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Existence and well-posedness of equilibrium for multi-leader-follower games with fuzzy constraint mappings

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Abstract

This article aims to research the multi-leader-follower games with fuzzy constraint mappings (MLFFGs), in which the feasible strategy mappings of the leaders are fuzzy mappings. By using the Fan-Glicksberg fixed point theorem, an existence theorem of equilibria for the MLFFGs under some conditions is obtained. Furthermore, with the assumption of finite rationality, we prove the stable results on the generalized well-posedness, the generalized Tykhonov well-posedness, and the generalized Hadamard well-posedness of fuzzy multi-leader-follower games.

Keywords: Fuzzy constraint mapping, multi-leader-follower game, well-posedness.

1 Introduction

Since Nash introduced the definition of Nash equilibrium in 1950, solving Nash equilibrium problems has attracted the attention of a great number of scholars in different fields [18]. The one-leader-follower games originated from the classic oligopolistic competition in the electricity market, which can be depicted as a dominant leader deciding the upper-level optimal problem, and the remaining players of the game, described as the followers, jointly optimizing lower-level problem [20]. When we model practical problems such as the regulation of electricity markets, in general, there are several dominant leaders at the same time. Therefore, for this situation, we need to consider the multi-leader-follower games(briefly, MLFGs) presented in [19], in which all leaders can predict the best option for followers in advance, then utilize the advantages of this predictive ability to choose the strategy that is the best for them. After all leaders have selected their strategies, the competition among all followers can be viewed as a generalized constrained game parameterized by the leader's strategy. Recently, we have seen more and more studies in MLFGs being published. For example, Leyffer and Munson [12] considered the properties of the solution set for MLFGs and studied various nonlinear optimization and nonlinear complementary formulas of linear programming with equilibrium constraints. Yu and Wang [28] considered, for simplicity, a class of MLFGs with only two leaders and a finite number of followers and show that there are equilibria for such games in the locally convex topological space. Ding [4] generalized the notion of MLFGs, furthermore constructed and proved several existence theorems for equilibrium in FC-uniform space, with no convex structure. Jia and Xiang et al. [7] discussed a kind of generalized multi-objective MLFGs in Hausdorff topological space, and obtained the existence conditions of weak Pareto-Nash equilibrium of the game. Based on the method of essential solutions, a few results about the genetic stability of multi-objective MLFGs were derived. Deng and Zuo [3] obtained an existence theorem for equilibria of MLFGs by using Ky Fan's inequality, proving that this class of MLFG problems is generalized Hadamard well-posedness as well as generalized Tykhonov well-posedness.

In the past few decades, the research of classical game theory has embarked on the path of systematisation and axiomatisation, and has achieved abundant research results. However, with the development of science and technology, as well as the complicated factors such as the ambiguity of people's understanding of the problem, the asymmetry of the information they possess and the uncertainty of the decision-making environment in the real game problems, it makes

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the research on fuzzy game theory and applications be naturally combined. Especially since fuzzy sets were proposed by Zadeh [31], it has not only been widely used in the fields of artificial intelligence, optimal control, etc., but also been combined with game theory by many researchers (cf. [5, 10, 13, 17, 23]), which has made the fuzzy game theory become an interesting issue in the current game theory research.

Up to now, there are two main research directions to deal with fuzzy information in classical non-cooperative game problems using fuzzy set theory. One is that the strategy sets, preference information, etc. of the players are represented by fuzzy sets, e.g., Zhou and Jia et al. [32] assumed that players have fuzzy constraint mappings and considered a class of generalized multi-objective games (GMFG) with fuzzy constraint mappings, constructed a GMFG problem space ε, and investigated the stability of the fuzzy equilibria of this game. Mao and Yang [16] researched the fuzzy equilibrium point of a game with fuzzy constraint mapping in a one-leader-follower game problem, in which the equilibrium set-value mapping is a fuzzy mapping, and proved the existence of fuzzy equilibrium point and slightly altruistic fuzzy equilibrium points for this type of game by using nonlinear theory. Another one is that the payoff values are fuzzy numbers, and the reader can refer to [6, 8, 9] for related research results. In recent years, there have also been a few research achievements using fuzzy set theory to study leader-follower games, e.g., Liu and Yang et al. [14] discussed the problem of equilibrium existence in single-leader-multi-follower games with fuzzy goals involving fuzzy parameters. Liu and Wang et al. [15] studied the case where the payoff function of the game players are fuzzy valued functions, the existence of Nash equilibria of leader-follower fuzzy games. However, we find a fact that, for leader-follower games, there is no relevant research that the constraint mapping of the players is fuzzy mappings. Therefore, referring to the multi-leader-follower generalized constrained multi-objective game of type (II) summarized in [4], we will consider the multi-leader-follower games with fuzzy constraint mappings, where the objective function of the followers is a single-valued function and the feasible constraint mappings of the leaders are fuzzy mappings.

Well-posedness is of great significance for discussing the stability of solutions for optimization problems. Tykhonov [21] first proposed the concept of Tykhonov well-posedness in 1966. After that, some scholars thought that there is another problem of continuous dependence problem on parameters, which is essentially a problem of stability of solutions, and they call it the Hadamard well-posedness problem. For related references, the reader may consult [11, 22, 24, 30, 29]. Besides that, Anderlini and Canning [2] introduced bounded rationality into the economic model, establishing a class of general games with parameterization of abstract rationality functions. Then, Yu [27] extended the results of Anderlini and Canning, linked the finite rationality model to well-posedness for nonlinear problems, and researched the well-posedness of various optimization problems.

Inspired by these research works, this article studies the multi-leader-follower games with fuzzy constraint mappings (MLFFGs) whose objective function is a single-valued mapping and gives the existence theorem of the fuzzy equilibria. Under the assumption of bounded rationality, we discuss the related well-posedness of MLFFGs. The details of this article can be divided into five parts. In Sect. 2, we will review the essential definitions and lemmas in theorem proving. The next section aims to introduce the main model of this article and prove that there will exist an equilibrium point when MLFFGs satisfy certain conditions. We establish the bounded rationality model B of MLFFGs in Sect. 4. Moreover, we can derive a few results of the generalized well-posedness, Tykhonov well-posedness and Hadamard well-posedness of the game δ. Sect. 5 is the summary of the article.

