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Strong admissible reduction of hesitant L-fuzzy automaton

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Abstract

In this article, after the definitions of the reduced hesitant L-fuzzy automaton (RHLFA) and the minimal hesitant L-fuzzy automaton; we convert a hesitant L-fuzzy automaton (HLFA) to a RHLFA by reducing the number of its states such that its language is equal to the original HLFA language. Then, by defining an equivalence relation on the monoid X^*, we construct an HLFA whose language is equal to the language of the transformed RHLFA, and we show that this HLFA is minimal. In conclusion, we delineate the criteria under which, the number of states in the minimal HLFA is equal to the number of states in the RHLFA.

Keywords: Hesitant set, hesitant automata, behavior, lattice.

1 Introduction

The idea of an automaton has its origins in the scientific determinism of the 17th century. Automata were introduced as the simplest machine for pattern recognition. Finite automata are the best-known computational devices. They are used to model a number of formal systems. Automata theory is closely related to formal language theory, as the automata are often classified by the class of formal languages they are able to recognize. Moreover, it is widely known that finite automata can solve a large number of problems in computer science, among which are the modeling of reactive systems, the design of hardware of digital systems, compilers, and the description of natural languages. In 1965, Zadeh [37] introduced the notion of a fuzzy subset of a set as a method for representing uncertainty. His ideas have been applied to a wide range of scientific areas. One such area is automata theory and language theory, first introduced by Wee [35]. Furthermore, Lee and Zadeh [15] introduced the concept of fuzzy finite state automata.

The uncertainty and ambiguity in the diagnosis process, force us to carry out mathematical computation in a fuzzy environment. Fuzzy automata are used to handle uncertainty systems more accurately where classical automata fail to cater to the circumstances. An ambiguous and confusing computation can be done in a constructive and useful environment thanks to fuzzy automata. Their importance has been proven by their ability to solve meaningful problems in learning systems and pattern recognition. As it is reported, fuzzy finite automata have many important applications; see [12, 20, 21, 24, 25, 29, 30, 31]. With the development of the various extensions for fuzzy sets, some generalizations for fuzzy automata have been presented, for example, interval-valued fuzzy automata [22, 23], bisimulation for BL-general fuzzy automata [25], Intuitionistic general fuzzy automata [24], and intuitionistic fuzzy automata [4].

Torra and Narukawa have proposed the concept of a hesitant fuzzy set to tackle the challenge of determining an element's membership in a set [32, 33], they first initiated this concept with a membership value of members to an arbitrary subset of $[0, 1]$. These memberships could include not only crisp values in [0, 1], but also interval values during the evaluating process in practice. Returning a list of membership values for each element in the domain is how an HFS is defined in terms of a function. Using the concept of a typical hesitant fuzzy set [2, 32], Costa and Bedregal [5] managed to bring the concept of nondeterministic typical hesitant fuzzy automation into the realm of automation, which generalizes the notion of fuzzy automata. It appears that a hesitant fuzzy automata system is

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able to effectively solve multimembership problems. Dehmiry, Mashinchi, and Mesiar [6] introduced the concept of hesitant L-fuzzy sets (HLFS) into mathematics by extending the interval $[0,1]$ to a lattice. Then the concept of the hesitant L-fuzzy automaton (HLFA) was established by Shamsizadeh, Zahedi, and Agheli [28]. In essence, HLFA is the marriage of classical deterministic automata and HLFSs. Suppose L is a distributive lattice, in principle some carriers are assigned to subsets of L by HLFSs. Using these assignments, the value of membership degree of acceptance of a word in HLFA is wider than the value of membership degree of acceptance of word in FA (subsets of a lattice instead of 0 and 1).

HLFA makes up for some shortcomings in applications that rely on fuzzy automata because assigning membership values to active states of a fuzzy automaton, resolves the multi-membership [28]. HLFA achieves a wider performance for the automata by expanding the range of the membership value from the interval $[0,1]$ to the set 2^L. This was our main motivation to first reduce the number of states of an HLFA in such a way that it preserves the original language, and then we set out to minimize it by preserving the language.

Minimization is a fundamental problem in automata theory. There are many papers on the minimization problem of fuzzy finite automata. For example, the minimization of a mealy type of fuzzy finite automata was discussed in [3], the minimization of fuzzy finite automata with crisp final states without outputs was studied in [1], and minimizing the deterministic finite automaton with fuzzy (final) states was studied in [18]. Myhill-Neroded's theorem has been extended to fuzzy regular language, and an algorithm was presented for minimizing the deterministic finite automaton with fuzzy (final) states [10, 14, 18]. The technique of minimizing fuzzy automata was introduced by Li and Qui [17] and a new automata system was constructed that could accurately handle multiple states at a time. A fuzzy automaton was reduced to another fuzzy automaton with m states that were functionally comparable to the original fuzzy automaton by using an ordered lattice. By Moghari and Zahedi [19] method during the minimization process, language, and behavior are both preserved. Although a variety of methods have been developed to minimize deterministic fuzzy automata while preserving the language [7, 8, 9, 10, 11, 13, 16, 17, 26, 27, 34, 36], our method is completely different from their method.

In this paper, the strong admissible reduction of hesitant L-fuzzy automaton (HLFA) is presented. This work has the following division: After the introduction, Section 2, presents all the basic definitions and notations. In section 3, after the definition of regular subset $\mathcal{L}$ from monoid X^*, in the following two ways, we define a reduced and minimal HLFA whose language is $\mathcal{L}$.

1. For acessible complete deterministic HLFA $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ whose language is $\mathcal{L}$, we define an equivalence relation ρ on Q. Then we denote a set of equivalence classes by $\frac{Q}{\rho}$, and by means of it we define an HLFA $\frac{\mathcal{A}}{\rho} = (\frac{Q}{\rho}, X, \delta_\rho, \iota_\rho, \tau_\rho)$ whose language is equal to the language $\mathcal{A}$, that is, $\mathcal{HL}_{\frac{\mathcal{A}}{\rho}} = \mathcal{HL}_{\mathcal{A}} = \mathcal{L}$. Moreover, we show that $\frac{\mathcal{A}}{\rho}$ is a RHLFA. If all its states are co-accessible, it is a reduced and minimal HLFA, but if some of its states are not co-accessible, by removing them, we arrive at a reduced and minimal HLFA whose language is $\mathcal{L}$ and we call this HLFA the strong admissible reduction of HLFA $\mathcal{A}$ for regular subset $\mathcal{L}$ from X^*.

2. For the regular subset $\mathcal{L}$ of monoid X^*, we introduce an equivalence relation $R_{\mathcal{L}}$ on X^*, and using it, we define a HLFA $\mathcal{A}_m = (\frac{X^*}{R_{\mathcal{L}}}, X, \iota_m, \delta_m, \tau_m)$, whose set of states is set $\frac{X^*}{R_{\mathcal{L}}}$ and its language is $\mathcal{L}$, that is, $\mathcal{HL}_{\mathcal{A}_m} = \mathcal{L}$. To show that $\mathcal{A}_m$ is minimal, we consider an arbitrary complete deterministic HLFA $\mathcal{A} = (Q, X, \iota, \delta, \tau)$, whose language is $\mathcal{L}$, that is, $\mathcal{HL}_{\mathcal{A}} = \mathcal{L}$. Then, by introducing an equivalence relation $R_{\mathcal{F}}$ on X^*, we show that the cardinality of $\frac{X^*}{R_{\mathcal{L}}}$ is less than or equal to the cardinality of $\frac{X^*}{R_{\mathcal{F}}}$, and considering that the cardinality of $\frac{X^*}{R_{\mathcal{F}}}$ is less than or equal to the cardinality of Q, we will conclude that the cardinality of $\frac{X^*}{R_{\mathcal{L}}}$ will be less than the cardinality of Q, that is, $|\frac{X^*}{R_{\mathcal{L}}}| \leq |\frac{X^*}{R_{\mathcal{F}}}| \leq |Q|$ and in this way we show that the HLFA $\mathcal{A}_m$ is minimal. Finally, we show that $\mathcal{A}_m$ is reduced.

2 Preliminaries

In this section, we present all the basic definitions and notations that are used throughout the text.

Definition 2.1. [6] *Let X denote a nonempty set of symbols and let $(L, \preceq_L)$ be an arbitrary lattice. Then a hesitant L-fuzzy set of X is a mapping*

$$\begin{aligned} H :& X \to 2^L \\ & x \mapsto H(x). \end{aligned}$$

The value $H(x)$ denotes the membership degree of x in H and is named hesitant L-fuzzy element. Where 2^L set is equal to the set of all subsets of the lattice L except the empty set.

Definition 2.2. [6] *For $A, B \in 2^L$, the function $\sqcup : 2^L \times 2^L \to 2^L$ is defined by:*

$$A \sqcup B = \{a \vee b \mid a \in A \text{ and } b \in B\},$$

and the function $\sqcap : 2^L \times 2^L \to 2^L$ is defined by:

$$A \sqcap B = \{a \wedge b \mid a \in A \text{ and } b \in B\}.$$

Definition 2.3. [20] *A fuzzy automaton is a 5-tuple $\mathcal{A} = (Q, X, \iota, \delta, \tau)$, where*

1. *Q is a finite nonempty set of states,*
2. *X is a finite nonempty set of input symbols,*
3. *ι is a fuzzy subset of Q, that is, $\iota : Q \to [0, 1]$, called the initial fuzzy states,*
4. *$\delta : Q \times X \to Q$ is a function, called the transition function,*
5. *τ is a fuzzy subset of Q, that is, $\tau : Q \to [0, 1]$, called the final fuzzy states.*

And the behavior of fuzzy automaton $\mathcal{A}$ is defined as follows:

$$\beta_{\mathcal{A}} : X^* \to [0, 1],$$
$$\beta_{\mathcal{A}}(u) = \vee_{q,p\in Q}(\iota(q) \wedge \delta^*(q, u, p) \wedge \tau(p)),$$

where $u \in X^*$.

Definition 2.4. [28] *Let L be a distributive lattice. Then a hesitant L-fuzzy automaton (HLFA) is a quintuple $\mathcal{A} = (Q, X, \iota, \delta, \tau)$, where*

1. *Q is a finite nonempty set of states,*
2. *X is a finite set of input symbols,*
3. *$\iota : Q \to 2^L$ is a hesitant L-fuzzy set on Q of initial states,*
4. *$\delta : Q \times X \times Q \to 2^L$ is a transition hesitant L-fuzzy set,*
5. *$\tau : Q \to 2^L$ is a hesitant L-fuzzy set of final states.*

Definition 2.5. *Let L be a distributive lattice and let $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ be an HLFA. Then for any $\beta \in L$, we let $\beta_L : Q \to 2^L$ be a hesitant L-fuzzy set, which is defined by $\beta_L(q) = \{\beta\}$ for every $q \in Q$.*

Definition 2.6. [28] *Let L be a distributive lattice. Then for an HLFA $\mathcal{A} = (Q, X, \iota, \delta, \tau)$, the hesitant L-fuzzy set δ is extended into a function $\delta^* : Q \times X^* \times Q \to 2^L$ using the recursion:*

$$\delta^*(q, \Lambda, p) = \begin{cases} 1_L & \text{if } q = p, \\ 0_L & \text{otherwise,} \end{cases}$$

and

$$\delta^*(q, ua, p) = \sqcup_{r\in Q}[\delta^*(q, u, r) \sqcap \delta(r, a, p))],$$

where $u \in X^*$ and $a \in X$.

Definition 2.7. [28] *Let L be a distributive lattice and let $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ be an HLFA. Then $\mathcal{A}$ computes the hesitant L-fuzzy behavior (HLFB) $\beta_{\mathcal{A}} : X^* \to 2^L$ defined by:*

$$\beta_{\mathcal{A}}(u) = \sqcup_{q,p\in Q}(\iota(q) \sqcap \delta^*(q, u, p) \sqcap \tau(p)),$$

where $u \in X^*$.

Definition 2.8. [28] *Let L be a distributive lattice and let $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ be an HLFA. Then we say that $u \in X^*$ is recognized by an HLFA $\mathcal{A}$ if $\beta_{\mathcal{A}}(u) \neq 0_L$. We display all the words that are recognized by HLFA $\mathcal{A}$ with $\mathcal{HL}_{\mathcal{A}}$, indicating $\mathcal{HL}_{\mathcal{A}} = \{u \in X^* | \beta_{\mathcal{A}}(u) \neq 0_L\}$. It is called the language $\mathcal{A}$.*

Definition 2.9. [28] *Let L be a distributive lattice and let $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ be an HLFA. Then we define a hesitant L-fuzzy set $\xi_{\mathcal{A}}$ on $X^* \times Q$, as follows:*

$$\xi_{\mathcal{A}} : X^* \times Q \to 2^L,$$
$$\xi_{\mathcal{A}}(u, q) = \sqcup\{i(p) \sqcap \delta^*(p, u, q) | p \in Q\}.$$

Definition 2.10. [28] *Let L be a distributive lattice and let $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ be an HLFA. Then we say $q \in Q$ is accessible, if there exists $u \in X^*$ such that $\xi_{\mathcal{A}}(u, q) \neq 0_L$.*

Definition 2.11. [28] *Let L be a distributive lattice and let $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ be an HLFA. Then $\mathcal{A}$ is called complete if for every $q \in Q$ and $a \in X$, there exist $p \in Q$ and $0 \neq \alpha \in L$ such that $\alpha \in \delta(q, a, p)$.*

Definition 2.12. [28] *Let L be a distributive lattice and let $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ be an HLFA. Then we say $\mathcal{A}$ is deterministic if the following conditions hold:*

1. *There exists unique $q_0 \in Q$ such that $\iota(q_0) \neq 0_L$.*
2. *For every $q \in Q$ and $a \in X$, there exists at most one $q' \in Q$ such that $\delta(q, a, q') \neq 0_L$.*

Notation 2.13. [28] *Let L be a distributive lattice and let $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ be an HLFA. Then*

1. *There exists a complete deterministic HLFA $\mathcal{A}'$ such that $\mathcal{HL}_{\mathcal{A}} = \mathcal{HL}_{\mathcal{A}'}$.*
2. *There exists an accessible complete deterministic HLFA $\mathcal{A}''$ such that $\mathcal{HL}_{\mathcal{A}} = \mathcal{HL}_{\mathcal{A}''}$.*

3 Strong admissible reduction of HLFA

At the beginning of this section, with the Example 3.1, we express the superiority of HLFA over FA, and with the Example 3.2, we show the use of this superiority in practical problems.

