \mid M \in Max(A), a \in M\}, \qquad M(a) = \{M \mid M \in Max(A), a \in M\}$$

Lemma 2.15. [2] *If $a \in A$, then $P_a = Ann(Ann(a))$.*

Lemma 2.16. [1] *Let a be an element of $B(A)$. Then $Ann(a) = (a^*]$.*

Definition 2.17. [2] *A proper ideal I of A is called a $Z-$ideal, if $M_a \subseteq I$, for each $a \in I$.*

• Every maximal ideal is a $Z-$ideal [2].

Theorem 2.18. [2] *Let I be a proper ideal of A. Then the following statements are equivalent:*
(i) I is a $Z-$ideal in A.
(ii) $M(a) \subseteq M(b)$ and $a \in I$, implies that $b \in I$.
(iii) $M(a) = M(b)$ and $a \in I$, implies that $b \in I$.

Theorem 2.19. [11] *For any A, the following are equivalent:*
(i) A is an $MV-$chain;
(ii) Any proper ideal of A is prime;
(iii) $\{0\}$ is a prime ideal;
(iv) $Spec(A)$ is linearly ordered.

Theorem 2.20. [7] *Let I be a proper ideal of A and $P \in Spec(A)$ such that $I \subseteq P$. Then there exists $P^* \in Min(I)$ such that $P^* \subseteq P$.*

Theorem 2.21. [11] *A is Hyperarchimedean if and only if $Spec(A) = Max(A)$.*

Theorem 2.22. [2] *If A is a semi-simple $MV-$algebra and $a \in A$, then $M_a \subseteq P_a$.*

3 Z_J-ideals in $MV-$algebras

In this section, we begin by providing definitions for Z_J-ideals and Z_{NJ}-ideals. We thoroughly examine their properties and proceed to discuss the conditions under which various ideals can be considered as Z_J-ideals or $Z_{NJ}-$ideals. The notion of Z_J-ideals is introduced as a generalization of $Z-$ideals, expanding the scope of their applicability. Additionally, we introduce the concept of $Z-$factors and demonstrate that the set of $Z-$factors of an ideal forms a lattice.

Definition 3.1. *If I and J are two ideals of A, then I is said to be a Z_J-ideal if $M_a \cap J \subseteq I$, for all $a \in I$.*

Obviously, if $J \subseteq I$, then I is always a Z_J-ideal. Whenever, for an ideal I, there exist an ideal $J \nsubseteq I$ such that I is a Z_J-ideal, we say that I is a $Z_{NJ}-$ideal and J is called a $Z-$factor of the ideal I.
It is clear that every $Z_{NJ}-$ideal is a Z_J-ideal.

Example 3.2. *(i) Let $A = \{0, a, b, c, d, e, f, 1\}$. Where $0 < a, b < 1$, $0 < a < c < e < 1$, $0 < b < d < f < 1$, $b < c$ and $d < e$. With the diagram below:*

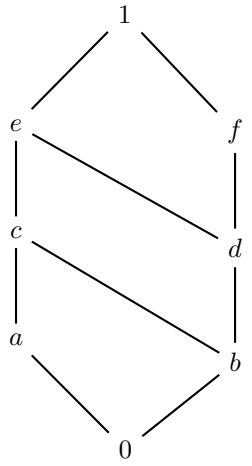


Figure 1: The non-linearly ordered $MV-$algebra.

Define $\oplus$ and $*$ as follows:

$\oplus$	0	a	b	c	d	e	f	1
0	0	a	b	c	d	e	f	1
a	a	a	c	c	e	e	1	1
b	b	c	d	e	f	1	f	1
c	c	c	e	e	1	1	1	1
d	d	e	f	1	f	1	f	1
e	e	e	1	1	1	1	1	1
f	f	1	f	1	f	1	f	1
1	1	1	1	1	1	1	1	1

$*$	0	a	b	c	d	e	f	1
	1	f	e	d	c	b	a	0

Then $(A, \oplus, *, 0)$ is an MV-algebra [9]. $I_0 = \{0\}$, $I_1 = \{0, a\}$, $I_2 = \{0, b, d, f\}$, $I_3 = A$ are ideals.
It is easy to see that I_1 is a $Z_{I_2}-$ideal and $I_2 \nsubseteq I_1$, so I_1 is a $Z_{NI_2}-$ideal. Also, I_1 is a $Z_{I_0}-$ideal and I_0 is a $Z_{NI_2}-$ideal.
(ii) Let $\mathbb{R}^*$ be a non-standard model of real numbers with natural order and ε be a positive infinitesimal element of $\mathbb{R}^*$. Let $\varepsilon^2 = \varepsilon.\varepsilon, ..., \varepsilon^n = \varepsilon.\varepsilon...\varepsilon(n - times)$, where is the usual product in the field $\mathbb{R}^*$; then $\varepsilon^i > 0$, for any $i \in \mathbb{N}$ and $\varepsilon^i \leqslant \varepsilon^j$, for $i > j$.
The unit interval $[0, 1]^* \subseteq \mathbb{R}^*$ is an $MV-$algebra with the operations: $x \oplus y = min\{1, x + y\}, x^* = 1 - x$. Let $\mathbb{N}$ be the ordered set of positive natural numbers. For every $n \in \mathbb{N}$, let E_n be the subalgebra of $[0, 1]^*$ generated by $\{\varepsilon, \varepsilon^2, ..., \varepsilon^n\}$ and E be the subalgebra $\bigcup_{n\in\mathbb{N}} E_n$ generated by $\{\varepsilon, \varepsilon^2, ..., \varepsilon^n, \ldots\}$. The ideals of E are $\{0\}, (\varepsilon], ..., (\varepsilon^i]..., $ where $i \in \mathbb{N}$ and

$(\varepsilon^i] \subseteq (\varepsilon^j]$, for any $i > j$ [5]. Put $I = (\varepsilon^i]$ and $J = (\varepsilon^j]$, if $i > j$, then I is not a Z_J-ideal ($Z_{NJ}-$ideal) and if $i \leq j$, then I is a Z_J-ideal. Also, consider $K = \{0\}$, then K is not a Z_J-ideal, for all $j \in \mathbb{N}$.

In Example 3.2 (ii), we can see $(\varepsilon^2]$ is a $Z_{(\varepsilon^4]}-$ideal but $(\varepsilon^2]$ is not a $Z-$ideal. Also, in Example 3.2 (i), I_0 is a Z_J-ideal for every ideal J of A, but is not a maximal (prime) ideal.

