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Curved splicing constructions of (quasi-)copulas with given opposite track sections

Q. Lou[1], H. M. Zhang[2] and S. Y. Ye[3]

[1,2,3] *School of Mathematics and Statistics, Guilin University of Technology, Guilin, Guangxi, 541006, China*
[2] *Guangxi College and University Key Laboratory of Applied Statistics, Guilin University of Technology, Guilin, Guangxi, 541006, China*

1758871233@qq.com, zhanghm@glut.edu.cn, 1261075681@qq.com

Abstract

In this paper, our focus is to construct (quasi-)copulas with a given opposite track section. Three different ways for constructing (quasi-)copulas with a given opposite track section are proposed and the best-possible bounds for all those (quasi-)copulas sharing a common opposite track section are also discussed. Drawing inspiration from the opposite diagonal splicing operation, we propose the curved splicing operation. We obtain that the curved splicing of any two quasi-copulas with a common opposite track section must be a quasi-copula. We also show several sufficient conditions for the curved splicing of any two copulas sharing a common opposite track section to be a copula. Furthermore, we discuss the characteristics of a copula constructed by the curved splicing operation. Finally, we construct two novel types of semilinear copulas, and by applying the curved splicing operation we obtain two additional types of semilinear copulas.

Keywords: Curved splicing, copulas, quasi-copulas, best-possible bounds, semilinear copulas.

1 Introduction

At present, many scholars have conducted extensive research on the dependence modeling analysis among random variables [1, 6, 7, 27]. According to Sklar's Theorem [32], copulas (and quasi-copulas more generally) are the functions of interest when studying the dependence between random variables, which combine the univariate marginal cumulative distribution functions $F_1, ..., F_n$ to form the joint cumulative distribution function H. Nowadays, many scholars in different fields use copulas to model and analyze the dependency relationships between variables, which makes copulas widely used in many fields [3, 14, 20, 34].

In theoretical research of (quasi-)copulas, a large amount of research was devoted to constructions of bivariate (quasi-)copulas with the pre-given information on some subdomains of the unit square. Previous studies have shown that the above subdomains may be one-dimensional, e.g., the diagonal section [29], the opposite diagonal section [13] and the curvilinear section [33], or two-dimensional, e.g., a limited number of discrete points in the unit square [31] and a rectangle of the unit square [28]. As the tail dependence coefficients of a copula are directly determined by their diagonal sections, the (quasi-)copulas with a given diagonal section have attracted significant attention[5, 15, 18, 29].

Now, many scholars have researched (quasi-)copulas with given diagonal or opposite diagonal sections. For example, Nelsen et al. [29] investigated the diagonal splicing operation to construct new copulas with given diagonal sections in 2008. One year later, Durante et al. [18] proposed a generalization of the diagonal splicing construction of copulas procedure, which results in a method for constructing symmetric copulas. At the same time, De Baets et al. [13] studied the opposite diagonal sections of quasi-copulas and the best-possible bounds for all those copulas sharing the common opposite diagonal section. Furthermore, the copulas with given diagonal and opposite diagonal sections were identified by de Amo et al. [8] in 2016.

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A significant amount of recent attention has been directed towards quasi-copulas with given track sections (also referred to as curvilinear sections) or opposite track sections. For example, Fernández-Sánchez and Úbeda-Flores [21] provided the conditions for the existence of the copulas whose track and opposite track sections are given. Based on the diagonal splicing operation, the curved splicing operation was developed and applied by Jwaid et al. in [26] to create new copulas that track sections are given. Subsequently, Zou et al. [36] established the best-possible bounds for all those (quasi-)copulas that share a common track section. Later in 2023, the curved splicing constructions of (quasi-)copulas sharing a common track section were further investigated by Xie et al. [33].

All three techniques described above are straightforward and effective for constructing new (quasi-)copulas with a given section. They essentially splice together two different functions defined on the different subareas of the unit square, where these two subareas form exactly the unit square. A large number of works have studied the above three techniques. However, few research has been done on curved splicing constructions of quasi-copulas with given opposite track sections.

The main aim of this paper is to generalize the opposite diagonal splicing operation to the curved context. Since only those copulas that share a common opposite track section can perform the curved splicing operation, we investigate the best-possible bounds for all those copulas with a common opposite track section, which allows us to easily construct the copulas with given opposite track sections that are associated with the corresponding best-possible bounds. In this paper, the copula that is obtained by the curved splicing operation is referred to as a *curved splice copula*. We study whether the characteristic of opposite symmetry w.r.t. a curve in the subdomain of the unit square acts in a similar way for curved splicing as conventional opposite symmetry does for opposite diagonal splicing. Furthermore, we develop two methods to generate semilinear copulas with a given opposite track section, which generalizes the type of semilinear copulas with a given opposite diagonal section [23] and can be readily used to generate some curved analogues of two types of copulas with given opposite diagonal sections, namely paralinear copulas [25] and biconic copulas [24].

This paper is structured as follows. Section 2 shows a few preliminaries regarding track and opposite track sections of quasi-copulas and the best-possible bounds of quasi-copulas sharing a common track section. Section 3 proposes three different ways for constructing (quasi-)copulas with a given opposite track section and discusses the best-possible bounds for all those quasi-copulas or copulas sharing a common opposite track section. In Section 4, we introduce and investigate the curved splicing of (quasi-)copulas and show several sufficient conditions for the curved splicing of any two copulas that share the common opposite track section to a copula. Section 5 discusses the characteristics of a *curved splice copula*. Section 6 introduces two methods to construct semilinear functions and applies the proposed operation to the members of proposed semilinear copulas, which results in two other types of semilinear copulas. Finally, we present the main conclusions in Section 7.

2 Preliminaries

2.1 Copula and quasi-copula

In this subsection, we present the definitions of copula and quasi-copula along with some of their associated theoretical outcomes.

Definition 2.1. *We call the bivariate function $C : [0,1]^2 \to [0,1]$ a copula if it satisfies:*

(C1) *For any $x \in [0,1]$, $C(0,x) = C(x,0) = 0$ and $C(1,x) = C(x,1) = x$ (Boundary conditions);*

(C2) *For any rectangle $H = [x_1, x_2] \times [y_1, y_2] \subset [0,1]^2$, $V_C(H) = C(x_2,y_2) + C(x_1,y_1) - C(x_2,y_1) - C(x_1,y_2) \geq 0$ (2-increasing property).*

Usually we call $V_C(H)$ the *C-volume* of the rectangle H. By Sklar's Theorem [32], for any joint cumulative distribution function G with marginals G_1 and G_2, it can be represented by a copula C and marginals G_1 and G_2, that is, $G(u,v) = C(G_1(u), G_2(v))$ for any $(u,v) \in \mathbb{R}^2$. When G_1 and G_2 are both continuous, the uniqueness of C is ensured. The copulas denoted as $M(x,y) = \min\{x,y\}$ and $W(x,y) = \max\{x+y-1, 0\}$ correspond to the Fréchet-Hoeffding upper bound and lower bound, respectively. It is known that the inequality $W(x,y) \leq C(x,y) \leq M(x,y)$ holds on $[0,1]^2$ for any copula C.

Definition 2.2. *We call the bivariate function $Q : [0,1]^2 \to [0,1]$ a quasi-copula if it satisfies:*

(Q1) *For any $x \in [0,1]$, $Q(0,x) = Q(x,0) = 0$ and $Q(1,x) = Q(x,1) = x$ (Boundary conditions);*

(Q2) *For any $x_1, x_2, y_1, y_2 \in [0,1]$ with $x_1 \leq x_2$ and $y_1 \leq y_2$, $Q(x_1,y_1) \leq Q(x_2,y_2)$ (Non-decreasing condition);*

(Q3) *For any $x_1, y_1, x_2, y_2 \in [0,1]$, $|Q(x_1,y_1) - Q(x_2,y_2)| \leq |x_1 - x_2| + |y_1 - y_2|$ (1-Lipschitz continuity condition).*

The quasi-copula is recognized as a broader concept compared to a copula [2]. In fact, every copula is a quasi-copula, but the reverse is not true, that is, there exist certain functions known as *proper* quasi-copulas that are not copulas (see [2, 4, 28]). An aggregation function A is a non-decreasing function defined on $[0,1]^2$ that satisfies $A(0,0)=0$ and $A(1,1)=1$ [17]. Hence, both of the above bivariate functions are special aggregation functions.

The following definition is referenced from [10].

Definition 2.3. *Let A be an aggregation function with neutral element 1. x-Flipping A yields the function $A^{x\text{-}flip}$ defined on $[0,1]^2$ by $A^{x\text{-}flip}(x,y)=x-A(x,1-y)$. Similarly, y-flipping A yields the function $A^{y\text{-}flip}$ defined on $[0,1]^2$ by $A^{y\text{-}flip}(x,y)=y-A(1-x,y)$.*

According to the results of [28] and [30], we can get that *x-flipping* of any quasi-copula Q yields a copula (respectively, *proper* quasi-copula) if and only if Q is a copula (respectively, *proper* quasi-copula). Furthermore, for any quasi-copula Q, we have that $(Q^{x\text{-}flip})^{x\text{-}flip}=Q$. Note that similar results can be obtained in the case of *y-flipping*.

2.2 (Quasi)-Copula with the given track section or opposite track section

We first outline the functions essential for our analysis, which are defined on $[0,1]$ with strict monotonicity. Let ϕ be an increasing homeomorphism with $\phi(0)=0$ and $\phi(1)=1$. Similarly, let ψ be a decreasing homeomorphism with $\psi(0)=1$ and $\psi(1)=0$. The graph of ϕ, represented as $Gr(\phi)$, constitutes a *track*, while the graph of ψ, represented as $Gr(\psi)$, represents, where $Gr(\phi)=\{(x,\phi(x))|x\in[0,1]\}$ and $Gr(\psi)=\{(x,\psi(x))|x\in[0,1]\})$. For simplicity, the track $Gr(\phi)$ (respectively, opposite track $Gr(\psi)$) is called the curve ϕ (respectively, curve ψ). Let Φ be the set of all increasing homeomorphisms and let Ψ be the set of all decreasing homeomorphisms. In this paper, for any $\psi\in\Psi$, the subareas of $[0,1]^2$ $D_{L,\psi}$ and $D_{U,\psi}$ are defined by $D_{L,\psi}=\{(x,y)\mid 0\le y\le\psi(x)\le 1\}$ and $D_{U,\psi}=\{(x,y)\mid 0\le\psi(x)\le y\le 1\}$, respectively.

Given that $\phi\in\Phi$, the track section $\Gamma_{Q,\phi}$ of any quasi-copula Q is given on $[0,1]$ by $\Gamma_{Q,\phi}(x)=Q(x,\phi(x))$. In a similar vein, given that $\psi\in\Psi$, the opposite track section $\omega_{Q,\psi}$ of any quasi-copula Q is given on $[0,1]$ by $\omega_{Q,\psi}(x)=Q(x,\psi(x))$. In accordance with the definition provided by [26], let $\phi,\varphi\in\Psi$ with $\varphi(x)\le\phi(x)$ for any $x\in[0,1]$, the set $U_{\phi,\varphi}$ given by $U_{\phi,\varphi}=\{(x,y)\mid 0\le\varphi(x)\le y\le\varphi^{\phi}(x)\le 1\}$ is termed as the φ-leaf with midrib ϕ, where $\varphi^{\phi}(x)=\phi\circ\varphi^{-1}\circ\phi(x)=\phi(\varphi^{-1}(\phi(x)))$. Moreover, the function F satisfying that $F(x,y)=F(\phi^{-1}(x),\phi(x))$ on $U_{\phi,\varphi}$ is called *ϕ-symmetric* on $U_{\phi,\varphi}$.

