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A note on "A novel defuzzification approach of type-2 fuzzy variable to solving matrix games: An application to plastic ban problem"

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Abstract

Seikh et al. (Iranian Journal of Fuzzy Systems, 18(5) (2021), 155-172) pointed out that there does not exist any approach to solve type-2 fuzzy variable matrix games (matrix games in which each element of the payoff matrix is represented by a type-2 fuzzy variable). To fill this gap, Seikh et al. proposed an approach to solve triangular type-2 fuzzy variable (TT2FV) matrix games (matrix games in which each element of the payoff matrix is represented by a TT2FV). Seikh et al. also solved a real-life TT2FV matrix game (plastic ban problem) to show the applicability of their proposed approach. In future, other researchers may use Seikh et al.'s approach to solve other real-life TT2FV matrix games. However, it is inappropriate to use Seikh et al.'s approach as it may fail to find a correct optimal solution of TT2FV matrix games. To validate this claim, a TT2FV matrix game is solved by Seikh et al.'s approach and it is shown that the obtained optimal solution is not correct. Also, it is pointed out that this inappropriateness occurs as a mathematically incorrect result is considered in Seikh et al.'s approach. Furthermore, Seikh et al.'s approach is modified to resolve its inappropriateness.

Keywords: Triangular type-2 fuzzy variable, matrix games, plastic ban problem.

1 Introduction

Seikh et al. [21] pointed out that although several approaches are proposed in the literature to solve fuzzy matrix games [4, 11, 12, 17, 22, 27], intuitionistic fuzzy matrix games [2, 3, 13, 24, 25, 26], hesitant matrix games [7, 23], neutrosophic matrix games [18, 19, 20] etc. However, there does not exist any approach to solve type-2 fuzzy variable matrix games. While, type-2 fuzzy variables [14] have been used in other research areas [5, 6, 8, 9, 10, 15, 16]. To fill this gap, Seikh et al. [21], firstly, proposed the following method for comparing two triangular type-2 fuzzy variables $\widetilde{\widetilde{\kappa}}_1 = \left(a_1^s, a_1^m, a_1^u; \tau_1^l, \tau_1^r\right)$ and $\widetilde{\widetilde{\kappa}}_2 = \left(a_2^s, a_2^m, a_2^u; \tau_2^l, \tau_2^r\right)$.

(i) $\widetilde{\widetilde{\kappa}}_1 \succcurlyeq \widetilde{\widetilde{\kappa}}_2$ if $R\left(\widetilde{\widetilde{\kappa}}_1\right) \geq R\left(\widetilde{\widetilde{\kappa}}_2\right)$.

(ii) $\widetilde{\widetilde{\kappa}}_1 \preccurlyeq \widetilde{\widetilde{\kappa}}_2$ if $R\left(\widetilde{\widetilde{\kappa}}_1\right) \leq R\left(\widetilde{\widetilde{\kappa}}_2\right)$.

(iii) $\widetilde{\widetilde{\kappa}}_1 \approx \widetilde{\widetilde{\kappa}}_2$ if $R\left(\widetilde{\widetilde{\kappa}}_1\right) = R\left(\widetilde{\widetilde{\kappa}}_2\right)$.

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where, $R\left(\widetilde{\widetilde{\kappa}}_i\right)$, $i = 1, 2$ is defined by (1) or (2) or (3),

$$R\left(\widetilde{\widetilde{\kappa}}_i\right) = \frac{1}{2}\left[\frac{a_i^s + 2a_i^m + a_i^u}{2} - \frac{a_i^u - a_i^m}{\tau_i^l} - \frac{a_i^m - a_i^s}{\left(\tau_i^l\right)^2}\left\{\left(1+\tau_i^l\right)\ln\left(\frac{1+\tau_i^l}{1+0.5\tau_i^l}\right) + \ln\left(1-0.5\tau_i^l\right)\right\}\right.$$
$$\left. + \frac{a_i^u - a_i^m}{\left(\tau_i^l\right)^2}\left\{\left(1+\tau_i^l\right)\ln\left(1+0.5\tau_i^l\right) + \ln\left(\frac{1-\tau_i^l}{1-0.5\tau_i^l}\right)\right\}\right], \tag{1}$$

or

$$R\left(\widetilde{\widetilde{\kappa}}_i\right) = \frac{1}{2}\left[\frac{a_i^s + 2a_i^m + a_i^u}{2} - \frac{a_i^u - a_i^m}{\tau_i^r} + \frac{a_i^m - a_i^s}{\left(\tau_i^r\right)^2}\left\{\left(1+\tau_i^r\right)\ln\left(\frac{1+0.5\tau_i^r}{1+\tau_i^r}\right) - \ln\left(1-0.5\tau_i^r\right)\right\}\right.$$
$$\left. + \frac{a_i^u - a_i^m}{\left(\tau_i^r\right)^2}\left\{\left(1+\tau_i^r\right)\ln\left(1+0.5\tau_i^r\right) - \ln\left(\frac{1-0.5\tau_i^r}{1-\tau_i^r}\right)\right\}\right], \tag{2}$$

or

$$R\left(\widetilde{\widetilde{\kappa}}_i\right) = \frac{1}{2}\left[\frac{\left(2\tau_i^r+1\right)\left(a_i^s + a_i^u\right) - 2a_i^m}{4\tau_i^r} - \frac{a_i^s - 2\left(2\tau_i^l+1\right)a_i^m + a_i^u}{4\tau_i^l} + \frac{1+\tau_i^r}{4\left(\tau_i^r\right)^2}\left\{\left(a_i^m - a_i^s\right)\ln\left(1+\tau_i^r\right)\right.\right.$$
$$\left.\left. + \quad \left(a_i^u - a_i^m\right)\ln\left(1+\tau_i^r\right)\right\} - \frac{1+\tau_i^l}{4\left(\tau_i^l\right)^2}\left\{\left(a_i^m - a_i^s\right)\ln\left(1+\tau_i^l\right) + \left(a_i^u - a_i^m\right)\ln\left(1+\tau_i^l\right)\right\}\right]. \tag{3}$$

Then, using the proposed comparing method, Seikh et al. [21] proposed an approach to solve TT2FV matrix games. Seikh et al. [21] also solved a real-life TT2FV matrix game (plastic ban problem) to show the applicability of their proposed approach. In this paper, it is pointed out that a mathematically incorrect result is considered in Seikh et al. [21]'s approach due to which Seikh et al. [21]'s approach fails to find correct optimal solution of TT2FV matrix games. Also, Seikh et al. [21]'s approach is modified to resolve its inappropriateness.