Lemma 3.3. *If $a, b \in A$, then the following sentences hold:*
(i) $a \in M_b$ if and only if $M_a \subseteq M_b$.
(ii) $M_a \cap M_b = M_{(a\wedge b)}$.
(iii) If $a \leq b$, then $M(b) \subseteq M(a)$.
(iv) If $a \leq b$, then $M_a \subseteq M_b$.
(v) $M_{(a\oplus b)} = M_a \vee M_b$.

Proof. (i) It is trivial.
(ii) Let $x \in M_a \cap M_b$, but $x \notin M_{(a\wedge b)}$. Then there exists $M \in Max(A)$ such that $x \notin M$ and $a \wedge b \in M$. Hence $a \in M$ or $b \in M$ and we get $x \notin M_a$ or $x \notin M_b$, which is a contradiction. Therefore, $M_a \cap M_b \subseteq M_{(a\wedge b)}$. Now, suppose that $x \in M_{(a\wedge b)}$ but $x \notin M_a \cap M_b$. Without loss of generality, suppose that $x \notin M_a$. So there exists $M \in Max(A)$ such that $x \notin M$ and $a \in M$. As $a \wedge b \leq a$, hence we conclude that $a \wedge b \in M$. Thus $x \notin M_{(a\wedge b)}$, which is a contradiction. Hence $M_{(a\wedge b)} \subseteq M_a \cap M_b$. Therefore, $M_{(a\wedge b)} = M_a \cap M_b$.
(iii) Suppose that $M \in M(b)$, hence $M \in Max(A)$ and $b \in M$. As M is an ideal and $a \leq b$, hence $a \in M$ and we conclude that $M \in M(a)$.
(iv) If $a \leq b$, then by part (iii), we get $M(b) \subseteq M(a)$.

$$M(b) = \{M \in Max(A) \mid b \in M\} \subseteq \{M \in Max(A) \mid a \in M\} = M(a).$$

Hence $\cap \{M \in Max(A) \mid a \in M\} \subseteq \cap \{M \in Max(A) \mid b \in M\}$. Thus $M_a \subseteq M_b$.
(v) Since $a, b \leq (a \oplus b)$, using part (iv), we deduce that $M_a \vee M_b \subseteq M_{(a\oplus b)}$. Also, it follows from Notation 2.8, that $a \oplus b \in M_a \vee M_b$, hence by part (i) of this lemma, $M_{(a\oplus b)} \subseteq M_a \vee M_b$. □

Lemma 3.4. *If I, J and K are three ideals in A, then $J \wedge (I \vee K) = (J \wedge I) \vee (J \wedge K)$.*

Proof. Suppose that $x \in (J \wedge I) \vee (J \wedge K)$, hence $x \leq a \oplus b$, for some $a \in J \wedge I$ and $b \in J \wedge K$. Since J is an ideal and $a, b \in J$, we get $a \oplus b \in J$. So $x \in J$. Also, $a \in I$ and $b \in K$ thus $x \in I \vee K$. Therefore, $(J \wedge I) \vee (J \wedge K) \subseteq J \wedge (I \vee K)$. Now, we show that $J \wedge (I \vee K) \subseteq (J \wedge I) \vee (J \wedge K)$. Assume that $x \in J \wedge (I \vee K)$ hence $x \in J$ and $x \in I \vee K$. Thus there are two elements $a \in I$ and $b \in K$ such that $x \leq a \oplus b$. Hence $x \wedge x \leq x \wedge (a \oplus b)$. By Lemma 2.2, we have $x \leq x \wedge (a \oplus b) \leq (x \wedge a) \oplus (x \wedge b)$. Therefore, $x \in (J \wedge I) \vee (J \wedge K)$. □

Proposition 3.5. *Let I, J and K be three ideals of A.*
(i) If I is a Z_J-ideal of A and $P \in Min(I)$, then P is a Z_J-ideal of A.
(ii) Let I and J be Z_K-ideals of A. Then $I \vee J$ is a Z_K-ideal of A.
(iii) Let $\{I_i\}_{i\in\Omega}$ be a non-empty family of Z_J-ideal. Then $\bigwedge_{i\in\Omega} I_i$ is a Z_J-ideal.
(iv) Let I be a Z_J-ideal and $K \subseteq J$. Then I is a Z_K-ideal.
(v) If $J = A$, then Z_A-ideals and $Z-$ideals are the same.
(vi) Every maximal ideal is a Z_J-ideal.
(vii) Every $Z-$ideal is a Z_J-ideal.
$(viii)$ If the zero ideal is a Z_J-ideal, then $J \cap Rad(A) = \{0\}$.
(ix) Let J be a non-zero ideal and $J \cap Rad(A) = \{0\}$. Then the zero ideal is a $Z_{NJ}-$ideal.

Proof. (i) Suppose that $a \in P$. By Theorem 2.9, there exists $r \in A \setminus P$ such that $a \wedge r \in I$. Since I is a Z_J-ideal, so $M_{(a\wedge r)} \cap J \subseteq I$. It follows from Lemma 3.3, that $M_a \cap M_r \cap J \subseteq I \subseteq P$. It is clear that $M_r \nsubseteq P$ and by Lemma 2.6, we conclude that $M_a \cap J \subseteq P$. Therefore, P is a $Z-$ideal.
(ii) Let $x \in I \vee J$. Hence $x \leq a \oplus b$, for some $a \in I$ and $b \in J$. Since I and J are Z_K-ideal, so we get $M_a \cap K \subseteq I$ and $M_b \cap K \subseteq J$. Always, $M_a \cap K \subseteq I \vee J$ and $M_b \cap K \subseteq I \vee J$, thus by Lemma 3.4, $(M_a \cap K) \vee (M_b \cap K) = (M_a \vee M_b) \cap K \subseteq I \vee J$. By Lemma 3.3, $M_{(a\oplus b)} \cap K = (M_a \vee M_b) \cap K \subseteq I \vee J$. We have $x \leq a \oplus b$, so again by Lemma 3.3, $M_x \subseteq M_{(a\oplus b)}$. Therefore, $M_x \cap K \subseteq I \vee J$ and $I \vee J$ is a Z_K-ideal.
(iii) Suppose that $a \in \bigwedge_{i\in\Omega} I_i$. Hence $a \in I_i$, for all $i \in \Omega$. Since I_i is a Z_J-ideal, so $M_a \cap J \subseteq I_i$, for all $i \in \Omega$. Obviously, $M_a \cap J \subseteq \bigwedge_{i\in\Omega} I_i$, for $a \in \bigwedge_{i\in\Omega} I_i$. Therefore, $\bigwedge_{i\in\Omega} I_i$ is a Z_J-ideal.