For any $\phi\in\Phi$, based on the relevant notation in [9] and [36], we call the function $\Gamma_{\phi}\colon[0,1]\to[0,1]$ a *ϕ-function* if it satisfies:

(d1) $\max\{0,x+\phi(x)-1\}\le\Gamma_{\phi}(x)\le\min\{x,\phi(x)\}$ for any $x\in[0,1]$;

(d2) $0\le\Gamma_{\phi}(x_2)-\Gamma_{\phi}(x_1)\le x_2-x_1+\phi(x_2)-\phi(x_1)$ for any $x_1,x_2\in[0,1]$ such that $x_1\le x_2$.

It is easy to see that the track section $\Gamma_{C,\phi}$ of any bivariate copula C qualifies as a *ϕ-function*. On the other hand, for any *ϕ-function* Γ_{ϕ}, there exists not less than one bivariate copula C whose track section $\Gamma_{C,\phi}$ is Γ_{ϕ}[9, 21].

Denote by $\mathbb{Q}_{\Gamma_{\phi}}$ (or $\mathbb{C}_{\Gamma_{\phi}}$) the set of all quasi-copulas (or copulas) with track section Γ_{ϕ}. For any *ϕ-function* Γ_{ϕ}, the sets $\mathbb{Q}_{\Gamma_{\phi}}$ and $\mathbb{C}_{\Gamma_{\phi}}$ are non-empty [21]. Given that Γ_{ϕ} is a *ϕ-function*, from [9] and [36], we know that the largest quasi-copula in $\mathbb{Q}_{\Gamma_{\phi}}$ is

$$A_{\Gamma_{\phi}}(x,y)=\begin{cases}\min\{y,x-\max\limits_{s\in[\phi^{-1}(y),x]}\{s-\Gamma_{\phi}(s)\}\}, & y<\phi(x),\\ \min\{x,y-\max\limits_{s\in[x,\phi^{-1}(y)]}\{\phi(s)-\Gamma_{\phi}(s)\}\}, & y\ge\phi(x),\end{cases}\tag{1}$$

and the smallest copula in $\mathbb{C}_{\Gamma_{\phi}}$ is

$$B_{\Gamma_{\phi}}(x,y)=\begin{cases}y-\min\limits_{s\in[\phi^{-1}(y),x]}\{\phi(s)-\Gamma_{\phi}(s)\}, & y<\phi(x),\\ x-\min\limits_{s\in[x,\phi^{-1}(y)]}\{s-\Gamma_{\phi}(s)\}, & y\ge\phi(x),\end{cases}\tag{2}$$

which is called the Bertino copula with track section Γ_{ϕ}. It should be noted that the smallest quasi-copula in $\mathbb{Q}_{\Gamma_{\phi}}$ is $B_{\Gamma_{\phi}}$ as well. The best-possible upper and lower bounds for $\mathbb{Q}_{\Gamma_{\phi}}$ are $A_{\Gamma_{\phi}}$ and $B_{\Gamma_{\phi}}$, respectively. Then, it follows that $B_{\Gamma_{\phi}}\le Q\le A_{\Gamma_{\phi}}$ for any quasi-copula $Q\in\mathbb{Q}_{\Gamma_{\phi}}$. Let $K_{\Gamma_{\phi}}$ be a bivariate function given by

$$K_{\Gamma_{\phi}}(x,y)=\min\{x,y,\frac{\Gamma_{\phi}(x)+\Gamma_{\phi}\left(\phi^{-1}(y)\right)}{2}\},\tag{3}$$

where $x,y\in[0,1]$. Then $K_{\Gamma_{\phi}}\in\mathbb{C}_{\Gamma_{\phi}}$ if and only if $2x-\Gamma_{\phi}(x)$ and $2x-\Gamma_{\phi}(\phi^{-1}(x))$ are non-decreasing on $[0,1]$. Moreover, if $K_{\Gamma_{\phi}}$ is a copula, then $K_{\Gamma_{\phi}}\ge C$ for all $C\in\mathbb{C}^{S}_{\Gamma_{\phi}}$, where $\mathbb{C}^{S}_{\Gamma_{\phi}}$ is the set of all copulas in $\mathbb{C}_{\Gamma_{\phi}}$ that are *ϕ-symmetric* on some leaf with midrib ϕ [36].

3 The best-possible bounds of (quasi-)copulas with a given opposite track section

For the purpose of further investigation, we first provided a required definition.

Definition 3.1. *Let $\psi \in \Psi$. We call the function $\omega_\psi : [0,1] \to [0,1)$ an opposite ψ-function if it satisfies:*

(d3) $\max\{0, x+\psi(x)-1\} \le \omega_\psi(x) \le \min\{x, \psi(x)\}$ *for all* $x \in [0,1]$;

(d4) $\psi(x_2) - \psi(x_1) \le \omega_\psi(x_2) - \omega_\psi(x_1) \le x_2 - x_1$ *for all* $x_1, x_2 \in [0,1]$ *such that* $x_1 \le x_2$.

Note that the opposite track section $\omega_{C,\psi}$ of any copula C qualifies as an *opposite ψ-function*. On the other hand, for any quasi-copula Q and *opposite ψ-function* ω_ψ defined on $[0,1]$ by $\omega(x) = Q(x, \psi(x))$, there exists not less than one copula C whose opposite track section $\omega_{C,\psi}$ is ω_ψ. This fact is the consequence of the result given in [30]. Denote by $\mathbb{Q}_{\omega_\psi}$ (or $\mathbb{C}_{\omega_\psi}$) the set of all quasi-copulas (or copulas) with opposite track section ω_ψ.

Let $\psi \in \Psi$ and ω_ψ be an *opposite ψ-function*. Clearly, $x - \omega_\psi(x)$ is a *ϕ-function*, where $\phi(x) = 1 - \psi(x)$ for all $x \in [0,1]$, which is an increasing homeomorphism. Let Q be any quasi-copula with opposite track section $\omega_\psi(x)$, then it holds that $x - \omega_\psi(x) = x - Q(x, \psi(x))$ is a *ϕ-function*, where $\phi(x) = 1 - \psi(x)$ for all $x \in [0,1]$.

Theorem 3.2. *Let ω_ψ be an opposite ψ-function and Γ_ϕ be a ϕ-function such that $\Gamma_\phi(x) = x - \omega_\psi(x)$ and $\phi(x) = 1 - \psi(x)$ for all $x \in [0,1]$. Then $Q \in \mathbb{Q}_{\omega_\psi}$ if and only if $Q^{x\text{-}flip} \in \mathbb{Q}_{\Gamma_\phi}$, $C \in \mathbb{C}_{\omega_\psi}$ if and only if $C^{x\text{-}flip} \in \mathbb{C}_{\Gamma_\phi}$.*

Proof. We only verify the necessity, since sufficiency can be proven similarly. Assume that Q is a quasi-copula in $\mathbb{Q}_{\omega_\psi}$, then it holds that $\omega_\psi(x) = Q(x, \psi(x))$ for all $x \in [0,1]$. Clearly, we have that $x - \omega_\psi(x) = x - Q(x, \psi(x)) = Q^{x\text{-}flip}(x, 1 - \psi(x))$ for all $x \in [0,1]$. Let $\phi(x) = 1 - \psi(x)$ and $\Gamma_\phi(x) = x - \omega_\psi(x)$, we can easily get that $\Gamma_\phi(x) = Q^{x\text{-}flip}(x, \phi(x))$ for all $x \in [0,1]$. Given that $\phi \in \Phi$ and Γ_ϕ is a *ϕ-function*, then it immediately follows that $Q^{x\text{-}flip} \in \mathbb{Q}_{\Gamma_\phi}$. Similarly, we can get that $C^{x\text{-}flip} \in \mathbb{C}_{\Gamma_\phi}$. □

Note that in Theorem 3.2, by replacing $\Gamma_\phi(x) = x - \omega_\psi(x)$ with $\Gamma_\phi(x) = \psi(1-x) - \omega_\psi(1-x)$, $\phi(x) = 1 - \psi(x)$ with $\phi(x) = \psi(1-x)$, and $Q^{x\text{-}flip}$ with $Q^{y\text{-}flip}$, similar results can be obtained. In the study that follows, we justify our results based on the former case only.

It is known that for any *ϕ-function* Γ_ϕ, there exists not less than one quasi-copula C whose track section $\Gamma_{C,\phi}$ is Γ_ϕ. From Theorem 3.2, it is easy to see that for any *opposite ψ-function* ω_ψ, there exists not less than one copula C whose opposite track section $\omega_{C,\psi}$ is ω_ψ. Clearly, for any *opposite ψ-function* ω_ψ, the sets $\mathbb{Q}_{\omega_\psi}$ and $\mathbb{C}_{\omega_\psi}$ are non-empty. Moreover, for any quasi-copula Q, we know that $Q^{x\text{-}flip}$ is unique. Thus, we get directly the following corollary from the above discussions.

Corollary 3.3. *Let ω_ψ be an opposite ψ-function and Γ_ϕ be a ϕ-function such that $\omega_\psi(x) = x - \Gamma_\phi(x)$ and $\psi(x) = 1 - \phi(x)$ for all $x \in [0,1]$, the cardinality of set $\mathbb{C}_{\omega_\psi}$ (respectively, $\mathbb{Q}_{\omega_\psi}$) is equal to that of set $\mathbb{C}_{\Gamma_\phi}$ (respectively, $\mathbb{Q}_{\Gamma_\phi}$).*

As in Figure 1(a), we refer to points (x, y) and $(\psi^{-1}(y), \psi(x))$ as each other's ψ-symmetric points. Let $\psi, \varphi \in \Psi$ with $\psi(x) \le \varphi(x)$ on $[0,1]$. In Figure 1(b), the curve φ^ψ is formed by all ψ-symmetric points of curve φ in $[0,1]^2$. For any $x \in [0,1]$, the ψ-symmetric point of $(x, \psi(x))$ is point $(\psi^{-1}(\varphi(x)), \psi(x))$. In other words, the curve φ^ψ is formed by all points $(x, \psi \circ \varphi^{-1} \circ \psi(x))$ in $[0,1]^2$. Clearly, the curve ψ is enclosed in the region formed by curves φ and φ^ψ.

We call the copula C an *opposite symmetric* copula if $C(x,y) = x + y - 1 + C(1-y, 1-x)$ holds on $[0,1]^2$ (see [13]). Clearly, for any $x, y \in [0,1]$, the point (x, y) is the symmetric point of $(1-y, 1-x)$ w.r.t. opposite diagonal. By replacing the opposite diagonal with a decreasing homeomorphism, we can generalize the concept of opposite symmetry for copulas.

Firstly, we provide the following definitions.