2 Preliminaries

Let D be a domain, μ_Ω is a mapping from the domain D to the interval $[0,1]$, and a fuzzy set Ω in D is characterized by the mapping μ_Ω, then μ_Ω is called the membership function of Ω. For any $a \in D, \mu_\Omega(a)$ represents the membership degree of element a to fuzzy set Ω. Assuming F is a topological space, take $\Theta \subset F$ and we shall use $P_0(\Theta)$ to represent the collection of all subsets of Θ. Moreover, Υ is a nonempty convex subset of Hausdorff topological vector space E. Employing $P_0(\Upsilon)$ and $\mathcal{F}(\Upsilon)$ to be the collection of all nonempty sets and the collection of all fuzzy sets over Υ, respectively. Define the fuzzing mapping $\mathcal{M} : \Theta \to \mathcal{F}(\Upsilon)$, for any $\theta \in \Theta, \mathcal{M}(\theta)$ is a fuzzy set in $\mathcal{F}(\Upsilon)$. For simplicity, we use $\mathcal{M}_\theta$ to replace $\mathcal{M}(\theta)$, then for any $\upsilon \in \Upsilon, \mathcal{F}_\theta(\upsilon)$ represents the membership degree of element υ to fuzzy set $\mathcal{M}(\theta)$. Let $\tilde{\Omega} \in \mathcal{F}(\Upsilon)$ be a fuzzy set and $\gamma \in [0,1]$, the γ-cut set of $\tilde{\Omega}$ is depicted as follows:

$$(\tilde{\Omega})_\gamma = \{\upsilon \in \Upsilon : \tilde{\Omega}(\upsilon) \geq \gamma\}.$$

Definition 2.1. [28] *Let Θ and Υ be two Hausdorff topological vector spaces and Ω be a convex subset of Θ, then set-valued mapping $\mathcal{M} : \Omega \to P_0(\Upsilon)$ is convex, if any convex combination $\sum_{k=1}^{n} \lambda_k \theta_k$ in Ω satisfies*

$$\sum_{k=1}^{n} \lambda_k \mathcal{M}(\theta_k) \subset \mathcal{M}(\sum_{k=1}^{n} \lambda_k \theta_k).$$

Definition 2.2. [10] *Let Θ and Υ be two Hausdorff topological vector spaces and Ω be a subset of Θ, a fuzzy mapping $\mathcal{M}:\Omega\to\mathcal{F}(\Upsilon)$ is said to be convex if $\mathcal{M}$ satisfies: for any $\theta\in\Omega,\upsilon_1,\upsilon_2\in\Upsilon,\alpha\in[0,1]$, such that $\mathcal{M}_\theta(\alpha\upsilon_1+(1-\alpha)\upsilon_2)\geq\min\{\mathcal{M}_\theta(\upsilon_1),\mathcal{M}_\theta(\upsilon_2)\}$.*

Definition 2.3. [25] *Let Θ and Υ be two Hausdorff topological spaces, the set-valued mapping $\mathcal{M}$ from Θ to $P_0(\Upsilon)$ is called to be upper semi-continuous, abbreviated as usc(or lower semi-continuous, abbreviated as lsc) if for each $\theta\in\Theta$ and any open set G in Υ, $G\supset\mathcal{M}(\theta)$(or $\mathcal{M}(\theta)\cap G\neq\varnothing$), there exist an open neighborhood $O(\theta)$ of θ, satisfying for any $\theta'\in O(\theta)$, $G\supset\mathcal{M}(\theta')$(or $G\cap\mathcal{M}(\theta')\neq\varnothing$).*

$\mathcal{M}$ is continuous, if $\mathcal{M}$ is both usc and lsc.

Lemma 2.4. [32] *Let Θ and Υ be two Hausdorff topological vector spaces and denote by Ω a nonempty set of Θ. We describe the fuzzy mapping $\mathcal{M}:\Omega\to\mathcal{F}(\Upsilon)$ and a function $\zeta:\Omega\to[0,1]$, $\tilde{\mathcal{M}}:\Omega\to P_0(\Upsilon),\theta\mapsto(\mathcal{M}_\theta)_{\zeta(\theta)}$ be a set-valued mapping deduce by $\mathcal{M}$, then:*

(1) *$(\mathcal{M}_\theta)_{\zeta(\theta)}$ is closed, if $\mathcal{M}$ is usc on Ω, for any $\theta\in\Omega$.*

(2) *$Graph(\tilde{\mathcal{M}})$ is closed, if $\mathcal{M}$ is usc on Ω and ζ is lsc on Ω.*

Lemma 2.5. [32] *Let Θ and Υ be two Hausdorff topological vector spaces and denote by Ω be a nonempty set of Θ, define a fuzzy mapping $\mathcal{M}$ from Ω to $\mathcal{F}(\Upsilon)$ and a function ζ from Ω to $[0,1]$. If $\mathcal{M}$ is convex, then for any $\theta\in\Omega,(\mathcal{M}_\theta)_{\zeta(\theta)}$ is also convex.*

Lemma 2.6. (Berge Maximum Theorem [25]) *Denote by Θ and Υ two Hausdorff topological spaces, let $g:\Theta\times\Upsilon\to R$ be continuous, and suppose $F:\Upsilon\to P_0(\Theta)$ is a continuous correspondence both these two topological spaces, for each $\upsilon\in\Upsilon,F(\upsilon)$ is a compact set. Then:*

(1) *$V(\upsilon)=\max\limits_{\theta\in F(\upsilon)}g(\theta,\upsilon)$ is a continuous function.*

(2) *The set-values correspondence $M(\upsilon)=\{\theta\in F(\upsilon):g(\theta,\upsilon)=V(\upsilon)\}$ in usc on Υ .*

Lemma 2.7. (Fan-Glicksberg Fixed Point Theorem [25]) *Denote by Ω a nonempty convex compact set in Hausdorff locally convex linear topological space E, set-valued correspondence $G:\Omega\to P_0(\Omega)$ satisfies for any $\omega\in\Omega,G(\omega)$ is a nonempty convex compact set, and G is usc on Ω, then there exist $\omega^*\in\Omega$, such that $\omega^*\in G(\omega^*)$.*

Lemma 2.8. [25] *Let Θ and Υ be Hausdorff topological spaces, $\{S_\alpha\}$ and S be a group of nonempty compact net and a nonempty compact set of Θ, respectively. $\{S_\alpha\}\to S$, $\{\upsilon^\alpha\}$ is a net of Υ, $\{\upsilon^\alpha\}\to\upsilon\in\Upsilon$, $\{f^\alpha(\omega,\upsilon)\}$ is a family of continuous functions net on $\Theta\times\Upsilon$, $\sup\limits_{(\omega,\upsilon)\in\Theta\times\Upsilon}|f^\alpha(\omega,\upsilon)-f(\omega,\upsilon)|\to0$, where $f(\omega,\upsilon)$ is a continuous function on $\Theta\times\Upsilon$, then $\max\limits_{\omega\in S_\alpha}f^\alpha(\omega,\upsilon^\alpha)\to\max\limits_{\omega\in S}f(\omega,\upsilon)$.*

Lemma 2.9. [1] *Denote by Θ and Υ two Hausdorff topological spaces, for the set-valued mapping $\varphi:\Theta\to\Upsilon$, for any $\omega\in\Theta$, the two following sentences are equivalent:*

(1) *φ is usc on ω and $\varphi(\omega)$ is compact.*

(2) *For any $\{(\omega_\alpha,\upsilon_\alpha)\}\in Graph(\varphi)$, that is for any α, satisfied $\{\upsilon_\alpha\}\in\varphi(\omega_\alpha)$, if $\omega_\alpha\to\omega$, then $\{\upsilon_\alpha\}$ has a limit point in $\varphi(\omega)$.*

Lemma 2.10. [1] *Let Θ and Υ be Hausdorff topological spaces, for the mapping $\varphi:\Theta\to\Upsilon$, we have the following two equivalent descriptions:*

(1) *φ is lsc at a point ω.*

(2) *If $\omega_\beta\to\omega$, for any $\upsilon\in\varphi(\omega)$, there exist a subsequence $\{\omega_{\beta_\delta}\}_{\delta\in\Delta}$ of $\{\omega_\beta\}$ and a net $\{\upsilon_\delta\}_{\delta\in\Delta}$ of Υ, such that $\upsilon_\delta\in\varphi(\omega_{\beta_\delta}),\upsilon_\delta\to\upsilon$, for any $\delta\in\Delta$.*