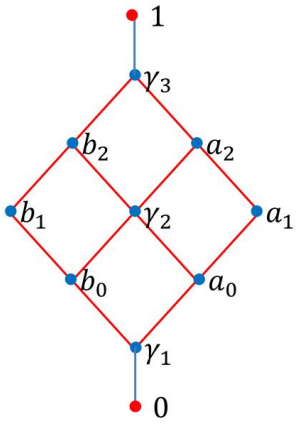


Figure 1: The distributive lattice L of Example 3.1

Example 3.1. *Let $Q = \{q_0, q_1, q_2, q_3, q_4\}$, let $X = \{a, b\}$, and let L be distributive lattice of Figure 1. Define $\iota : Q \to 2^L$ and $\delta^* : Q \times X^* \times Q \to 2^L$, $\tau : Q \to 2^L$ as follows:*

$$\iota(q_0) = \{\gamma_1, \gamma_2, \gamma_3\},$$

$$\begin{aligned}
\delta(q_0, a, q_1) &= \{a_0, a_1\}, & \delta(q_0, b, q_2) &= \{b_0, b_2\},\\
\delta(q_1, a, q_3) &= \{a_1, \gamma_1\}, & \delta(q_1, b, q_1) &= \{b_1, b_2\},\\
\delta(q_2, a, q_4) &= \{a_2, \gamma_1\}, & \delta(q_2, b, q_2) &= \{b_2, \gamma_2\},\\
\delta(q_3, a, q_3) &= \{a_2, a_1\}, & \delta(q_3, b, q_3) &= \{0, b_2\},\\
\delta(q_4, a, q_4) &= \{a_1\}, & \delta(q_4, b, q_4) &= \{b_0\},\\
\tau(q_1) &= \{\gamma_3\}, & \tau(q_2) &= \{\gamma_1, \gamma_2, \gamma_3\}.
\end{aligned}$$

It is clear that $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ is an accessible complete deterministic HLFA and the state diagram HLFA $\mathcal{A}$ is Figure 2 and the language HLFA $\mathcal{A}$ is the set $ab^* \cup b^+$.

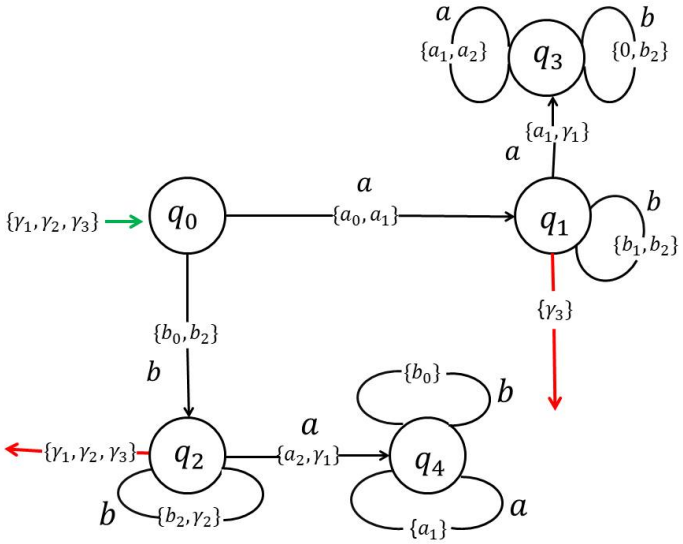


Figure 2: The state diagram HLFA $\mathcal{A}$ of Example 3.1

As can be seen from Figure 2, a path for the word ab^2 is as follows: The path starts through the hesitant L-fuzzy set of ι with the initial state q_0 and the hesitant membership degree $\{\gamma_1, \gamma_2, \gamma_3\}$. Then it is transferred from the initial state q_0 to the state q_1 through the letter a with the transition hesitant L-fuzzy set $\delta(q_0, a, q_1)$ and with hesitant membership degree $\{a_0, a_1\}$, next it is transferred from the state q_1 to the state q_1 through the letter b with the transition hesitant L-fuzzy set $\delta(q_1, b, q_1)$ and with hesitant membership degree $\{b_1, b_2\}$, then it is transferred from the state q_1 to the state q_1 through the letter b with the transition hesitant L-fuzzy set $\delta(q_1, b, q_1)$ and with hesitant membership degree $\{b_1, b_2\}$. Finally, it is transferred to the final state q_1 with hesitant membership degree $\{\gamma_3\}$ through the final hesitant L-fuzzy set τ. Therefore, it can be said that the word ab^2 is explained through the path and according to the behavior of the HLFA $\mathcal{A}$, it is recognized by the following hesitant membership degree.

$$\begin{aligned}
\beta_{\mathcal{A}}(ab^2) &= \iota(q_0) \sqcap \delta(q_0, ab^2, q_1) \sqcap \tau(q_1)\\
&= \iota(q_0) \sqcap \delta(q_0, a, q_1) \sqcap \delta(q_1, b, q_1) \sqcap \delta(q_1, b, q_1) \sqcap \tau(q_1)\\
&= \{\gamma_1, \gamma_2, \gamma_3\} \sqcap \{a_0, a_1\} \sqcap \{b_1, b_2\} \sqcap \{b_1, b_2\} \sqcap \{\gamma_3\}\\
&= \{\gamma_1, a_0\}.
\end{aligned}$$

As it is clear from Example 3.1, a word is recognized in an HLFA with multiple degree of membership, so it seems that an HLFA is a more suitable tool than a fuzzy automaton for solving multiple problems.

Now by an example, we will show the use of this advantage in solving practical problems. Incorporating the hesitant L-fuzzy automaton into the Hazard Analysis of Road Accidents (HARA) introduces an innovative method that improves the evaluation of road accident risks by utilizing hesitant fuzzy logic principles, where $L = [0, 1]$. This technique applies hesitant L-fuzzy automaton within the HARA framework to achieve a more detailed and resilient

analysis of road hazards. By embedding hesitant L-fuzzy automata into HARA, researchers and practitioners are equipped with enhanced tools for a deeper insight into hazard analysis related to road accidents, thereby increasing the precision of hazard evaluations. This advanced approach holds significant potential for progressing the study of road accident hazards and bettering risk reduction measures in areas susceptible to accidents.

Example 3.2. *Let the fuzzy automaton $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ as Figure 3, where*

$$Q = \{q_0, q_1, q_2, p_1, \ldots, p_4, s_1, \ldots, s_4, r_1, \ldots, r_8\},$$

is the set of states and $X = \{f_1, f_2, l_1, l_2, e_1, m_1, m_2\}$ is the set of alphabet. Let f_1 and f_2 represent night driving and day driving, respectively, l_1 and l_2 represent mountain roads and desert roads, respectively. Also, e_1 represents here are traffic signs also m_1 and m_2 represent using a mobile phone while driving and eating while driving, respectively. We consider q_0 as the only initial state, we define $\iota(q_0) = 1$. We consider the states $r_1, \ldots, r_8$ as the final states and consider $\tau(r_1) = \ldots = \tau(r_8) = 1$.

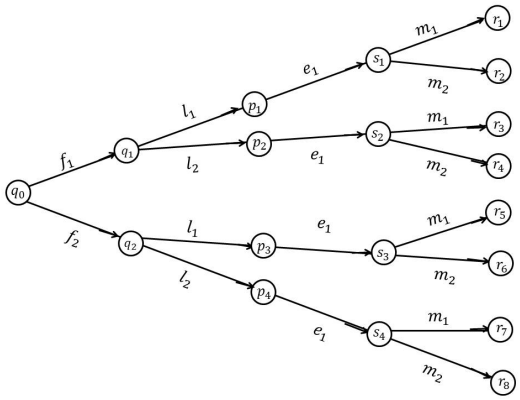


Figure 3: The state diagram of FA, Example 3.2

For the sake of simplicity, instead of using the transition $\delta(q, u, p)$, we use u. Let us assume that we have taken the opinions of 3 experts about the objects as follows:

.	Expert 1	Expert 2	Expert 3
f_1	0.2	0.8	0.9
f_2	0.4	1	0.5
l_1	0.5	0.5	0.9
l_2	0.8	0.7	0.5
e_1	0	0.8	0.9
m_1	0.4	0.6	0.5
m_2	0.8	0.7	0.9

If we consider the opinions of the first expert, the degree of hazard analysis of road accidents for all paths is zero and the information is lost. So, it is not a good choice.
If we use the t-norm "minimum" to obtain a value for every transition, we have the values of transitions as follows:

f_1	0.2
f_2	0.4
l_1	0.5
l_2	0.5
e_1	0
m_1	0.4
m_2	0.7

The values of all of the paths are zero, too. If we use the average function to obtain a value for every transition we have the values as follows:

f_1	0.63
f_2	0.63
l_1	0.63
l_2	0.67
e_1	0.57
m_1	0.5
m_2	0.8

Therefore, the value of each path based on the automaton fuzzy behavior defined in Definition 2.3 is as follows:

$word \;\; x_1x_2\dots x_n$	$\beta_{\mathcal{A}}(x_1x_2\dots x_n)$
$f_1l_1e_1m_1$	0.5
$f_1l_1e_1m_2$	0.57
$f_1l_2e_1m_1$	0.5
$f_1l_2e_1m_2$	0.57
$f_2l_1e_1m_1$	0.5
$f_2l_1e_1m_2$	0.57
$f_2l_2e_1m_1$	0.5
$f_2l_2e_1m_2$	0.57

We can see that the paths' values are similar and repeated between the two sets. I think that by using fuzzy automata, we cannot make the right decision, and information is lost. Now, let Figure 3 as the state diagram hesitant L-fuzzy automaton $\mathcal{A} = (Q, X, \iota, \delta, \tau)$. *Where* $\iota : Q \to 2^L$ *and* $\tau : Q \to 2^L$ *as follows:*

$$\iota(q_0) = \{1\},$$
$$\tau(s_1) = \ldots = \tau(s_8) = \{1\}.$$

$\delta^* : Q \times X^* \times Q \to 2^L$, *which is the opinion of experts, is as follows: (Instead of transition* $\delta^*(q, u, p)$, *we use* u.*)*

f_1	{0.2, 0.8, 0.9}
f_2	{0.4, 1, 0.5}
l_1	{0.5, 0.5, 0.9}
l_2	{0.8, 0.7, 0.5}
e_1	{0, 0.8, 0.9}
m_1	{0.4, 0.5, 0.6}
m_2	{0.8, 0.7, 0.9}

Therefore, the value of each path based on the hesitant L-fuzzy behavior defined in Definition 2.7 is as follows:

$word \;\; x_1x_2\dots x_n$	$\beta_{\mathcal{A}}(x_1x_2\dots x_n)$
$f_1l_1e_1m_1$	$\{0, 0.2, 0.4, 0.5, 0.6\}$
$f_1l_1e_1m_2$	$\{0, 0.2, 0.5, 0.7, 0.8, 0.9\}$
$f_1l_2e_1m_1$	$\{0, 0.2, 0.4, 0.5, 0.6\}$
$f_1l_2e_1m_2$	$\{0, 0.2, 0.5, 0.7, 0.8\}$
$f_2l_1e_1m_1$	$\{0, 0.4, 0.5, 0.6\}$
$f_2l_1e_1m_2$	$\{0, 0.4, 0.5, 0.7, 0.8, 0.9\}$
$f_2l_2e_1m_1$	$\{0, 0.4, 0.5, 0.6\}$
$f_2l_2e_1m_2$	*{0, 0.4, 0.5, 0.7, 0.8}*

We used the operations of the hesitant L-fuzzy automaton to prevent the loss of expert information and the involvement of all experts' opinions.
Incorporating hesitant L-fuzzy automaton into the evaluation of road accident risks marks a pioneering method that could substantially improve the assessment of hazards associated with road accidents. The hesitant L-fuzzy automaton offers a robust mechanism for depicting and managing ambiguous information with greater flexibility and detail than conventional approaches. Utilizing this advanced methodology in the analysis of road accident hazards enables both researchers and practitioners to more accurately address the complexities and uncertainties typical of road accident scenarios.

In Example 3.2, we demonstrated the use of HLFA's superiority over FA in solving real-world problems. Therefore, reduced and minimal of an HLFA is of particular importance.

In the following section, for an accessible complete deterministic HLFA $\mathcal{A} = (Q, X, \iota, \delta, \tau)$, we define an equivalence relation ρ on Q, and using it, we define a RHLFA. Then we clarify it by an example. Moreover, we denote a set of equivalence classes by $\frac{Q}{\rho}$ and by means of it, we define the HLFA $\frac{\mathcal{A}}{\rho} = (\frac{Q}{\rho}, X, \delta_\rho, \iota_\rho, \tau_\rho)$ whose language is equal to the language $\mathcal{A}$, that is, $\mathcal{HL}_{\frac{\mathcal{A}}{\rho}} = \mathcal{HL}_{\mathcal{A}}$. In last, we show that $\frac{\mathcal{A}}{\rho}$ is a RHLFA, and we clarify it by an example.

In this paper, we assume that L is a distributive lattice such that for every $A, B \in 2^L$, if $A \neq 0_L, B \neq 0_L$, then $A \sqcap B \neq 0_L$.

Remark 3.3. *All the chains and the lattice in the Example 3.1, which is not chain, but has the above property (For every $A, B \in 2^L$, if $A \neq 0_L, B \neq 0_L$, then $A \sqcap B \neq 0_L$). Below we show that some lattices do not meet the above condition. Consider the lattice of Figure 4,*

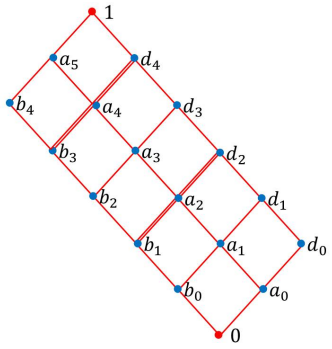


Figure 4: The distributive lattice L

then we have $\{a_0, d_0\} \neq 0_L$, $\{b_3, b_1\} \neq 0_L$, $\{a_0, d_0\} \sqcap \{b_3, b_1\} = 0_L$.