(iv) Assume that I is a Z_J-ideal. This means that for all $a \in I$, $M_a \cap J \subseteq I$. By the given hypothesis, we have $K \subseteq J$. This implies that $M_a \cap K \subseteq M_a \cap J$. Therefore, I is a Z_k-ideal.
(v) Let I be a Z_A-ideal. Then $M_a \cap A \subseteq I$, for all $a \in I$. Always, $M_a \cap A = M_a$, hence I is a $Z-$ideal.
(vi) Suppose that M is a maximal ideal of A. For all $a \in M$, we have $M_a \subseteq M$. On the other hand, for all $J \in Id(A)$, we always have $M_a \cap J \subseteq M_a \subseteq M$, as the intersection of M_a with any ideal J is always contained within M. Therefore, we can conclude that M is a Z_J-ideal.
(vii) Let I be a $Z-$ideal. Then $M_a \subseteq I$, for all $a \in I$. It is clear that $M_a \cap J \subseteq M_a$, for all $J \in Id(A)$. Hence, I is a Z_J-ideal.
$(viii)$ Suppose that the zero ideal is a Z_J-ideal. This means that $M_0 \cap J \subseteq \{0\}$. Since $M_0 = Rad(A)$, so $Rad(A) \cap J = \{0\}$.
(ix) By hypothesis, $J \cap Rad(A) = \{0\}$. Hence, the zero ideal is a Z_J-ideal. Since $J \neq \{0\}$, so the zero ideal is a $Z_{NJ}-$ideal. □

Corollary 3.6. *Let J be an ideal of A and $\Sigma = \{I \in Id(A) \mid I \text{ is a } Z_J - ideal\}$. Then*
(i) $(\Sigma, \subseteq)$ is a poset.
(ii) $(\Sigma, \vee, \wedge)$ is a lattice.

Proof. By Proposition 3.5, parts (ii) and (iii), it is clear. □

In Example 3.2 (i), for the ideal $J = I_1$, the set of all Z_J-ideals is $\Sigma = \{I_0, I_1, I_2, I_3\}$ that forms a lattice.

Proposition 3.7. *Let P be a prime ideal of A and J be a proper ideal of A. Then the following conditions are equivalent:*
(i) P is a Z_J-ideal.
(ii) P is either a $Z-$ideal or $J \subseteq P$.

Proof. $(i) \Rightarrow (ii)$ Suppose that $J \nsubseteq P$, we prove that P is a $Z-$ideal. Since P is a Z_J-ideal, so $M_a \cap J \subseteq P$, for all $a \in P$. By Lemma 2.6, we deduce that $M_a \subseteq P$, for all $a \in P$. Therefore P is a Z_J-ideal.
$(ii) \Rightarrow (i)$ It is clear. □

Corollary 3.8. *Let I and J be two ideals of A.*
(i) If I is a $Z_{NJ}-$ideal of A, then there exists $P \in Min(I)$ such that P is a $Z-$ideal of A.
(ii) Every prime $Z_{NJ}-$ideal is a $Z-$ideal.
(iii) Let A be an $MV-$chain. Then every proper $Z_{NJ}-$ideal is a $Z-$ideal

Proof. (i) I is a $Z_{NJ}-$ideal, hence $J \nsubseteq I$ and I is a Z_J-ideal. Since $J \nsubseteq I$, so there exists $x \in J \setminus I$. Obviously, if we take $K := (x]$ thus I is a Z_K-ideal. Also, $x \notin I$ hence by Theorem 2.14, we deduce that there exists $Q \in Spec(A)$ containing I such that $x \notin Q$. It follows from Theorem 2.20, that there exists $P \in Min(I)$ such that $P \subseteq Q$. Clearly, $x \notin P$, so $K \nsubseteq P$. By Proposition 3.5 (i) and Proposition 3.7, we conclude that P is a $Z-$ideal.
(ii) By Proposition 3.7, it is clear.
(iii) Suppose that I is a proper Z_J-ideal such that $J \nsubseteq I$. Hence, $M_a \cap J \subseteq I$, for all $a \in I$. Since A is a $MV-$chain, by Theorem 2.19, I is a prime ideal. Also, by part (ii) of this corollary, we conclude that $M_a \subseteq I$, for all $a \in I$. Therefore I is a $Z-$ideal. □

Note: Let I be an ideal of A. We define:

$$I_z = \{b \in A \mid \ b \in M_a, \ for \ some \ a \in I\}.$$

Lemma 3.9. *Let I and J be ideals of A. Then the following sentences hold:*
(i) I_z is a $Z-$ideal of A.
(ii) $I \subseteq I_z$.
(iii) $I_z = \cap \{ Q \mid I \subseteq Q \text{ and } Q \text{ is a } Z-ideal \}$.
(iv) $(I \cap J)_z = I_z \cap J_z$.
(v) If $I \subseteq J$, then $I_z \subseteq J_z$.
(vi) If I is a $Z-$ideal, then $I = I_z$.

Proof. (i) First, we prove that I_z is an ideal. It is clear that $0 \in I_z$. Suppose that $b, c \in I_z$, hence $b \in M_a$, for some $a \in I$ and $c \in M_t$, for some $t \in I$. By Lemma 3.3, we have $M_b \subseteq M_a$ and $M_c \subseteq M_t$. Thus $M_b \vee M_c = M_{(b \oplus c)} \subseteq M_{(a \oplus t)}$. Since I is an ideal, so $a \oplus t \in I$ and we deduce that $b \oplus c \in I_z$. Also, if $b \leq c$ and $c \in I_z$, then $c \in M_a$, for some $a \in I$. So, $b \in M_a$ for some $a \in I$, that is, $b \in I_z$. Now, we show that I_z is a $Z-$ideal. This is proven by Theorem 2.18. Let $M(a) \subseteq M(b)$ and $a \in I_z$. Then $M_b \subseteq M_a$ and there exists $t \in I$ such that $a \in M_t$. Hence $M_a \subseteq M_t$ and we get $b \in M_t$.