Lemma 2.11. [1] *Products of mappings have the following two statements:*

(1) *The product of a family of usc mappings with compact values is also a usc mapping with compact values.*

(2) *The product of a finite family of lsc mappings is also lsc.*

3 Multi-leader-follower games with fuzzy constrained mappings and its equilibrium existence theorems

This section will first depict the model for the MLFFGs considered in this article:

$$\delta=((M_k,P_k,p_k,\phi_k)_{k\in K},N,T).$$

Where, the MLFFGs has q leaders and n followers, with $K = \{1, 2, \ldots, q\}$ denoting the set of leaders and $L = \{1, \ldots, n\}$ denoting the set of followers. Let M_k and Y_l be the strategy sets of the k-th leader and l-th follower, respectively. Let $-k = K \setminus \{k\}$, then $M_{-k} = \prod_{i \in K, i \neq k} M_i$ and denote the leader's strategy profile $m \in M = \prod_{k \in K} M_k$. Let the fuzzy mapping $P_k : M_{-k} \to \mathcal{F}(M_k)$ be the feasible constraint mapping for the k-th leader, $p_k : M_{-k} \to [0, 1]$ be the feasibility level mapping, $\phi_k : M \times N \to R$ be the objective mapping for the k-th leader, where $N = \prod_{l \in L} Y_l$ represents the Cartesian product of the strategy spaces of all followers. For each $l \in L$, $C_l : M \times Y_{-l} \to P_0(Y_l)$ be a constrained correspondence and $\eta_l : M \times Y_l \times Y_{-l} \to R$ be an objective mappings of the l-follower. Let $T : M_k \times M_{-k} \to N$ denote the equilibrium mapping of the follower problem (a generalized game parameterized by the leader's strategies), i.e., for any $(m_k, m_{-k}) \in M_k \times M_{-k}, T(m_k, m_{-k})$ is the set of equilibrium points of the followers problem. More specifically, for each $(y_l, y_{-l}) \in T(m_k, m_{-k})$, we have the fact that $(y_l, y_{-l}) \in N$ such that for each $l \in L$, there exists

$$y_l \in C_l(m, y_{-l}), \quad \eta_l(m, y_l, y_{-l}) = \min_{u_l \in C_l(m, y_{-l})} \eta_l(m, u_l, y_{-l}).$$

Definition 3.1. *The strategy profile $(m^*, n^*) = (m_1^*, m_2^*, \ldots m_q^*, n_1^*, n_2^*, \ldots n_q^*) \in M \times N^q$ is called an equilibrium point of a MLFFG, if for any $k \in K, (m_k^*, n_k^*)$ satifies:*
(1) *$n_k^* \in T(m_k, m_{-k}^*)$, i.e., n_k^* is an equilibrium of the generalized game of followers parameterized by the leader's strategies.*
(2) *(m_k^*, n_k^*) is an optimal solution to the k-th leader's problem:*

$$\phi_k(m_k^*, m_{-k}^*, n_k^*) = \min \phi_k(m_k, m_{-k}^*, n_k), \forall m_k \in \tilde{P}_k(m_{-k}^*), n_k \in T(m_k, m_{-k}^*).$$

Where $\tilde{P}_k(m_{-k}) = (P_{k(m_{-k})})_{p_k(m_{-k})} = \left\{ m_k \in M_k : P_{k(m_{-k})}(m_k) \geq p_k(m_{-k}) \right\}$.

Remark 3.2. *In* [4]*, Ding proposed the classical model of multi-leader-follower games with generalized constrained multiobjective, where the leader's constraint mapping is certain, which makes the feasible strategy sets of leaders have well-defined boundaries, and for the leaders, the feasibility degree of each feasible strategy is 1. However, in the MLFFGs considered in this paper, the leader's constraint mapping is a fuzzy mapping, and strategy set M_k is a fuzzy set. It is clear that when the feasibility level $p_k(m_{-k})$ is 1, the constraint mapping P_k degenerates into the classical constraint mapping in* [4]*. Therefore, the application of MLFFGs is broader than that of the classical multi-leader-follower games. Compared to the* [3]*, we consider the situation where there exists a feasible constraint mapping between followers rather than without a constraint mapping.*

Theorem 3.3. *Let M_k and N be nonempty convex compact sets in locally convex Hausdorff topological vector spaces R^{n_k} and R^l , respectively, if the game δ satisfies:*
(1) *For each $k \in K, \phi_k : M \times N \to R$ is continuous.*
(2) *The fuzzy mapping $P_k : M_{-k} \to \mathcal{F}(M_k)$ is usc with convex value, for any $m_{-k} \in M_{-k}, P_k(m_{-k})$ is nonempty convex. $p_k : M_{-k} \to [0, 1]$ is lsc.*
(3) *For any $m_{-k} \in M_{-k}, (m_k, n_k) \to \phi_k(m_k, m_{-k}, n_k)$ is quasi-convex.*
(4) *The set-value mapping $T : M_k \times M_{-k} \to P_0(N)$ is continuous, and for any $(m_k, m_{-k}) \in M_k \times M_{-k}, T(m_k, m_{-k})$ is nonempty compact convex value.*
(5) *The set-valued mapping defined by for any $m_{-k} \in M_{-k}, w_k \to T(w_k, m_{-k})$ is convex.*
Then, there exists equilibrium point for MLFFG δ.

Proof. (1) First, define set-valued mappings $F_k : M_{-k} \times N \to P_0(M_k \times N)$ as follows:

$$\begin{aligned} F_k(m_{-k}, n_{-k}) = \Big\{ & m_k \in \tilde{P}_k(m_{-k}), n_k \in T(m_k, m_{-k}) : \phi_k(m_k, m_{-k}, n_k) \\ & = \min \phi_k(\omega_k, m_{-k}, v_k), \forall \omega_k \in \tilde{P}_k(m_{-k}), v_k \in T(m_k, m_{-k}) \}. \end{aligned}$$

Since $P_k : M_{-k} \to \mathcal{F}(M_k)$ is usc, by Lemma 2.4, we know $\tilde{P}_k(m_{-k})$ is compact, T is continuous and $T(m_k, m_{-k})$ is compact, thus $T(m_k, m_{-k}) = \{T(a_k, m_{-k}) : a_k \in \tilde{P}_k(m_{-k})\}$ is compact. Again, since ϕ_k is continuous, then $F_k(m_{-k}, n_{-k}) \neq \emptyset$. And it's easy to prove $F_k(m_{-k}, n_{-k})$ is closed, therefore, $F_k(m_{-k}, n_{-k})$ is compact.