Definition 3.4. *Let $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ be an accessible complete deterministic HLFA. We define the relation ρ on Q by $q_1 \rho q_2$ if and only if the following two sets are equal:*
$\{w \in X^* | \delta^*(q_1, w, p) \sqcap \tau(p) \neq 0_L$, for some $p \in Q\} = \{w \in X^* | \delta^*(q_2, w, p) \sqcap \tau(p) \neq 0_L$, for some $p \in Q\}$.

Definition 3.5. *Let $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ be an HLFA. Then we say $\mathcal{A}$ is reduced if $q_i \rho q_j$ implies that $q_i = q_j$, for every $q_i, q_j \in Q$.*

Example 3.6. *The HLFA of Example 3.1 is not reduced. Because $\{w \in X^* | \delta^*(q_1, w, p) \sqcap \tau(p) \neq 0_L$, for some $p \in Q\} = b^* = \{w \in X^* | \delta^*(q_2, w, p) \sqcap \tau(p) \neq 0_L$, for some $p \in Q\}$, so $q_1 \rho q_2$ while $q_1 \neq q_2$. And $\{w \in X^* | \delta^*(q_3, w, p) \sqcap \tau(p) \neq 0_L$, for some $p \in Q\} = \varnothing = \{w \in X^* | \delta^*(q_4, w, p) \sqcap \tau(p) \neq 0_L$, for some $p \in Q\}$, so $q_3 \rho q_4$ while $q_3 \neq q_4$. Consequently, we can say $\mathcal{A}$ is not reduced.*

Example 3.7. *Let $Q = \{q_0, q_1, q_2\}$, let $X = \{a, b\}$, and let $L = [0, 1]$. Define $\iota : Q \to 2^L$ and $\delta^* : Q \times X^* \times Q \to 2^L$, $\tau : Q \to 2^L$ as follows:*

$$\iota(q_0) = \{0, \tfrac{1}{2}, [\tfrac{1}{4}, \tfrac{2}{3}]\},$$

$$\begin{aligned}
\delta(q_0, a, q_1) &= \{0, \tfrac{1}{3}, [\tfrac{1}{3}, \tfrac{2}{3}]\}, & \delta(q_0, b, q_2) &= \{\tfrac{1}{3}, \tfrac{1}{2}\},\\
\delta(q_1, a, q_1) &= \{\tfrac{1}{3}, \tfrac{1}{2}\}, & \delta(q_1, b, q_1) &= \{\tfrac{1}{4}, \tfrac{1}{3}\},\\
\delta(q_2, a, q_2) &= \{\tfrac{1}{4}, \tfrac{1}{2}\}, & \delta(q_2, b, q_2) &= \{0, \tfrac{1}{3}\},\\
\tau(q_1) &= \{\tfrac{1}{3}\}, & \tau(q_2) &= \{\tfrac{1}{2}, \tfrac{1}{4}\}.
\end{aligned}$$

It is clear that $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ is an accessible complete deterministic HLFA. The state diagram HLFA $\mathcal{A}$ is Figure 5 and the language HLFA $\mathcal{A}$ is the set $aX^* \cup bX^*$.

As is clear the HLFA $\mathcal{A}$ is not reduced. Because $\{w \in X^* | \delta^*(q_1, w, p) \sqcap \tau(p) \neq 0_L$, for some $p \in Q\} = X^* = \{w \in X^* | \delta^*(q_2, w, p) \sqcap \tau(p) \neq 0_L$, for some $p \in Q\}$, so $q_1 \rho q_2$ while $q_1 \neq q_2$. Consequently, we can say $\mathcal{A}$ is not reduced.

In the following, we will explain the method of reduction of an HLFA.

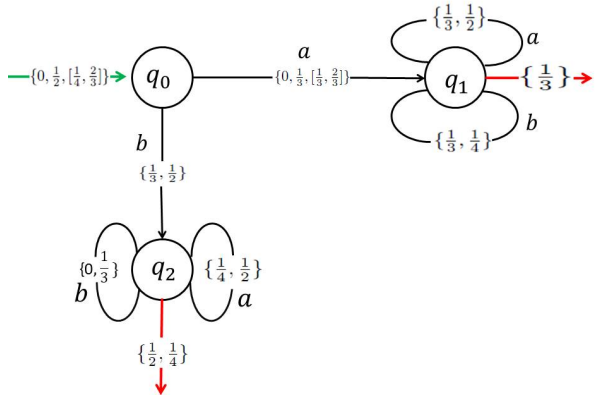


Figure 5: The state diagram HLFA $\mathcal{A}$ of Example 3.7

Definition 3.8. *Let $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ be an accessible complete deterministic HLFA. Suppose $q_i \in Q$ and suppose that ρ is the equivalence relation defined in Definition 3.4 and for any $q_i \in Q, q_i\rho$ denotes the equivalence class by q_i under ρ and suppose $a \in X$. Then, we define $\frac{\mathcal{A}}{\rho} = (\frac{Q}{\rho}, X, \delta_\rho, \iota_\rho, \tau_\rho)$, where $\frac{Q}{\rho} = \{q_i\rho | q_i \in Q\}$,*
$\iota_\rho : \frac{Q}{\rho} \to 2^L$ is defined by

$$\iota_\rho(q_0\rho) = \sqcup_{p \in q_0\rho} \iota(p),$$

$\delta_\rho : \frac{Q}{\rho} \times X \times \frac{Q}{\rho} \to 2^L$ is defined by

$$\delta_\rho(q_i\rho, a, q_j\rho) = \begin{cases} \sqcup_{q\rho q_i, p\rho q_j} \delta(q, a, p) & \text{if } \sqcup_{p\rho q_j} \delta(q_i, a, p) \neq 0_L, \\ 0_L & \text{otherwise,} \end{cases}$$

and $\tau_\rho : \frac{Q}{\rho} \to 2^L$ is defined by

$$\tau_\rho(q\rho) = \sqcup_{p \in q\rho} \tau(p),$$

for $q, p, q_i, q_j \in Q$.

Theorem 3.9. *Let $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ be an accessible complete deterministic HLFA. Then $\frac{\mathcal{A}}{\rho} = (\frac{Q}{\rho}, X, \delta_\rho, \iota_\rho, \tau_\rho)$ defined in Definition 3.8 is an HLFA.*

Proof. Since Q is finite, $\frac{Q}{\rho}$ is finite. Now, it suffices to show that δ_ρ, ι_ρ, and τ_ρ are well defined. First, we show that ι_ρ is well defined. For this purpose, we let $q_0\rho = q_0'\rho$, where q_0 is initial states of $\mathcal{A}$ and

$$\iota_\rho(q_0\rho) = \sqcup_{p \in q_0\rho} \iota(p) = \sqcup_{p \in q_0'\rho} \iota(p) = \iota_\rho(q_0'\rho).$$

Therefore $\iota_\rho(q_0\rho) = \iota_\rho(q_0'\rho)$. Now, we show that δ_ρ is well defined. Let $q_1\rho, q_2\rho, p_1\rho, p_2\rho \in \frac{Q}{\rho}$ and let $a, b \in X$. Let $a = b$, let $q_1\rho = p_1\rho$, and let $q_2\rho = p_2\rho$. Then $q_1\rho p_1$ and $q_2\rho p_2$. Therefore $A = \{w \in X^* | \delta^*(q_1, w, p) \sqcap \tau(p) \neq 0_L$, for some $p \in Q\} = \{w \in X^* | \delta^*(p_1, w, p) \sqcap \tau(p) \neq 0_L$, for some $p \in Q\} = B$. Let $\delta_\rho(q_1\rho, a, q_2\rho) \neq 0_L$. By the definition of $\delta_\rho(q_1\rho, a, q_2\rho)$, there exists $q' \in Q$ such that $q'\rho q_2$ and $\delta(q_1, a, q') \neq 0_L$. Because $\mathcal{A}$ is deterministic, q' is unique. Since $A = B$, we have $w \in A$ if and only if $w \in B$, for every $w \in X^*$. This holds in particular for every $w = at \in X^*$, that is $\delta^*(q_1, at, p) \sqcap \tau(p) \neq 0_L$ for some $p \in Q$, if and only if, $\delta^*(p_1, at, p) \sqcap \tau(p) \neq 0_L$ for some $p \in Q$. Since $\mathcal{A}$ is complete and deterministic for unique $p', q' \in Q$, we have $\delta(q_1, a, q') \sqcap \delta^*(q', t, p) \sqcap \tau(p) \neq 0_L$, for some $p \in Q$ if and only if $\delta(p_1, a, p') \sqcap \delta^*(p', t, p) \sqcap \tau(p) \neq 0_L$, for some $p \in Q$. Then , we have $\delta(q_1, a, q') \neq 0_L$ and $\delta(p_1, a, p') \neq 0_L$, so $\delta^*(q', t, p) \sqcap \tau(p) \neq 0_L$, for some $p \in Q$ if and only if $\delta^*(p', t, p) \sqcap \tau(p) \neq 0_L$, for some $p \in Q$, and that means that $\{t \in X^* | \delta^*(q', t, p) \sqcap \tau(p) \neq 0_L$, for some $p \in Q\} = \{t \in X^* | \delta^*(p', t, p) \sqcap \tau(p) \neq 0_L$, for some $p \in Q\}$. So, $p'\rho q'$, and according to the above discussions, $q'\rho q_2$ and $q_2\rho p_2$. Therefore $p'\rho p_2$, and since $\delta(p_1, a, p') \neq 0_L$ and δ is well defined, we have $\delta(p_1, b, p') = \delta(p_1, a, p') \neq 0_L$. So $\delta_\rho(p_1\rho, b, p_2\rho) \neq 0_L$. Hence $\delta_\rho(q_1\rho, a, q_2\rho) = \sqcup_{q\rho q_1, p\rho q_2} \delta(q, a, p) =$

$\sqcup_{q\rho p_1, p\rho p_2}\delta(q,a,p) = \sqcup_{q\rho p_1, p\rho p_2}\delta(q,b,p) = \delta_\rho(p_1\rho, b, p_2\rho)$. Consequently δ_ρ is well defined. Also we show that τ_ρ is well defined. For this purpose, we let $q_1\rho = q_2\rho$, so

$$\tau_\rho(q_1\rho) = \sqcup_{p\in q_1\rho}\tau(p) = \sqcup_{p\in q_2\rho}\tau(p) = \tau_\rho(q_2\rho).$$

Hence τ_ρ is well defined. Consequently, $\frac{\mathcal{A}}{\rho} = (\frac{Q}{\rho}, X, \delta_\rho, \iota_\rho, \tau_\rho)$ is an HLFA. It is clear that the HLFA $\frac{\mathcal{A}}{\rho}$ is also an accessible complete deterministic HLFA. □

Theorem 3.10. *Suppose that $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ is an accessible complete deterministic HLFA and that $\frac{\mathcal{A}}{\rho} = (\frac{Q}{\rho}, X, \delta_\rho, \iota_\rho, \tau_\rho)$ is defined in Definition 3.8. Then for every $w \in X^*$, $\beta_{\mathcal{A}}(w) \neq 0_L$ if and only if $\beta_{\frac{\mathcal{A}}{\rho}}(w) \neq 0_L$.*

Proof. Let $w \in X^*$ and $\beta_{\frac{\mathcal{A}}{\rho}}(w) \neq \{0\}$. Then there exist $p\rho, q\rho \in \frac{Q}{\rho}$ such that $\iota_\rho(p\rho) \sqcap \delta^*_\rho(p\rho, w, q\rho) \sqcap \tau_\rho(q\rho) \neq 0_L$. Therefore $\iota_\rho(p\rho) \neq 0_L$, $\delta^*_\rho(p\rho, w, q\rho) \neq 0_L$ and $\tau_\rho(q\rho) \neq 0_L$. Since $\delta^*_\rho(p\rho, w, q\rho) \neq 0_L$, then there exists $q' \in Q$ such that $q'\rho q$ and $\delta^*(p, w, q') \neq 0_L$. Since $\tau_\rho(q\rho) \neq 0_L$, there exists $r \in Q$ such that $r\rho q$ and $\tau(r) \neq 0_L$. Because $r\rho q$ and $q\rho q'$, we have $\tau(q') \neq 0_L$. So $\delta^*(p, w, q') \sqcap \tau(q') \neq 0_L$. Now, since $\iota_\rho(p\rho) \neq 0_L$, by Definition 3.8, we have $p\rho q_0$. So by Definition 3.4 and also $\delta^*(p, w, q') \sqcap \tau(q') \neq 0_L$, there exists $q'' \in Q$ such that $\delta^*(q_0, w, q'') \sqcap \tau(q'') \neq 0_L$. Therefore $\iota(q_0) \sqcap \delta^*(q_0, w, q'') \sqcap \tau(q'') \neq 0_L$. Hence $\beta_{\mathcal{A}}(w) \neq \{0\}$.