Definition 3.4. *Let $\psi, \varphi \in \Psi$ with $\varphi(x) \ge \psi(x)$ on $[0,1]$. We say that the set $U_{\psi,\varphi}$ defined by*

$$U_{\psi,\varphi} = \{(x,y) \mid 0 \le \varphi^\psi(x) \le y \le \varphi(x) \le 1\},$$

is a φ-leaf with midrib ψ, where φ^ψ is given on $[0,1]$ by $\varphi^\psi(x) = \psi \circ \varphi^{-1} \circ \psi(x)$. If ψ is the opposite diagonal, we call $U_{\psi,\varphi}$ the φ-hairpin.

Definition 3.5. *Let $U_{\psi,\varphi}$ be a φ-leaf with midrib ψ. We call the copula C an opposite symmetric copula w.r.t. ψ on $U_{\psi,\varphi}$ (or ψ-opposite symmetric copula on $U_{\psi,\varphi}$) if*

$$C(x,y) = x - \psi^{-1}(y) + C(\psi^{-1}(y), \psi(x)), \tag{4}$$

holds on $U_{\psi,\varphi}$. Note that if ψ is the opposite diagonal and (4) is satisfied on $[0,1]^2$, then copula C is called opposite symmetric.

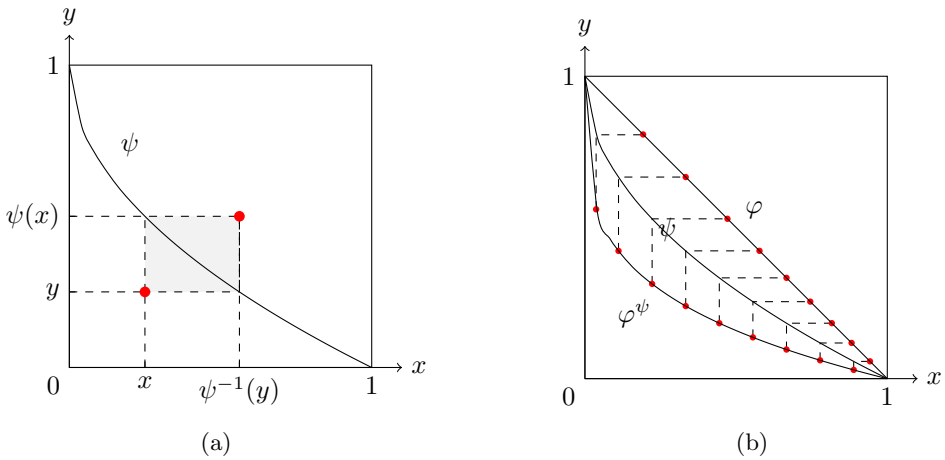


Figure 1: (a) Rectangle with (x, y) and $(\psi^{-1}(y), \psi(x))$ as vertices. (b) φ-leaf with midrib ψ.

Definition 3.5 further generalizes the concept of opposite symmetry of a copula, that is, the difference in copula values at points (x, y) and $(\psi^{-1}(y), \psi(x))$ can be simply expressed by the difference between x and $\psi^{-1}(y)$.

Next, we propose three different ways for constructing (quasi-)copulas whose opposite track sections are given. Moreover, the best-possible bounds for $\mathbb{Q}_{\omega_\psi}$ and $\mathbb{C}_{\omega_\psi}$ are also discussed.

Theorem 3.6. *Let $\psi \in \Psi$ and ω_ψ be an opposite ψ-function. Let A_{ω_ψ}, B_{ω_ψ} and K_{ω_ψ} be three bivariate functions defined on $[0,1]^2$ by*

$$A_{\omega_\psi}(x,y) = \begin{cases} \max\{x+y-1, \max\limits_{s\in[\psi^{-1}(y),x]}\{\omega_\psi(s)\}\}, & y > \psi(x), \\ \max\{0, x+y-1+\max\limits_{s\in[x,\psi^{-1}(y)]}\{1-\psi(s)+\omega_\psi(s)-s\}\}, & y \le \psi(x), \end{cases} \tag{5}$$

$$B_{\omega_\psi}(x,y) = \begin{cases} x+y-1+\min\limits_{s\in[\psi^{-1}(y),x]}\{1-\psi(s)+\omega_\psi(s)-s\}, & y > \psi(x), \\ \min\limits_{s\in[x,\psi^{-1}(y)]}\{\omega_\psi(s)\}, & y \le \psi(x), \end{cases} \tag{6}$$

and

$$K_{\omega_\psi}(x,y) = \max\{0, x+y-1, \frac{x+\omega_\psi(x)-\psi^{-1}(y)+\omega_\psi(\psi^{-1}(y))}{2}\}, \tag{7}$$

respectively. It then follows that the following statements are true.

(i) *$A_{\omega_\psi} \in \mathbb{Q}_{\omega_\psi}$, $B_{\omega_\psi} \in \mathbb{C}_{\omega_\psi}$ and $K_{\omega_\psi} \in \mathbb{C}_{\omega_\psi}$ if and only if $x+\omega_\psi(x)$ and $2x-\psi^{-1}(1-x)+\omega_\psi(\psi^{-1}(1-x))$ are non-decreasing on $[0,1]$.*

(ii) *For any $Q \in \mathbb{Q}_{\omega_\psi}$, $A_{\omega_\psi} \le Q \le B_{\omega_\psi}$, and for any $C \in \mathbb{C}_{\omega_\psi}$, $A_{\omega_\psi} \le C \le B_{\omega_\psi}$.*

(iii) *Let $\mathbb{C}^S_{\omega_\psi}$ be the set of all copulas in $\mathbb{C}_{\omega_\psi}$ that are ψ-opposite symmetric on some leaf with midrib ψ. If K_{ω_ψ} is a copula, then $K_{\omega_\psi} \le C$ for all $C \in \mathbb{C}^S_{\omega_\psi}$.*

Proof. Let $\psi \in \Psi$ and ω_ψ be an *opposite ψ-function*. Let $x \in [0,1]$, we can easily get that the function $\phi(x) = 1-\psi(x)$ is in Φ and the function $x-\omega_\psi(x)$ is a *ϕ-function*, which we denote by Γ_ϕ. In this case, *x-flipping* of functions A_{ω_ψ}, B_{ω_ψ} and K_{ω_ψ} yields the functions A_{Γ_ϕ} of type (1), B_{Γ_ϕ} of type (2) and K_{Γ_ϕ} of type (3), that is, $A^{x\text{-}flip}_{\omega_\psi} = A_{\Gamma_\phi}$, $B^{x\text{-}flip}_{\omega_\psi} = B_{\Gamma_\phi}$ and $K^{x\text{-}flip}_{\omega_\psi} = K_{\Gamma_\phi}$ on $[0,1]^2$. For any *ϕ-function* Γ_ϕ, it is known that $A_{\Gamma_\phi} \in \mathbb{Q}_{\Gamma_\phi}$ and $B_{\Gamma_\phi} \in \mathbb{C}_{\Gamma_\phi}$, while $K_{\Gamma_\phi} \in \mathbb{C}_{\Gamma_\phi}$ if and only if $2x-\Gamma_\phi(x)$ and $2x-\Gamma_\phi(\phi^{-1}(x))$ are non-decreasing on $[0,1]$.

(i). According to Theorem 3.2, we know that $A_{\omega_\psi} \in \mathbb{Q}_{\omega_\psi}$, $B_{\omega_\phi} \in \mathbb{C}_{\omega_\psi}$ and $K_{\omega_\phi} \in \mathbb{C}_{\omega_\psi}$ if and only if $A^{x\text{-}flip}_{\omega_\psi} \in \mathbb{Q}_{\Gamma_\phi}$, $B^{x\text{-}flip}_{\omega_\psi} \in \mathbb{C}_{\Gamma_\phi}$ and $K^{x\text{-}flip}_{\omega_\psi} \in \mathbb{C}_{\Gamma_\phi}$, where $\phi(x) = 1-\psi(x)$ and $\Gamma_\phi = x-\omega_{\psi(x)}$ on $[0,1]$. In other words, $A_{\omega_\psi} \in \mathbb{Q}_{\omega_\psi}$, $B_{\omega_\phi} \in \mathbb{C}_{\omega_\psi}$ and $K_{\omega_\phi} \in \mathbb{C}_{\omega_\psi}$ if and only if $A_{\Gamma_\phi} \in \mathbb{Q}_{\Gamma_\phi}$, $B_{\Gamma_\phi} \in \mathbb{C}_{\Gamma_\phi}$ and $K_{\Gamma_\phi} \in \mathbb{C}_{\Gamma_\phi}$. Thus, for any *opposite ψ-function* ω_ψ, it immediately follows that $A_{\omega_\psi} \in \mathbb{Q}_{\omega_\psi}$ and $B_{\omega_\phi} \in \mathbb{C}_{\omega_\psi}$. It is easy to see that the necessary and sufficient conditions of (3) and (7) to be copulas are equivalent and $\psi^{-1}(x) = \phi^{-1}(1-x)$ for all $x \in [0,1]$. Then, it immediately follows that $K_{\omega_\psi} \in \mathbb{C}_{\omega_\psi}$ if and only if $x+\omega_\psi(x)$ and $2x-\psi^{-1}(1-x)+\omega_\psi(\psi^{-1}(1-x))$ are non-decreasing on $[0,1]$.

(ii). For any quasi-copula $Q \in \mathbb{Q}_{\Gamma_\phi}$, it is known that $B_{\Gamma_\phi} \le Q \le A_{\Gamma_\phi}$. According to Theorem 3.2, for any $Q \in \mathbb{Q}_{\omega_\psi}$ and $x, y \in [0,1]$, we can easily get that $B_{\Gamma_\phi}(x, 1-y) \le Q^{x\text{-}flip}(x, 1-y) \le A_{\Gamma_\phi}(x, 1-y)$, where $\phi(x) = 1-\psi(x)$ and $\Gamma_\phi(x) = x - \omega_\psi(x)$ on $[0,1]$. Then it follows that $A^{x\text{-}flip}_{\Gamma_\phi}(x,y) \le Q(x,y) \le B^{x\text{-}flip}_{\Gamma_\phi}(x,y)$ for any $(x,y) \in [0,1]^2$. Denote by A_{ω_ψ} and B_{ω_ψ} the $A^{x\text{-}flip}_{\Gamma_\phi}$ and $B^{x\text{-}flip}_{\Gamma_\phi}$, respectively. Then for any $Q \in \mathbb{Q}_{\omega_\psi}$, it immediately follows that $A_{\omega_\psi} \le Q \le B_{\omega_\psi}$. Similarly, for any $C \in \mathbb{C}_{\omega_\psi}$, we have that $A_{\omega_\psi} \le C \le B_{\omega_\psi}$.