This paper is organized as follows; In Section 2, Seikh et al. [21]'s approach is discussed in a brief manner. In Section 3, a numerical example is considered to show that Seikh et al. [21]'s approach fails to find correct optimal solution of TT2FV matrix games. Origin of Seikh et al. [21]'s approach is discussed in Section 4. In Section 5, the reason for the inappropriateness of Seikh et al. [21]'s approach is discussed. In Section 6, the required modification is suggested to resolve the inappropriateness of Seikh et al. [21]'s approach. In Section 7, Seikh et al. [21]'s approach is modified to resolve its inappropriateness. In Section 8, correct optimal solution of TT2FV matrix game, considered to show the inappropriateness of Seikh et al. [21]'s approach, is obtained by the modified approach. Section 9 concludes the paper.

2 Seikh et al.'s approach

Let $\left(\widetilde{A}, X, Y\right)$ be a TT2FV matrix game, where,

(i) $\widetilde{A} = \left(\widetilde{\widetilde{\kappa}}_{ij}\right)_{m\times n}$ represents the TT2FV payoff matrix of player I, where $\widetilde{\widetilde{\kappa}}_{ij} = \left(a_{ij}^s, a_{ij}^m, a_{ij}^u; \tau_{ij}^l, \tau_{ij}^r\right)$ is a TT2FV.

(ii) $X = \{(x_1, x_2, \ldots, x_m) : x_i \geq 0,\ \sum_{i=1}^m x_i = 1\}$ represents the set of strategies of player I.

(iii) $Y = \left\{(y_1, y_2, \ldots, y_n) : y_j \geq 0,\ \sum_{j=1}^n y_j = 1\right\}$ represents the set of strategies of player II.

Then, using Seikh et al. [21]'s approach, optimal strategy/strategies of player I, minimum expected gain of player I, optimal strategy/ strategies of player II and maximum expected loss of player II can be obtained as follows:

Step 1: Using the expression (1) or (2) or (3), transform the TT2FV payoff matrix $\widetilde{A} = \left(\widetilde{\widetilde{\kappa}}_{ij}\right)_{m\times n}$ into the crisp payoff matrix $A = \left(R\left(\widetilde{\widetilde{\kappa}}_{ij}\right)\right)_{m\times n}$.

Step 2: Solve the crisp linear programming problem (CLPP) (P1).

Problem (P1)

$Maximize\ \{w_1\}$
Subject to
$\sum_{i=1}^{m} R\left(\widetilde{\widetilde{\kappa}}_{ij}\right) x_i \geq w_1,\ j = 1, 2, \ldots, n,$
$\sum_{i=1}^{m} x_i = 1,\ x_i \geq 0,\ i = 1, 2, \ldots, m.$
In the obtained optimal solution $(x_1, x_2, \ldots, x_m, w_1)$,

(i) $(x_1, x_2, \ldots, x_m)$ represents the optimal strategy of player I.

(ii) w_1 represents the minimum expected gain of player I.

Step 3: Solve the CLPP (P2).
Problem (P2)
Minimize $\{w_2\}$
Subject to
$\sum_{j=1}^{n} R\left(\widetilde{\widetilde{\kappa}}_{ij}\right) y_j \leq w_2,\ i = 1, 2, \ldots, m,$
$\sum_{j=1}^{n} y_j = 1,\ y_j \geq 0,\ j = 1, 2, \ldots, n.$
In the obtained optimal solution $(y_1, y_2, \ldots, y_n, w_2)$,

(i) $(y_1, y_2, \ldots, y_n)$ represents the optimal strategy of player II.

(ii) w_2 represents the maximum expected loss of player II.

3 Inappropriateness of Seikh et al.'s approach

In this paper, it is pointed out that a mathematically incorrect result is considered in Seikh et al. [21]'s approach due to which Seikh et al. [21]'s approach fails to find correct optimal solution of TT2FV matrix games. In this section, this claim is validated with the help of a numerical example.

Let $\left(\widetilde{A}, X, Y\right)$ be a TT2FV matrix game where,

(i) $\widetilde{A} = \left(\widetilde{\widetilde{\kappa}}_{ij}\right)_{2\times 2} = \begin{pmatrix} (175, 180, 190; 0.4, 0.2) & (175, 180, 190; 0.2, 0.1) \\ (175, 180, 190; 0.3, 0.1) & (175, 180, 190; 0.6, 0.3) \end{pmatrix}$ represents the TT2FV payoff matrix of player I.

(ii) $X = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0,\ x_1 + x_2 = 1\}$ represents the set of strategies of player I.

(iii) $Y = \{(y_1, y_2) : y_1 \geq 0,\ y_2 \geq 0, y_1 + y_1 = 1\}$ represents the set of strategies of player II.

Then, according to the literature [1].

(i) Each optimal solution $(x_1, x_2, \ldots, x_m)$ of the TT2FV non-linear programming problem (NLPP) (P3) will represent an optimal strategy of player I.