For any $(m_k^1, n_k^1), (m_k^2, n_k^2) \in F_k(m_{-k}, n_{-k})$, then $m_k^1, m_k^2 \in \tilde{P}_k(m_{-k})$, $n_k^1 \in T(m_k^1, m_{-k}), n_k^2 \in T(m_k^2, m_{-k})$ and

$$\begin{aligned} \phi_k(m_k^1, m_{-k}, n_k^1) &= \min_{w_k \in \tilde{P}_k(m_{-k}), v_k \in T(w_k, m_{-k})} \phi_k(\omega_k, m_{-k}, v_k), \\ \phi_k(m_k^2, m_{-k}, n_k^2) &= \min_{w_k \in \tilde{P}_k(m_{-k}), v_k \in T(w_k, m_{-k})} \phi_k(\omega_k, m_{-k}, v_k). \end{aligned}$$

Hence, for any $\omega_k \in \tilde{P}_k(m_{-k}), v_k \in T(\omega_k, m_{-k})$, we have

$$\phi_k(m_k^1, m_{-k}, n_k^1) = \phi_k(m_k^2, m_{-k}, n_k^2) \leq \phi_k(\omega_k, m_{-k}, v_k).$$

For any $\alpha \in (0,1)$, since the convexity of P_k, and by Lemma 2.5, for each $m_{-k} \in M_{-k}$, $\tilde{P}_k(m_{-k})$ is convex on M_k, i.e., $\alpha m_k^1 + (1-\alpha)m_k^2 \in \tilde{P}_k(m_{-k})$. By the assumption that, we have

$$\alpha n_k^1 + (1-\alpha)n_k^2 \in \alpha T(m_k^1, m_{-k}) + (1-\alpha)T(m_k^2, m_{-k}) \subset T(\alpha m_k^1 + (1-\alpha)m_k^2, m_{-k}).$$

Thus,

$$\phi_k(\alpha m_k^1 + (1-\alpha)m_k^2, m_{-k}, \alpha v_k^1 + (1-\alpha)v_k^2) \leq \max\{\phi_k(m_k^1, m_{-k}, n_k^1), \phi_k(m_k^2, m_{-k}, n_k^2)\} \leq \phi_k(\omega_k, m_{-k}, v_k).$$

Then, $F_k(m_{-k}, n_{-k})$ is convex.

The second step, we need to prove the set-value mappings

$$H(m_{-k}) = \{(\omega_k, v_k) \in M_k \times N : \omega_k \in \tilde{P}_k(m_{-k}), v_k \in T(\omega_k, m_{-k})\},$$

is continuous.

$\forall m_{-k}^n \in M_{-k}, n = 1, 2, 3, \ldots, m_{-k}^n \to m_{-k}, \forall(\omega_k^n, v_k^n) \in H(m_{-k}^n), (\omega_k^n, v_k^n) \to (\omega_k, v_k) \in \tilde{P}_k(m_{-k}) \times T(\omega_k, m_{-k})$. Noting that T is usc, $T(\omega_k, m_{-k})$ is compact, $\omega_k^n \to \omega_k, v_k^n \in T(\omega_k, m_{-k}), v_k^n \to v_k$, Lemma 2.9 implies that $v_k \in T(\omega_k, m_{-k})$. Thus $(\omega_k, v_k) \in \tilde{P}_k(m_{-k}) \times T(\omega_k, m_{-k})$, that is $(\omega_k, v_k) \in H(m_{-k})$. Hence the graph of the set-value H is closed, since $\tilde{P}_k(m_{-k}) \times T(\omega_k, m_{-k})$ is compact, H is usc.

Additionally, assume for any $m_{-k}^n \in M_{-k}, n = 1, 2, 3, \cdots$ and $m_{-k}^n \to m_{-k}$, for any $(w_k, v_k) \in H(m_{-k}^n)$, then $\omega_k \in \tilde{P}_k(m_{-k}), v_k \in T(\omega_k, m_{-k})$. Since T is lower continuous, by Lemma 2.10, there exists $v_k^n \in T(\omega_k, m_{-k}^n)$ with $v_k^n \to v_k$. It indicates $(w_k, v_k^n) \in H(m_{-i}^n)$ and $(w_k^n, v_k^n) \to (w_k, v_k)$, also by Lemma 2.10, H is lsc, and therefore H is continuous.

From the above proof, we know H is continuous with $H(m_{-k})$ is compact, by Lemma 2.6, F_k is usc, define set-value mapping $F : M_k \times M_{-k} \times N^q \to P_0(M_k \times M_{-k} \times N^q), F(m,n) = \prod_{k\in K} F_k(m_{-k}, n_{-k})$, it's clear that $F_k(m_{-k}, n_{-k})$ is upper continuous with nonempty convex compact values, combining Lemma 2.11, F is usc with convex-valued and compact-valued, by Lemma 2.7, there exist $(m^*, n^*) \in M \times N^n$, satisfies $(m^*, n^*) \in F(m^*, n^*) = \prod_{k\in K} F_k(m_{-k}^*, n_{-k}^*)$ and $(m_k^*, n_k^*) \in F_k(m_{-k}^*, n_{-k}^*)$, we obtain that:

$$\phi_k(m_k^*, m_{-k}^*, n_k^*) = \min \phi_k(m_k, m_{-k}^*, n_k), \forall m_k \in \tilde{P}_k(m_{-k}^*), n_k \in T(m_k, m_{-k}^*).$$

□

Remark 3.4. *Theorem 3.3 provides the existence theorem of equilibrium point for MLFFGs in Hausdorff topological vector spaces, which is different from the existence theorem of equilibrium point given in [4] in locally finitely continuous topological spaces(brief, FC-uniform spaces). This also shows that Theorem 3.3 of this paper is a new result.*

Example 3.5. *Consider an MLFFG $\delta = ((M_k, P_k, p_k, \phi_k)_{k\in K}, N, T)$ with two leaders and two followers, where $K = \{1,2\}$, $L = \{1,2\}$. The strategy sets of leader 1 and 2 are $M_1 = M_2 = [0,2]$, and the strategy sets for followers 1 and 2 are $Y_1 = Y_2 = [0,1]$, respectively. Let $P_k : [0,2] \to \mathcal{F}([0,2])$ be the fuzzy constrain mappings, where*

$$P_{1(m_2)}(m_1) = \begin{cases} \frac{1}{2}, & m_2 \in [0,2], m_1 \in [0,1] \\ 0, & m_2 \in [0,2], m_1 \in (1,2] \end{cases}, \quad P_{2(m_1)}(m_2) = \begin{cases} \frac{1}{2}, & m_1 \in [0,2], m_2 \in [0,1] \\ 0, & m_1 \in [0,2], m_2 \in (1,2] \end{cases}.$$

And $p_k : [0,2] \to [0,1] : p_1(m_2) = p_2(m_1) = \frac{1}{3}$, then we know $\tilde{P}_1(m_2) = (P_{1(m_2)})_{p_1(m_2)} = [0,1]$ and $\tilde{P}_2(m_1) = (P_{2(m_1)})_{p_2(m_1)} = [0,1]$.