Now, let $\beta_{\mathcal{A}}(w) \neq \{0\}$. Then there exist $p, q \in Q$ such that $\iota(q) \sqcap \delta^*(q, w, p) \sqcap \tau(p) \neq 0_L$, so $\iota(q) \neq 0_L$, $\delta^*(q, w, p) \neq 0_L$, and $\tau(p) \neq 0_L$. Therefore $\iota_\rho(q\rho) \neq 0_L$, $\delta^*_\rho(q\rho, w, p\rho) \neq 0_L$, and $\tau_\rho(p\rho) \neq 0_L$. Hence $\beta_{\frac{\mathcal{A}}{\rho}}(w) \neq \{0\}$. □

Corollary 3.11. *Suppose that $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ is an accessible complete deterministic HLFA and that $\frac{\mathcal{A}}{\rho} = (\frac{Q}{\rho}, X, \delta_\rho, \iota_\rho, \tau_\rho)$ is defined in Definition 3.8. Then $\mathcal{HL}_{\frac{\mathcal{A}}{\rho}} = \mathcal{HL}_{\mathcal{A}}$.*

Proof. The proof is clear by Definition 2.8 and Theorem 3.10. □

Theorem 3.12. *Let $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ be an accessible complete deterministic HLFA. Then $\frac{\mathcal{A}}{\rho} = (\frac{Q}{\rho}, X, \delta_\rho, \iota_\rho, \tau_\rho)$ defined in Definition 3.8 is reduced.*

Proof. Consider the equivalence relation ρ' on $\frac{Q}{\rho}$ similar to the equivalence relation ρ on Q. Let $q_1\rho\rho' q_2\rho$, where $q_1, q_2 \in Q$. We have to show that $q_1\rho = q_2\rho$, for this it is enough to show that $q_1\rho q_2$, that is $A = \{w \in X^* | \delta^*(q_1, w, p) \sqcap \tau(p) \neq 0_L$, for some $p \in Q\} = \{w \in X^* | \delta^*(q_2, w, p) \sqcap \tau(p) \neq 0_L$, for some $p \in Q\} = B$. Let $w \in A$. Then there exists $p_1 \in Q$ such that $\delta^*(q_1, w, p_1) \sqcap \tau(p_1) \neq 0_L$. So $\delta^*(q_1, w, p_1) \neq 0_L$ and $\tau(p_1) \neq 0_L$. Therefore $\delta^*_\rho(q_1\rho, w, p_1\rho) \neq 0_L$ and $\tau_\rho(p_1\rho) \neq 0_L$. Then $\delta^*_\rho(q_1\rho, w, p_1\rho) \sqcap \tau_\rho(p_1\rho) \neq 0_L$. Since $q_1\rho\rho' q_2\rho$, by Definition 3.8, we have $\delta^*_\rho(q_2\rho, w, p_2\rho) \sqcap \tau_\rho(p_2\rho) \neq 0_L$ for some $p_2\rho \in \frac{Q}{\rho}$. Then $\delta^*_\rho(q_2\rho, w, p_2\rho) \neq 0_L$ and $\tau_\rho(p_2\rho) \neq 0_L$. So, there exists $p' \in Q$ such that $p'\rho p_2$ and $\delta(q_2, w, p') \neq 0_L$. Also, we have $\tau_\rho(p_2\rho) \neq 0_L$. Then there exists $r \in Q$ such that $\tau(r) \neq 0_L$, where $r\rho p_2$. Since $r\rho p_2$ and $p_2\rho p'$, then $\tau(p') \neq 0_L$. So, $\delta(q_2, w, p') \sqcap \tau(p') \neq 0_L$ and that means that $w \in B$. Consequently $A \subseteq B$. We show in a similar way that $B \subseteq A$. Hence $A = B$. Consequently $q_1\rho q_2$. Therefore $\frac{\mathcal{A}}{\rho} = (\frac{Q}{\rho}, X, \delta_\rho, \iota_\rho, \tau_\rho)$ is reduced. □

Remark 3.13. *Let $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ be an accessible complete deterministic HLFA. If $p\rho q$, then $\tau(p) \neq \{0\}$ if and only if $\tau(q) \neq \{0\}$.*

Corollary 3.14. *For each HLFA $\mathcal{A}$ there exists a RHLFA $\mathcal{B}$ such that $\mathcal{HL}_{\mathcal{A}} = \mathcal{HL}_{\mathcal{B}}$.*

Proof. By Note 2.13, for every HLFA $\mathcal{A}$, there exists an accessible complete deterministic HLFA $\mathcal{A}'$ such that $\mathcal{HL}_{\mathcal{A}} = \mathcal{HL}_{\mathcal{A}'}$, and by Corollary 3.11, for every accessible complete deterministic HLFA $\mathcal{A}'$, there exists a RHLFA $\frac{\mathcal{A}'}{\rho}$ such that $\mathcal{HL}_{\mathcal{A}'} = \mathcal{HL}_{\frac{\mathcal{A}'}{\rho}}$, so we can say that for every HLFA $\mathcal{A}$, there exists a RHLFA $\frac{\mathcal{A}'}{\rho}$ such that $\mathcal{HL}_{\mathcal{A}} = \mathcal{HL}_{\frac{\mathcal{A}'}{\rho}}$. Therefore, it is enough that we select RHLFA $\mathcal{B}$ equal to RHLFA $\frac{\mathcal{A}'}{\rho}$. □

Example 3.15. *Let $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ be the HLFA defined in Example 3.1. Then we calculate the RHLFA $\frac{\mathcal{A}}{\rho}$ by Definition 3.8. Since in Example 3.1, $q_1\rho q_2$ and $q_3\rho q_4$, we have $q_1\rho = q_2\rho$ and $q_3\rho = q_4\rho$. We set $q_1\rho = q_2\rho = q\rho$ and*

$q_3\rho = q_4\rho = q'\rho$. *So we have* $\frac{Q}{\rho} = \{q_0\rho, q\rho, q'\rho\}$ *and we have*

$$\begin{aligned}
\iota_\rho(q_0\rho) &= \iota(q_0) = \{\gamma_1, \gamma_2, \gamma_3\},\\
\delta_\rho(q_0\rho, a, q\rho) &= \sqcup_{q\rho q_0, p\rho q}\delta(q, a, p) = \delta(q_0, a, q_1) \sqcup \delta(q_0, a, q_2) = \{a_0, a_1\} \sqcup 0_L = \{a_0, a_1\},\\
\delta_\rho(q_0\rho, b, q\rho) &= \sqcup_{q\rho q_0, p\rho q}\delta(q, b, p) = \delta(q_0, b, q_1) \sqcup \delta(q_0, b, q_2) = 0_L \sqcup \{b_0, b_2\} = \{b_0, b_2\},\\
\delta_\rho(q\rho, a, q'\rho) &= \sqcup_{p\rho q, p'\rho q'}\delta(p, a, p') = \delta(q_1, a, q_3) \sqcup \delta(q_1, a, q_4) \sqcup \delta(q_2, a, q_3) \sqcup \delta(q_2, a, q_4)\\
&= \{a_1, \gamma_1\} \sqcup 0_L \sqcup 0_L \sqcup \{a_2, \gamma_1\} = \{\gamma_1, a_1, a_2\},\\
\delta_\rho(q\rho, b, q\rho) &= \sqcup_{p\rho q, p'\rho q}\delta(p, b, p') = \delta(q_1, b, q_1) \sqcup \delta(q_1, b, q_2) \sqcup \delta(q_2, b, q_1) \sqcup \delta(q_2, b, q_2)\\
&= \{b_1, b_2\} \sqcup 0_L \sqcup 0_L \sqcup \{b_2, \gamma_2\} = \{b_2\},\\
\delta_\rho(q'\rho, a, q'\rho) &= \sqcup_{p\rho q', p'\rho q'}\delta(p, a, p') = \delta(q_3, a, q_3) \sqcup \delta(q_3, a, q_4) \sqcup \delta(q_4, a, q_3) \sqcup \delta(q_4, a, q_4)\\
&= \{a_1, a_2\} \sqcup 0_L \sqcup 0_L \sqcup \{a_1\} = \{a_1, a_2\},\\
\delta_\rho(q'\rho, b, q'\rho) &= \sqcup_{p\rho q', p'\rho q'}\delta(p, b, p') = \delta(q_3, b, q_3) \sqcup \delta(q_3, b, q_4) \sqcup \delta(q_4, b, q_3) \sqcup \delta(q_4, b, q_4)\\
&= \{0, b_2\} \sqcup 0_L \sqcup 0_L \sqcup \{b_0\} = \{b_0, b_2\},\\
\tau_\rho(q\rho) &= \tau(q_1) \sqcup \tau(q_2) = \{\gamma_3\} \sqcup \{\gamma_1, \gamma_2, \gamma_3\} = \{\gamma_3\}.
\end{aligned}$$

The state diagram RHLFA $\frac{\mathcal{A}}{\rho}$ *is Figure 6. As is clear the language HLFA* $\frac{\mathcal{A}}{\rho}$ *is the set* $ab^* \cup b^+$ *and the HLFA* $\frac{\mathcal{A}}{\rho}$ *is*

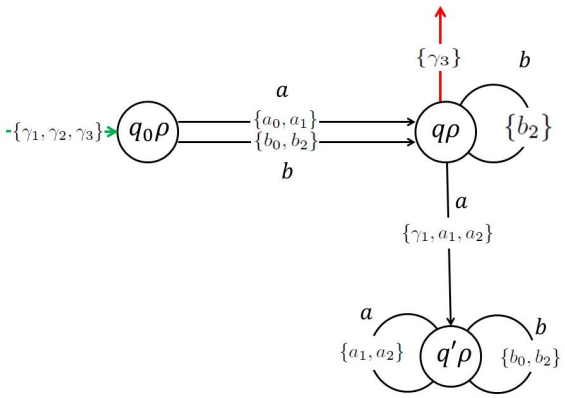


Figure 6: The state diagram HLFA $\frac{\mathcal{A}}{\rho}$ of Example 3.15

reduced.

Example 3.16. *Let* $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ *be the HLFA defined in Example 3.7. Then we calculate the RHLFA* $\frac{\mathcal{A}}{\rho}$ *by Definition 3.8. Since in Example 3.7,* $q_1\rho q_2$*, we have* $q_1\rho = q_2\rho$*. We set* $q_1\rho = q_2\rho = q\rho$*. So we have* $\frac{Q}{\rho} = \{q_0\rho, q\rho\}$ *and we have,*

$$\begin{aligned}
\iota_\rho(q_0\rho) &= \iota(q_0) = \{0, \tfrac{1}{2}, [\tfrac{1}{4}, \tfrac{2}{3}]\},\\
\delta_\rho(q_0\rho, a, q\rho) &= \sqcup_{q\rho q_0, p\rho q}\delta(q, a, p) = \delta(q_0, a, q_1) \sqcup \delta(q_0, a, q_2) = \{0, \tfrac{1}{3}, [\tfrac{1}{3}, \tfrac{2}{3}]\} \sqcup 0_L = \{0, \tfrac{1}{3}, [\tfrac{1}{3}, \tfrac{2}{3}]\},\\
\delta_\rho(q_0\rho, b, q\rho) &= \sqcup_{q\rho q_0, p\rho q}\delta(q, b, p) = \delta(q_0, b, q_1) \sqcup \delta(q_0, b, q_2) = 0_L \sqcup \{\tfrac{1}{3}, \tfrac{1}{2}\} = \{\tfrac{1}{3}, \tfrac{1}{2}\},\\
\delta_\rho(q\rho, a, q\rho) &= \sqcup_{p\rho q, p'\rho q}\delta(p, a, p') = \delta(q_1, a, q_1) \sqcup \delta(q_1, a, q_2) \sqcup \delta(q_2, a, q_1) \sqcup \delta(q_2, a, q_2)\\
&= \{\tfrac{1}{3}, \tfrac{1}{2}\} \sqcup 0_L \sqcup 0_L \sqcup \{\tfrac{1}{4}, \tfrac{1}{2}\} = \{\tfrac{1}{3}, \tfrac{1}{2}\},\\
\delta_\rho(q\rho, b, q\rho) &= \sqcup_{p\rho q, p'\rho q}\delta(p, b, p') = \delta(q_1, b, q_1) \sqcup \delta(q_1, b, q_2) \sqcup \delta(q_2, b, q_1) \sqcup \delta(q_2, b, q_2)\\
&= \{\tfrac{1}{3}, \tfrac{1}{4}\} \sqcup 0_L \sqcup 0_L \sqcup \{0, \tfrac{1}{3}\} = \{\tfrac{1}{4}, \tfrac{1}{3}\},\\
\tau_\rho(q\rho) &= \tau(q_1) \sqcup \tau(q_2) = \{\tfrac{1}{3}\} \sqcup \{\tfrac{1}{2}, \tfrac{1}{4}\} = \{\tfrac{1}{3}, \tfrac{1}{2}\}.
\end{aligned}$$

The state diagram HLFA $\frac{\mathcal{A}}{\rho}$ *is Figure 7. As is clear the language HLFA* $\frac{\mathcal{A}}{\rho}$ *is the set* $aX^* \cup bX^*$ *and the HLFA* $\frac{\mathcal{A}}{\rho}$ *is reduced.*

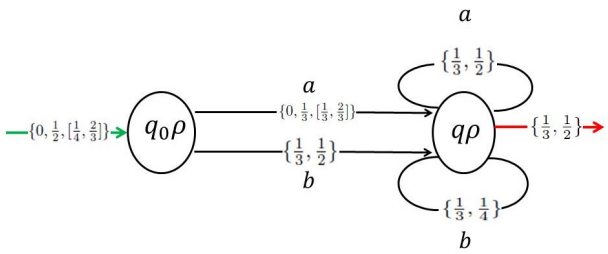


Figure 7: The state diagram HLFA $\frac{\mathcal{A}}{\rho}$ of Example 3.16

With the method described in [28], an HLFA can be transformed into an accessible complete deterministic HLFA so that their languages are equal. Using the method described in Definition 3.8, we turned an accessible complete deterministic HLFA into a RHLFA by reducing the number of states so that their languages are also equal. In the following, after defining the regular subset $\mathcal{L}$ of monoid X^*, we introduce the HLFA $\mathcal{A}_m$ with the following properties:

1. $\mathcal{HL}_{\mathcal{A}_m} = \mathcal{L}$,
2. For every complete deterministic HLFA $\mathcal{A}$ with $\mathcal{HL}_{\mathcal{A}} = \mathcal{L}$, the cardinality of the set of states $\mathcal{A}_m$ is less than or equal to the cardinality of the set of states $\mathcal{A}$, that is $\mathcal{A}_m$ is minimal.
3. $\mathcal{A}_m$ is a RHLFA.

We call this HLFA, strong admissible reduction of the HLFA $\mathcal{A}$ for regular subset $\mathcal{L}$ from monoid X^*. And we will state under what conditions the cardinality of the set of states of the RHLFA $\frac{\mathcal{A}}{\rho}$ defined in Definition 3.8 whose language is $\mathcal{L}$ is equal to the cardinality of the set of states of the HLFA $\mathcal{A}_m$ whose language is $\mathcal{L}$ and thus $\frac{\mathcal{A}}{\rho}$ is also a strong admissible reduction of the HLFA $\mathcal{A}$ for regular subset $\mathcal{L}$ from monoid X^* under the stated conditions.