(iii). Assuming that C is a copula in $\mathbb{C}^S_{\omega_\psi}$ which is *ψ-opposite symmetric* on $U_{\psi,\varphi_1} = \{(x,y) | 0 \le \varphi_1^\psi(x) \le y \le \varphi_1(x) \le 1\}$, then it follows that $C(x,y) = x - \psi^{-1}(y) + C(\psi^{-1}(y), \psi(x))$ for any $(x,y) \in U_{\psi,\varphi_1}$. For any $(x,y) \in U_{\psi,\varphi_1}$, it is easy to get that

$$C^{x\text{-}flip}(x, 1-y) = \psi^{-1}(y) - C(\psi^{-1}(y), \psi(x)) = C^{x\text{-}flip}(\psi^{-1}(y), 1-\psi(x)).$$

Clearly, if $(x,y) \in U_{\psi,\varphi_1}$, then $(x, 1-y)$ is in the subarea $U = \{(x,y) \mid 0 \le 1-\varphi_1(x) \le y \le 1-\varphi_1^\phi(x) \le 1\}$, which is a φ_2-leaf with midrib ϕ, where $\varphi_2(x) = 1-\varphi_1(x)$ and $\phi(x) = 1-\psi(x)$. Now let's use U_{ϕ,φ_2} to represent U. Let $\phi(x) = 1-\psi(x)$ and $\Gamma_\phi(x) = x - \omega_\psi(x)$ on $[0,1]$. Then it immediately follows that

$$C^{x\text{-}flip}(x,y) = C^{x\text{-}flip}(\phi^{-1}(y), \phi(x)),$$

for any $(x,y) \in U_{\phi,\varphi_2}$, and $C^{x\text{-}flip}(x, \phi(x)) = \Gamma_\phi(x)$ for any $x \in [0,1]$. Thus, it is clear that $C^{x\text{-}flip}$ is a copula in $\mathbb{C}_{\Gamma_\phi}$ and *ϕ-symmetric* on U_{ϕ,φ_2}, which implies that $C^{x\text{-}flip} \in \mathbb{C}^S_{\Gamma_\phi}$.

Summarizing the above, for any $C \in \mathbb{C}^S_{\omega_\psi}$, it is easy to see that $C^{x\text{-}flip} \in \mathbb{C}^S_{\Gamma_\phi}$, where $\mathbb{C}^S_{\Gamma_\phi}$ is the set of all copulas in $\mathbb{C}_{\Gamma_\phi}$ that are *ϕ-symmetric* on some leaf with midrib ϕ. When K_{ω_ψ} is a copula, it is known that $K^{x\text{-}flip}_{\omega_\psi} = K_{\Gamma_\phi} \ge C$ for all $C \in \mathbb{C}^S_{\Gamma_\phi}$. Thus, we can easily get that $K_{\omega_\psi} \le C$ for all $C \in \mathbb{C}^S_{\omega_\psi}$. □

From Theorem 3.6, we can know that the best-possible lower and upper bounds for $\mathbb{Q}_{\omega_\psi}$ are A_{ω_ψ} of type (5) and B_{ω_ψ} of type (6), respectively. Moreover, if A_{ω_ψ} is a copula, then the best-possible lower and upper bounds for $\mathbb{C}_{\omega_\psi}$ are also A_{ω_ψ} and B_{ω_ψ}, respectively.

Example 3.7. *Let $\psi(x) = 1-x^\beta$ and $\omega_\psi(x) = x - x^\beta$, where $1 < \beta \le 2$. Then both $x + \omega_\psi(x) = 2x - x^\beta$ and $2x - \psi^{-1}(1-x) + \omega_\psi(\psi^{-1}(1-x)) = x$ are increasing on [0,1]. Thus we can get that*

$$A_{\omega_\psi}(x,y) = \begin{cases} \max\{x+y-1, \max\limits_{s \in [(1-y)^{1/\beta}, x]} \{s - s^\beta\}\}, & y > 1-x^\beta, \\ \max\{0, x+y-1\}, & y \le 1-x^\beta, \end{cases}$$

is a quasi-copula in $\mathbb{Q}_{\omega_\psi}$,

$$B_{\omega_\psi}(x,y) = \begin{cases} x+y-1, & y > 1-x^\beta, \\ \min\limits_{s \in [x, (1-y)^{1/\beta}]} \{s - s^\beta\}, & y \le 1-x^\beta, \end{cases}$$

and

$$K_{\omega_\psi}(x,y) = \max\{0, x+y-1, \frac{2x - x^\beta - 1 + y}{2}\},$$

are two copulas in $\mathbb{C}_{\omega_\psi}$. Since $W(x, 1-x^\beta) = \max\{0, x-x^\beta\} = x - x^\beta$, it immediately holds that $A_{\omega_\psi} = W$. From Theorem 3.6(ii), it follows that $A_{\omega_\psi} \le K_{\omega_\psi} \le B_{\omega_\psi}$.

4 Curved splicing of (quasi)-copulas

First, we give the definition of the curved splicing of two bivariate functions.

Definition 4.1. *Let $\psi \in \Psi$, F_1 and F_2 be two bivariate functions with $F_1(x, \psi(x)) = F_2(x, \psi(x))$ on $[0,1]$. The curved splicing of the functions F_1 and F_2 is called the ψ-splice of F_1 and F_2 and is given by*

$$F_1 \square_\psi F_2(x,y) = \begin{cases} F_1(x,y), & y \ge \psi(x), \\ F_2(x,y), & y < \psi(x). \end{cases} \tag{8}$$

Clearly, the copula A_{ω_ψ} of type (5), the quasi-copula B_{ω_ψ} of type (6) and the copula K_{ω_ψ} of type (7) can be obtained by the curved splicing operation. Note that when $\psi(x) = 1-x$, the curved splicing operation reduces to the opposite diagonal splicing operation discussed in [13]. Next, the curved splicing operation is applied to quasi-copulas Q_1 and Q_2 that are in $\mathbb{C}_{\omega_\psi}$.

Theorem 4.2. *If Q_1 and Q_2 are two different quasi-copulas in $\mathbb{Q}_{\omega_\psi}$, then both ψ-splices $Q_1\square_\psi Q_2$ and $Q_2\square_\psi Q_1$ are quasi-copulas in $\mathbb{Q}_{\omega_\psi}$.*

Proof. Clearly, both $Q_1\square_\psi Q_2$ and $Q_2\square_\psi Q_1$ satisfy the conditions (Q1), (Q2) and (Q3) on $D_{U,\psi}$ and $D_{L,\psi}$. Moreover, for ψ-spliec $Q_1\square_\psi Q_2$, we can get that $Q_1\square_\psi Q_2(x,\psi(x)) = \omega_\psi(x)$. Let $(x_1,y_1)\in D_{U,\psi}$ and $(x_2,y_2)\in D_{L,\psi}$, for ψ-splice $Q_1\square_\psi Q_2$ and any $x_0\in[x_1,x_2]$, then it follows that

$$\begin{aligned}|Q_1(x_1,y_1)-Q_2(x_2,y_2)| &= |Q_1(x_1,y_1)-Q_1(x_0,\psi(x_0))+Q_2(x_0,\psi(x_0))-Q_2(x_2,y_2)|\\ &\le |Q_1(x_1,y_1)-Q_1(x_0,\psi(x_0))|+|Q_2(x_0,\psi(x_0))-Q_2(x_2,y_2)|.\end{aligned}$$

After some calculations, we have that the condition (Q3) holds for any $(x_1,y_1)\in D_{U,\psi}$ and $(x_2,y_2)\in D_{L,\psi}$. Putting all things together, it immediately follows that $Q_1\square_\psi Q_2$ is a quasi-copula in Q_{ω_ψ}. Similarly, $Q_2\square_\psi Q_1$ is also a quasi-copula in Q_{ω_ψ}. □

Clearly, the ψ-splice of any two copulas in $\mathbb{C}_{\omega_\psi}$ is also a quasi-copula. Further, one may ask under which conditions two copulas in $\mathbb{C}_{\omega_\psi}$, when spliced together by way of (8), can yield a copula. The following theorem provides the obvious answer.

Theorem 4.3. *Let $\psi\in\Psi$ and ω_ψ be an opposite ψ-function. Let C_1 and C_2 be two copulas in $\mathbb{C}_{\omega_\psi}$. Then the ψ-splice $C_1\square_\psi C_2$ is a copula if and only if for any rectangle $H=[x_1,x_2]\times[\psi(x_2),\psi(x_1)]$, it holds that*

$$\omega_\psi(x_1)+\omega_\psi(x_2)\le C_1(x_2,\psi(x_1))+C_2(x_1,\psi(x_2)). \tag{9}$$

Proof. Clearly, ψ-splice $C_1\square_\psi C_2$ satisfies conditions (C1). Since C_1 and C_2 are both copulas, $C_1\square_\psi C_2$ is 2-increasing on $D_{L,\psi}$ and $D_{U,\psi}$. It is easy to see that $C_1\square_\psi C_2$ is a copula if and only if (9) holds for all $x_1,x_2\in[0,1]$. □

Example 4.4. *Consider the decreasing homeomorphism $\psi(x)=1-x^2$ and copulas*

$$C_1(x,y)=\begin{cases} x(1-x^{-1/2}(1-y)), & y\ge 1-x,\\ x(1-(1-y)^{1/2}), & y\le 1-x,\end{cases} \tag{10}$$

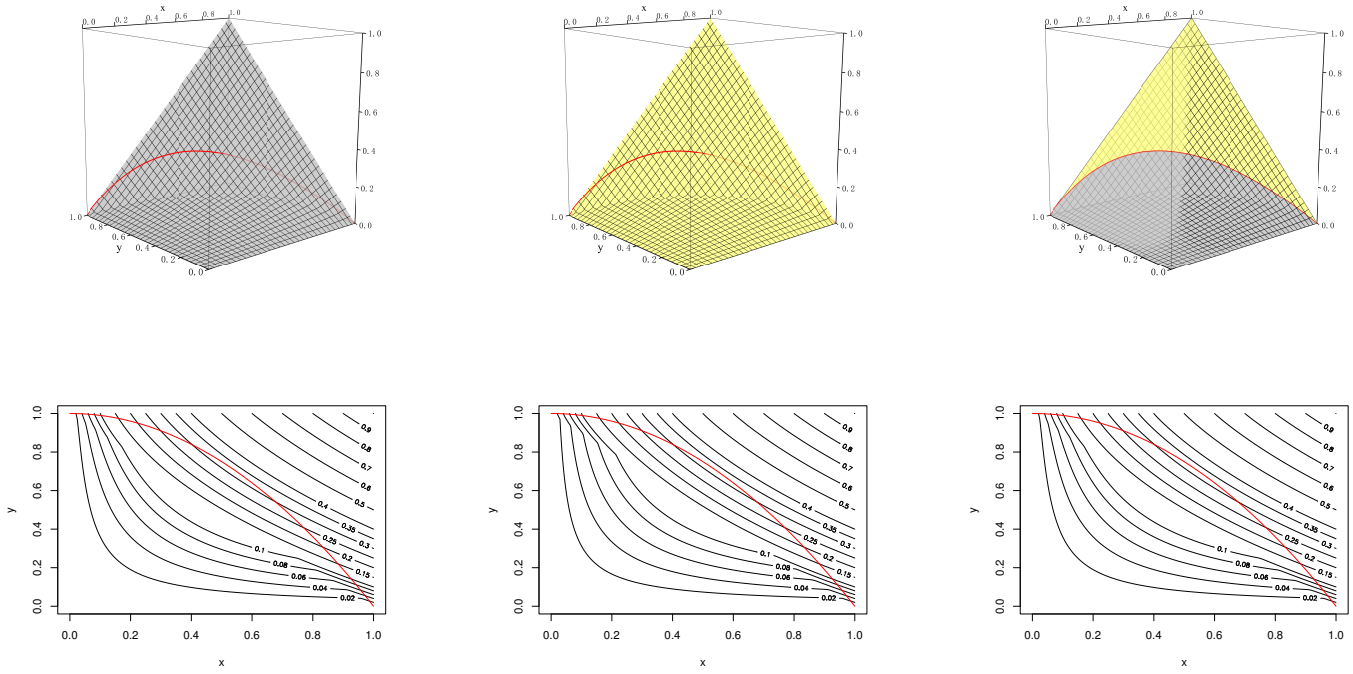
$$C_2(x,y)=\begin{cases} x(1-x^{-1/2}(1-y)), & y\ge 1-x^2,\\ x(1-(1-y)^{3/4}), & y\le 1-x^2,\end{cases} \tag{11}$$

$$C_3(x,y)=\begin{cases} x(1-\frac{(1-y)(x+y-1)}{xy}-\frac{1-x}{xy}(1-y)^{5/4}(1-(1-y)^{1/4}+(1-y)^{1/2})), & y\ge 1-x,\\ x(1-\frac{1-x-y}{1-x}-\frac{y}{1-x}x^{1/4}(1-x^{1/4}+x^{1/2})), & y\le 1-x.\end{cases} \tag{12}$$