Therefore $b \in I_z$.
(ii) Suppose that $a \in I$. Since $a \in M_a$, so we obtain $a \in I_z$.
(iii) Let $K = \cap \{ Q \mid I \subseteq Q$ and Q is a $Z-$ideal $\}$. Then by parts (i) and (ii) of this lemma, it is clear that $K \subseteq I_z$. By contrary, if $I_z \nsubseteq K$. Suppose that $b \in I_z$ but $b \notin K$. Hence there exists a $Z-$ideal Q such that $I \subseteq Q$ and $b \notin Q$. According to Theorem 2.18 and using the equivalent definition, so there is not $a \in Q$ such that $M_b \subseteq M_a$. We deduce that $b \notin M_a$, thus $b \notin I_z$, which is a contradiction.
(iv) Assume that $b \in (I \cap J)_z$, hence there exists $a \in I \cap J$ such that $b \in M_a$. Since $a \in I$ and $b \in M_a$, so $b \in I_z$. Similarly, $b \in J_z$. Therefore $(I \cap J)_z \subseteq I_z \cap J_z$. Now, suppose that $b \in I_z \cap J_z$. Then there exist $a \in I$ and $c \in J$ such that $b \in M_a$ and $b \in M_c$. Obviously, $a \wedge c \in I \cap J$ and $b \in M_a \cap M_c = M_{(a \wedge c)}$. We conclude that $b \in (I \cap J)_z$ and $I_z \cap J_z \subseteq (I \cap J)_z$.
(v) It is clear.
(vi) By part (iii) of this lemma, the proof is evident. □

Example 3.10. *Let $I_1 = \{0, a\}$ be the ideal from Example 3.2 (i), it is easy to clarify $(I_1)_z = \{0, a\}$. (Because I_1 is a $Z-$ideal.) Also, for the ideal $I = (\varepsilon^4]$ in Example 3.2 (ii), $I_z = (\varepsilon]$. On the other hand, $I \subseteq I_z$.*

In the following, we give equivalent definitions for Z_J-ideals and $Z_{NJ}-$ideals.

Proposition 3.11. *Suppose that I and J are ideals of A. The following are equivalent:*
(i) I is a Z_J-ideal.
(ii) $I_z \cap J \subseteq I$.
(iii) There exists a $Z-$ideal K containing I such that $K \cap J \subseteq I$.
(iv) For each $a \in I$ and $b \in J$ if $M_b \subseteq M_a$, then $b \in I$.

Proof. $(i) \Rightarrow (ii)$ Let $b \in I_z \cap J$. Then there exists $a \in I$ such that $b \in M_a$, hence $M_b \subseteq M_a$. Since I is a Z_J-ideal, so $M_a \cap J \subseteq I$. We have $b \in M_b \cap J \subseteq M_a \cap J \subseteq I$, thus $b \in I$.
$(ii) \Rightarrow (iii)$ Consider $K := I_z$.
$(iii) \Rightarrow (iv)$ By hypothesis $a \in I$ and $I \subseteq K$, thus $a \in K$. Since K is a $Z-$ideal, so $M_a \subseteq K$. We have $b \in J$ and $b \in M_b$ hence $b \in M_b \cap J$. Therefore $b \in M_b \cap J \subseteq M_a \cap J \subseteq K \cap J \subseteq I$. As a result $b \in I$.
$(iv) \Rightarrow (i)$ We show that $M_a \cap J \subseteq I$, for all $a \in I$. Assume that $b \in M_a \cap J$, then $M_b \subseteq M_a$ and $b \in J$. By (iv), we conclude that $b \in I$. Hence I is a Z_J-ideal. □

Proposition 3.12. *Let $I, J \in Id(A)$. Then I is a Z_J-ideal ($Z_{NJ}-$ideal) if and only if $I \cap J$ is a Z_J-ideal ($Z_{NJ}-$ideal).*

Proof. Let $I \cap J$ be a Z_J-ideal. By Proposition 3.11, we have $(I \cap J)_z \cap J \subseteq I \cap J$. Also, by Lemma 3.9, we get $(I \cap J)_z \cap J = I_z \cap J_z \cap J = I_z \cap J$ (since $J \subseteq J_z$). On the other hand, $I_z \cap J = (I \cap J)_z \cap J \subseteq I \cap J \subseteq I$. Therefore, I is a Z_J-ideal. Additionally, by the hypothesis, $I \cap J \subsetneq J$. This implies that $J \nsubseteq I$ and we can conclude that I is a $Z_{NJ}-$ideal. The other side is clear. □

Proposition 3.13. *An ideal I of A is a $Z_{NJ}-$ideal if and only if there exists $b \in A \setminus I$ such that $(b] \cap M_a \subseteq I$, for all $a \in I$.*

Proof. Let I be a $Z_{NJ}-$ideal. Then I is a Z_J-ideal, for some ideal J of A such that $J \nsubseteq I$. It is enough to take $b \in J \setminus I$.
Conversely, by hypothesis there exists $b \in A \setminus I$, we take $J = (b]$, it is clear that I is a Z_J-ideal and $J \nsubseteq I$. Therefore I is a $Z_{NJ}-$ideal. □

Proposition 3.14. *Let I, J, K and L be ideals in A. Then the following sentences hold:*
(i) I is a Z_J-ideal if and only if I is a $Z_{I \vee J}-$ideal.
(ii) Let J be a $Z-$ideal and $I \subseteq J$. Then I is a Z_J-ideal if and only if I is a $Z-$ideal.
(iii) If I is a $Z-$ideal, then J is a Z_I-ideal if and only if $I \cap J$ is a $Z-$ideal.
(iv) $I \cap J$ is a Z_J-ideal and is a Z_I-ideal if and only if I is a Z_J-ideal and J is a Z_I-ideal.
(v) Let M be a maximal ideal of A. Then $I \cap M$ is a $Z-$ideal if and only if I is a $Z-$ideal.
(vi) If $I \subseteq J$ and I is a Z_J-ideal and J is a Z_K-ideal, then I is a Z_K-ideal.
(vii) If $I \subseteq J$ and $L \subseteq K$ and I is a Z_J-ideal and L is a Z_K-ideal, then $I \cap L$ is a $Z_{J \cap K}-$ideal.
$(viii)$ $I_z \cap J$ is the smallest Z_J-ideal containing $I \cap J$.
(ix) $J \cap Rad(A)$ is the smallest Z_J-ideal.
(x) If I is a Z_J-ideal and a Z_K-ideal, then I is a $Z_{J \vee K}-$ideal.