Problem (P3)

$$\text{Maximize} \left\{ \begin{array}{c} Minimize \left(\widetilde{\widetilde{\kappa}}_{11} x_1 y_1 + \widetilde{\widetilde{\kappa}}_{21} x_2 y_1 + \widetilde{\widetilde{\kappa}}_{12} x_1 y_2 + \widetilde{\widetilde{\kappa}}_{22} x_2 y_2\right) \\ Subject\ to \\ \sum_{j=1}^{n} y_j = 1,\ y_j \geq 0,\ j = 1, 2, \ldots, n \end{array} \right\}$$

Subject to
$\sum_{i=1}^{m} x_i = 1,\ x_i \geq 0,\ i = 1, 2, \ldots, m.$

(ii) Each optimal solution $(y_1, y_2, \ldots, y_n)$ of the TT2FV NLPP (P4) will represent an optimal strategy of player II.

Problem (P4)

$$\text{Minimize} \left\{ \begin{array}{c} Maximize \left(\widetilde{\widetilde{\kappa}}_{11} x_1 y_1 + \widetilde{\widetilde{\kappa}}_{21} x_2 y_1 + \widetilde{\widetilde{\kappa}}_{12} x_1 y_2 + \widetilde{\widetilde{\kappa}}_{22} x_2 y_2\right) \\ Subject\ to \\ \sum_{i=1}^{m} x_i = 1,\ x_i \geq 0,\ i = 1, 2, \ldots, m \end{array} \right\}$$

Subject to

$\sum_{j=1}^{n} y_j = 1,\ y_j \geq 0,\ j = 1, 2, \ldots, n.$

Since, for the considered TT2FV matrix game,
$\left(\widetilde{\widetilde{\kappa}}_{11}x_1y_1 + \widetilde{\widetilde{\kappa}}_{21}x_2y_1 + \widetilde{\widetilde{\kappa}}_{12}x_1y_2 + \widetilde{\widetilde{\kappa}}_{22}x_2y_2\right)$
$= (175x_1y_1 + 175x_2y_1 + 175x_1y_2 + 175x_2y_2, 180x_1y_1 + 180x_2y_1 + 180x_1y_2 + 180x_2y_2, 190x_1y_1 + 190x_2y_1 + 190x_1y_2 +$
$190x_2y_2; maximum\{0.4, 0.2, 0.3, 0.6\},\ minimum\{0.2, 0.1, 0.1, 0.3\})$
$= (175, 180, 190; 0.6, 0.1).$

So, the objective functions of the TT2FV NLPPs (P3) and (P4) are independent from x_1, x_2, y_1 and y_2. Hence,

(i) $\left(\widetilde{\widetilde{\kappa}}_{11}x_1y_1 + \widetilde{\widetilde{\kappa}}_{21}x_2y_1 + \widetilde{\widetilde{\kappa}}_{12}x_1y_2 + \widetilde{\widetilde{\kappa}}_{22}x_2y_2\right) = (175, 180, 190; 0.6, 0.1)$ will represent the minimum expected gain of player I and the maximum expected loss of player II.

(ii) Each element (x_1, x_2) of the set $X = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0,\ x_1 + x_2 = 1\}$ will represent an optimal strategy for player I.

(iii) Each element (y_1, y_2) of the set $Y = \{(y_1, y_2) : y_1 \geq 0,\ y_2 \geq 0, y_1 + y_2 = 1\}$ will represent an optimal strategy for player II.

Therefore, on solving the considered TT2FV matrix game by the existing approach [21]

(i) $w_1 = w_2 = R\left(\widetilde{\widetilde{\kappa}}_{11}x_1y_1 + \widetilde{\widetilde{\kappa}}_{21}x_2y_1 + \widetilde{\widetilde{\kappa}}_{12}x_1y_2 + \widetilde{\widetilde{\kappa}}_{22}x_2y_2\right) = R(175, 180, 190; 0.6, 0.1)$ should be obtained.

(ii) Each element (x_1, x_2) of the set $X = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0,\ x_1 + x_2 = 1\}$ should represent an optimal strategy for player I.

(iii) Each element (y_1, y_2) of the set $Y = \{(y_1, y_2) : y_1 \geq 0,\ y_2 \geq 0, y_1 + y_2 = 1\}$ should represent an optimal strategy for player II.

However, in this section, it is shown that if the considered TT2FV matrix game is solved with the existing approach [21] by considering the expression (1) to evaluate the values of $R\left(\widetilde{\widetilde{\kappa}}_{ij}\right)$. Then,

(i) $w_1 = w_2 = 164.945$ is obtained instead of $w_1 = w_2 = R\left(\widetilde{\widetilde{\kappa}}_{11}x_1y_1 + \widetilde{\widetilde{\kappa}}_{21}x_2y_1 + \widetilde{\widetilde{\kappa}}_{12}x_1y_2 + \widetilde{\widetilde{\kappa}}_{22}x_2y_2\right)$
$= R(175, 180, 190; 0.6, 0.1) = 171.144.$

(ii) Only one element $(x_1, x_2) = (0.4062, 0.5938)$ of the set $X = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0,\ x_1 + x_2 = 1\}$ represents the optimal strategy for player I instead of each element (x_1, x_2) of the set $X = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0,\ x_1 + x_2 = 1\}$.

(iii) Only one element $(y_1, y_2) = (0.7571, 0.2429)$ of the set $Y = \{(y_1, y_2) : y_1 \geq 0,\ y_2 \geq 0, y_1 + y_2 = 1\}$ represents the optimal strategy for player II instead of each element (y_1, y_2) of the set $Y = \{(y_1, y_2) : y_1 \geq 0,\ y_2 \geq 0, y_1 + y_2 = 1\}$.