Note that x-flipping of C_1 and C_2 yields two copulas that belong to the Marshall-Olkin family of copulas [27] and x-flipping of C_3 yields the upper semilinear copula in [16]. Thus, we can easily get that all three functions C_1, C_2 and C_3 are copulas with the common opposite track section ω_ψ, that is, $C_i(x,1-x^2)=x-x^{5/2}$ for $i\in\{1,2,3\}$. Moreover, C_1 and C_3 are both opposite symmetric copulas. To verify the pairwise splicing of C_1, C_2 and C_3 as copulas is a straightforward matter, that is, confirming whether it satisfies inequality (9)*. For instance, in the case of $C_1\square_\psi C_2$, the inequality* (9) *becomes $(x_2-x_1)(x_2^{2/3}-x_1^{2/3})\ge 0$, and since this inequality holds for any $0\le x_1\le x_2\le 1$, $C_1\square_\psi C_2$ is a copula. Proceeding similarly, we can obtain that the ψ-splices $C_1\square_\psi C_2$, $C_2\square_\psi C_1$, $C_3\square_\psi C_2$, and $C_3\square_\psi C_1$ all are copulas, while both $C_2\square_\psi C_3$ and $C_1\square_\psi C_3$ are not.*

In geometrical terms, the ψ-splice of two copulas in $\mathbb{C}_{\omega_\psi}$ is the result of the recombination of the two copulas in different regions. Moreover, similar results can also be observed in contour lines, that is, the curved splicing of copulas and the curved splicing of their contour lines occur simultaneously. The wireframe plots and contour plots of the copulas C_1, C_3 and $C_3\square_\psi C_1$ are shown in Figure 2, where the red line is formed by all points $(x,1-x^2)$ in $[0,1]^2$ for each contour plot, while it is formed by all points $(x,1-x^2,x-x^{5/2})$ in $[0,1]^3$ for each wireframe plot. It is clear that subfigure (c) is same as subfigures (a) on $D_{L,\psi}$ and (b) on $D_{U,\psi}$.

The above example illustrates that ψ-splice $C_1\square_\psi C_2$ cannot be restricted to a copula even if both C_1 and C_2 are copulas, which may be a *proper* quasi-copula. Also, when both Q_1 and Q_2 are *proper* quasi-copulas in $\mathbb{Q}_{\omega_\psi}$, ψ-splice $Q_1\square_\psi Q_2$ cannot be restricted to a *proper* quasi-copula, that is, $Q_1\square_\psi Q_2$ may be a copula.



(a) (b) (c)

Figure 2: Wireframe plots (top) and contour plots (bottom) of the copulas C_1 (left), C_3 (middle) and $C_3\square_\psi C_1$ (right).

Example 4.5. *Let ψ be a decreasing homeomorphism given by $\psi(x) = 1-x^2$. Consider the functions $Q_1, Q_2 : [0,1]^2 \to [0,1]$ given by*

$$Q_1(x,y) = \begin{cases} x(1-x^{-1/2}(1-y)), & y \geq 1-x^2, \\ x(1-\frac{(1-y)(x+y-1)}{xy} - \frac{1-x}{xy}(1-y)^{5/4}(1-(1-y)^{1/4}+(1-y)^{1/2})), & 1-x \leq y \leq 1-x^2, \\ x(1-\frac{1-x-y}{1-x} - \frac{y}{1-x}x^{1/4}(1-x^{1/4}+x^{1/2})), & y \leq 1-x, \end{cases}$$

and

$$Q_2(x,y) = \begin{cases} \max\{x+y-1, \min\limits_{s\in[\sqrt{1-y},x]}\{s-s^{5/2}\}\}, & y \geq 1-x^2, \\ x(1-(1-y)^{3/4}), & y \leq 1-x^2. \end{cases}$$

To prove that both Q_1 and Q_2 are quasi-copulas is a straightforward process, we omit it. Let rectangle $H = [x_1, x_2] \times [1-x_2^2, 1-x_1^2]$. Choosing $x_1 = 0.45$, $x_2 = 0.525$, it yields that

$$V_{Q_1}(H) = \omega_{Q_1,\psi}(x_1) + \omega_{Q_1,\psi}(x_2) - Q_1(x_1, 1-x_2^2) - Q_1(x_2, 1-x_1^2) = -0.0623 < 0.$$

Moreover, choosing $x_1 = 0.6$ and $x_2 = 0.65$, it yields that

$$V_{Q_2}(H) = Q_2(x_1, \psi(x_2)) + Q_2(x_2, \psi(x_1)) - \omega_{Q_2,\psi}(x_1) - \omega_{Q_2,\psi}(x_2) = -0.0249 < 0.$$

Then both Q_1 and Q_2 are proper quasi-copulas. In this case, the ψ-splice $Q_1\square_\psi Q_2$ is expressed as

$$Q_1\square_\psi Q_2(x,y) = \begin{cases} x(1-x^{-1/2}(1-y)), & y \geq 1-x^2, \\ x(1-(1-y)^{3/4}), & y < 1-x^2, \end{cases}$$

which is the copula C_2 in Example 4.4.

Note that the ψ-splice of a copula and a *proper* quasi-copula (and vice versa) can also be a copula. Considering the copulas C_1, C_2 in Example 4.4 and the *proper* quasi-copulas Q_1, Q_2 in Example 4.5, we can easily get that ψ-splices $C_1\square_\psi Q_2$ and $Q_1\square_\psi C_2$ are both copulas.

It is not difficult to see that there are countless opposite tracks in the unit square. For any known copula, the information on each opposite track is known, that is, the opposite track section corresponding to each opposite track

is known. It follows that we can obtain an infinite number of *opposite ψ-functions* from a copula, which means that we can construct an infinite number of copulas B_{ω_ψ} of type (6). For any $\psi \in \Psi$ and any known copula C, we know that the function $\omega_\psi = \omega_{C,\psi}$ is an *opposite ψ-function*, then the function B_{ω_ψ} of type (6) is a copula. At this point, copulas C and B_{ω_ψ} have common opposite track section ω_ψ and $C \leq B_{\omega_\psi}$. According to Theorem 4.3, we can easily get that the ψ-splices $C\Box_\psi B_{\omega_\psi}$ and $B_{\omega_\psi}\Box_\psi C$ are both copulas with opposite track section ω_ψ.

In the following, two new sufficient conditions to guarantee that ψ-splices $C_1\Box_\psi C_2$ and $C_2\Box_\psi C_1$ are both copulas are provided.

Proposition 4.6. *Let ω_ψ be an opposite ψ-function and K_{ω_ψ} be a function as given by* (7).

(i) *Suppose C_1, C_2 are two copulas in $\mathbb{C}_{\omega_\psi}$ with $\min\{C_1(x,y), C_2(x,y)\} \geq K_{\omega_\psi}(x,y)$ for all $(x,y) \in [0,1]^2$, then ψ-splices $C_1\Box_\psi C_2$ and $C_2\Box_\psi C_1$ are both copulas in $\mathbb{C}_{\omega_\psi}$.*

(ii) *Suppose C_1 and C_2 are two copulas in $\mathbb{C}_{\omega_\psi}$ with $C_1(x,y) \leq C_2(x,y)$ for all $(x,y) \in [0,1]^2$, then ψ-splices $C_1\Box_\psi C_2$ and $C_2\Box_\psi C_1$ are both copulas in $\mathbb{C}_{\omega_\psi}$.*

Proof. (i) Clearly, for any copula C, the inequality $\max\{0, x+y-1\} \leq C(x,y)$ holds on $[0,1]^2$. Then the inequality

$$\min\{C_1(x,y), C_2(x,y)\} \geq K_{\omega_\psi}(x,y) = \max\{0, x+y-1, \frac{x+\omega_\psi(x)-\psi^{-1}(y)+\omega_\psi(\psi^{-1}(y))}{2}\},$$

implies that

$$\min\{C_1(x,y), C_2(x,y)\} \geq \frac{x+\omega_\psi(x)-\psi^{-1}(y)+\omega_\psi(\psi^{-1}(y))}{2},$$

for all $(x,y) \in [0,1]^2$. Some elementary manipulations yield that inequality (9) holds on $[0,1]^2$ in cases of ψ-splices $C_1\Box_\psi C_2$ and $C_2\Box_\psi C_1$.

(ii) Since C_1 is a copula, for any rectangle $H = [x_1, x_2] \times [\psi(x_2), \psi(x_1)]$, we have that

$$C_1(x_1, \psi(x_2)) + C_1(x_2, \psi(x_1)) \geq \omega_\psi(x_1) + \omega_\psi(x_2).$$

For any $x, y \in [0,1]$, since $C_1(x,y) \leq C_2(x,y)$, it yields that inequality (9) holds in cases of ψ-splices $C_1\Box_\psi C_2$ and $C_2\Box_\psi C_1$. □

Note that the function K_{ω_ψ} in Proposition 4.6 is not necessarily a copula. In the following, the example as an application of Proposition 4.6 is provided.