Proof. (i) By Lemma 3.9, we have $I \subseteq I_z$ and by Lemma 3.4, we have $I_z \cap (I \vee J) = (I_z \cap I) \vee (I_z \cap J)$. Since I is a Z_J-ideal, so $I_z \cap J \subseteq I$. Thus, we obtain $I_z \cap (I \vee J) = (I_z \cap I) \vee (I_z \cap J) \subseteq I \vee I = I$. It follows from Proposition 3.11, that I is a $Z_{I \vee J}-$ideal. The converse is obvious.
(ii) Assume that I is a Z_J-ideal, hence for all $a \in I$, $M_a \cap J \subseteq I$. Since J is a $Z-$ideal and $a \in I \subseteq J$, so we have $M_a \subseteq J$. We deduce that $M_a = M_a \cap J \subseteq I$. So I is a $Z-$ideal. As every $Z-$ideal is a Z_J-ideal, the converse is clear.
(iii) First, we show that $I \cap J$ is a $Z-$ideal. Since J is a Z_I-ideal, so by Proposition 3.12, we deduce that $I \cap J$ is a Z_I-ideal. By hypothesis, I is a $Z-$ideal and $I \cap J \subseteq I$, hence by part (ii) of this proposition, we conclude that $I \cap J$ is a $Z-$ideal.
Conversely, it follows from Proposition 3.12 and part (ii) of Proposition 3.14.
(iv) By Proposition 3.12, it is clear.
(v) M is a maximal ideal, hence M is a $Z-$ideal. Since I is a $Z-$ideal, so $I \cap M$ is a $Z-$ideal. Now suppose that $I \cap M$ is a $Z-$ideal. If $I \subseteq M$, then $I = I \cap M$ is a $Z-$ideal. Assume that $I \nsubseteq M$, since $I \cap M$ is a $Z-$ideal and M is a $Z-$ideal, so by part (iii), we conclude that I is a Z_M-ideal. Moreover, by part (i), we obtain that I is a $Z_{I \vee M}-$ideal. Hence I is a Z_A-ideal. Therefore I is a $Z-$ideal.
(vi) By hypothesis and Lemma 3.9, we have $I_z \cap J \subseteq I$ and $J_z \cap K \subseteq J$. Also, $I_z \subseteq J_z$ hence we get $I_z \cap K = I_z \cap J_z \cap K \subseteq I_z \cap J \subseteq I$. It means that I is a Z_K-ideal.
(vii) It follows from Proposition 3.11, that $I_z \cap J \subseteq I$ and $L_z \cap K \subseteq L$. So, we have

$$(I \cap L)_z \cap (J \cap K) = I_z \cap L_z \cap J \cap K = (I_z \cap J) \cap (L_z \cap K) \subseteq I \cap L.$$

$(viii)$ We know that, $I \subseteq I_z$, so $I \cap J \subseteq I_z \cap J$. It is clear that $(I_z)_z = I_z$, hence $(I_z \cap J)_z \cap J \subseteq I_z \cap J$ and we conclude that $I_z \cap J$ is a Z_J-ideal. Now, suppose that there exists an ideal K of A such that it is a Z_J-ideal contains $I \cap J$. Hence $I_z \cap J = I_z \cap J_z \cap J = (I \cap J)_z \cap J \subseteq K_z \cap J \subseteq K$.
(ix) Suppose that I is a Z_J-ideal, thus $J \cap Rad(A) \subseteq J \cap I_z \subseteq I$.
(x) We have $I_z \cap (J \vee K) = (I_z \cap J) \vee (I_z \cap K) \subseteq I$. Hence I is a $Z_{J \vee K}-$ideal. □

Corollary 3.15. *Let J be a $Z-$factor of an ideal I. Then $I \vee J$ is a $Z-$factor of I containing I.*

Proof. Since I is a Z_J-ideal, so by Proposition 3.14 (i), I is a $Z_{I \vee J}-$ideal. Thus, $I \vee J$ is a $Z-$factor of I. Obviously, $I \subseteq I \vee J$. □

Corollary 3.16. *Let I be an ideal of A. Consider:*

$$\Omega = \{J \in Id(A) \mid J \text{ is a } Z-\text{factor of } I\}.$$

(i) $(\Omega, \subseteq)$ *is a poset.*
(ii) $(\Omega, \vee, \wedge)$ *is a lattice.*

In Example 3.2 (ii), for the ideal $I = (\varepsilon^2]$, the set of all $Z-$factors of I is $\Omega = \{(\varepsilon^2], (\varepsilon^3], ..., \{0\}\}$ that forms a lattice.

Proposition 3.17. *Let I, J be ideals and P, Q be prime ideals of A. Then*
(i) *If $I \cap P$ is a Z_J-ideal ($Z_{NJ}-$ideal), then either I is a Z_J-ideal ($Z_{NJ}-$ideal) or P.*
(ii) *If $P \cap Q$ is a $Z_{NJ}-$ideal, then either P is a $Z-$ideal or Q.*

Proof. (i) Obviously, if $I \subseteq P$, then I is a Z_J-ideal ($Z_{NJ}-$ideal.) Assume that $I \nsubseteq P$, hence there exists $a \in I \setminus P$ and for all $b \in P$, we have $a \wedge b \in I \cap P$. Since $I \cap P$ is a Z_J-ideal, so $M_{(a \wedge b)} \cap J \subseteq I \cap P$. On the other hand, $M_a \cap M_b \cap J \subseteq P$ and $M_a \nsubseteq P$. By Lemma 2.6, we conclude that $M_b \cap J \subseteq P$, for all $b \in P$. Thus, P is a Z_J-ideal. Since $I \cap P$ is a $Z_{NJ}-$ideal, so there is an ideal J such that $J \nsubseteq I \cap P$ and $(I \cap P)_z \cap J \subseteq I \cap P$. Hence $(I_z \cap J) \cap P_z \subseteq P$, by Lemma 2.6, we have $I_z \cap J \subseteq P$ or $P_z \subseteq P$. We consider two cases:
Case 1: If $I_z \cap J \subseteq P$, then $J \subseteq P$. (Since $I \nsubseteq P$, so $I_z \nsubseteq P$.) On the other hand, $J \subseteq P \subseteq P_z$, hence $J \cap P_z = J$. Since $I \cap P$ is a Z_J-ideal, so $I_z \cap J = I_z \cap J \cap P_z = (I \cap P)_z \cap J \subseteq I \cap P \subseteq I$ implies that I is a Z_J-ideal. Now, we show that $J \nsubseteq I$. We have $J \subseteq P$, if $J \subseteq I$, then $J \subseteq I \cap P$, which is a contradiction. Therefore I is a $Z_{NJ}-$ideal.
Case 2: If $P_z \subseteq P$, then $P_z = P$ and P is a $Z-$ideal. It follows from Proposition 3.7, that P is a $Z_{NJ}-$ideal.
(ii) It follows from Proposition 3.7 and part (i). □

However, the converse of Proposition 3.17, is not true in general as we can see in the following example.