Using the existing approach [21], the considered TT2FV matrix game can be solved as follows:

Step 1: According to the Step 1 of the existing approach [21], there is a need to transform the TT2FV payoff matrix $\widetilde{A} = \left(\widetilde{\widetilde{\kappa}}_{ij}\right)_{2\times 2} = \begin{pmatrix} (175, 180, 190; 0.4, 0.2) & (175, 180, 190; 0.2, 0.1) \\ (175, 180, 190; 0.3, 0.1) & (175, 180, 190; 0.6, 0.3) \end{pmatrix}$ into the crisp payoff matrix

$A = \begin{pmatrix} R(175, 180, 190; 0.4, 0.2) & R(175, 180, 190; 0.2, 0.1) \\ R(175, 180, 190; 0.3, 0.1) & R(175, 180, 190; 0.6, 0.3) \end{pmatrix}.$
It can be easily verified that on using the expression (1),

(i) $R(175, 180, 190; 0.4, 0.2) = 167.851,$

(ii) $R(175, 180, 190; 0.2, 0.1) = 155.883,$

(iii) $R(175, 180, 190; 0.3, 0.1) = 162.956,$

(iv) $R(175, 180, 190; 0.6, 0.3) = 171.144.$

Hence,

$$A = \begin{pmatrix} 167.851 & 155.883 \\ 162.956 & 171.144 \end{pmatrix}.$$

Step 2: According to Step 2 of Seikh et al. [21]'s approach, in each optimal solution (x_1, x_2, w_1) of the CLPP (P5),

(i) (x_1, x_2) will represent an optimal strategy of player I.

(ii) w_1 will represent the minimum expected gain of player I.

Problem (P5)
Maximize $\{w_1\}$
Subject to
$167.851\ x_1 + 162.956\ x_2 \geq w_1,$

$155.883\ x_1 + 171.144\ x_2 \geq w_1,$

$x_1 + x_2 = 1,\ x_1 \geq 0, x_2 \geq 0.$

It can be easily verified that on solving the CLPP (P5), the unique optimal solution i.e., $(x_1, x_2, w_1) = (0.4062, 0.5938, 164.944)$ is obtained. Hence,

(i) The unique optimal strategy of player I is $(x_1, x_2) = (0.4062, 0.5938)$.

(ii) The minimum expected gain of player I is $w_1 = 164.944$.

Step 3: According to Step 3 of Seikh et al. [21]'s approach, in each optimal solution (y_1, y_2, w_2) of the CLPP (P6),

(i) (y_1, y_2) will represent an optimal strategy of player II.

(ii) w_2 will represent the maximum expected loss of player II.

Problem (P6)
Minimize $\{w_2\}$
Subject to
$167.851\ y_1 + 155.883\ y_2 \leq w_2,$

$162.956\ y_1 + 171.144\ y_2 \leq w_2,$

$y_1 + y_2 = 1,\ y_1 \geq 0, y_2 \geq 0.$

It can be easily verified that on solving the CLPP (P6), the unique optimal solution i.e., $(y_1, y_2, w_2) = (0.7571,\ 0.2429, 164.944)$ is obtained. Hence,

(i) The unique optimal strategy of player II is $(y_1, y_2) = (0.7571,\ 0.2429)$.

(ii) The maximum expected loss of player II is $w_2 = 164.944$.

Remark 3.1. *It can be easily verified that if in Step 1 of Section 3, the expression (2) or (3) is used to evaluate the values of* $R\left(\widetilde{\widetilde{\kappa}}_{ij}\right)$. *Then, also a unique optimal strategy instead of infinite number of optimal strategies will be obtained for player I and player II.*

4 Origin of Seikh et al.'s approach

It is obvious from Section 3 that Seikh et al. [21]'s approach fails to find the correct optimal solution of TT2FV matrix games. In this paper, it is claimed that this inappropriateness occurs as a mathematically incorrect result is considered in Seikh et al. [21]'s approach. Since, to point out the considered mathematically incorrect result, there is a need to

discuss the origin of CLPPs (P1) and (P2). So, the same is discussed in this section.

The origin of the CLPPs (P1) and (P2) is as follows:

Step 1: To find an optimal strategy $(x_1, x_2, \ldots, x_m)$ of player I is equivalent to find an optimal solution of the TT2FV NLPP (P7). Also, to find an optimal strategy $(y_1, y_2, \ldots, y_n)$ of player II is equivalent to find an optimal solution of the TT2FV NLPP (P8).

Problem (P7)

$$\text{Maximize} \left\{ \begin{array}{c} Minimize \left(\sum_{j=1}^{n} \sum_{i=1}^{m} \widetilde{\widetilde{\kappa}}_{ij} x_i y_j \right) \\ Subject\ to \\ \sum_{j=1}^{n} y_j = 1,\ y_j \geq 0,\ j = 1, 2, \ldots, n \end{array} \right\}$$

Subject to

$\sum_{i=1}^{m} x_i = 1,\ x_i \geq 0,\ i = 1, 2, \ldots, m.$

Problem (P8)

$$\text{Minimize} \left\{ \begin{array}{c} Maximize \left(\sum_{j=1}^{n} \sum_{i=1}^{m} \widetilde{\widetilde{\kappa}}_{ij} x_i y_j \right) \\ Subject\ to \\ \sum_{i=1}^{m} x_i = 1,\ x_i \geq 0,\ i = 1, 2, \ldots, m \end{array} \right\}$$

Subject to

$\sum_{j=1}^{n} y_j = 1,\ y_j \geq 0,\ j = 1, 2, \ldots, n.$

Step 2: Using the comparing method, $\widetilde{A}_1 \succcurlyeq \widetilde{A}_2$ if $R\left(\widetilde{A}_1\right) \geq R\left(\widetilde{A}_2\right)$ where, $R\left(\widetilde{A}_i\right),\ i = 1, 2$ can be obtained by the expression (1) or (2) or (3), discussed in Section 1, to find an optimal solution of the TT2FV NLPPs (P7) and (P8) is equivalent to find an optimal solution of the crisp NLPPs (P9) and (P10) respectively.