Let the constrainted correspondence of followers 1 and 2 be $C_1, C_2 : [0,2] \times [0,2] \times [0,1] \to P_0([0,1])$:

$$C_1(m_1, m_2, y_2) = \begin{cases} [\frac{1}{2}, 1], & y_2 \in [0, \frac{1}{3}), \ (m_1, m_2) \in [0, \frac{1}{2}) \times [0,1] \\ [0, \frac{1}{3}], & y_2 \in [\frac{1}{3}, 1], \ (m_1, m_2) \in [\frac{1}{2}, 1] \times [0,1] \\ [\frac{1}{3}, \frac{1}{2}], & y_2 \in [\frac{1}{3}, 1], \ (m_1, m_2) \in [0, \frac{1}{2}) \times [0,1] \\ [\frac{1}{2}, 1], & y_2 \in [0, \frac{1}{3}), \ (m_1, m_2) \in [\frac{1}{2}, 1] \times [0,1] \end{cases},$$

$$C_2(m_1,m_2,y_1)=\begin{cases}[0,\frac{1}{6}],\ y_1\in[\frac{2}{3},1],\ (m_1,m_2)\in[0,1]\times[\frac{1}{2},1]\\ [\frac{1}{6},\frac{1}{3}],\ y_1\in[\frac{2}{3},1],\ (m_1,m_2)\in[0,1]\times[0,\frac{1}{2})\\ [\frac{1}{3},\frac{2}{3}],\ y_1\in[0,\frac{2}{3}),\ (m_1,m_2)\in[0,1]\times[0,\frac{1}{2})\\ [\frac{2}{3},1],\ y_1\in[0,\frac{2}{3}),\ (m_1,m_2)\in[0,1]\times[\frac{1}{2},1]\end{cases},$$

and let the object mappings $\eta_1,\eta_2:[0,2]\times[0,2]\times[0,1]\times[0,1]\to R$ *of the followers be*

$$\eta_1(m_1,m_2,y_1,y_2)=\begin{cases}m_1+m_2+4y_1-3y_2, & y_1\in[0,\frac{1}{2}],\ \ y_2\in[\frac{1}{3},1],\ \ \forall(m_1,m_2)\in M\\ m_1-m_2+y_1+\frac{y_2}{3}, & y_1\in[\frac{1}{2},\frac{2}{3}),\ \ y_2\in[\frac{1}{3},1],\ \ \forall(m_1,m_2)\in M\\ m_1+m_2+y_1-y_2, & y_1\in[\frac{2}{3},1],\ \ y_2\in[0,\frac{2}{3}],\ \ \forall(m_1,m_2)\in M\end{cases},$$

$$\eta_2(m_1,m_2,y_1,y_2)=\begin{cases}4-m_1-m_2, & y_1\in[0,\frac{1}{2}],\ \ y_2\in[\frac{1}{3},1],\ \ \forall(m_1,m_2)\in M\\ 3-m_1-m_2+y_1-y_2, & y_1\in(\frac{1}{2},1],\ \ y_2\in[0,\frac{1}{3}),\ \ \forall(m_1,m_2)\in M\end{cases},$$

respectively. It is obvious that the equilibrium points of the generalized game parameterized by leader's strategies are $(1,0)$ *and* $(\frac{1}{2},\frac{1}{3})$*, which also means that* $\forall(m_1,m_2)\in M$*,* $T(m_1,m_2)=\{(1,0),(\frac{1}{2},\frac{1}{3})\}$.

Let the objective functions of leader 1 and 2 be $\phi_1(m,n)=m_1+m_2-y_1-y_2$ *and* $\phi_2(m,n)=m_1+m_2+y_1+y_2,\forall(m,n)\in M\times N$*, respectively. Next, we need to find the point* $(m_1^*,m_2^*,n_1^*,n_2^*)$ *which satisfies*

$$\phi_1(m_1^*,m_2^*,n_1^*)=\min\phi_1(m_1,m_2^*,n_1),\forall m_1\in\tilde{P}_1(m_2),n_1\in T(m_1,m_2^*),$$

$$\phi_2(m_1^*,m_2^*,n_2^*)=\min\phi_2(m_1^*,m_2,n_2),\forall m_2\in\tilde{P}_2(m_1),n_2\in T(m_1^*,m_2).$$

It is not difficult to prove that $\delta=((M_k,P_k,p_k,\phi_k)_{k\in K},N,T)$ *satisfies all conditions in Theorem 3.3 and if* $(m_1,n_1)=(0,(1,0)),(m_2,n_2)=(0,(\frac{1}{2},\frac{1}{3}))$*,* $\phi_1(m_1,m_2^*,n_1)$ *and* $\phi_2(m_1^*,m_2,n_2)$ *can achieve the minimum value, respectively. Thus, the equilibrium point of* δ *is* $(m_1^*,m_2^*,n_1^*,n_2^*)=(0,0,(1,0),(\frac{1}{2},\frac{1}{3}))$.

4 Some well-posedness of the fuzzy multi-leader-follower games

Let M_k,N be complete compact metric space and Δ be the problem space of MLFFG $\delta=((M_k,P_k,p_k,\phi_k)_{k\in K},N,T)$ (briefly, $\delta=(P_k,p_k,\phi_k,T)$), defined as follows:

$$\Delta=\left\{\delta=(\mathrm{P_k},\mathrm{p_k},\phi_\mathrm{k},\mathrm{T}):\begin{array}{l}\cdot\,P_k:\ M_{-k}\to\mathcal{F}(M_k)\text{ is usc with convex values, }p_k:\ M_{-k}\to[0,1]\text{ is lsc.}\\ \cdot\,\phi_k:\ M\times N\to\ R\text{ is a continuous mapping, for any }k\in K.\\ \cdot\,(m_k,n_k)\to\phi_k(m_k,m_{-k},n_k)\text{ is quasi}-\text{convex, for any }m_{-k}\in M_{-k}.\\ \cdot\,T:\ M_k\times M_{-k}\to P_0(N)\text{ is continuous with nonempty compact convex value.}\\ \cdot\,w_k\to T(w_k,m_{-k})\text{ is convex, for any }m_{-k}\in M_{-k}.\\ \cdot\,\text{For all }k\in K,\text{ there exist }(m^*,n^*)\in M\times N^q\text{ such that}\\ \quad\phi_k(m_k^*,m_{-k}^*,n_k^*)=\min\phi_k(m_k,m_{-k}^*,n_k),\forall m_k\in\tilde{P}_k(m_{-k}^*),n_k\in T(m_k,m_{-k}^*).\end{array}\right\}$$

Consider the finite rationality model of the MLFFGs as: $B=(\Delta,Z=M\times N^n,G,\Phi)$, where Δ is the game space, for each $\delta\in\Delta,\delta$ is a MLFFG. Z is the action space, for each z in Z, z denotes a strategy profile of the MLFFGs, define the feasibility mapping $G(\delta,z)=G(z)=\prod\limits_{k\in K}F_k(m_{-k},n_{-k})$, then the behavioral mapping induced by G is $g(\delta)=\{z\in Z:z\in G(\delta,z)\}=\{z\in Z:z\in G(z)\}$. And the rationality mapping of the MLFFGs is defined by:

$$\Phi(\delta,z)=\max_{1\le k\le q}[\phi_k(m_k,m_{-k},n_k)-\min_{w_k\in\tilde{P}_k(m_{-k}),v_k\in T(w_k,m_{-k})}\phi_k(w_k,m_{-k},v_k)].$$

Employ $E(\delta)$ to express the equilibrium point set of the MLFFGs, obviously $E(\delta)=\{z\in G(z):\Phi(\delta,z)=0\}$.

For any $\delta_1=(P_{1k},p_{1k},\phi_{1k},T_1),\delta_2=(P_{2k},p_{2k},\phi_{2k},T_2)\in\Delta$, define the distance on Δ by:

$$\begin{aligned}\rho(\delta_1,\delta_2)=&\sup_{(m,n)\in M\times N}\sum_{k\in K}\mid\phi_{1k}(m,n)-\phi_{2k}(m,n)\mid+\sup_{m_{-k}\in M_{-k}}\sum_{k\in K}\mid p_{1k}(m_{-k})-p_{2k}(m_{-k})\mid\\&+\sup_{m\in M}\sum_{k\in K}\mid P_{1k(m_{-k})}(m_k)-P_{2k(m_{-k})}(m_k)\mid+\sup_{(m,n)\in M\times N}h(T_1(m),T_2(m)),\end{aligned}$$

where h is the Hausdorff distance of $M\times N$.