Example 4.7. *When functions B_{ω_ψ} of type* (6) *and K_{ω_ψ} of type* (7) *are both copulas in $\mathbb{C}_{\omega_\psi}$, for any $C \in \mathbb{C}_{\omega_\psi}$, we can easily get that $B_{\omega_\psi}\Box_\psi C$, $C\Box_\psi B_{\omega_\psi}$, $K_{\omega_\psi}\Box_\psi B_{\omega_\psi}$ and $B_{\omega_\psi}\Box_\psi K_{\omega_\psi}$ are both copulas.*

Corollary 4.8. *Let C_1, C_2 be two copulas in $\mathbb{C}^S_{\omega_\psi}$, then ψ-splices $C_1\Box_\psi C_2$ and $C_2\Box_\psi C_1$ are both copulas in $\mathbb{C}_{\omega_\psi}$.*

Proof. Given that $C_1, C_2 \in \mathbb{C}^S_{\omega_\psi}$, according to Theorem 3.6(iii), we can get that $\min\{C_1(x,y), C_2(x,y)\} \geq K_{\omega_\psi}(x,y)$ for any $(x,y) \in [0,1]^2$. Therefore, Corollary 4.8 is held by Proposition 4.6(i). □

Example 4.9. *Consider the decreasing homeomorphism $\psi(x) = 1 - x^{3/2}$ and the opposite ψ-function $\omega_\psi(x) = x - x^2$. Let C_4 and C_5 be two bivariate functions on $[0,1]^2$ that are given by*

$$C_4(x,y) = \begin{cases} x(1-x^{-1/2}(1-y)), & y \geq 1-x^{3/2}, \\ x(1-x^{1/2}(1-y)^{1/3}), & 1-x \leq y \leq 1-x^{3/2}, \\ x(1-(1-y)^{5/6}), & y \leq 1-x, \end{cases}$$

and

$$C_5(x,y) = \max\{0, x+y-1, \frac{2x-x^2-(1-y)^{4/3}}{2}\}.$$

Verifying that C_4 is a copula is a straightforward process, we omit it. Clearly, C_4 is in $\mathbb{C}_{\omega_\psi}$ and is ψ-opposite symmetric on the φ_1-leaf U_{ψ,φ_1} with midrib ψ, where φ_1 is given by $\varphi_1(x) = 1-x^{9/4}$. Note that $\varphi_1^{\psi}(x) = 1-x$ on $[0,1]$. Suppose that the function $k(x,y) = [2x-x^2-(1-y)^{4/3}]/2$, it is not difficult to verify that $k(x,y)$ can be seen as an increasing, 1-Lipschitz continuous and supermodular function defined on $[0,1]^2$. Thus, C_5 can be obtained by truncating by W. From [11], we can easily get that C_5 is a copula in $\mathbb{C}_{\omega_\psi}$. Moreover, C_5 is ψ-opposite symmetric on the U_{ψ,φ_2}, where φ_2 can be given by the equation

$$x = \varphi_2^{-1}(y) = \sqrt{2(1-y)-(1-y)^{4/3}}.$$

Defining $\varphi = \min\{\varphi_1, \varphi_2\}$, then it follows that both C_4 and C_5 are ψ-opposite symmetric on $U_{\psi,\varphi}$. Thus we can conclude that by Corollary 4.8 the ψ-splices $C_5\Box_\psi C_4$ and $C_4\Box_\psi C_5$ are both copulas in $\in \mathbb{C}_{\omega_\psi}$.

The following example shows that the conditions stated in Proposition 4.6 and Corollary 4.8 are sufficient but not necessary.

Example 4.10. (i) *Consider copulas C_1 and C_2 again in Example 4.4. Note that C_1 and C_2 are in $\mathbb{C}_{\omega_\psi}$, where $\omega_\psi(x) = x - x^{5/2}$ and $\psi(x) = 1 - x^2$.*

From Example 4.4, we know that $C_1\Box_\psi C_2$ and $C_2\Box_\psi C_1$ are both copulas in $\mathbb{C}_{\omega_\psi}$. It is not difficult to see that C_1 and C_2 are not in $\mathbb{C}^S_{\omega_\psi}$. Clearly, the condition of Corollary 4.8 does not hold. Moreover, $\min\{C_1(0.5, 0.7), C_2(0.5, 0.7)\} = 0.5\min\{1 - 0.3 \times 0.5^{-1/2}, 1 - 0.3^{3/4}\} \approx 0.287868 < K_{\omega_\psi}(0.5, 0.7) = \max\{0, 0.2, (1 - 0.5^{5/2} - 0.3^{5/4})/2\} \approx 0.300599$. Then, the condition of Proposition 4.6(i) isn't true in this case.

(ii) *Consider copulas C_1 and C_3 in Example 4.4. Given that C_1, C_3 and $C_3\Box_\psi C_1$ are copulas in $\mathbb{C}_{\omega_\psi}$. In this case, $C_3(0.6, 0.9) - C_1(0.6, 0.9) \approx 0.00150$ and $C_3(0.16, 0.9) - C_1(0.16, 0.9) \approx -0.00623$. Then the condition of Proposition 4.6(ii) isn't true in this case.*

Suppose that C_1 and C_2 are two *opposite symmetric* copulas in $\mathbb{C}_{\omega_\psi}$, it is easy to see that by opposite symmetry C_1 and C_2 also share the common opposite track section ω_{ψ^*}, where $\psi^*(x) = 1 - \psi^{-1}(1 - x)$ and $\omega_{\psi^*}(x) = C_i(x, \psi^*(x))$ for any $i \in \{1, 2\}$. It is not difficult to see that curves ψ and ψ^* are symmetric w.r.t opposite diagonal (see Figure 3(a)). We immediately obtain the following result as the corollary of the *opposite symmetric* copula.

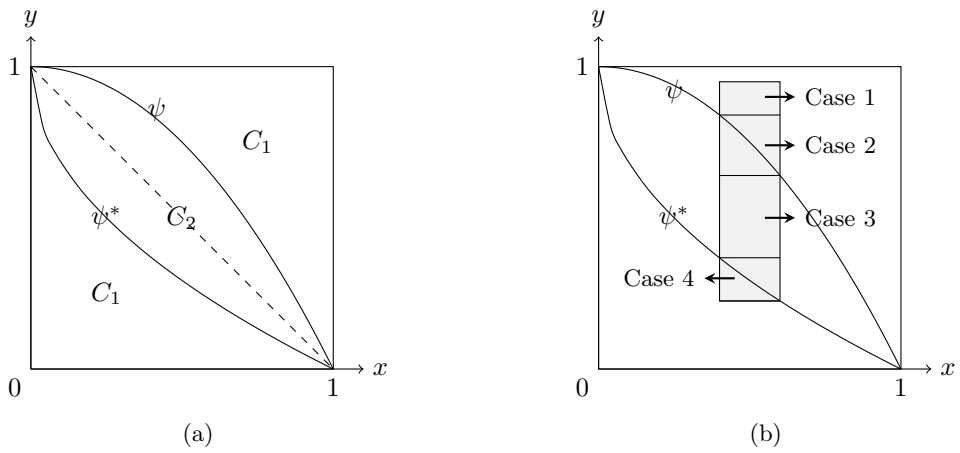


Figure 3: (a) Graph of ψ-splice $C_1\Box_{\psi^*}C_2\Box_\psi C_1$. (b) Four special cases of rectangle of H.

Corollary 4.11. *Let C_1 and C_2 be two opposite symmetric copulas in $\mathbb{C}_{\omega_\psi}$, then the ψ-splice $C_1\Box_\psi C_2$ is a copula if and only if ψ^*-splice $C_2\Box_{\psi^*}C_1$ is a copula.*

Proof. We only verify the necessity since sufficiency can be proved similarly. If $C_1\Box_\psi C_2$ is a copula, the equality (9) holds for all $0 \le x_1 \le x_2 \le 1$. Given that curves ψ and ψ^* are symmetric w.r.t. the opposite diagonal of $[0, 1]^2$, then it follows from the opposite symmetry of C_1 and C_2 that $C_2\Box_{\psi^*}C_1$ is a copula. □

We may assume that $\psi^*(x) \le \psi(x)$ on $[0, 1]$, then the curve ψ and ψ^* co-constitute a shape of the hairpin, called ψ-hairpin. We combine the ψ^*-splice and ψ-splice of the *opposite symmetric* copulas C_1 and C_2 that share the common opposite track section ω_ψ, which we call the ψ-hairpin splice and denoted as $C_1\Box_\psi C_2\Box_{\psi^*}C_1$, in the following way:

$$C_1\Box_\psi C_2\Box_{\psi^*}C_1 = \begin{cases} C_1, & y \le \psi^*(x), \\ C_2, & \psi^*(x) \le y \le \psi(x), \\ C_1, & y \ge \psi(x). \end{cases} \tag{13}$$

The graph of $C_1\Box_\psi C_2\Box_{\psi^*}C_1$ is shown as Figure 3(a).

Next, we provide a necessary and sufficient condition for the ψ-hairpin splice $C_1\Box_\psi C_2\Box_{\psi^*}C_1$ defined by (13) to be a copula.

Proposition 4.12. *Let C_1 and C_2 be two opposite symmetric copulas in $\mathbb{C}_{\omega_\psi}$, then the ψ-hairpin splice $C_1\Box_\psi C_2\Box_{\psi^*}C_1$ is a copula if and only if $C_1\Box_\psi C_2$ (or $C_2\Box_{\psi^*}C_1$) is a copula.*

Proof. Assuming that $D_{C_2} = \{(x,y) \mid 0 \le \psi^*(x) \le y \le \psi(x) \le 1\}$ and $D_{C_1} = [0,1]^2/D_{C_2}$. Clearly, ψ-hairpin splice $C_1\square_\psi C_2\square_{\psi^*}C_1$ satisfies boundary conditions (C1). Thus, We only need to verify that the ψ-hairpin splice $C_1\square_\psi C_2\square_{\psi^*}C_1$ satisfies the 2-increasing property.

Four special cases are considered (see Figure 3(b)):

- Case 1: $H \subset D_{C_1}$.
- Case 2: H is a rectangle with $y_1 = \psi(x_2)$ and $y_2 = \psi(x_1)$.
- Case 3: $H \subset D_{C_2}$.
- Case 4: H is a rectangle with $y_1 = \psi^*(x_2)$ and $y_2 = \psi^*(x_1)$.

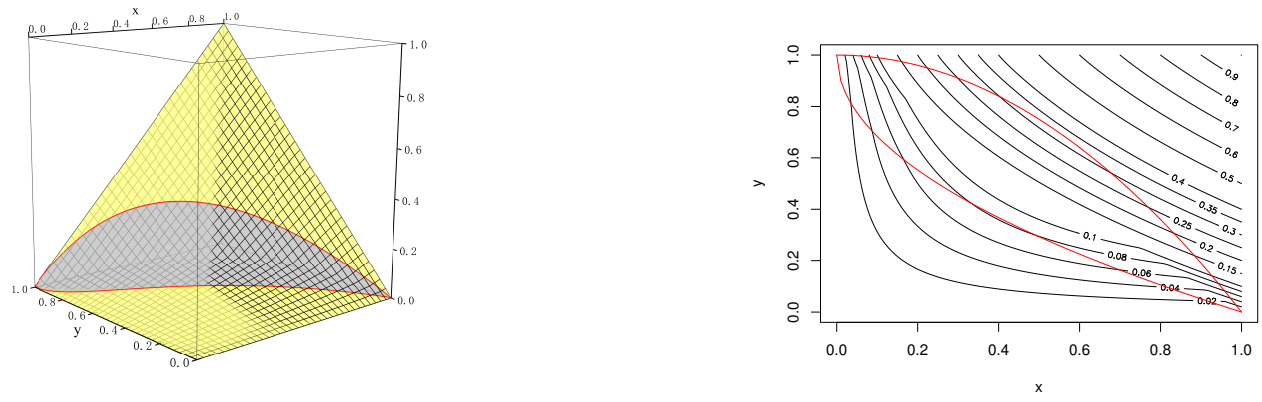
For any rectangle $H \subset [0,1]^2$, it can be subdivided into several sub-rectangles H_i belonging to one of above four cases, which implies that the value $V_{C_1\square_\psi C_2\square_{\psi^*}C_1}(H)$ is simply the aggregate of the values $V_{C_1\square_\psi C_2\square_{\psi^*}C_1}(H_i)$ corresponding to these subdivisions, where $i \in \{1,2,3,4\}$ Thus, the ψ-hairpin splice $C_1\square_\psi C_2\square_{\psi^*}C_1$ is a copula if and only if the equality (9) holds for all $0 \le x_1 \le x_2 \le 1$ in above four cases.