Problem (P9)

$$\text{Maximize} \left\{ \begin{array}{c} Minimize\ \left(R\left(\sum_{j=1}^{n} \sum_{i=1}^{m} \widetilde{\widetilde{\kappa}}_{ij} x_i y_j \right) \right) \\ Subject\ to \\ \sum_{j=1}^{n} y_j = 1,\ y_j \geq 0,\ j = 1, 2, \ldots, n \end{array} \right\}$$

Subject to

$\sum_{i=1}^{m} x_i = 1,\ x_i \geq 0,\ i = 1, 2, \ldots, m.$

Problem (P10)

$$\text{Minimize} \left\{ \begin{array}{c} Maximize \left(R\left(\sum_{j=1}^{n} \sum_{i=1}^{m} \widetilde{\widetilde{\kappa}}_{ij} x_i y_j \right) \right) \\ Subject\ to \\ \sum_{i=1}^{m} x_i = 1,\ x_i \geq 0,\ i = 1, 2, \ldots, m \end{array} \right\}$$

Subject to

$\sum_{j=1}^{n} y_j = 1,\ y_j \geq 0,\ j = 1, 2, \ldots, n.$

Step 3: Using the expression (4), to find an optimal solution of the crisp NLPPs (P9) and (P10) is equivalent to find an optimal solution of the crisp NLPPs (P11) and (P12), respectively.

$$R\left(\sum_{j=1}^{n} \sum_{i=1}^{m} \widetilde{\widetilde{\kappa}}_{ij} x_i y_j \right) = \sum_{j=1}^{n} \sum_{i=1}^{m} R\left(\widetilde{\widetilde{\kappa}}_{ij} \right) x_i y_j. \tag{4}$$

Problem (P11)

$$\text{Maximize} \left\{ \begin{array}{c} Minimize \left(\sum_{j=1}^{n} \sum_{i=1}^{m} R\left(\widetilde{\widetilde{\kappa}}_{ij} \right) x_i y_j \right) \\ Subject\ to \\ \sum_{j=1}^{n} y_j = 1,\ y_j \geq 0,\ j = 1, 2, \ldots, n \end{array} \right\}$$

Subject to

$\sum_{i=1}^{m} x_i = 1,\ x_i \geq 0,\ i = 1, 2, \ldots, m.$

Problem (P12)

Minimize $\left\{ \begin{array}{c} Maximize \left(\sum_{j=1}^{n} \sum_{i=1}^{m} R\left(\widetilde{\widetilde{\kappa}}_{ij}\right) x_i y_j \right) \\ Subject\ to \\ \sum_{i=1}^{m} x_i = 1,\ x_i \geq 0,\ i = 1, 2, \ldots, m \end{array} \right\}$
Subject to
$\sum_{j=1}^{n} y_j = 1,\ y_j \geq 0,\ j = 1, 2, \ldots, n.$

Step 4: Since, in the CLPP (P11), $\sum_{j=1}^{n} \sum_{i=1}^{m} R\left(\widetilde{\widetilde{\kappa}}_{ij}\right) x_i y_j$ represents the convex linear combination of $\sum_{i=1}^{m} R\left(\widetilde{\widetilde{\kappa}}_{i1}\right) x_i, \sum_{i=1}^{m} R\left(\widetilde{\widetilde{\kappa}}_{i2}\right) x_i, \ldots, \sum_{i=1}^{m} R\left(\widetilde{\widetilde{\kappa}}_{in}\right) x_i$ and in the CLPP (P12), $\sum_{j=1}^{n} \sum_{i=1}^{m} R\left(\widetilde{\widetilde{\kappa}}_{ij}\right) x_i y_j$ represents the convex linear combination of $\sum_{j=1}^{n} R\left(\widetilde{\widetilde{\kappa}}_{1j}\right) y_j, \sum_{j=1}^{n} R\left(\widetilde{\widetilde{\kappa}}_{2j}\right) y_j, \ldots, \sum_{j=1}^{n} R\left(\widetilde{\widetilde{\kappa}}_{mj}\right) y_j$. So, to find an optimal solution of the crisp NLPPs (P11) and (P12) is equivalent to find an optimal solution of the crisp NLPPs (P13) and (P14) respectively.

Problem (P13)
Maximize $\left\{ Minimum_{1 \leq j \leq n} \left(\left(\sum_{i=1}^{m} R\left(\widetilde{\widetilde{\kappa}}_{i1}\right) x_i \right), \left(\sum_{i=1}^{m} R\left(\widetilde{\widetilde{\kappa}}_{i2}\right) x_i \right), \ldots, \left(\sum_{i=1}^{m} R\left(\widetilde{\widetilde{\kappa}}_{in}\right) x_i \right) \right) \right\}$
Subject to
$\sum_{i=1}^{m} x_i = 1,\ x_i \geq 0,\ i = 1, 2, \ldots, m.$

Problem (P14)
Minimize $\left\{ Maximum_{1 \leq i \leq m} \left(\left(\sum_{j=1}^{n} R\left(\widetilde{\widetilde{\kappa}}_{1j}\right) y_j \right), \left(\sum_{j=1}^{n} R\left(\widetilde{\widetilde{\kappa}}_{2j}\right) y_j \right), \ldots, \left(\sum_{j=1}^{n} R\left(\widetilde{\widetilde{\kappa}}_{mj}\right) y_j \right) \right) \right\}$
Subject to
$\sum_{j=1}^{n} y_j = 1,\ y_j \geq 0,\ j = 1, 2, \ldots, n.$

Step 5: Assuming $Minimum_{1 \leq j \leq n} \left(\left(\sum_{i=1}^{m} R\left(\widetilde{\widetilde{\kappa}}_{i1}\right) x_i \right), \left(\sum_{i=1}^{m} R\left(\widetilde{\widetilde{\kappa}}_{i2}\right) x_i \right), \ldots, \left(\sum_{i=1}^{m} R\left(\widetilde{\widetilde{\kappa}}_{in}\right) x_i \right) \right) = w_1$, and $Maximum_{1 \leq i \leq m} \left(\left(\sum_{j=1}^{n} R\left(\widetilde{\widetilde{\kappa}}_{1j}\right) y_j \right), \left(\sum_{j=1}^{n} R\left(\widetilde{\widetilde{\kappa}}_{2j}\right) y_j \right), \ldots, \left(\sum_{j=1}^{n} R\left(\widetilde{\widetilde{\kappa}}_{mj}\right) y_j \right) \right) = w_2$ to find an optimal solution of the crisp NLPP (P13) and (P14) is equivalent to find an optimal solution of the CLPPs (P1) and (P2) respectively.