Example 3.18. (i) *Consider $C = \{0, c, 2c, 3c, ..., 1-2c, 1-c, 1\}$ be the $MV-$algebra by the following operations:*
if $x = nc$ and $y = mc$, then $x \oplus y := (m+n)c$,
if $x = 1 - nc$ and $y = 1 - mc$, then $x \oplus y := 1$,

if $x = nc$ *and* $y = 1 - mc$ *and* $m \leq n$, *then* $x \oplus y := 1$,
if $x = nc$ *and* $y = 1 - mc$ *and* $n < m$, *then* $x \oplus y := 1 - (m - n)c$,
if $x = 1 - mc$ *and* $y = nc$ *and* $m \leq n$, *then* $x \oplus y := 1$,
if $x = 1 - mc$ *and* $y = nc$ *and* $n < m$, *then* $x \oplus y := 1 - (m - n)c$,
if $x = nc$, *then* $x^* := 1 - nc$,
if $x = 1 - nc$, *then* $x^* := nc$.
The structure $(C, \oplus, *, 0)$ *is called the Chang' MV−algebra [11]. It has three ideals:* $\{0\}$, $\{0, c, 2c, 3c, ...\}$ *and* C.
Consider $I = \{0\}$ *and* $P = \{0, c, 2c, 3c, ...\}$. *Then* P *is a prime* Z_P-*ideal but* $I \cap P = I$ *is not a* Z_P-*ideal.*
(ii) *In Example 3.2* (ii), *put* $I = (\varepsilon^4]$, $J = (\varepsilon^2]$ *and* $K = (\varepsilon]$. *It is clear that,* I *and* K *are prime ideals of* A. *Also,* K *is a* $Z-$*ideal, but* $I \cap K = I$ *is not a* Z_J-*ideal.*

Note: A Z_J-ideal I of A whenever $J \nsubseteq I$ is called a maximal Z_J-ideal if there is no Z_J-ideal which contains I properly.
In Example 3.2 (i), we can see that I_1, for any $J \in Id(A)$ such that $J \nsubseteq I_1$, the ideal I_1 is a maximal Z_J-ideal.

Theorem 3.19. *Let* I *and* J *be ideals of* A. *If* I *is a maximal* Z_J-*ideal, then* I *is a maximal element of the set of all prime* $Z-$*ideals.*

Proof. Since $J \cap I_z \subseteq I_z$, so I_z is a Z_J-ideal. Also, it is clear that $J \nsubseteq I_z$. (If $J \subseteq I_z$, then $J \cap I_z = J \subseteq I$, which is a contradiction.) By maximality of I, we conclude that $I = I_z$. Hence I is a $Z-$ideal of A. Since $J \nsubseteq I$ and $I = \bigcap\{P \in Spec(A) | I \subseteq P\}$, so there exists $P \in Spec(A)$ such that $J \nsubseteq P$. Also, $J \cap I_z \subseteq I \subseteq P$, thus P is a Z_J-ideal and we get $I = P$. □

Note: The ideal J is called a maximal $Z-$factor of the ideal I, if there is no $Z-$factor of I, which contains J properly.
In Example 3.2 (ii), where $I = (\varepsilon^3]$. The set of all $Z-$factors of I is $\{(\varepsilon^3], (\varepsilon^4], ..., \{0\}\}$. Therefore, the maximal $Z-$factor is $(\varepsilon^3]$.

Theorem 3.20. *If maximal* $Z-$*factor of* I *exists, then it is of the form*

$$\{a \in A \mid M_r \cap (a] \subseteq I, \text{ for all } r \in I\}.$$

Proof. Assume that J is a maximal $Z-$factor of I. Consider:

$$K = \{a \in A \mid M_r \cap (a] \subseteq I, \text{ for all } r \in I\}.$$

K is an ideal. Obviously, $0 \in K$. Let $a, b \in K$. Then $M_r \cap (a] \subseteq I$ and $M_r \cap (b] \subseteq I$, for all $r \in I$. Thus $(M_r \cap (a]) \vee (M_r \cap (b]) = M_r \cap (a \oplus b] \subseteq I$. Therefore $a \oplus b \in K$. If $a \leq b$ and $b \in K$, then $(a] \subseteq (b]$ and $M_r \cap (b] \subseteq I$, for all $r \in I$. Hence $M_r \cap (a] \subseteq I$ and $a \in K$. We want to prove I is a Z_K-ideal. Let $x \in M_t \cap K$ for all $t \in I$. Then by Lemma 3.3, $M_x \subseteq M_t$. Since $x \in K$, so we deduce that $M_t \cap (x] \subseteq I$. Obviously, $x \in M_x \cap (x]$ and $M_x \cap (x] \subseteq M_t \cap (x]$. Therefore $x \in I$. Now, we show that $J \subseteq K$. Let $x \in J$. Then $M_r \cap (x] \subseteq M_r \cap J$. Since I is a Z_J-ideal and $r \in I$, so $M_r \cap (x] \subseteq I$, for all $r \in I$. Hence $x \in K$ and therefore $J \subseteq K$. By maximality of J, we conclude that $J = K$. □

Proposition 3.21. *Let* I, J *be ideals of* A *and* I *be a* Z_J-*ideal. Then* $(I : x)$ *is a* Z_J-*ideal of* A, *for all* $x \in A \setminus I$.

Proof. Following Proposition 3.11, assume that $M_b \subseteq M_a$ and $a \in (I : x)$ and $b \in J$. Since I is a Z_J-ideal and $a \wedge x \in I$, so we get $M_{(a \wedge x)} \cap J \subseteq I$. Also, $b \wedge x \leq b$ and J is an ideal, hence $b \wedge x \in J$. It is clear that $M_{(b \wedge x)} \cap J = M_b \cap M_x \cap J \subseteq M_a \cap M_x \cap J = M_{(a \wedge x)} \cap J \subseteq I$ and $b \wedge x \in M_{(b \wedge x)} \cap J$. Thus, we conclude that $b \wedge x \in I$ and $b \in (I : x)$. □

Corollary 3.22. *Let* I, J *be ideals of* A *such that* I *be a* Z_J-*ideal and* X *be a subset of* A *such that* $X \nsubseteq I$. *Then* $(I : X) = \{a \in A \mid a \wedge x \in I, \text{ for all } x \in X\}$ *is a* Z_J-*ideal.*

In the following theorem, we show that if J is not a $Z-$ideal and $Rad(A)$ does not contain J, then there is always a non-trivial Z_J-ideal.