Clearly, if $C_1\square_\psi C_2\square_{\psi^*}C_1$ is a copula, then it immediately follows that $C_1\square_\psi C_2$ and $C_2\square_{\psi^*}C_1$ are copulas. Conversely, if $C_1\square_\psi C_2$ (or $C_2\square_{\psi^*}C_1$) is a copula, it can be easily obtained that the ψ-hairpin splice $C_1\square_\psi C_2\square_{\psi^*}C_1$ is a copula. □

Example 4.13. *Consider again the copulas C_1 of type* (10) *and C_3 of type* (12)*. Given that C_1 and C_3 are both opposite symmetric copulas, then the ψ-hairpin splice $C_3\square_\psi C_1\square_{\psi^*}C_3$ can be explicitly given by*

$$C_3\square_\psi C_1\square_{\psi^*}C_3 = \begin{cases} x\left(1 - \frac{(1-y)(x+y-1)}{xy} - \frac{1-x}{xy}(1-y)^{5/4}(1-(1-y)^{1/4}+(1-y)^{1/2})\right), & y \ge 1-x^2, \\ x(1-x^{-1/2}(1-y)), & 1-x \le y \le 1-x^2, \\ x(1-(1-y)^{1/2}), & 1-x^{1/2} \le y \le 1-x, \\ x\left(1 - \frac{1-x-y}{1-x} - \frac{y}{1-x}x^{1/4}(1-x^{1/4}+x^{1/2})\right), & y \le 1-x^{1/2}, \end{cases}$$

where $\psi^*(x) = 1-x^{1/2}$ and $\psi = 1-x^2$. According to Example 4.4, we get that $C_3\square_\psi C_1$ is a copula, then it holds that the ψ-hairpin splice $C_3\square_\psi C_1\square_{\psi^*}C_3$ is also a (opposite symmetric) copula. Figure 4 shows the wireframe plot (a) and Contour plot (b) of $C_3\square_{\psi^*}C_1\square_\psi C_3$, and it can be seen that ψ-hairpin splice $C_3\square_\psi C_1\square_{\psi^*}C_3$ has a more complex structure than that of ψ-splice $C_3\square_\psi C_1$.



(a) (b)

Figure 4: Wireframe plot (a) and Contour plot (b) of $C_3\square_{\psi^*}C_1\square_\psi C_3$.

5 The characteristics of a curved splice copula

In this section, we study the density, absolute continuity and tail dependence coefficients (*upper-lower, lower-upper*) of a *curved splice copula*.

5.1 The density function and absolute continuity of a curved splice copula

In this subsection, we concentrate on the *curved splice copula* $C_1\square_\psi C_2$ that is obtained by splicing two absolutely continuous copulas C_1 and C_2 that share the common opposite track section ω_ψ. Given that C_1 and C_2 are two different copulas in $\mathbb{C}_{\omega_\psi}$ with the density functions c_1 and c_2, respectively. Consider the function $c_1\square_\psi c_2$ given by

$$c_1\square_\psi c_2(x,y)=\begin{cases} c_1(x,y), & y\geq\psi(x),\\ c_2(x,y), & y<\psi(x),\end{cases} \tag{14}$$

where $\psi\in\Psi$. Note that $c_1\square_\psi c_2$ is not necessarily a density function defined on $[0,1]^2$, since $C_1\square_\psi C_2$ may be a *proper* quasi-copula. Conversely, if the curved splice $C_1\square_\psi C_2$ is a copula, then $c_1\square_\psi c_2$ must be a density function defined on $[0,1]^2$. It is known that a copula is a cumulative distribution function whose marginal cumulative distributions are all uniformly distributed on $[0,1]$. If $c_1\square_\psi c_2$ is a density function on $[0,1]^2$, then it follows that $C_1\square_\psi C_2$ is a cumulative distribution function on $[0,1]^2$. Furthermore, for any $x,y\in[0,1]$, we have that $C_1\square_\psi C_2(1,y)=y$ and $C_1\square_\psi C_2(x,1)=x$, which implies that the marginal cumulative distributions of cumulative distribution function $C_1\square_\psi C_2$ are all uniform on $[0,1]$, in other words, $C_1\square_\psi C_2$ is a copula. Hence, According to the above discussions, we can directly get the following theorem.

Theorem 5.1. *Let C_1 and C_2 be two different copulas in $\mathbb{C}_{\omega_\psi}$ with the density functions c_1 and c_2, respectively. Then curved splice $C_1\square_\psi C_2$ is a copula if and only if $c_1\square_\psi c_2$ is a density function on $[0,1]^2$.*

The following preliminary lemma is from [29] .

Lemma 5.2. *Let c be a density function defined on $[0,1]^2$. Then, c is the density function of a copula if and only if*

$$\int_0^1 c(x,y)\mathrm{d}y=1 \quad \textit{and} \quad \int_0^1 c(x,y)\mathrm{d}x=1,$$

hold for any $x\in[0,1]$ and $y\in[0,1]$, respectively.

Next, we provide a necessary and sufficient condition for the absolute continuity of the *curved splice copula* C_1 and C_2.

Theorem 5.3. *Let $\psi\in\Psi$, ω_ψ be an opposite ψ-function. Let C_1 and C_2 be two absolutely continuous copulas in $\mathbb{C}_{\omega_\psi}$ with density functions c_1 and c_2, respectively. Then the ψ-splice $C_1\square_\psi C_2$ is an absolutely continuous copula if and only if*

$$\int_0^{\psi(x)} c_1(x,y)\mathrm{d}y=\int_0^{\psi(x)} c_2(x,y)\mathrm{d}y \quad \textit{for any } x\in[0,1]. \tag{15}$$

In this case, $C_1\square_\psi C_2$ is a copula in $\mathbb{C}_{\omega_\psi}$ with density function $c_1\square_\psi c_2$ almost everywhere.

Proof. Assume that $C_1\square_\psi C_2$ is an absolutely continuous copula, then it immediately follows that its density function must be equal to $c_1\square_\psi c_2$ almost everywhere. Consequently, by applying Lemma 5.2, we can get that

$$\int_0^1 \left(c_1\square_\psi c_2\right)(x,y)\mathrm{d}y=\int_0^{\psi(x)} c_2(x,y)\mathrm{d}y+\int_{\psi(x)}^1 c_1(x,y)\mathrm{d}y=1=\int_0^1 c_1(x,y)\mathrm{d}y,$$

for all $x\in[0,1]$. Then condition (15) holds immediately.

Conversely, assume that condition (15) holds. Then, integrating with respect to x yields

$$\int_0^1\int_0^{\psi(x)} c_1(x,y)\mathrm{d}y\ \mathrm{d}x=\int_0^1\int_0^{\psi(x)} c_2(x,y)\mathrm{d}y\ \mathrm{d}x,$$

which means that C_1 and C_2 distribute the same probability mass on $D_{L,\psi}$ and $D_{U,\psi}$. By this, we can easily get that

$$\int_0^1\int_0^1 (c_1\square_\psi c_2)(x,y)\mathrm{d}y\ \mathrm{d}x=1.$$

Then, $c_1\square_\psi c_2$ is a density function on $[0,1]^2$. By Theorem 5.1, it immediately follows that $c_1\square_\psi c_2$ is a density function of a copula.

For $i \in \{1,2\}$, we know that $\omega_\psi(s) = C_i(s,\psi(s))$ for all $s \in [0,1]$. Then for any $s \in [0,1]$, we can write that

$$\omega_\psi(s) = C_i(s,\psi(s)) = s - \int_{\psi(s)}^{1}\int_{0}^{\psi^{-1}(y)} c_i(x,y)\mathrm{d}x\ \mathrm{d}y - \int_0^s \int_{\psi(x)}^1 c_i(x,y)\mathrm{d}y\ \mathrm{d}x,$$

where $i \in \{1,2\}$. Differentiating the above expression concerning the variable s we have

$$\omega'_\psi(s) = 1 + \psi'(s)\int_0^s c_i(x,\psi(s))\mathrm{d}x - \int_{\psi(s)}^1 c_i(s,y)\mathrm{d}y,\ i \in \{1,2\},$$

almost everywhere in $[0,1]$. Combining the above equality with (15) we can obtain that

$$\int_0^{\psi^{-1}(y)} c_1(x,y)\mathrm{d}x = \int_0^{\psi^{-1}(y)} c_2(x,y)\mathrm{d}x \text{ for all } y \in [0,1]. \tag{16}$$

Finally, by applying (15) and (16), it yields that

$$\int_0^x\int_0^y \left(c_1 \square_\psi c_2\right)(s,t)\mathrm{d}t\ \mathrm{d}s = \left(C_1 \square_\psi C_2\right)(x,y),$$

for any $x, y \in [0,1]$, that is, $C_1\square_\psi C_2$ is an absolutely continuous copula with density function $c_1\square_\psi c_2$. □

Note that if c is a density fuction of copula C, then it immediately follows that

$$\int_0^{\psi(x)} c(x,y)\mathrm{d}y = \frac{\partial C}{\partial x}(x,\psi(x)),$$

on $[0,1]$. Thus, condition (15) can be equivalently expressed as the following expression:

$$\frac{\partial C_1}{\partial x}(x,\psi(x)) = \frac{\partial C_2}{\partial x}(x,\psi(x)) \text{ on } [0,1]. \tag{17}$$

As an application of the above result, we first consider two parametric families of copulas

$$C_\theta(x,y) = \exp\left(-\left[(-\ln x)^\theta + (-\ln y)^\theta\right]^{1/\theta}\right), \tag{18}$$

and

$$C_\beta(x,y) = \exp\left(\ln(xy) - \beta\frac{\ln x \ln y}{\ln(xy)}\right), \tag{19}$$

where $\theta \geq 1$ and $\beta \in [0,1]$. Note that the parametric family of copulas C_θ is known as Gumbel-Hougaard family (see [28]) and the parametric family of copulas C_β can be obtained from [29].