Theorem 4.1. (Δ, ρ) *is a complete metric space.*

Proof. Assume $\{\delta_n\}$ is any *Cauchy* sequence in Δ, and $\forall \varepsilon > 0$, there exists a positive integer $N(\varepsilon)$, for any $\alpha, \beta \geq N(\varepsilon)$, satisfying

$$\rho(\delta_\alpha, \delta_\beta) = \sup_{(m,n)\in M\times N} \sum_{k\in K} \mid \phi_{\alpha k}(m,n) - \phi_{\beta k}(m,n) \mid + \sup_{m_{-k}\in M_{-k}} \sum_{k\in K} \mid p_{\alpha k}(m_{-k}) - p_{\beta k}(m_{-k}) \mid$$
$$+ \sup_{m\in M} \sum_{k\in K} \mid P_{\alpha k(m_{-k})}(m_k) - P_{\beta k(m_{-k})}(m_k) \mid + \sup_{(m,n)\in M\times N} h(T_\alpha(m), T_\beta(m)) \leq \varepsilon.$$

For any $(m,n) \in M \times N$, any $k \in K$, there exists $\phi_k(m,n), P_{k(m_{-k})}(m_k)$ and $p_k(m_{-k})$ such that $\lim\limits_{\beta\to\infty} \phi_{\beta k}(m,n) = \phi_k(m,n)$, $\lim\limits_{\beta\to\infty} P_{\beta k(m_{-k})}(m_k) = P_{k(m_{-k})}(m_k)$, $\lim\limits_{\beta\to\infty} p_{\beta k}(m_{-k}) = p_k(m_{-k})$, respectively. Obviously, there exists a compact set $T(m) \subset N$, satisfying $\lim\limits_{\beta\to\infty} T_\beta(m) = T(m)$, for any $\beta \geq N(\varepsilon)$, then

$$\sup_{(m,n)\in M\times N} \sum_{k\in K} \mid \phi_{\alpha k}(m,n) - \phi_k(m,n) \mid + \sup_{m_{-k}\in M_{-k}} \sum_{k\in K} \mid p_{\alpha k}(m_{-k}) - p_k(m_{-k}) \mid$$
$$+ \sup_{m\in M} \sum_{k\in K} \mid P_{\alpha k(m_{-k})}(m_k) - P_{k(m_{-k})}(m_k) \mid + \sup_{(m,n)\in M\times N} h(T_\alpha(m), T(m)) \leq \varepsilon.$$

It can be easily verified that $\phi_k(m,n)$ is continuous on $M \times N$, P_k is usc on M_{-k} with convex values, p_k is lsc on M_{-k}, and T is continuous on M. Since $(m_k, n_k) \to \phi_{\beta k}(m_k, m_{-k}, n_k)$ is quasi-convex, for any $(m_k^1, n_k^1), (m_k^2, n_k^2) \in M_k \times N, \lambda \in (0,1)$, we have $\phi_{lk}(\lambda m_k^1 + (1-\lambda)m_k^2, m_{-k}, \lambda n_k^1 + (1-\lambda)n_k^2) \leq \max\{\phi_{lk}(m_k^1, m_{-k}, n_k^1), \phi_{lk}(m_k^2, m_{-k}, n_k^2)\}$, and since $\phi_{lk}(m_k, m_{-k}, n_k) \to \phi_k(m_k, m_{-k}, n_k)$, we conclude that

$$\phi_k(\lambda m_k^1 + (1-\lambda)m_k^2, m_{-k}, \lambda n_k^1 + (1-\lambda)n_k^2) \leq \max\{\phi_k(m_k^1, m_{-k}, n_k^1), \phi_k(m_k^2, m_{-k}, n_k^2)\}.$$

Therefore, for any $(m_k, n_k) \in M_k \times N, (m_k, n_k) \to \phi_k(m_k, m_{-k}, n_k)$ is quasi-convex. Similarly, since $T(m)$ is continuous, this indicates that for each $m_{-k} \in M_{-k}, \omega_k \to T(\omega_k, m_{-k})$ is convex.

From $\delta_l = (P_{lk}, p_{lk}, \phi_{lk}, T_l) \in \Delta$, we know that there exists $(m_1^l, m_2^l, \ldots, m_q^l, n_1^l, \ldots, n_q^l) \in M \times N^q$, such that $\phi_{lk}(m_k^l, m_{-k}^l, n_k^l) = \min \phi_{lk}(m_k, m_{-k}^l, n_k), \forall m_k \in \tilde{P}_k(m_{-k}^l), n_k \in T_l(m_k, m_{-k}^l)$.

Next, we prove the completeness of $\tilde{P}_k(m_{-k})$, by contradiction, suppose that for any $k \in K, m_k^l \in \tilde{P}_{lk}(m_{-k}), m_k^l \to m_k, m_k \notin \tilde{P}_{lk}(m_{-k})$, then $P_{k(m_{-k})}(m_k) < p_k(m_{-k})$, there exists $\kappa > 0$ satisfying $p_k(m_{-k}) - P_{k(m_{-k})}(m_k) > \kappa > 0$. On the other hand, since $P_{lk} \to P_k, p_{lk} \to p_k$, when l is sufficiently large, such that $p_k^l(m_{-k}^l) - P^l_{k(m_{-k}^l)}(m_{-k}^l) > p_k(m_{-k}^l) - P_{k(m_{-k}^l)}(m_{-k}^l) - \kappa/2 > 0$. It's apparent that $m_k^l \in \tilde{P}_k(m_{-k}^l)$, which is contradictory.

Note that $\tilde{P}_k(m_{-k}), N$ is nonempty compact and complete, suppose that $(m_k^l, n_k^l) \to (m_k^*, n_k^*)$, $m_k^* \in \tilde{P}_k(m_{-k}^*), n_k \in N$. Thus $h(T_l(m_k, m_{-k}^l), T(m_k, m_{-k}^*)) \leq h(T_l(m_k, m_{-k}^l), T(m_k, m_{-k}^l)) + h(T(m_k, m_{-k}^l), T(m_k, m_{-k}^*)) \to 0$, that is, $h(T_l(m_k, m_{-k}^l), T(m_k, m_{-k}^l)) \to 0$. Denote d the distance of the Hausdorff linear topological space, we have

$$d(n_k^*, G(m_k, m_{-k}^*)) \leq d(n_k^*, n_{-k}^l)) + d(n_k^l, G(m_k^l, m_{-k}^l) + h(G_l(m_k^l, m_{-k}^l), G(m_k, m_{-k}^*)).$$