Theorem 3.23. *Let* J *be an ideal of* A *such that* $J \nsubseteq Rad(A)$ *and it is not a* $Z-$*ideal. Then there exists an ideal* I *such that* $I \subsetneqq J$ *and is a* Z_J-*ideal but not a* $Z-$*ideal.*

Proof. Since $J \nsubseteq Rad(A)$, so there exists a maximal ideal M in A such that $J \nsubseteq M$. Consider $I := J \cap M$, it is clear that $I \subsetneqq J$ (if $I = J$, then $J \cap M = J$ and $J \subseteq M$, which is a contradiction). M is a maximal ideal, so it is a $Z-$ideal. Also, it is a Z_J-ideal and by Proposition 3.12, $J \cap M$ is a Z_J-ideal. Therefore, I is a Z_J-ideal. We know that M is maximal and J is not a $Z-$ideal, so by Proposition 3.14, part (v) we conclude that I is not a $Z-$ideal. □

To summarize, if $J \nsubseteq Rad(A)$, then there are abundant Z_J-ideals in A. Consider a maximal ideal K such that $J \nsubseteq K$, we know that K is a $Z-$ideal. Therefore, for all $k \in K$, $M_k \cap J$ is a Z_J-ideal and in fact it is a $Z_{NJ}-$ideal. However, if $J \subseteq Rad(A)$, then there are only trivial Z_J-ideals. In fact, if K is a non-trivial Z_J-ideal, then $J \nsubseteq K$ and $M_a \cap J \subseteq K$, for all $a \in K$. On the other hand, $J = Rad(A) \cap J \subseteq M_a \cap J \subseteq K$ and it contradicts with $J \nsubseteq K$. As a result $J \nsubseteq Rad(A)$ if and only if there exists a non-trivial Z_J-ideal.
Now, we are going to study Z_J-ideals in Hyperarchimedean and semi-simple $MV-$algebras and get some results.
Note: In Example 3.2 (ii), the zero ideal is not a Z_J-ideal. Next, we characterize $MV-$algebras in which the zero ideal is a Z_J-ideal.

Proposition 3.24. *Let A be Hyperarchimedean and $J \in Id(A)$. Then every proper ideal is a Z_J-ideal.*

Proof. Suppose that I is a proper ideal. For all $a \in I$;

$$\{\ P \in Spec(A)\ | I \subseteq P\} \subseteq \{\ P \in Spec(A) \mid a \in P\},$$

$$\bigcap \{\ P \in Spec(A) \mid a \in P\} \subseteq \bigcap \{\ P \in Spec(A) \mid I \subseteq P\}.$$

By Theorem 2.21 and by Corollary 2.5,

$$M_a = \bigcap \{\ P \in Spec(A) \mid a \in P\} \subseteq \bigcap \{\ P \in Spec(A) \mid I \subseteq P\} = I.$$

We always have $M_a \cap J \subseteq M_a$, for all $a \in I$. On the other hand, $M_a \cap J \subseteq M_a \subseteq I$, for all $a \in I$. Therefore, I is a Z_J-ideal. □

Proposition 3.25. *Let A be a semi-simple $MV-$algebra, X be a subset of A and J be an ideal of A. Then the following hold:*
(i) The zero ideal is a Z_J-ideal. In particular, if J is a non-zero ideal, then the zero ideal is a $Z_{NJ}-$ideal.
(ii) $Ann(X)$ is a Z_J-ideal.
(iii) Every minimal prime ideal of A is a Z_J-ideal.

Proof. (i) Since A is semi-simple, so $M_0 = Rad(A) = \{0\}$. For each ideal J of A, we have $M_0 \cap J \subseteq \{0\}$, hence the zero ideal is a Z_J-ideal.
(ii) By part (i) of this proposition, the zero ideal is a Z_J-ideal. Put $I := \{0\}$. Now by Corollary 3.22, $(\{0\} : X)$ is a Z_J-ideal. Obviously, $(\{0\} : X) = Ann(X)$. Thus $Ann(X)$ is a Z_J-ideal.
(iii) In Proposition 3.5 (i), take $I = \{0\}$ and by part (i) of this proposition, we conclude that every minimal prime ideal is a Z_J-ideal. □

Theorem 3.26. *Let A be a semi-simple $MV-$algebra and I be an ideal of A such that $Ann(I) \neq 0$. Then there is an ideal J of A such that I is a $Z_{NJ}-$ideal of A.*

Proof. Since A is semi-simple, so $Rad(A) = 0$. We first prove that $M_a \cap Ann(I) \subseteq Rad(A)$, for all $a \in I$. Let $b \in M_a \cap Ann(I)$. Then $b \in M_a$ and $b \wedge a = 0$. Hence $M_b \subseteq M_a$. We get $b \in M_b = M_b \cap M_a = M_{(a \wedge b)} = M_0 = Rad(A)$. Take $J = Ann(I)$. Moreover, $M_a \cap Ann(I) = Rad(A) = \{0\} \subseteq I$, for all $a \in I$. Thus I is a Z_J-ideal. Now, we show that $J \nsubseteq I$. By contrary, suppose that $Ann(I) \subseteq I$ and let $0 \neq a \in Ann(I)$. Hence $a \in I$ and $a \wedge t = 0$ for all $t \in I$. Hence $a \wedge a = 0$, as a result $a = 0$, which is a contradiction. Therefore I is a $Z_{NJ}-$ideal. □

Corollary 3.27. (i) *Let $I, J \in Id(A)$. Then every ideal I of a semi-simple $MV-$algebra A is a $Z_{NJ}-$ideal or an essential ideal.*
(ii) Every ideal of a finite $MV-$algebra A is a $Z_{NJ}-$ideal or an essential ideal.