For this purpose, we introduce an equivalence relation $R_{\mathcal{L}}$ on X^*. Then we consider its set of equivalence classes as $\frac{X^*}{R_{\mathcal{L}}}$ and define **an** HLFA $\mathcal{A}_m = (\frac{X^*}{R_{\mathcal{L}}}, X, \iota_m, \delta_m, \tau_m)$. Using it, then we show $\mathcal{HL}_{\mathcal{A}_m} = \mathcal{L}$. To show that $\mathcal{A}_m$ is minimal, we consider an arbitrary complete deterministic HLFA $\mathcal{A} = (Q, X, \iota, \delta, \tau)$, whose language is $\mathcal{L}$, that is, $\mathcal{HL}_{\mathcal{A}} = \mathcal{L}$. Then, by introducing an equivalence relation $R_{\mathcal{F}}$ on X^*, we show that the cardinality of $\frac{X^*}{R_{\mathcal{L}}}$ is less than or equal to the cardinality of $\frac{X^*}{R_{\mathcal{F}}}$, and considering that the cardinality of $\frac{X^*}{R_{\mathcal{F}}}$ is less than or equal to the cardinality of Q, we will conclude that the cardinality of $\frac{X^*}{R_{\mathcal{L}}}$ will be less than the cardinality of Q, that is, $|\frac{X^*}{R_{\mathcal{L}}}| \leq |\frac{X^*}{R_{\mathcal{F}}}| \leq |Q|$. Next, we show that $\mathcal{A}_m$ is reduced, then we show with an example that the cardinality of the set of states of the HLFA $\mathcal{A}_m$ is less than the cardinality of the set of states of the RHLFA $\frac{\mathcal{A}}{\rho}$ defined in Definition 3.8, whose language is $\mathcal{L}$. Finally, we show by defining the co-accessible state that if all states of an HLFA $\frac{\mathcal{A}}{\rho}$ be co-accessible, then $\frac{\mathcal{A}}{\rho}$, also, is a strong admissible reduction of the HLFA $\mathcal{A}$ for regular subset $\mathcal{L}$ from monoid X^*.

Definition 3.17. *Let $\mathcal{L}$ be a subset of monoid X^*. Then $\mathcal{L}$ is called regular, if there exists an HLFA $\mathcal{A}$ such that $\mathcal{HL}_{\mathcal{A}} = \mathcal{L}$.*

Definition 3.18. *Let $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ and $\mathcal{A}' = (Q', X, \iota', \delta', \tau')$ be two HLFAs. We say that $|\mathcal{A}| \leq |\mathcal{A}'|$ if and only if $|Q| \leq |Q'|$.*

Definition 3.19. *Let $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ be an HLFA. We call $\mathcal{A}$ the strong admissible reduction of an HLFA $\mathcal{B}$ for regular subset $\mathcal{L}$ from X^*, if*

1. *$\mathcal{B}$ be an accessible complete deterministic HLFA such that $\mathcal{HL}_{\mathcal{A}} = \mathcal{HL}_{\mathcal{B}} = \mathcal{L}$,*
2. *for every complete deterministic HLFA $\mathcal{A}'$ with $\mathcal{HL}_{\mathcal{A}'} = \mathcal{L}$, the cardinality of the set of states $\mathcal{A}$ is less than or equal to the cardinality of the set of states $\mathcal{A}'$, i.e. $\mathcal{A}$ be minimal,*
3. *$\mathcal{A}$ be a RHLFA.*

Definition 3.20. *Suppose that X is a nonempty set and suppose $\mathcal{L}$ is a subset of X^*. We define the equivalence relation $R_{\mathcal{L}}$ on X^*, for any two strings u and v in X^*, $(uR_{\mathcal{L}}v)$ if and only if $(uz \in \mathcal{L}$ if and only if $vz \in \mathcal{L}$, for every $z \in X^*)$.*

Theorem 3.21. *Suppose that X is a nonempty set of symbols. Then for every complete deterministic HLFA $\mathcal{A} = (Q, X, \iota, \delta, \tau)$, there exists an equivalence relation R_F on X^* such that the number of classes of equivalence relation R_F is not more than the number of states of $\mathcal{A}$. That is, $|\frac{X^*}{R_{\mathcal{F}}}| \leq |Q|$.*

Proof. Suppose that X is a nonempty set of symbols and that $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ is a complete deterministic HLFA. We define the relation R_F on X^*, for any two strings u and v in X^*, (uR_Fv) if and only if (there exists $q \in Q$ such that $\delta^*(q_0, u, q) \neq 0_L$ if and only if $\delta^*(q_0, v, q) \neq 0_L$). The reflexivity and symmetric properties of the relation R_F are clear. We show that the relation R_F is transitive. For this purpose, we assume that

$$\begin{cases} uR_Fv \\ and \\ vR_Ft \end{cases} \text{Hence,} \begin{cases} \text{there exists } q \in Q \text{ such that } \delta^*(q_0, u, q) \neq 0_L \text{ if and only if } \delta^*(q_0, v, q) \neq 0_L, \\ and \\ \text{there exists } q' \in Q \text{ such that } \delta^*(q_0, v, q') \neq 0_L \text{ if and only if } \delta^*(q_0, t, q') \neq 0_L. \end{cases}$$

Since $\mathcal{A}$ is complete deterministic, there exists unique $q \in Q$ such that $\delta^*(q_0, v, q) \neq 0_L$, so $q = q'$. Therefore there exists $q \in Q$ such that $\delta^*(q_0, u, q) \neq 0_L$ if and only if $\delta^*(q_0, t, q) \neq 0_L$. Consequently, uR_Ft and R_F is an equivalence relation. It is clear that the number of classes of equivalence relation R_F is not more than the number of states of $\mathcal{A}$. That is, $|\frac{X^*}{R_{\mathcal{F}}}| \leq |Q|$. □

Corollary 3.22. *If a complete deterministic HLFA $\mathcal{A}$ be accessible, then the number of classes of equivalence relation R_F is equal the number of states of $\mathcal{A}$. That is, $|\frac{X^*}{R_{\mathcal{F}}}| = |Q|$.*

Proof. Since $\mathcal{A}$ is complete deterministic, for every $u \in X^*$, there exists a uniqe $q \in Q$ such that $\delta^*(q_0, u, q) \neq 0_L$ and from Definition R_F, for every $w \in X^*$ that $\delta^*(q_0, w, q) \neq 0_L$, we have $[u]_{R_F} = [w]_{R_F}$. Therefore, we can say that for every state $q \in Q$ that exists $u \in X^*$ such that $\delta^*(q_0, u, q) \neq 0_L$, we have an equivalence class. Now since the complete deterministic HLFA $\mathcal{A}$ is accessible, for every $q \in Q$ there exists one $u \in X^*$ such that $\delta^*(q_0, u, q) \neq 0_L$, therefore the number of classes of equivalence relation R_F is equal to the number of states of $\mathcal{A}$. □

Theorem 3.23. *Let $\mathcal{L}$ be a regular subset of monoid X^* and let $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ be a complete deterministic HLFA such that its language is $\mathcal{L}$, that is, $\mathcal{HL}_{\mathcal{A}} = \mathcal{L}$. Then for every equivalence class $[w]_{R_F}$ of R_F, we have an equivalence class $[w]_{R_{\mathcal{L}}}$ of $R_{\mathcal{L}}$ such that $[w]_{R_F} \subseteq [w]_{R_{\mathcal{L}}}$. Each equivalence class $[w]_{R_{\mathcal{L}}}$ of relation $R_{\mathcal{L}}$ is a finite union of equivalence classes of R_F. That is, $|\frac{X^*}{R_{\mathcal{L}}}| \leq |\frac{X^*}{R_{\mathcal{F}}}|$.*

Proof. Let $[w]_{R_F}$ be an equivalence class of R_F and let $u \in [w]_{R_F}$. Then uR_Fw. Since $\mathcal{A}$ is complete, for every $y \in X^*$, there exists $q \in Q$ such that $\delta^*(q_0, uy, q) \neq 0_L$. So $\sqcup_{r \in Q}[\delta^*(q_0, u, r) \sqcap \delta^*(r, y, q))] \neq 0_L$, then there exists $p \in Q$ such that $\delta^*(q_0, u, p) \neq 0_L$ and $\delta^*(p, y, q) \neq 0_L$. Now, since uR_Fw, we have $\delta^*(q_0, w, p) \neq 0_L$. So $\delta^*(q_0, w, p) \sqcap \delta^*(p, y, q) \neq 0_L$. Therefore $\delta^*(q_0, wy, q) \neq 0_L$. Now, we have the following cases:

1. $\tau(q) \neq 0_L$, then
$$\begin{cases} \delta^*(q_0, uy, q) \sqcap \tau(q) \neq 0_L, \\ \\ \delta^*(q_0, wy, q) \sqcap \tau(q) \neq 0_L. \end{cases}$$
Thus $uy, wy \in \mathcal{L}$.
2. $\tau(q) = 0_L$, then
$$\begin{cases} \delta^*(q_0, uy, q) \sqcap \tau(q) = 0_L, \\ \\ \delta^*(q_0, wy, q) \sqcap \tau(q) = 0_L. \end{cases}$$
Thus $uy, wy \notin \mathcal{L}$.

Consequently, $uy \in \mathcal{L}$ if and only if $wy \in \mathcal{L}$, for every $y \in X^*$. So $u \in [w]_{R_{\mathcal{L}}}$ and $[w]_{R_F} \subseteq [w]_{R_{\mathcal{L}}}$.
Now, let $[w]_{R_{\mathcal{L}}}$ be an equivalence class of $R_{\mathcal{L}}$. It is clear that if $w \notin \mathcal{L}$, then $u \notin \mathcal{L}$, for every $uR_{\mathcal{L}}w$. Also, if $w \in \mathcal{L}$, then $u \in \mathcal{L}$, for every $uR_{\mathcal{L}}w$. Now, let $w \notin \mathcal{L}$. Since $\mathcal{A}$ is complete and deterministic, then $\delta^*(q_0, w, q) \neq 0_L$ for unique $q \in Q$. So $\tau(q) = 0_L$. The set $S = \{q \in Q | \delta^*(q_0, u, q) \neq 0_L, u \in [w]_{R_{\mathcal{L}}}\}$ is finite, since Q is finite. Hence the equivalence class $[w]_{R_{\mathcal{L}}}$ of $R_{\mathcal{L}}$ is a finite union of the equivalence classes $[w]_{R_F}$ of R_F, where $q \in S$. □

Definition 3.24. *Suppose that $\mathcal{L}$ is a subset of monoid X^*, that $R_{\mathcal{L}}$ is the equivalence relation defined in Definition 3.20, and suppose for any $w \in X^*, [w]$ denotes the equivalence class by w under $R_{\mathcal{L}}$ and suppose $a \in X$ and suppose $A, B, C \in 2^L$, where $A \neq 0_L$, $B \neq 0_L$, $C \neq 0_L$. We define $\mathcal{A}_m = (\frac{X^*}{R_{\mathcal{L}}}, X, \iota_m, \delta_m, \tau_m)$, where $\iota_m : \frac{X^*}{R_{\mathcal{L}}} \to 2^L$ defined by*

$$\iota_m([w]) = \begin{cases} B & \text{if } w = \Lambda, \\ 0_L & \text{if } w \neq \Lambda, \end{cases}$$

$\delta_m : \frac{X^*}{R_{\mathcal{L}}} \times X \times \frac{X^*}{R_{\mathcal{L}}} \to 2^L$ defined by

$$\delta_m([w], a, [u]) = \begin{cases} A & \text{if } [wa] = [u] \neq \varnothing, \\ 0_L & \text{otherwise}, \end{cases}$$

and $\tau_m : \frac{X^*}{R_{\mathcal{L}}} \to 2^L$ defined by

$$\tau_m([w]) = \begin{cases} C & \text{if } w \in \mathcal{L}, \\ 0_L & \text{otherwise}. \end{cases}$$

The following theorem is one of the original theorems of this section because with the help of this theorem, we can find a minimal and reduced HLFA for the regular subset $\mathcal{L}$ of monoid X^*.

Theorem 3.25. *Let $\mathcal{L}$ be a regular subset of monoid X^* and let $\mathcal{B}$ be an accessible complete deterministic HLFA such that $\mathcal{HL}_{\mathcal{B}} = \mathcal{L}$. Then the HLFA $\mathcal{A}_m$ defined in Definition 3.24 is the strong admissible reduction of HLFA $\mathcal{B}$ for the regular subset $\mathcal{L}$ from X^*.*

Proof. First, we show that $\mathcal{A}_m$ is an HLFA. Then we show that $\mathcal{HL}_{\mathcal{A}m} = \mathcal{L}$, and $|\mathcal{A}_m| \leq |\mathcal{A}|$ for every complete deterministic HLFA $\mathcal{A}$ such that $\mathcal{HL}_{\mathcal{A}} = \mathcal{L}$. In last, we prove $\mathcal{A}_m$ is reduced.

Since $\mathcal{L}$ is a regular subset of X^*, there exists an HLFA $\mathcal{A}'$ such that $\mathcal{HL}_{\mathcal{A}'} = \mathcal{L}$. By Note 2.13, for HLFA $\mathcal{A}'$, there exists an accessible complete deterministic HLFA $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ such that $\mathcal{HL}_{\mathcal{A}} = \mathcal{HL}_{\mathcal{A}'} = \mathcal{L}$. Consider $\frac{X^*}{R_{\mathcal{L}}}$ the set of equivalence classes of $R_{\mathcal{L}}$. According to Theorem 3.23, equivalence classes of $R_{\mathcal{L}}$ are finite union of equivalence classes of R_F, and by Theorem 3.21, the number of equivalence classes of R_F is not more than the number of states of $\mathcal{A}$, so the number of equivalence classes of $R_{\mathcal{L}}$ are finite. That is, the set $\frac{X^*}{R_{\mathcal{L}}}$ is finite. It is easy to see that $\iota_m, \delta_m, \tau_m$ are well defined. Consequently, $\mathcal{A}_m$ is an HLFA.