Since $n_k^l \to n_k^*, n_k^l \in T_l(x_k^l, x_{-k}^l)$, we know that $d(n_k^*, T(m_k, m_{-k}^*)) \to 0$, thus $n_k^* \in T(m_k, m_{-k}^*)$. For any $w_k \in \tilde{P}_k(m_{-k})$, because of the completeness of $\tilde{P}_k(m_{-k})$, there exist $w_k^l \to w_k$, in addition, we have proved that $h(T_l(m_k, m_k^l), T(m_k, m_{-k}^*)) \to 0$, hence, there exists $v_k^l \to v_k$, then $\mid \phi_{lk}(m_k^l, m_{-k}^l, n_k^l) - \phi_k(m_k^*, m_{-k}^*, n_k^*) \mid \leq \mid \phi_{lk}(m_k^l, m_{-k}^l, n_k^l) - \phi_k(m_k^l, m_{-k}^l, n_k^l) \mid + \mid \phi_k(m_k^l, m_{-k}^l, n_k^l) - \phi_k(m_k^*, m_{-i}^*, n_k^*) \mid \to 0$. That is $\phi_{lk}(m^l, n_k^l) \to \phi_k(m^*, n_k^*)$ and

$$\phi_k(w_k, m_{-k}^*, v_k) = \lim_{l\to\infty} \phi_k(w_k^l, m_{-k}^*, v_k^l) = \lim_{l\to\infty} \phi_{mk}(w_k^l, m_{-k}^*, v_k^l)$$
$$\geq \lim_{l\to\infty} \phi_{mk}(m_k^l, m_{-k}^l, n_k^l) = \phi_k(m_k^*, m_{-k}^*, n_k^*).$$

Therefore, for any $k \in K, \phi_k(m^*, n^*) = \min \phi_k(m_k, m_{-k}^*, n_k), m_k \in \tilde{P}_k(m_{-k}^*), n_k \in T(m_k, m_{-k}^*)$, and thus $\delta = (P_k, p_k, \phi_k, T) \in \Delta, (\Delta, \rho)$ is complete metric space. □

Theorem 4.2. $\Phi(\delta, z)$ *is continuous on* (δ, z)*, for any* $(\delta, z) \in (\Delta, Z)$.

Proof. To prove that Theorem 4.1 holds, i.e., to prove that for any $\varepsilon > 0, \forall \delta_l = (P_{lk}, p_{lk}, \phi_{lk}, T_l) \in \Lambda, \delta_l \to \delta$, for any $z^l \in Z, z^l \to z$, there exist $\Phi(\delta_l, z^l) \to \Phi(\delta, z)$. For any $k \in K, m^l_{-k} \to m_{-k}$. It can be verified that $h(T_l(w_k, m^l_{-k}), T(w_k, m_{-k})) \leq h(T_l(w_k, m^l_{-k}), T(w_k, m^l_{-k})) + h(T(w_k, m^l_{-k}), T(w_k, m_{-k}) \to 0$. That is, $T_l(m_k, m^l_{-k}) \to T(m_k, m_{-k})$.

By Berge Maximum Theorem, we know that

$$\min_{w_k \in \tilde{P}_k(m_{-k}), v_k \in T(w_k, m_{-k})} \phi_k(w_k, m_{-k}, v_k),$$

is continuous, and by Lemma 2.8, there exists

$$\min \phi_{lk}(w_k, m_{-lk}, v_k) \to \min \phi_k(w_k, m_{-k}, v_k), \forall w_k \in \tilde{P}_k(m_{-k}), v_k \in T(w_k, m_{-k}).$$

Since ϕ_k is continuous on $\tilde{P}_k(m_{-k}) \times T(m_k, m_{-k})$, thus

$$\phi_{lk}(m_{lk}, m_{-k}, n_{lk}) \to \phi_k(m_{lk}, m_{-k}, n_{lk}).$$

The proof of Theorem 4.1 leads to the following conclusion

$$\phi_{lk}(m_{lk}, m_{-lk}, n_{lk}) \to \phi_k(m_k, m_{-k}, n_k).$$

In summary, it can be seen that

$$\begin{aligned} &\max_{1 \leq k \leq n} [\phi_{lk}(m_{lk}, m_{-lk}, n_{lk}) - \min_{w_k \in \tilde{P}_k(m_{-k}), v_k \in T(w_k, m_{-k})} \phi_{lk}(w_k, m_{-lk}, v_k)] \\ &\to \max_{1 \leq k \leq n} [\phi_k(m_k, m_{-k}, n_k) - \min_{w_k \in \tilde{P}_k(m_{-k}), v_k \in T(w_k, m_{-k})} \phi_k(w_k, m_{-k}, v_k)]. \end{aligned}$$

Thus $\Phi(\delta_l, z^l) \to \Phi(\delta, z)$, $\Phi(\delta, z)$ is continuous. □

Definition 4.3. [26] *If for any $\delta_\alpha \in \Delta, \delta_\alpha \to \delta$, for any $z_\alpha \in E(\delta_\alpha, \varepsilon_\alpha)$, where $\varepsilon_\alpha \to 0$, there exists a subsequence $\{z_{\alpha_l}\}$ of $\{z_\alpha\}$, such that $z_{\alpha_l} \to z \in E(\delta)$, then the game δ is said to be generalized well-posedness(briefly, G-wp). If $E(\delta) = \{z\}$ is a single-point set, there exists $z_\alpha \to z$, and then the game δ is well-posedness(briefly, wp).*

Definition 4.4. [26] *If for any $z_\alpha \in E(\delta, \varepsilon_\alpha)$, where $\varepsilon_\alpha \to 0$, there exists a subsequence $\{z_{\alpha_l}\}$ of $\{z_\alpha\}$, such that $z_{\alpha_l} \to z \in E(\delta)$, then the game δ is said to be generalized Tykhonov well-posedness(briefly, GT-wp). If $E(\delta) = \{z\}$ is a single point set, there exists $z_\alpha \to z$, and then the game δ is Tykhonov well-posedness(briefly, T-wp).*

Definition 4.5. [26] *If for any $\delta_\alpha \in \Delta, \delta_\alpha \to \delta, \forall z_\alpha \in E(\delta_\alpha)$, there exists a subsequence $\{z_{\alpha_l}\}$ of $\{z_\alpha\}$, such that $z_{\alpha_l} \to z \in E(\delta)$, then the game δ is said to be generalized Hadamard well-posedness(briefly, GH-wp). If $E(\delta) = \{z\}$ is a single point set, there exists $z_\alpha \to z$, and then game δ is Hadamard well-posedness(briefly, H-wp).*

Theorem 4.6. (1) *For any $\delta \in \Delta$, δ is G-wp.*
(2) *If $E(\delta) = \{z\}$ (a single-point set), δ is wp.*

Proof. (1) For any $\delta_\alpha \in \Delta, \delta_\alpha \to \delta, \forall z_\alpha \in E(\delta_\alpha, \varepsilon_\alpha)$, where $\varepsilon_\alpha \to 0$, then $z_\alpha \in k(\delta_\alpha), \Phi(\delta_\alpha, z_\alpha) \leq \varepsilon_\alpha$, since g is usc with compact value, by Lemma 2.8, there exists a subsequence $\{z_{\alpha_l}\}$ of $\{z_\alpha\}$, such that $z_{\alpha_l} \to z \in g(\delta)$. Again, by $\Phi(\delta, Z)$ is lsc on (δ, z), we have $\{(\delta, z) \in \Delta \times Z : \Phi(\delta, z) - \varepsilon \leq 0\}$ is closed, suppose that $l \to \infty$, we obtain that $\Phi(\delta, z) \leq 0$, this indicate $z \in E(\delta)$. δ is G-wp.
(2) By contradiction: let $z_\alpha \nrightarrow z$, then there exists an open neighborhood O of z and a subsequence $\{z_{\alpha_l}\}$ of $\{z_\alpha\}$, that satisfies $z_{\alpha_l} \notin O$. Since $E(\delta) = \{z\}$, from the proof of (1), there exists $\{z_{\alpha_l}\} \to z$, which contradicts the fact that $z_{\alpha_l} \notin O$. □