5 Reason for the inappropriateness of Seikh et al.'s approach

It is obvious from Section 4 that in Step 3, the expression (4) is used to transform

(i) The crisp NLPP (P9) into the crisp NLPP (P11).

(ii) The crisp NLPP (P10) into the crisp NLPP (P12).

However, in this section, a numerical example is considered to show that the expression (4) is not valid. Therefore, it is mathematically incorrect to use the expression (4) for transforming

(i) The crisp NLPP (P9) into the crisp NLPP (P11).

(ii) The crisp NLPP (P10) into the crisp NLPP (P12).

It is pertinent to mention that as the expression (4) is not valid. So,

(i) The crisp NLPP (P9) is not equivalent to the crisp NLPP (P11) and hence, the crisp NLPP (P13) is not equivalent to the CLPP (P1). Therefore, it is mathematically incorrect to assume that to find an optimal solution of the crisp NLPP (P13) is equivalent to find an optimal solution of the CLPP (P1).

(ii) The crisp NLPP (P10) is not equivalent to the crisp NLPP (P12) and hence the crisp NLPP (P14) is not equivalent to the CLPP (P2). Therefore, it is mathematically incorrect to assume that to find an optimal solution of the crisp NLPP (P14) is equivalent to find an optimal solution of the CLPP (P2).

The following example clearly indicates that the expression (4) is not valid.
Let, $\widetilde{\widetilde{\kappa}}_{11} = (175, 180, 190; 0.4, 0.2), \widetilde{\widetilde{\kappa}}_{12} = (175, 180, 190; 0.2, 0.1)$,

$\widetilde{\widetilde{\kappa}}_{21} = (175, 180, 190; 0.3, 0.1), \widetilde{\widetilde{\kappa}}_{22} = (175, 180, 190; 0.6, 0.3), x_1 = 1, \quad x_2 = 0,\ y_1 = 1,\ y_2 = 0.$
If the expression (1) is used to define $R\left(\widetilde{\widetilde{\kappa}}_{ij}\right)$ then we have

$$\begin{aligned} R\left(\sum_{j=1}^{2}\sum_{i=1}^{2}\widetilde{\widetilde{\kappa}}_{ij}x_iy_j\right) &= R(175x_1y_1 + 175x_2y_1 + 175x_1y_2 + 175x_2y_2, 180x_1y_1 + 180x_2y_1 + 180x_1y_2 + 180x_2y_2 \\ &\quad 190x_1y_1 + 190x_2y_1 + 190x_1y_2 + 190x_2y_2; maximum\{0.4, 0.2, 0.3, 0.6\},\ minimum\{0.2, 0.1, 0.1, 0.3\}) \\ &= R(175, 180, 190; 0.6, 0.1) = 171.144. \end{aligned} \tag{5}$$

While,

$$\begin{aligned} \sum_{j=1}^{2}\sum_{i=1}^{2} R\left(\widetilde{\widetilde{\kappa}}_{ij}\right)x_iy_j &= R(175, 180, 190; 0.4, 0.2)\ x_1y_1 + R(175, 180, 190; 0.2, 0.1)x_1y_2 \\ &\quad + R(175, 180, 190; 0.3, 0.1)\ x_2y_1 + R(175, 180, 190; 0.6, 0.3)\ x_2y_2 \\ &= 167.851\ x_1y_1 + 155.883\ x_1y_2 + 162.956\ x_2y_1 + 171.144\ x_2y_2 \\ &= 167.851. \end{aligned} \tag{6}$$

It is obvious from (5) and (6) that $R\left(\sum_{j=1}^{2}\sum_{i=1}^{2}\widetilde{\widetilde{\kappa}}_{ij}x_iy_j\right) \neq \sum_{j=1}^{2}\sum_{i=1}^{2} R\left(\widetilde{\widetilde{\kappa}}_{ij}\right)x_iy_j$.
Hence, the expression (4) is not valid.
Similarly, it can be easily proved that

(i) If the expression (2) is used to define $R(\widetilde{\kappa}_{ij})$ then the expression (4) is not valid.

(ii) If the expression (3) is used to define $R(\widetilde{\kappa}_{ij})$ then the expression (4) is not valid.

6 Valid expression corresponding to existing expression

It is obvious from Section 5 that the expression (4) is not valid. In this section, it is proved that the expression (7) represents the valid expression corresponding to the existing expression (4).

$$R\left(\sum_{j=1}^{n}\sum_{i=1}^{m}\widetilde{\widetilde{\kappa}}_{ij}x_iy_j\right) = \sum_{j=1}^{n}\sum_{i=1}^{m} R\left(\widetilde{\widetilde{\kappa}}_{ijM}\right)x_iy_j$$

where,

$$\widetilde{\widetilde{\kappa}}_{ij} = \left(a_{ij}^s, a_{ij}^m, a_{ij}^u; \tau_{ij}^l, \tau_{ij}^r\right) \text{ and}$$

$$\widetilde{\widetilde{\kappa}}_{ijM} = \left(a_{ij}^s, a_{ij}^m, a_{ij}^u; maximum_{\substack{1 \le i \le m \\ 1 \le j \le n}}\left\{\tau_{ij}^l\right\}, minimum_{\substack{1 \le i \le m \\ 1 \le j \le n}}\left\{\tau_{ij}^r\right\}\right)$$