Now, we show $\mathcal{HL}_{\mathcal{A}m} = \mathcal{L}$ and we show that $\mathcal{A}_m$ is minimal. Let $w \in \mathcal{L}$. Then $\iota_m([\Lambda]) \neq 0_L$, $\delta_m^*([\Lambda], w, [w]) \neq 0_L$, and $\tau_m([w]) \neq 0_L$. Therefore $\iota_m([\Lambda]) \sqcap \delta_m^*([\Lambda], w, [w]) \sqcap \tau_m([w]) \neq 0_L$, so $w \in \mathcal{HL}_{\mathcal{A}_m}$.
Let $w \in \mathcal{HL}_{\mathcal{A}_m}$. Then there exists $[u] \in \frac{X^*}{R_{\mathcal{L}}}$ such that $\iota_m([\Lambda]) \sqcap \delta_m^*([\Lambda], w, [u]) \sqcap \tau_m([u]) \neq 0_L$, so $\delta_m^*([\Lambda], w, [u]) \neq 0_L$ and $\tau_m([u]) \neq 0_L$. Since $\delta_m^*([\Lambda], w, [u]) \neq 0_L$, we have $[\Lambda w] = [u]$. So $[w] = [u]$, hence $\tau_m([w]) = \tau_m([u]) \neq 0_L$. Since $\tau_m([w]) \neq 0_L$, we have $w \in \mathcal{L}$. Hence $\mathcal{HL}_{\mathcal{A}_m} = \mathcal{L}$.
To show that $\mathcal{A}_m$ is minimal, we assume that $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ is an arbitrary complete deterministic HLFA such that $\mathcal{HL}_{\mathcal{A}} = \mathcal{L}$. By considering Theorem 3.23, the number of equivalence classes of $R_{\mathcal{L}}$ is not more than the number equivalence classes of R_F. Also, by Theorem 3.21, the number of equivalence classes of R_F is not more than the number of states of the HLFA $\mathcal{A}$, that is, $|\frac{X^*}{R_{\mathcal{L}}}| \leq |\frac{X^*}{R_{\mathcal{F}}}| \leq |Q|$. So $|\mathcal{A}_m| \leq |\mathcal{A}|$.

Now, it suffices to show that $\mathcal{A}_m$ is reduced. Consider the equivalence relation ρ' on $\frac{X^*}{R_{\mathcal{L}}}$ similar to the relation ρ on Q. Let $[w_1]_{R_{\mathcal{L}}} \rho' [w_2]_{R_{\mathcal{L}}}$, where $[w_1]_{R_{\mathcal{L}}}, [w_2]_{R_{\mathcal{L}}} \in \frac{X^*}{R_{\mathcal{L}}}$. We have to show that $[w_1]_{R_{\mathcal{L}}} = [w_2]_{R_{\mathcal{L}}}$. We suppose that $[w_1]_{R_{\mathcal{L}}} \rho' [w_2]_{R_{\mathcal{L}}}$. Therefore
$A = \{w \in X^* | \delta_m^*([w_1], w, [u]) \sqcap \tau_m([u]) \neq 0_L$, for some $[u] \in \frac{X^*}{R_{\mathcal{L}}}\} = \{w \in X^* | \delta_m^*([w_2], w, [u]) \sqcap \tau_m([u]) \neq 0_L$, for some $[u] \in \frac{X^*}{R_{\mathcal{L}}}\} = B$.

So $w \in A$ if and only if $w \in B$, for every $w \in X^*$. This holds in particular for $\Lambda \in X^*$, that is, $\Lambda \in A$ if and only if $\Lambda \in B$. Then there exists $v_1 \in X^*$ such that $\delta_m^*([w_1], \Lambda, [v_1]) \sqcap \tau_m([v_1]) \neq 0_L$ if and only if there exists $v_2 \in X^*$ such that $\delta_m^*([w_2], \Lambda, [v_2]) \sqcap \tau_m([v_2]) \neq 0_L$. Therefor there exists $v_1 \in X^*$ such that

$$\begin{cases} \delta_m^*([w_1], \Lambda, [v_1]) \neq 0_L, \\ \tau_m([v_1]) \neq 0_L, \end{cases}$$

if and only if there exists $v_2 \in X^*$ such that

$$\begin{cases} \delta_m^*([w_2], \Lambda, [v_2]) \neq 0_L, \\ \tau_m([v_2]) \neq 0_L. \end{cases}$$

So by the definitions of δ_m^* and τ_m, from Theorem 3.25, there exists $v_1 \in X^*$ such that

$$\begin{cases} [w_1]_{R_\mathcal{L}} = [v_1]_{R_\mathcal{L}}, \\ v_1 \in \mathcal{L}, \end{cases}$$

if and only if there exists $v_2 \in X^*$ such that

$$\begin{cases} [w_2]_{R_\mathcal{L}} = [v_2]_{R_\mathcal{L}}, \\ v_2 \in \mathcal{L}. \end{cases}$$

Then there exists $v_1 \in X^*$ such that

$$\begin{cases} w_1 R_\mathcal{L} v_1, \\ v_1 \in \mathcal{L}, \end{cases}$$

if and only if there exists $v_2 \in X^*$ such that

$$\begin{cases} w_2 R_\mathcal{L} v_2, \\ v_2 \in \mathcal{L}. \end{cases}$$

Since $v_1 \in \mathcal{L}$ and $v_2 \in \mathcal{L}$, therefore for every $z \in X^*$, $w_1 z, v_1 z \in \mathcal{L}$ if and only if $w_2 z, v_2 z \in \mathcal{L}$. Therefore $w_1 z, w_2 z \in \mathcal{L}$ or $w_1 z, w_2 z \notin \mathcal{L}$. Then $w_1 R_\mathcal{L} w_2$. Hence $[w_1]_{R_\mathcal{L}} = [w_2]_{R_\mathcal{L}}$. □

Example 3.26. *Consider regular subset $\mathcal{L} = ab^* \cup b^+$ of X^*, where $X = \{a, b\}$. We use the method described in Definition 3.24 and construct an HLFA $\mathcal{A}_m$ whose language is $\mathcal{L}$ and we show that the cardinality of the set of states of the HLFA $\mathcal{A}_m$ is less than the cardinality of the set of states of the RHLFA $\frac{\mathcal{A}}{\rho}$ of Example 3.15, whose language is $\mathcal{L} = ab^* \cup b^+$. According to Definition 3.24, we have $\iota_m([\Lambda]) \neq 0_L$, so $[\Lambda]_{R_\mathcal{L}}$ is initial state. Now, the initial state of $[\Lambda]$ affects the letter a from the alphabet set X and it is transferred to the state of $[w]$. To calculate the $[w]$ state, we have the method mentioned in Definition 3.24*

$$\begin{aligned} [w]_{R_\mathcal{L}} = [\Lambda a]_{R_\mathcal{L}} = [a]_{R_\mathcal{L}} &= \{v \in X^* | v R_\mathcal{L} a\} \\ &= \{v \in X^* | vz \in \mathcal{L} \text{ if and only if } az \in \mathcal{L}, \text{ for every } z \in X^*\} \\ &= \mathcal{L}. \end{aligned}$$

So $[w]_{R_\mathcal{L}} = [\Lambda a]_{R_\mathcal{L}} = [a]_{R_\mathcal{L}} = \mathcal{L}$. Consequently $\delta_m([\Lambda], a, [w]) = \delta_m([\Lambda], a, \mathcal{L}) = A \neq 0_L$.
the initial state of $[\Lambda]$ affects the letter b from the alphabet set X and it is transferred to the state of $[w']$. To calculate the $[w']$ state, we have the method mentioned in Definition 3.24

$$\begin{aligned} [w']_{R_\mathcal{L}} = [\Lambda b]_{R_\mathcal{L}} = [b]_{R_\mathcal{L}} &= \{v \in X^* | v R_\mathcal{L} b\} \\ &= \{v \in X^* | vz \in \mathcal{L} \text{ if and only if } bz \in \mathcal{L}, \text{ for every } z \in X^*\} \\ &= \mathcal{L}. \end{aligned}$$

So $[w']_{R_\mathcal{L}} = [\Lambda b]_{R_\mathcal{L}} = [b]_{R_\mathcal{L}} = \mathcal{L} \neq \varnothing$. Consequently $\delta_m([\Lambda], b, [w']) = \delta_m([\Lambda], b, \mathcal{L}) = A \neq 0_L$. Therefore, the initial state of $[\Lambda]$ is transferred to the $\mathcal{L}$ state by affecting the letters a and b from the X alphabet set. Next, the state of $[b] = \mathcal{L}$ affects the letter a from the alphabet set X and it is transferred to the state of $[t]$. To calculate the $[t]$ state, we have the method mentioned in Definition 3.24

$$\begin{aligned} [t] = [ba]_{R_\mathcal{L}} &= \{v \in X^* | v R_\mathcal{L} ba\} \\ &= \{v \in X^* | vz \in \mathcal{L} \text{ if and only if } baz \in \mathcal{L}, \text{ for every } z \in X^*\} \\ &= \varnothing. \end{aligned}$$

So $[t]_{R_\mathcal{L}} = [ba]_{R_\mathcal{L}} = \varnothing$. Consequently $\delta_m([b], a, [t]) = \delta_m(\mathcal{L}, a, [t]) = 0_L$. Also the state of $[b] = \mathcal{L}$ affects the letter b from the alphabet set X and it is transferred to the state of $[t']$. To calculate the $[t']$ state, we have the method mentioned in Definition 3.24

$$\begin{aligned} [t'] = [bb]_{R_\mathcal{L}} &= \{v \in X^* | v R_\mathcal{L} bb\} \\ &= \{v \in X^* | vz \in \mathcal{L} \text{ if and only if } bbz \in \mathcal{L}, \text{ for every } z \in X^*\} \\ &= \mathcal{L}. \end{aligned}$$

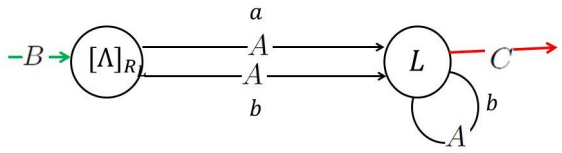


Figure 8: The state diagram HLFA $\mathcal{A}_m$ of Example 3.26

So $[t']_{R_\mathcal{L}} = [bb]_{R_\mathcal{L}} = [b]_{R_\mathcal{L}} = \mathcal{L} \neq \varnothing$. *Consequently* $\delta_m([b], b, [t']) = \delta_m(\mathcal{L}, b, \mathcal{L}) = A \neq 0_L$. *Finally,* $\tau_m([u]) \neq 0_L$, *and from the Definition of* τ_m *we have* $[u]_{R_\mathcal{L}} = \mathcal{L}$. *So the state diagram HLFA* $\mathcal{A}_m$ *is Figure 8.*

Since $[a]_{R_\mathcal{L}} = [w]_{R_\mathcal{L}} = [b]_{R_\mathcal{L}} = [w']_{R_\mathcal{L}} = [t']_{R_\mathcal{L}} = [u]_{R_\mathcal{L}} = \mathcal{L}$ *and* $[t]_{R_\mathcal{L}} = [ba]_{R_\mathcal{L}} = \varnothing$ *we can say*

$$\begin{aligned} &\iota_m([\Lambda]) = B, \\ \delta_m([\Lambda], a, \mathcal{L}) = A, &\qquad\qquad \delta_m([\Lambda], b, \mathcal{L}) = A, \\ \delta_m(\mathcal{L}, a, [t]) = 0_L, &\qquad\qquad \delta_m(\mathcal{L}, b, \mathcal{L}) = A, \\ &\tau_m(\mathcal{L}) = C. \end{aligned}$$

As it is clear, the state diagram of RHLFA $\frac{\mathcal{A}}{\rho}$ *of Example 3.15 is not the same as the state diagram of HLFA* $\mathcal{A}_m$ *of Example 3.26. Consequently the RHLFA* $\frac{\mathcal{A}}{\rho}$ *(Example 3.15) is not strong admissible reduction of the HLFA* $\mathcal{A}$ *(Example 3.1) for regular subset* $\mathcal{L} = ab^* \cup b^+$ *from* X^*, *where* $X = \{a, b\}$.

The following example shows that sometimes **a** RHLFA $\frac{\mathcal{A}}{\rho}$ may also be strong admissible reduction of **an** HLFA $\mathcal{A}$ for regular subset $\mathcal{L}$ from monoid X^*.

Example 3.27. *Consider regular subset* $\mathcal{L} = aX^* \cup bX^*$ *of* X^*, *where* $X = \{a, b\}$. *We use the method described in Definition 3.24 and construct an HLFA* $\mathcal{A}_m$ *whose language is* $\mathcal{L}$ *and we show that both the HLFA* $\mathcal{A}_m$ *and the RHLFA* $\frac{\mathcal{A}}{\rho}$ *(Example 3.16) are strong admissible reduction of the HLFA* $\mathcal{A}$ *(Example 3.7) for regular subset* $\mathcal{L} = aX^* \cup bX^*$ *from* X^*, *where* $X = \{a, b\}$. *According to Definition 3.24, we have* $\iota_m([\Lambda]) \neq 0_L$, *so* $[\Lambda]_{R_\mathcal{L}}$ *is initial state. Now, the initial state of* $[\Lambda]$ *affects the letter* a *from the alphabet set* X *and it is transferred to the state of* $[w]$. *To calculate the* $[w]$ *state, we have the method mentioned in Definition 3.24*

$$\begin{aligned} [w]_{R_\mathcal{L}} = [\Lambda a]_{R_\mathcal{L}} = [a]_{R_\mathcal{L}} &= \{v \in X^* | vR_\mathcal{L}a\} \\ &= \{v \in X^* | vz \in \mathcal{L} \ \text{ if and only if } \ az \in \mathcal{L}, \ \text{for every } z \in X^*\} \\ &= \mathcal{L}. \end{aligned}$$

So $[w]_{R_\mathcal{L}} = [\Lambda a]_{R_\mathcal{L}} = [a]_{R_\mathcal{L}} = \mathcal{L}$. *Consequently* $\delta_m([\Lambda], a, [w]) = \delta_m([\Lambda], a, \mathcal{L}) = A \neq 0$.
the initial state of $[\Lambda]$ *affects the letter* b *from the alphabet set* X *and it is transferred to the state of* $[w']$. *To calculate the* $[w']$ *state, we have the method mentioned in Definition 3.24*

$$\begin{aligned} [w']_{R_\mathcal{L}} = [\Lambda b]_{R_\mathcal{L}} = [b]_{R_\mathcal{L}} &= \{v \in X^* | vR_\mathcal{L}b\} \\ &= \{v \in X^* | vz \in \mathcal{L} \ \text{ if and only if } \ bz \in \mathcal{L}, \ \text{for every } z \in X^*\} \\ &= \mathcal{L}. \end{aligned}$$