Theorem 4.7. (1) *For any $\delta \in \Delta$, δ is GT-wp.*
(2) *If $E(\delta) = \{z\}$ (a single-point set), then δ is T-wp.*

Proof. (1) For any $z_\alpha \in E(\delta, \varepsilon_\alpha)$, there exists $\Phi(\delta, z_\alpha) \leq \varepsilon_\alpha$, where $\varepsilon_\alpha \to 0$, since g is usc with compact value, by Lemma 2.9, there exists a subsequence $\{z_{\alpha_l}\}$ of $\{z_\alpha\}$, such that $z_{\alpha_l} \to z \in g(\delta)$. Since $\Phi(\delta, Z)$ is continuous, assume that $l \to \infty$, we have $\Phi(\delta, z) = 0$, this indicate that $z \in E(\delta)$. δ is GT-wp.
(2) Same as the proof of above Theorem 4.6 (2). □

Theorem 4.8. (1) *For any $\delta \in \Delta$, δ is GH-wp.*
(2) *If $E(\delta) = \{z\}$ (a single-point set), δ is H-wp.*

Proof. Same as the proof of above Theorem 4.7. □

Example 4.9. *Consider the related well-posedness of δ in Example 3.5, let sequence $\delta_\alpha = (P_k^\alpha, p_k^\alpha, \phi_k^\alpha, T^\alpha)$ be*

$$P^\alpha_{1(m_2)}(m_1) = \begin{cases} \frac{1}{2} + \frac{m_2}{\alpha}, & m_2 \in [0,2], m_1 \in [0,1] \\ 0, & m_2 \in [0,2], m_1 \in (1,2] \end{cases}, \quad P^\alpha_{2(m_1)}(m_2) = \begin{cases} \frac{1}{2} + \frac{m_1}{\alpha}, & m_1 \in [0,2], m_2 \in [0,1] \\ 0, & m_1 \in [0,2], m_2 \in (1,2] \end{cases}$$

and $p_1^\alpha(m_2) = \frac{1}{3} + \frac{m_2}{2n}$, and $p_2^\alpha(m_1) = \frac{1}{3} + \frac{m_1}{2n}$. Similarly, we obtain $\tilde{P}_2^\alpha(m_1) = \tilde{P}_1^\alpha(m_2) = [0,1]$.

For δ_α, let the object mapping sequence $\eta_1^\alpha : [0,2] \times [0,2] \times [0,1] \times [0,1] \to R$ be

$$\eta_1^\alpha(m_1, m_2, y_1, y_2) = \begin{cases} m_1 + m_2 + 4y_1 - 3y_2 + 4(1 - \frac{2y_1}{\alpha}) - \frac{3y_2}{\alpha}, & y_1 \in [0, \frac{1}{2}], \quad y_2 \in [\frac{1}{3}, 1], \quad \forall (m_1, m_2) \in M \\ m_1 - m_2 + y_1 + \frac{y_2}{3} + \frac{y_1}{\alpha} + \frac{y_2}{\alpha}, & y_1 \in [\frac{1}{2}, \frac{2}{3}), \quad y_2 \in [\frac{1}{3}, 1], \quad \forall (m_1, m_2) \in M \\ m_1 + m_2 + y_1 - y_2 + \frac{y_1}{\alpha} - \frac{1-y_2}{\alpha}, & y_1 \in [\frac{2}{3}, 1], \quad y_2 \in [0, \frac{2}{3}], \quad \forall (m_1, m_2) \in M \end{cases},$$

$$\eta_2^\alpha(m_1, m_2, y_1, y_2) = \begin{cases} 4 - m_1 - m_2, & y_1 \in [0, \frac{1}{2}], \quad y_2 \in [\frac{1}{3}, 1], \quad \forall (m_1, m_2) \in M \\ 3 - m_1 - m_2 + y_1 - y_2 + \frac{y_1}{\alpha} - \frac{1-y_2}{\alpha}, & y_1 \in (\frac{1}{2}, 1], \quad y_2 \in [0, \frac{1}{3}), \quad \forall (m_1, m_2) \in M \end{cases},$$

when $\delta_\alpha \to \delta$, we obtain $T^\alpha(m) = \{(1,0), (\frac{1}{2}, \frac{1}{3})\}$.

Consider the objective function sequence of the leader in δ_α as follows:

$$\phi_1^\alpha(m,n) = m_1 + m_2 - y_1 - y_2 + \frac{1}{\alpha}\big[m_1 + y_2 + (1 - y_1)\big],$$
$$\phi_2^\alpha(m,n) = m_1 + m_2 + y_1 + y_2 + \frac{1}{\alpha}\big[m_2 + 2y_1 + (1 - 3y_2)\big], \forall (m,n) \in M \times N$$

It is easy to verify that $\delta_\alpha = (P_k^\alpha, p_k^\alpha, \phi_k^\alpha, T^\alpha)$, if $z_\alpha \to z$, there exists $z_\alpha = (0,0,(1,0),(\frac{1}{2},\frac{1}{3})) = z \in E(\delta)$. Therefore, δ is said to be Hadamard well-posedness(H-wp). Similarly, we can demonstrate that δ is wp and T-wp.

5 Conclusions

Classical non-cooperative game theory assumes that all players can know exactly the information of the game, but in reality, there are uncertain factors that make it impossible for the players to assess the outcome of the strategy profiles or the preferences of each player, which implies the limitations of classical game theory. Fuzzy sets provide an effective tool for analyzing this situation, expanding the application of game theory in various fields. There are two methods for dealing with the fuzziness of games using fuzzy set theory, based on the fuzzy preferences, fuzzy strategies, etc. of the players and based on the assessment of fuzzy payoffs.

Therefore, we have considered the situation where the strategy sets of the players are often uncertain and established the multi-leader-follower games with fuzzy constraint mappings(MLFFGs). Next, we proved the existence of equilibrium for the MLFFGs under certain conditions. At last, we constructed the bounded rationality model $B = (\Delta, Z = M \times N^q, G, \Phi)$ of the MLFFGs and the well-posedness of fuzzy equilibrium of the game was studied. Within the framework of well-posedness of nonlinear problems, it has been proved that every MLFFG in Δ is G-wp, GT-wp, and GH-wp.

Applying fuzzy set theory to classical leader-follower games, the fuzzy game model obtained in this paper extends the application of leader-follower games. Still, in this paper, we only use one of the two approaches provided by fuzzy sets, i.e., the fuzzy strategy-based approach. Therefore, in future research, we will work on the case where fuzzy mapping and fuzzy payoffs are applied to the leader-follower game simultaneously. Secondly, for MLFFGs, we have mainly discussed one class of its equilibrium. Will there be another definition of equilibrium point that satisfies an idea similar to Berge equilibrium, i.e., if the situation is favourable to each player, he will get the maximum payoff, and by obligation or willingness, the other players will choose the strategy that is favourable to him. This is another research issue that we are interested in.

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