Proof. (i) First, assume that I is not an essential ideal, that is, $Ann(I) \neq 0$. By Proposition 3.26, we conclude that there exists an ideal J such that I is a $Z_{NJ}-$ideal. Now, suppose that I is not a $Z_{NJ}-$ideal in A. Again by Proposition 3.26, we have A is not semi-simple or $Ann(I) = 0$. By hypothesis A is semi-simple, hence $Ann(I) = 0$ and I is an essential ideal.
(ii) It follows from Notation 2.7. □

Proposition 3.28. *The generated ideal by any idempotent element in semi-simple $MV-$algebra is a $Z-$ideal (Z_J-ideal).*

Proof. Let a be an element belonging to $B(A)$ and $I = (a]$ be a principal ideal of A. Suppose that $x \in I$, hence $x \leq a$ and by Theorem 2.22 and Lemma 3.3, we get $M_x \subseteq M_a \subseteq P_a$. Since $a \in B(A)$, so by Lemma 2.16 and Lemma 2.15, $P_a = Ann(Ann(a)) = Ann((a^*]) = (a]$. We have $M_x \subseteq P_a = (a]$. Therefore, I is a $Z-$ideal. Also, for every ideal J, always $M_x \cap J \subseteq M_x \subseteq P_a = (a]$. Hence I is a Z_J-ideal. □

Next, we present a summary of the relationship between these new ideals in a diagram. In the figure below, the meaning of "S" and "Zero" is that;
S:= Semi-simple, Zero:= The zero ideal.

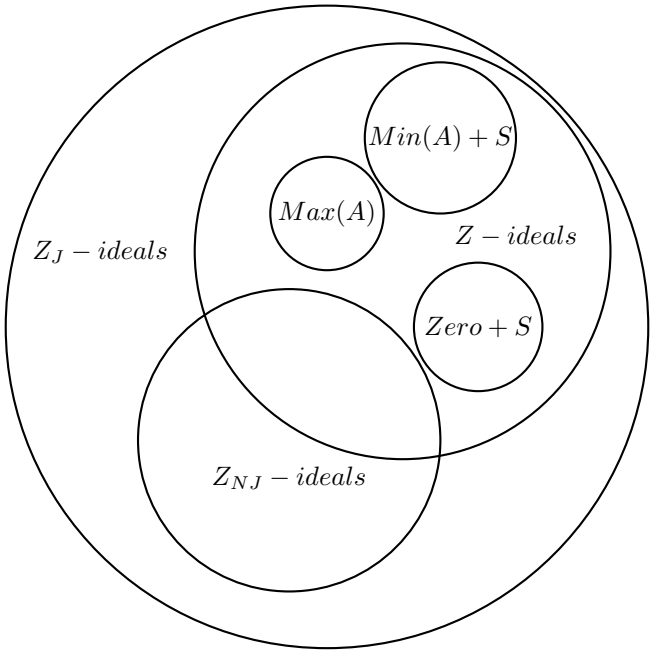


Figure 2: The relationships between Z_J-ideals and the other ideals in $MV-$algebras.

4 Conclusions

The defined ideals exhibit a higher level of generality compared to $Z-$ideals. It has been established that every $Z-$ideal is a Z_J-ideal and every $Z_{NJ}-$ideal is a Z_J-ideal. Furthermore, it has been demonstrated that every prime $Z_{NJ}-$ideal is a $Z-$ideal. We have successfully proven that the collection of all Z_J-ideals forms a lattice. We have conducted a comprehensive investigation of these ideals within various $MV-$algebras. It has been established that if the intersection of two prime ideals is a $Z_{NJ}-$ideal, then at least one of them is a $Z-$ideal. Additionally, we have shown that every prime ideal is a Z_J-ideal if and only if it is either a $Z-$ideal or contains the ideal J. Moreover, it has been concluded that there exists a minimal prime ideal over every $Z_{NJ}-$ideal.
In this paper, we have examined the features of these ideals in order to provide a foundation for future research. Some of the potential future directions we are considering include studying quotients and other types of extensions. For example, we invite interested researchers to explore the generalization of $Z^\circ-$ideals in $MV-$algebras, building upon the study of $Z^\circ-$ideals presented in this work on $MV-$algebras.

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References

[1] M. Bedrood, F. Sajadian, A. Borumand Saeid, *$Z^\circ-$ideals in $MV-$algebras of continuous function*, Filomat, **36**(7) (2022), 2439-2450. https://doi.org/10.2298/FIL2207439B

[2] M. Bedrood, F. Sajadian, G. Lenzi, A. Borumand Saeid, *$Z^\circ-$ideals and $Z-$ideals in $MV-$algebras*, Bulletin of the Belgian Mathematical Society-Simon Stevin, **30** (2023), 51-65. https://doi.org/10.36045/j.bbms.211109

[3] C. C. Chang, *Algebraic analysis of many valued logic*, Transactions of the AMS - American Mathematical Society, **88** (1958), 467-490. https://www.ams.org/journals/tran/1958-088-02/S0002-9947-1958-0094302-9/S0002-9947-1958-0094302-9.pdf

[4] R. Cignoli, I. M. L. D'Ottaviano, D. Mundici, *Algebraic foundations of many-valued reasoning*, Kluwer Academic, Dordrecht, 2000. https://doi.org/10.1007/978-94-015-9480-6

[5] A. Filipoiu, G. Georgescu, A. Lettieri, *Maximal $MV-$algebras*, Mathware, Soft Computing, **4** (1997), 53-62. http://eudml.org/doc/39098

[6] F. Forouzesh, F. Sajadian, M. Bedrood, *Inverse topology in $MV-$algebras*, Mathematica Bohemica, **144**(3) (2019), 273-285. https://doi.org/10.21136/MB.2018.0117-17

[7] F. Forouzesh, F. Sajadian, M. Bedrood, *Some results in types of extensions of $MV-$algebras*, Publications de l'lnstitut Mathematique, **105**(119) (2019), 161-177. https://doi.org/10.2298/PIM1919161F

[8] C. S. Hoo, *Maximal and essential ideals of $MV-$algebras*, Mathware Soft Computing, **2** (1995), 181-196. https://upcommons.upc.edu/bitstream/handle/2099/2471/hoo.pdf

[9] A. Iorgulescu, *Algebras of logic as BCK-algebras*, Bucharest University of Economics Bucharest, Romania, 2008.

[10] A. H. Movahed, M. Bedrood, A. Borumand Saeid, *On $2-$absorbing ideals in $MV-$algebras*, Submitted.

[11] D. Piciu, *Algebras of fuzzy logic*, Ed. Universitaria Craiova, 2007.