So $[w']_{R_\mathcal{L}} = [\Lambda b]_{R_\mathcal{L}} = [b]_{R_\mathcal{L}} = \mathcal{L} \neq \varnothing$. *Consequently* $\delta_m([\Lambda], b, [w']) = \delta_m([\Lambda], b, \mathcal{L}) = A \neq 0_L$. *Therefore, the initial state of* $[\Lambda]$ *is transferred to the* $\mathcal{L}$ *state by affecting the letters* a *and* b *from the* X *alphabet set. Next, the state of* $[b] = \mathcal{L}$ *affects the letter* a *from the alphabet set* X *and it is transferred to the state of* $[t]$. *To calculate the* $[t]$ *state, we have the method mentioned in Definition 3.24*

$$\begin{aligned} [t] = [ba]_{R_\mathcal{L}} &= \{v \in X^* | vR_\mathcal{L}ba\} \\ &= \{v \in X^* | vz \in \mathcal{L} \ \text{ if and only if } \ baz \in \mathcal{L}, \ \text{for every } z \in X^*\} \\ &= \mathcal{L}. \end{aligned}$$

So $[t]_{R_\mathcal{L}} = [ba]_{R_\mathcal{L}} = \mathcal{L}$. *Consequently* $\delta_m([b], a, [t]) = \delta_m(\mathcal{L}, a, [t]) = \delta_m(\mathcal{L}, a, \mathcal{L}) = A$. *Also the state of* $[b] = \mathcal{L}$ *affects the letter* b *from the alphabet set* X *and it is transferred to the state of* $[t']$. *To calculate the* $[t']$ *state, we have the method mentioned in Definition 3.24*

$$\begin{aligned}[t'] = [bb]_{R_\mathcal{L}} &= \{v \in X^* | vR_\mathcal{L}bb\} \\ &= \{v \in X^* | vz \in \mathcal{L} \text{ if and only if } bbz \in \mathcal{L}, \text{ for every } z \in X^*\} \\ &= \mathcal{L}.\end{aligned}$$

So $[t']_{R_\mathcal{L}} = [bb]_{R_\mathcal{L}} = [b]_{R_\mathcal{L}} = \mathcal{L} \neq \varnothing$. *Consequently* $\delta_m([b], b, [t']) = \delta_m(\mathcal{L}, b, \mathcal{L}) = A \neq 0_L$. *Finally,* $\tau_m([u]) \neq 0_L$, *and from the Definition of* τ_m *we have* $[u]_{R_\mathcal{L}} = \mathcal{L}$. *Since* $[a]_{R_\mathcal{L}} = [b]_{R_\mathcal{L}} = [w]_{R_\mathcal{L}} = [w']_{R_\mathcal{L}} = [t]_{R_\mathcal{L}} = [u]_{R_\mathcal{L}} = \mathcal{L}$, *we can say*

$$\begin{gathered}\iota_m([\Lambda]) = B, \\ \delta_m([\Lambda], a, \mathcal{L}) = A, \qquad \delta_m([\Lambda], b, \mathcal{L}) = A, \\ \delta_m(\mathcal{L}, a, \mathcal{L}) = A, \qquad \delta_m(\mathcal{L}, b, \mathcal{L}) = A, \\ \tau_m(\mathcal{L}) = C.\end{gathered}$$

So the state diagram HLFA $\mathcal{A}_m$ *is Figure 9. As it is clear, the state diagram of RHLFA* $\frac{\mathcal{A}}{\rho}$ *of Example 3.16 is the same*

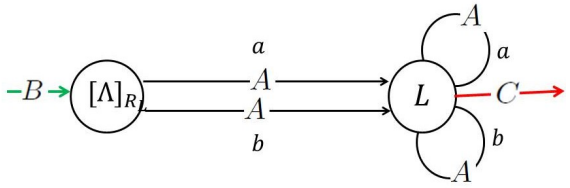


Figure 9: The state diagram HLFA $\mathcal{A}_m$ of Example 3.27

as the state diagram of HLFA $\mathcal{A}_m$ *of Example 3.27 and this similarity of two state diagrams, is not accidental. Also, the RHLFA* $\frac{\mathcal{A}}{\rho}$ *is a strong admissible reduction of the HLFA* $\mathcal{A}$ *(Example 3.7) for regular subset* $\mathcal{L} = aX^* \cup bX^*$ *from* X^*, *where* $X = \{a, b\}$.

In the following, we will state under what conditions a RHLFA $\frac{\mathcal{A}}{\rho}$ is a strong admissible reduction of an HLFA $\mathcal{A}$ for regular subset $\mathcal{L}$ from monoid X^*.

Definition 3.28. *Let* $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ *be an HLFA. Then we define a hesitant L-fuzzy set* $\mathcal{K}_\mathcal{A}$ *on* $X^* \times Q$, *as follows:*

$$\begin{aligned}\mathcal{K}_\mathcal{A} : X^* \times Q &\to 2^L \\ \mathcal{K}_\mathcal{A}(u, q) &= \sqcup\{\delta^*(q, u, p) \sqcap \tau(p) | p \in Q\}\end{aligned}$$

Definition 3.29. *Let* $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ *be an HLFA. Then we say* $q \in Q$ *is co-accessible, if there exists* $u \in X^*$ *such that* $\mathcal{K}_\mathcal{A}(u, q) \neq 0_L$. *If all states of the HLFA* $\mathcal{A}$ *are co-accessible, then* $\mathcal{A}$ *is also co-accessible.*

The following theorem is one of the original theorems of this section because with the help of this theorem we can find a minimal and reduced HLFA for the regular subset $\mathcal{L}$ of the monoid X^* in a way other than the one used in Theorem 3.25.

Theorem 3.30. *Let* $\mathcal{L}$ *be a regular subset of monoid* X^*. *Then every accessible complete deterministic RHLFA that all its states are co-accessible and its language is* $\mathcal{L}$ *(*$\mathcal{HL}_\mathcal{A} = \mathcal{L}$*), is minimal.*

Proof. According to Corollary 3.22, we showed that if $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ is an accessible complete deterministic HLFA, then the equivalence relation $R_\mathcal{F}$ exists on X^*, so that the cardinality of the set of states of the HLFA $\mathcal{A}$ is equal to the cardinality of the set of equivalence classes of the relation $R_\mathcal{F}$, (that is, $|\frac{X^*}{R_\mathcal{F}}| = |Q|$), and in Theorem 3.25 we showed that $\mathcal{A}_m$ is a minimal HLFA whose language is $\mathcal{L}$. Now we show that if the HLFA $\mathcal{A}$ is also reduced, then the cardinality of the set of states the HLFA $\mathcal{A}_m$ is equal to the cardinality of the set of equivalence classes of the relation $R_\mathcal{F}$ (that is, $|\frac{X^*}{R_\mathcal{F}}| = |\frac{X^*}{R_\mathcal{L}}|$). Therefore, we can conclude that ,$|Q| = |\frac{X^*}{R_\mathcal{F}}| = |\frac{X^*}{R_\mathcal{L}}|$. In Theorem 3.23, we showed that if $\mathcal{A}$ is a complete deterministic HLFA, then for every equivalence class $[w]_{R_F}$ of R_F, we have an equivalence class $[w]_{R_\mathcal{L}}$ of $R_\mathcal{L}$ such that $[w]_{R_F} \subseteq [w]_{R_\mathcal{L}}$. Now we show that if $\mathcal{A}$ is also the reduced and co-accessible, then $[w]_{R_\mathcal{L}} \subseteq [w]_{R_F}$.

We assume $u \in [w]_{R_{\mathcal{L}}}$, therefore $wR_{\mathcal{L}}u$ and from the definition of $R_{\mathcal{L}}$ we have ($uz \in \mathcal{L}$ if and only if $wz \in \mathcal{L}$, for every $z \in X^*$). Since $\mathcal{L}$ is the language of HLFA $\mathcal{A}$, then we have ($\beta_A(uz) \neq 0_L$ if and only if $\beta_A(wz) \neq 0_L$ for every $z \in X^*$), so $\big(\sqcup_{p\in Q}\,(\iota(q_0) \sqcap \delta^*(q_0, uz, p) \sqcap \tau(p)) \neq 0_L$ if and only if $\sqcup_{p\in Q}(\iota(q_0) \sqcap \delta^*(q_0, wz, p) \sqcap \tau(p)) \neq 0_L$ for every $z \in X^*\big)$. Consequently, $\big($ $(\delta^*(q_0, uz, p) \sqcap \tau(p) \neq 0_L$ for some $p \in Q)$ if and only if $(\delta^*(q_0, wz, p) \sqcap \tau(p) \neq 0_L$ for some $p \in Q)$ for every $z \in X^*\big)$. Since $\mathcal{A}$ is complete and deterministic, then there exist unique q_1 and q_2 such that $\delta^*(q_0, u, q_1) \neq 0_L$ and $\delta^*(q_0, w, q_2) \neq 0_L$, so it can say $\big($ $(\delta^*(q_0, u, q_1) \sqcap \delta^*(q_1, z, p) \sqcap \tau(p) \neq 0_L$ for some $p \in Q)$ if and only if $(\delta^*(q_0, w, q_2) \sqcap \delta^*(q_2, z, p) \sqcap \tau(p) \neq 0_L$ for some $p \in Q)$ for every $z \in X^*\big)$. Since all states of $\mathcal{A}$ are co-accessible, thus the states q_1 and q_2 are co-accessible, then $\big($ $(\delta^*(q_1, z, p) \sqcap \tau(p) \neq 0_L$ for some $p \in Q)$ if and only if $(\delta^*(q_2, z, p) \sqcap \tau(p) \neq 0_L$ for some $p \in Q)$ for every $z \in X^*\big)$. Consequently, we can say $\{z \in X^* | \delta^*(q_1, z, p) \sqcap \tau(p) \neq 0_L$, for some $p \in Q\} = \{z \in X^* | \delta^*(q_2, z, p) \sqcap \tau(p) \neq 0_L$, for some $p \in Q\}$. Therefore $q_1 \rho q_2$ and because $\mathcal{A}$ is reduced, we have $q_1 = q_2$. Therefore for $q_1 = q_2 = q$ we have $\delta^*(q_0, u, q) \neq 0_L$ and $\delta^*(q_0, w, q) \neq 0_L$ and that means that $u \in [w]_{R_F}$. So we have $[w]_{R_L} \subseteq [w]_{R_F}$ and Consequently $|\frac{X^*}{R_{\mathcal{F}}}| = |\frac{X^*}{R_{\mathcal{L}}}|$. Since $|Q| = |\frac{X^*}{R_{\mathcal{F}}}|$, we have $|Q| = |\frac{X^*}{R_{\mathcal{F}}}| = |\frac{X^*}{R_{\mathcal{L}}}|$, And since the HLFA $\mathcal{A}_m$ is minimal, we can conclude that the HLFA $\mathcal{A}$ is also minimal. □

Theorem 3.31. *Let $\mathcal{A} = (Q, X, \iota, \delta, \tau)$ be an HLFA. Then there exists a co-accessible HLFA $\mathcal{A}_c$ such that its language is equal to the language of HLFA $\mathcal{A}$, that is $\mathcal{HL}_{\mathcal{A}} = \mathcal{HL}_{\mathcal{A}_c}$.*

Proof. It is enough to show $\beta_{\mathcal{A}}(x) = \beta_{\mathcal{A}_c}(x)$, where $\beta_{\mathcal{A}}(x) \neq \{0\}$, for some $x \in X^*$. Consider the HLFA $\mathcal{A}_c$ as follows: $\mathcal{A}_c = (Q_c, X, \iota_c, \delta_c, \tau_c)$ such that $Q_c = \{q \in Q |$ q is co-accessible state$\}$, ι_c is the restriction of ι over Q_c, δ_c is the restriction of δ over $Q_c \times X \times Q_c$, and τ_c is the restriction of τ over Q_c. Now, consider $\beta_{\mathcal{A}}(x) = \sqcup_{p,q\in Q}\iota(p) \sqcap \delta^*(p, x, q) \sqcap \tau(q) \neq \{0\}$. Without loss of generality, let $Q = \{p_1, p_2, ..., p_k, q_1, q_2, ..., q_k\}$, not necessarily distinct. Then

$$\begin{aligned}\beta_{\mathcal{A}}(x) = \sqcup_{p,q\in Q}\iota(p) \sqcap \delta^*(p, x, q) \sqcap \tau(q) &= (\iota(p_1) \sqcap \delta^*(p_1, x, q_1) \sqcap \tau(q_1)) \\ &\sqcup (\iota(p_2) \sqcap \delta^*(p_2, x, q_2) \sqcap \tau(q_2)) \sqcup ... \sqcup (\iota(p_k) \sqcap \delta^*(p_k, x, q_k) \sqcap \tau(q_k)).\end{aligned}$$

If $\delta^*(p_i, x, q_i) \sqcap \tau(q_i) = \{0\}$, for $i \in \{1, ..., k\}$, then $\iota(p_i) \sqcap \delta^*(p_i, x, q_i) \sqcap \tau(q_i) = \{0\}$ and $\{0\} \sqcup H = H$, where $H \in 2^L$. Without loss of generality, let $\delta^*(p_i, x, q_i) \sqcap \tau(q_i) = \{0\}$, for $1 \leq i \leq r$ and $\delta^*(p_j, x, q_j) \sqcap \tau(q_j) \neq \{0\}$, for $r < j \leq k$. Then by using $\beta_{\mathcal{A}}(x) \neq \{0\}$ we have

$$\begin{aligned}\beta_{\mathcal{A}}(x) &= (\iota(p_1) \sqcap \delta^*(p_1, x, q_1) \sqcap \tau(q_1)) \sqcup (\iota(p_2) \sqcap \delta^*(p_2, x, q_2) \sqcap \tau(q_2)) \\ &\qquad \sqcup ... \sqcup (\iota(p_k) \sqcap \delta^*(p_k, x, q_k) \sqcap \tau(q_k)) \\ &= (\iota(p_{r+1}) \sqcap \delta^*(p_{r+1}, x, q_{r+1}) \sqcap \tau(q_{r+1})) \sqcup (\iota(p_{r+2}) \sqcap \delta^*(p_{r+2}, x, q_{r+2}) \sqcap \tau(q_{r+2})) \\ &\qquad \sqcup ... \sqcup (\iota(p_k) \sqcap \delta^*(p_k, x, q_k) \sqcap \tau(q_k)) \\ &= (\iota_c(p_{r+1}) \sqcap \delta_c^*(p_{r+1}, x, q_{r+1}) \sqcap \tau_c(q_{r+1})) \sqcup ... \sqcup (\iota_c(p_k) \sqcap \delta_c^*(p_k, x, q_k) \sqcap \tau_c(q_k)) \\ &= \beta_{\mathcal{A}_c}(x).\end{aligned}$$