The following proof clearly indicates that the expression (7) is valid.
If the expression (1) is used to define

$$R\left(\widetilde{\widetilde{\kappa}}_{ij}\right) \text{ and } R\left(\widetilde{\widetilde{\kappa}}_{ijM}\right),$$

then

$$\begin{aligned} &R\left(\sum_{j=1}^{n}\sum_{i=1}^{m}\widetilde{\widetilde{\kappa}}_{ij}x_iy_j\right) \\ &= R\left(\sum_{j=1}^{n}\sum_{i=1}^{m}\left(a_{ij}^s, a_{ij}^m, a_{ij}^u; \tau_{ij}^l, \tau_{ij}^r\right)x_iy_j\right) \\ &= R\left(\sum_{j=1}^{n}\sum_{i=1}^{m}a_{ij}^s x_iy_j, \sum_{j=1}^{n}\sum_{i=1}^{m}a_{ij}^m x_iy_j, \sum_{j=1}^{n}\sum_{i=1}^{m}a_{ij}^u x_iy_j; maximum_{\substack{1 \le i \le m \\ 1 \le j \le n}}\left\{\tau_{ij}^l\right\}, minimum_{\substack{1 \le i \le m \\ 1 \le j \le n}}\left\{\tau_{ij}^r\right\}\right) \end{aligned}$$

$$= \frac{1}{2}\left[\frac{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^{s} + 2\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^{m} + \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^{u}}{2} - \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^{u} - \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^{m}}{maximum_{\substack{1\le i\le m\\ 1\le j\le n}}\left\{\tau_{ij}^{l}\right\}}\right.$$

$$- \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^{m} - \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^{s}}{\left(maximum_{\substack{1\le i\le m\\ 1\le j\le n}}\left\{\tau_{ij}^{l}\right\}\right)^{2}}\left\{\left(1 + maximum_{\substack{1\le i\le m\\ 1\le j\le n}}\left\{\tau_{ij}^{l}\right\}\right)\ln\left(\frac{1 + maximum_{\substack{1\le i\le m\\ 1\le j\le n}}\left\{\tau_{ij}^{l}\right\}}{1 + 0.5\left(maximum_{\substack{1\le i\le m\\ 1\le j\le n}}\left\{\tau_{ij}^{l}\right\}\right)}\right)\right.$$

$$\left. + \ln\left(1 - 0.5\left(maximum_{\substack{1\le i\le m\\ 1\le j\le n}}\left\{\tau_{ij}^{l}\right\}\right)\right)\right\}$$

$$+ \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^{u} - \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^{m}}{\left(maximum_{\substack{1\le i\le m\\ 1\le j\le n}}\left\{\tau_{ij}^{l}\right\}\right)^{2}}\left\{\left(1 + maximum_{\substack{1\le i\le m\\ 1\le j\le n}}\left\{\tau_{ij}^{l}\right\}\right)\ln\left(1 + 0.5\left(maximum_{\substack{1\le i\le m\\ 1\le j\le n}}\left\{\tau_{ij}^{l}\right\}\right)\right)\right.$$

$$\left. + \ln\left(\frac{1 - maximum_{\substack{1\le i\le m\\ 1\le j\le n}}\left\{\tau_{ij}^{l}\right\}}{1 - 0.5\left(maximum_{\substack{1\le i\le m\\ 1\le j\le n}}\left\{\tau_{ij}^{l}\right\}\right)}\right)\right\}\Bigg] x_i y_j$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{m} R\left(a_{ij}^{s}, a_{ij}^{m}, a_{ij}^{u}; maximum_{\substack{1\le i\le m\\ 1\le j\le n}}\left\{\tau_{ij}^{l}\right\}, minimum_{\substack{1\le i\le m\\ 1\le j\le n}}\left\{\tau_{ij}^{r}\right\}\right) x_i y_j$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{m} R\left(\widetilde{\widetilde{\kappa}}_{ijM}\right) x_i y_j$$

Hence, the expression (7) is valid.
Similarly, it can be easily proved that

(i) If the expression (2) is used to define $R\left(\widetilde{\widetilde{\kappa}}_{ijM}\right)$ then the expression (7) is valid.

(ii) If the expression (3) is used to define $R\left(\widetilde{\widetilde{\kappa}}_{ijM}\right)$ then the expression (7) is valid.

7 Modified approach

In this section, Seikh et al. [21]'s approach is modified by replacing the existing expression (4) with the valid expression (7).

The steps of modified approach are as follows:
Step 1: Using the expression (1) or (2) or (3), transform the TT2FV payoff matrix

$$\widetilde{A} = \left(\widetilde{\widetilde{\kappa}}_{ij}\right)_{m\times n},$$

into the crisp payoff matrix

$$A = \left(R\left(\widetilde{\widetilde{\kappa}}_{ijM}\right)\right)_{m\times n} = \left(R\left(a_{ij}^{s}, a_{ij}^{m}, a_{ij}^{u}; maximum_{\substack{1\le i\le m\\ 1\le j\le n}}\left\{\tau_{ij}^{l}\right\}, minimum_{\substack{1\le i\le m\\ 1\le j\le n}}\left\{\tau_{ij}^{r}\right\}\right)\right)_{m\times n}.$$

Step 2: Solve the CLPP (P15).

Problem (P15)
Maximize $\{w_1\}$
Subject to
$\sum_{i=1}^{m} R\left(\widetilde{\widetilde{\kappa}}_{ijM}\right) x_i \ge w_1, \quad j = 1, 2, \ldots, n,$
$\sum_{i=1}^{m} x_i = 1,\ x_i \ge 0, \quad i = 1, 2, \ldots, m.$
In the obtained optimal solution $(x_1, x_2, \ldots, x_m, w_1)$,

(i) $(x_1, x_2, \ldots, x_m)$ represents the optimal strategy of player I.

(ii) w_1 represents the minimum expected gain of player I.

Step 3: Solve the CLPP (P16).

Problem (P16)
Minimize $\{w_2\}$
Subject to
$\sum_{j=1}^{n} R\left(\widetilde{\widetilde{\kappa}}_{ijM}\right) y_j \leq w_2, \quad i = 1, 2, \ldots, m,$
$\sum_{j=1}^{n} y_j = 1, \;\; y_j \geq 0, \quad j = 1, 2, \ldots, n.$
In the obtained optimal solution $(y_1, y_2, \ldots, y_n, w_2)$,

(i) $(y_1, y_2, \ldots, y_n)$ represents the optimal strategy of player II.