Hence, the claim holds. □

Example 3.32. *Let $\frac{\mathcal{A}}{\rho}$ be the HLFA defined in Example 3.15. Then we calculate the HLFA $(\frac{\mathcal{A}}{\rho})_c$ by Theorem 3.31. We have $(\frac{Q}{\rho})_c = \{q_0\rho, q\rho\}$ and we have*

$$\begin{aligned}(\iota_\rho(q_0\rho))_c &= \iota(q_0) = \{\gamma_1, \gamma_2, \gamma_3\}, \\ (\delta_\rho(q_0\rho, a, q\rho))_c &= \sqcup_{q\rho q_0, p\rho q}\delta(q, a, p) = \delta(q_0, a, q_1) \sqcup \delta(q_0, a, q_2) = \{a_0, a_1\} \sqcup 0_L = \{a_0, a_1\}, \\ (\delta_\rho(q_0\rho, b, q\rho))_c &= \sqcup_{q\rho q_0, p\rho q}\delta(q, b, p) = \delta(q_0, b, q_1) \sqcup \delta(q_0, b, q_2) = 0_L \sqcup \{b_0, b_2\} = \{b_0, b_2\}, \\ (\delta_\rho(q\rho, b, q\rho))_c &= \sqcup_{p\rho q, p'\rho q}\delta(p, b, p') = \delta(q_1, b, q_1) \sqcup \delta(q_1, b, q_2) \sqcup \delta(q_2, b, q_1) \sqcup \delta(q_2, b, q_2) \\ &= \{b_1, b_2\} \sqcup 0_L \sqcup 0_L \sqcup \{b_2, \gamma_2\} = \{b_2\}, \\ (\tau_\rho(q\rho))_c &= \tau(q_1) \sqcup \tau(q_2) = \{\gamma_3\} \sqcup \{\gamma_1, \gamma_2, \gamma_3\} = \{\gamma_3\}.\end{aligned}$$

The state diagram RHLFA $(\frac{\mathcal{A}}{\rho})_c$ is Figure 10. As is clear the language HLFA $(\frac{\mathcal{A}}{\rho})_c$ is the set $ab^* \cup b^+$ and the HLFA $(\frac{\mathcal{A}}{\rho})_c$ is reduced.

As it is clear, the state diagram of the RHLFA $(\frac{\mathcal{A}}{\rho})_c$ of Example 3.32 is the same as the state diagram of the HLFA $\mathcal{A}_m$ of Example 3.26. Consequently the RHLFA $(\frac{\mathcal{A}}{\rho})_c$ (Example 3.32) is strong admissible reduction of the HLFA $\mathcal{A}$ (Example 3.6) for regular subset $\mathcal{L} = ab^* \cup b^+$ from X^*, where $X = \{a, b\}$.

Remark 3.33. *Suppose $\mathcal{A}$ is an accessible complete deterministic RHLFA that some of its states are not co-accessible, then by removing these states, we have a reduced and minimal HLFA.*

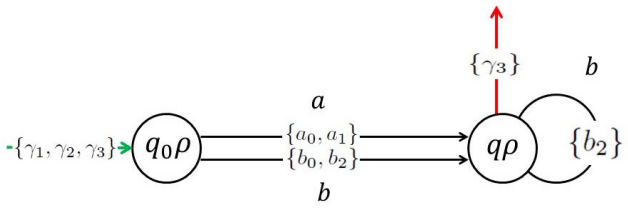


Figure 10: The state diagram HLFA $(\frac{\mathcal{A}}{\rho})_c$ of Example 3.32

4 Conclusions

In this paper, for the regular subset $\mathcal{L}$ of the monoid X^*, we defined a reduced and minimal HLFA whose language is $\mathcal{L}$, in the following two ways,

1. For an acessible complete deterministic HLFA $\mathcal{A}$ whose language is $\mathcal{L}$, we defined an equivalence relation ρ on Q. Then we denoted a set of equivalence classes by $\frac{Q}{\rho}$, and by means of it we defined an HLFA $\frac{\mathcal{A}}{\rho}$ whose language is equal to the language $\mathcal{A}$. Furthermore, we showed that $\frac{\mathcal{A}}{\rho}$ is a RHLFA. If all its states were co-accessible, it would be a reduced and minimal HLFA, but if some of its states were not co-accessible, by removing them, we would arrive at a reduced and minimal HLFA whose language is $\mathcal{L}$.

2. For the regular subset $\mathcal{L}$ of monoid X^*, we introduced an equivalence relation $R_{\mathcal{L}}$ on X^*, and using it, we defined an HLFA $\mathcal{A}_m$, whose set of states is the set $\frac{X^*}{R_{\mathcal{L}}}$ and its language is $\mathcal{L}$. To show that $\mathcal{A}_m$ is minimal, we considered an arbitrary complete deterministic HLFA $\mathcal{A}$, whose language is $\mathcal{L}$. Then, by introducing an equivalence relation $R_{\mathcal{F}}$ on X^*, we showed that the cardinality of $\frac{X^*}{R_{\mathcal{L}}}$ is less than or equal to the cardinality of $\frac{X^*}{R_{\mathcal{F}}}$, and considering that the cardinality of $\frac{X^*}{R_{\mathcal{F}}}$ is less than or equal to the cardinality of Q, we came to the conclusion that the cardinality of $\frac{X^*}{R_{\mathcal{L}}}$ is less than or equal the cardinality of Q, that is, $|\frac{X^*}{R_{\mathcal{L}}}| \leq |\frac{X^*}{R_{\mathcal{F}}}| \leq |Q|$ and thus we showed that the HLFA $\mathcal{A}_m$ is minimal. Finally, we showed that $\mathcal{A}_m$ is reduced.

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References

[1] N. C. Basak, A. Gupta, *On quotient machines of a fuzzy automata and the minimal machine*, Fuzzy Sets and Systems, **125**(2) (2002), 223-229. https://doi.org/10.1016/S0165-0114(01)00064-1

[2] B. Bedregal, R. Reiser, H. Bustince, C. Lopez-Molina, V. Torra, *Aggregation functions for typical hesitant fuzzy elements and the action of automorphisms*, Information Sciences, **255** (2014), 82-99. https://doi.org/10.1016/j.ins.2013.08.024

[3] W. Cheng, Z. W. Mo, *Minimization algorithm of fuzzy finite automata*, Fuzzy Sets and Systems, **141** (2004), 439-448. https://doi.org/10.1016/S0165-0114(02)00607-3

[4] A. Choubey, K. Ravi, *Intuitionistic fuzzy automata and intuitionistic fuzzy regular expressions*, Journal of Applied Mathematics and Informatics, **27**(1-2) (2009), 409-417.

[5] V. S. Costa, B. Bedregal, *On typical hesitant fuzzy automata*, Soft Computing, **24**(12) (2020), 8725-8736. https://doi.org/10.1007/s00500-020-04896-7

[6] A. H. Dehmiry, M. Mashinchi, R. Mesiar, *Hesitant L-fuzzy sets*, International Journal of Intelligent Systems, **33** (2018), 1027-1042. https://doi.org/10.1002/int.21910

[7] J. R. Gonzalez de Mendivil, *Conditions for minimal fuzzy deterministic finite automata via brzozowski's procedure*, IEEE Transactions on Fuzzy Systems, **26** (2018), 2409-2420. https://doi.org/10.1109/TFUZZ.2017.2775601

[8] A. G. D. M. Grau, S. Stanimirovic, F. Farina, *Minimal determinization algorithm for fuzzy automata*, IEEE Transactions on Fuzzy Systems, **31** (2023), 3812-3822. https://doi.org/10.1109/TFUZZ.2023.3268406

[9] S. Halamish, O. Kupferman, *Minimizing deterministic lattice automata*, ACM Transactions on Computational Logic, **6604** (2015), 199-213. https://doi.org/10.1007/978-3-642-19805-2_14

[10] J. Ignjatovic, M. Ciric, S. Bogdanovic, T. Petkovic, *Myhill-Nerode type theory for fuzzy languages and automata*, Fuzzy Sets and Systems, **161** (2010), 1288-1324. https://doi.org/10.1016/j.fss.2009.06.007

[11] Z. Jancic, M. Ciric, *Brzozowski type determinization for fuzzy automata*, Fuzzy Sets and Systems, **249** (2014), 73-82. https://doi.org/10.1016/j.fss.2014.02.021

[12] Y. B. Jun, *Quotient structures of intuitionistic fuzzy finite state machines*, Information Sciences, **177** (2007), 4977-4986. https://doi.org/10.1016/j.ins.2007.06.008

[13] V. Ladimir, V. Topencharov, K. Peeva, *Equivalence, reduction and minimization on finite fuzzy automata*, Journal of Mathematica Analysis and Applications, **84** (1981), 270-281. https://doi.org/10.1016/0022-247X(81)90165-7

[14] H. S. Lee, *Minimizing fuzzy finite automata*, 2000. Fuzzy Systems, FUZZ IEEE 2000. The Ninth IEEE International Conference on, **1** (2000), 65-70. https://doi.org/10.1109/FUZZY.2000.838635

[15] E. T. Lee, L. A. Zadeh, *Note on fuzzy languages*, Information Sciences, **1** (1969), 421-434. https://doi.org/10.1016/0020-0255(69)90025-5

[16] Y. Li, W. Pedrycz, *Minimization of lattice finite automata and its application to the decomposition of lattice languages*, Fuzzy Sets and Systems, **158** (2007), 1423-1436. https://doi.org/10.1016/j.fss.2007.03.003

[17] L. Li, D. Qiu. *On the state minimization of fuzzy automata*, IEEE Transactions on Fuzzy Systems, **23** (2015), 434-443. https://doi.org/10.1109/TFUZZ.2014.2315620

[18] A. Mateescu, A. Salomaa, K. Salomaa, S. Yu, *Lexical analysis with a simple finite fuzzy automaton model*, Journal of Universal Computer Science, **1** (1995), 292-311. https://doi.org/10.1007/978-3-642-80350-5_28

[19] S. Moghari, M. M. Zahedi, *Minimization of deterministic fuzzy tree automata*, Journal of Fuzzy Set-Valued Analysis, **2014** (2014), 1-18. https://doi.org/10.5899/2014/jfsva-00191

[20] J. N. Mordeson, D. S. Malik, *Fuzzy automata and languages theory and applications*, Chapman Hall, CRC Boca Raton, London, New York, Washington DC, 2002.

[21] D. Qiu, *Supervisory control of fuzzy discrete event systems: A formal approach*, IEEE Transactions on Systems, Man and Cybernetics Part B, **35** (2005), 72-88. https://doi.org/10.1109/TSMCB.2004.840457

[22] K. Ravi, A. Choubey, *Interval-valued fuzzy regular language*, Journal of Applied Mathematics and Informatics, **28**(3-4) (2010), 639-649.

[23] K. Ravi, A. Choubey, *Myhill-Nerode theorem for interval-valued fuzzy regular language*, In: AIP Conference Proceedings. American Institute of Physics, Chandigarh, **1324**(1) (2010), 30-33. https://doi.org/10.1063/1.3526220

[24] M. Shamsizadeh, M. M. Zahedi, *Intuitionistic general fuzzy automata*, Soft Computing, **20** (2016), 3505-3519. https://doi.org/10.1007/s00500-015-1969-x

[25] M. Shamsizadeh, M. M. Zahedi, *Bisimulation of type 2 for BL-general fuzzy automata*, Soft Computing, **23** (2019), 9843-9852. https://doi.org/10.1007/s00500-019-03812-y

[26] M. Shamsizadeh, M. M. Zahedi, *On reduced fuzzy multiset finite automata*, Soft Computing, **22** (2022), 13381-13390. https://doi.org/10.1007/s00500-022-07549-z

[27] M. Shamsizadeh, M. M. Zahedi, Kh. Abolpour, *Reduction of BL-general L-fuzzy automata*, Iranian Journal of Mathematical Sciences and Informatics, **18**(1) (2023), 131-144. https://doi.org/10.52547/ijmsi.18.1.131

[28] M. Shamsizadeh, M. M. Zahedi, M. J. Agheli Goki, *Minimization of hesitant L-fuzzy automata*, Fuzzy Set and System, **477** (2024), 108793. https://doi.org/10.1016/j.fss.2023.108793

[29] A. K. Srivastava, S. P. Tiwari, *IF-topologies and IF-automata*, Soft Computing, **14** (2010), 571-578. https://doi.org/10.1007/s00500-009-0427-z

[30] S. P. Tiwari, Sh. Sharan, *Fuzzy automata based on lattice-ordered monoid with algebraic and topological aspects*, Fuzzy Information and Engineering, **4** (2012), 155-164. https://doi.org/10.1007/s12543-012-0108-y

[31] S. P. Tiwari, A. K. Srivastava, *On a decomposition of fuzzy automata*, Fuzzy Sets and Systems, **151** (2005), 503-511. https://doi.org/10.1016/j.fss.2004.06.014

[32] V. Torra, *Hesitant fuzzy sets*, International Journal of Intelligent Systems, **25** (2010), 529-539. https://doi.org/10.1002/int.20418

[33] V. Torra, Y. Narukawa, *On hesitant fuzzy sets and decision*, 2009 IEEE International Conference on Fuzzy Systems. IEEE, (2009), 1378-1382. https://doi.org/10.1109/FUZZY.2009.5276884

[34] A. Tripathi, R. Kaur, *A review of state minimization and state reduction techniques in fuzzy automata*, AIP Conference Proceedings, **2061** (2019), 020-033. https://doi.org/10.1063/1.5086655

[35] W. G. Wee, *On generalization of adaptive algorithm and application of the fuzzy sets concept to pattern classification*, Ph.D. Thesis, Purdue University, Lafayette, IN, 1967.

[36] L. Wu, D. Qiu, *Automata theory based on complete residuated lattice-valued logic: Reduction and minimization*, Fuzzy Sets and Systems, **161** (2010), 1635-1656. https://doi.org/10.1016/j.fss.2009.12.011

[37] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965,) 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X