(ii) w_2 represents the maximum expected loss of player II.

8 Illustrative example

In Section 3, a TT2FV matrix game is considered and it is shown that on solving considered TT2FV matrix game,

(i) $w_1 = w_2 = 171.144$ should be obtained.

(ii) Each element (x_1, x_2) of the set $X = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0, \; x_1 + x_2 = 1\}$ should represent an optimal strategy for player I.

(iii) Each element (y_1, y_2) of the set $Y = \{(y_1, y_2) : y_1 \geq 0, \; y_2 \geq 0, y_1 + y_2 = 1\}$ should represent an optimal strategy for player II.

While, on solving the considered TT2FV matrix game by the existing approach [21].

(i) $w_1 = w_2 = 164.945$ is obtained.

(ii) Only one element $(x_1, x_2) = (0.4062, 0.5938)$ of the set $X = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0, \; x_1 + x_2 = 1\}$ represents the optimal strategy for player I.

(iii) Only one element $(y_1, y_2) = (0.7571, 0.2429)$ of the set $Y = \{(y_1, y_2) : y_1 \geq 0, \; y_2 \geq 0, y_1 + y_2 = 1\}$ represents the optimal strategy for player II.

In this section, it is shown that the correct results are obtained on solving the considered TT2FV matrix game by modified approach.

Using the modified approach, the considered TT2FV matrix game can be solved as follows:

Step 1: According to the Step 1 of the modified approach, there is need to transform the TT2FV payoff matrix

$\widetilde{A} = \begin{pmatrix} (175, 180, 190; 0.4, 0.2) & (175, 180, 190; 0.2, 0.1) \\ (175, 180, 190; 0.3, 0.1) & (175, 180, 190; 0.6, 0.3) \end{pmatrix}$ into the crisp payoff matrix,

$A = \begin{pmatrix} R(175, 180, 190; 0.6, 0.1) & R(175, 180, 190; 0.6, 0.1) \\ R(175, 180, 190; 0.6, 0.1) & R(175, 180, 190; 0.6, 0.1) \end{pmatrix}$, where, 0.6 and 0.1 represents $maximum\,\{0.4, 0.2, 0.3, 0.6\}$ and $minimum\,\{(0.2, 0.1, 0.1, 0.3)\}$ respectively. It can be easily verified that on using the expression (1),

$$R(175, 180, 190; 0.6, 0.1) = 171.144.$$

Hence, $A = \begin{pmatrix} 171.144 & 171.144 \\ 171.144 & 171.144 \end{pmatrix}$.

Step 2: According to Step 2 of the modified approach, in each optimal solution (x_1, x_2, w_1) of the CLPP (P17),

(i) (x_1, x_2) will represent an optimal strategy of player I.

(ii) w_1 will represent the minimum expected gain of player I.

Problem (P17)
Maximize $\{w_1\}$
Subject to
$171.144\ x_1 + 171.144\ x_2 \geq w_1$,
$171.144\ x_1 + 171.144\ x_2 \geq w_1$,
$x_1 + x_2 = 1,\ x_1 \geq 0, x_2 \geq 0$.
Since, $x_1 + x_2 = 1$ so to find an optimal solution of the CLPP (P17) is equivalent to find an optimal solution of the CLPP (P18).

Problem (P18)
Maximize $\{w_1\}$
Subject to
$171.144 \geq w_1$,
$171.144 \geq w_1$,
$x_1 + x_2 = 1,\ x_1 \geq 0, x_2 \geq 0$.
It can be easily verified that each element of the set $\{(x_1, x_2, w_1) : x_1 \geq 0, x_2 \geq 0,\ x_1 + x_2 = 1,\ w_1 = 171.144\}$ represents an optimal solution of the CLPP (P18). Hence,

(i) Each element of the set $X = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0,\ x_1 + x_2 = 1\}$ represents an optimal strategy of player I.

(ii) The minimum expected gain of player I is $w_1 = 171.144$.

Step 3: According to Step 3 of the modified approach, in each optimal solution (y_1, y_2, w_2) of the CLPP (P19),

(i) (y_1, y_2) will represent an optimal strategy of player II.

(ii) w_2 will represent the maximum expected loss of player II.

Problem (P19)
Minimize $\{w_2\}$
Subject to
$171.144\ y_1 + 171.144\ y_2 \leq w_2$,
$171.144\ y_1 + 171.144\ y_2 \leq w_2$,
$y_1 + y_2 = 1,\ y_1 \geq 0, y_2 \geq 0$.
Since, $y_1 + y_2 = 1$ so to find an optimal solution of the CLPP (P19) is equivalent to find an optimal solution of the CLPP (P20).

Problem (P20)
Minimize $\{w_2\}$
Subject to
$171.144\ \leq w_2$,
$171.144\ \leq w_2$,
$y_1 + y_2 = 1,\ y_1 \geq 0, y_2 \geq 0$.
It can be easily verified that each element of the set $\{(y_1, y_2, w_2) : y_1 \geq 0, y_2 \geq 0,\ y_1 + y_2 = 1,\ w_2 = 171.144\}$ represents an optimal solution of the CLPP (P20). Hence,

(i) Each element of the set $Y = \{(y_1, y_2) : y_1 \geq 0, y_2 \geq 0,\ y_1 + y_2 = 1\}$ represents an optimal strategy of player II.

(ii) The maximum expected loss of player II is $w_2 = 171.144$.

9 Conclusion

A numerical example is considered to show that Seikh et al. [21]'s approach fails to find correct optimal solution of TT2FV matrix games. Also, it is pointed out that this inappropriateness occurs due to considering a mathematically incorrect result in the origin of Seikh et al. [21]'s approach. Furthermore, Seikh et al. [21]'s approach is modified to resolve its inappropriateness. Finally, a correct optimal solution of the considered numerical example is obtained by the modified approach.

It is pertinent to mention that as the proposed approach cannot be used to solve bi-matrix games. To overcome this limitation of the proposed method may be considered as an open research problem.

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