Example 5.4. *Consider x-flipping of copulas C_θ and C_β that are defined by* (18) *and* (19)*, respectively. Let $C_\theta^{x\text{-}flip}(x,y) = x - C_\theta(x,1-y)$ and $C_\beta^{x\text{-}flip}(x,y) = x - C_\beta(x,1-y)$. Clearly, $C_\theta^{x\text{-}flip}$ and $C_\beta^{x\text{-}flip}$ are two absolutely continuous copulas. Let the curve $\psi(x) = 1 - x^{0.5}$, it is not difficult to obtain that $\omega_{C_\theta^{x\text{-}flip},\psi}(x) = x - x^{(1+0.5^\theta)^{1/\theta}}$ and $\omega_{C_\beta^{x\text{-}flip},\psi}(x) = x - x^{1.5-0.5\beta/1.5}$. Then it follows that*

$$\frac{\partial C_\theta^{x\text{-}flip}}{\partial x}(x,\psi(x)) = 1 - \frac{\omega_{C_\theta^{x\text{-}flip},\psi}(x)}{x}(1+0.5^\theta)^{1/\theta-1},$$

and

$$\frac{\partial C_\beta^{x\text{-}flip}}{\partial x}(x,\psi(x)) = 1 - \frac{\omega_{C_\beta^{x\text{-}flip},\psi}(x)}{x}(1-\frac{0.5^2}{1.5^2}\beta),$$

for any $x \in [0,1]$. From (17) *and Theorem 5.3, it follows that for any $\beta \in [0,1]$, when θ satisfies the equality $(1+0.5^\theta)^{1/\theta} = 1.5 - 0.5\beta/1.5$ and $(1+0.45^\theta)^{1/\theta-1} = 1 - 0.5^2\beta/1.5^2$, $C_\theta^{x\text{-}flip}\square_\psi C_\beta^{x\text{-}flip}$ and $C_\beta^{x\text{-}flip}\square_\psi C_\theta^{x\text{-}flip}$ are two absolutely continuous copulas. For example, $\beta = 0.910264$, $\theta = 1.594142$, $C_\theta^{x\text{-}flip}\square_\psi C_\beta^{x\text{-}flip}$ and $C_\beta^{x\text{-}flip}\square_\psi C_\theta^{x\text{-}flip}$ are both absolutely continuous copulas with the common opposite track section $\omega_\psi(x) = x - x^{1.196597}$.*

Note that for any absolutely continuous copulas C_1 and C_2 in $\mathbb{C}_{\omega_\psi}$, the *curved splice copula* $C_1\square_\psi C_2$ is absolutely continuous on $D_{L,\psi}$ and $D_{U,\psi}$, which implies that if *curved splice copula* $C_1\square_\psi C_2$ has a singular component, then it must happen on the opposite track $Gr(\psi)$. Hence, we provide the following corollary without proof since it follows directly from Theorem 1.1 of [22], which shows that the *curved splice copula* $C_1\square_\psi C_2$ may have a singular component along the opposite track $Gr(\psi)$ even if C_1 and C_2 are both absolutely continuous.

Corollary 5.5. *Let C_1 and C_2 be two absolutely continuous copulas in $\mathbb{C}_{\omega_\psi}$ with density functions c_1 and c_2, respectively. Suppose that $C_1\square_\psi C_2$ is a curved splice copula and the random vector $(X,Y) \sim C_1\square_\psi C_2$. If $\Lambda_\psi = \{(x,\psi(x)) \in [0,1]^2 | (\partial C_1/\partial x)(x,\psi(x)) > (\partial C_2/\partial x)(x,\psi(x))\} \neq \emptyset$, then the curved splice copula $C_1\square_\psi C_2$ has a singular component along the opposite track $Gr(\psi)$. Furthermore, the singular component is*

$$\mathbb{P}_{Gr(\psi)} = \int_0^1 \frac{\partial C_1}{\partial x}(x,\psi(x)) - \frac{\partial C_2}{\partial x}(x,\psi(x))\mathrm{d}x.$$

Example 5.6. *Let C_1 and C_2 be two bivariate functions defined on $[0,1]^2$ by $C_1(x,y) = xy+(y-1+x^2)(x-x^2)(y-y^2)$ and $C_2 = xy$. By using Theorem 3.2.8 in [28], we can know that C_1 is an absolutely continuous copula with a cubic vertical section. Clearly, C_2 is an absolutely continuous copula as well. Let $\psi(x) = 1-x^2$, then we have that $\omega_{C_1,\psi} = \omega_{C_2,\psi} = x - x^3$. It is not difficult to see that the ψ-splices $C_2\square_\psi C_1$ and $C_1\square_\psi C_2$ are not absolutely continuous copulas since $C_2\square_\psi C_1$ is a proper quasi-copula ($V_{C_2\square_\psi C_1}([0.5,0.6]\times[0.64,0.75]) \approx -0.024$) and $C_1\square_\psi C_2$ is a copula with a singular componment on the opposite track $Gr(\psi)$, that is, $C_1\square_\psi C_2$ spreads a mass of 13/420 on $Gr(\psi)$.*

5.2 The tail dependence coefficient of a curved splice copula

Tail dependence is a crucial concept in the fields of financial and actuarial risk management, used to delineate the interdependence among different risks ([19, 35]). Different tail dependencies are often necessary when modeling extreme events using copulas. The *upper-lower* and *lower-upper* tail dependence coefficients of copula C are determined by its opposite diagonal section ω_C[13]. For a copula C, the *upper-lower* and the *lower-upper* tail dependence coefficients λ_{UL} and λ_{LU} are defined by

$$\lambda_{UL}(C) = 1 - \lim_{s\to 1^-}\frac{C(s,1-s)}{1-s} = 1+\omega_C'(1^-), \tag{20}$$

and

$$\lambda_{LU}(C) = 1 - \lim_{s\to 1^-}\frac{C(1-s,s)}{1-s} = 1-\omega_C'(0^+),$$

(given that the limits exist). Let C be a copula that is associated with random vector (U,V) whose marginal cumulative distribution functions are F_1 and F_2, then λ_{UL} (respectively, λ_{LU}) refers to the asymptotic probability of occurrence of the extreme event $\{V \le F_2^{-1}(1-s) | U \ge F_1^{-1}(s)\}$ (respectively, $\{V \ge F_2^{-1}(s) | U \le F_1^{-1}(1-s)\}$, where $s \to 1^-$. If $\lambda_{UL} > 0$ (respectively, $\lambda_{LU} > 0$), we say that there exists *upper-lower* (respectively, the *lower-upper*) tail dependence between random variables U and V.

In the following, the relationships between the tail dependence coefficients of *the curved splice copula* $C_1\square_\psi C_2$ and the one of copulas C_1 and C_2 are shown.

Proposition 5.7. *Let $\psi \in \Psi$ and ψ-splice $C_1\square_\psi C_2$ be a copula with given opposite track section ω_ψ, where $C_1, C_2 \in \mathbb{C}_{\omega_\psi}$.*

(i) *Suppose that $\omega_{C_1}'(1^-)$ and $\omega_{C_2}'(1^-)$ exist. If there exists $\xi_1 \in (0,1]$ such that for any $x \in [1-\xi_1, 1]$, $\psi(x) \le 1-x$, then $\lambda_{UL}(C_1\square_\psi C_2) = 1+\omega_{C_1}'(1^-)$. And if there exists $\xi_2 \in (0,1]$ such that for any $x \in [1-\xi_2, 1]$, $\psi(x) \ge 1-x$, then $\lambda_{UL}(C_1\square_\psi C_2) = 1+\omega_{C_2}'(1^-)$.*

(ii) *Suppose that $\omega_{C_1}'(0^+)$ and $\omega_{C_2}'(0^+)$ exist. If there exists $\xi_1 \in (0,1]$ such that for any $x \in [0, \xi_1]$, $\psi(x) \le 1-x$, then $\lambda_{LU}(C_1\square_\psi C_2) = 1-\omega_{C_1}'(0^+)$. And if there exists $\xi_2 \in (0,1]$ such that for any $x \in [0, \xi_2]$, $\psi(x) \ge 1-x$, then $\lambda_{LU}(C_1\square_\psi C_2) = 1-\omega_{C_2}'(0^+)$.*

Proof. (i) Assume that there exists $\xi_1 \in (0,1]$ such that for any $x \in [1-\xi_1, 1]$, $\psi(x) \le 1-x$, then we have that $C_1\square_\psi C_2(x,1-x) = C_1(x,1-x)$ for $x \to 1^-$. According to (20), it follows that

$$\lambda_{UL}(C_1\square_\psi C_2) = \lambda_{UL}(C_1) = 1+\omega_{C_1}'(1^-).$$

Similarly, we can get that $\lambda_{UL}(C_1\square_\psi C_2) = \lambda_{UL}(C_2) = 1+\omega_{C_2}'(1^-)$ when there exist $\xi_2 \in (0,1]$ such that for any $x \in [0,\xi_2]$, $\psi(x) \ge 1-x$.

(ii) It can proved similarly, we omit it. □

Note that if the first condition of Proposition 5.7(i) holds, we can get that $\psi'(1^-) \geq -1$ when $\psi'(1^-)$ exists. Otherwise, if the second condition of Proposition 5.7(i) holds, we can get that $\psi'(1) \leq -1$ when $\psi'(1^-)$ exists. Thus, if $\psi'(1^-)$ exist, from Proposition 5.7, we can obtain that the *upper-lower* tail dependence coefficient of the *curved splice copula* $C_1 \square_\psi C_2$ is given by

$$\lambda_{UL}(C_1 \square_\psi C_2) = \begin{cases} 1 + \omega'_{C_1}(1^-), & \psi'(1^-) \geq -1, \\ 1 + \omega'_{C_2}(1^-), & \psi'(1^-) \leq -1. \end{cases}$$

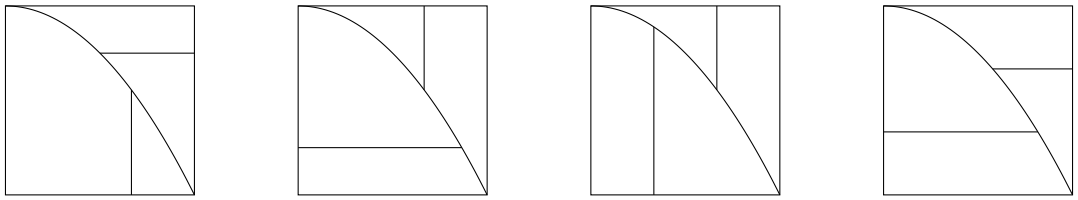
Similarly, if $\psi'(0^+)$ exists, the *lower-upper* tail dependence coefficient of the *curved splice copula* $C_1 \square_\psi C_2$ is given by

$$\lambda_{LU}(C_1 \square_\psi C_2) = \begin{cases} 1 - \omega'_{C_1}(0^+), & \psi'(0^+) \leq -1, \\ 1 - \omega'_{C_2}(0^+), & \psi'(0^+) \geq -1. \end{cases}$$

6 Semilinear copulas with a given opposite track section

Semilinear copulas are usually bivariate functions defined on $[0,1]^2$ that, given the value of one variable, are the linear functions of another variable[12, 16, 23, 26]. The linear interpolation method is widely used to construct semilinear copulas. For instance, the upper-lower (respectively, the lower-upper) semilinear copula with a given opposite diagonal section ω can be obtained by linear interpolation along the segments that connect the opposite diagonal with the right and lower (respectively, the left and upper) sides of the unit square. Moreover, the vertical (respectively, horizontal) semilinear copulas with a given opposite diagonal section can be obtained by linear interpolation along the segments that connect the opposite diagonal with the lower and upper side (respectively, left and right side) [23]. In what follows, we will adhere to a similar framework.

Replacing the opposite diagonal by an opposite track $Gr(\psi)$ (or curve ψ), we can obtain the schemes to construct the above four of types semilinear copulas, which are depicted in Figure 5.



(a) Upper-lower (b) Lower-upper (c) Vertical (d) Horizontal

Figure 5: Four linear interpolation schemes.

Let $\psi \in \Psi$ and ω_ψ be an *opposite ψ-function*. We call the function $C^l_{\omega_\psi}$ defined on $[0,1]^2$ by (21) the upper-lower semilinear function with *opposite ψ-function* ω_ψ (with the convention $0/0 =: 0$).

$$C^l_{\omega_\psi}(x,y) = \begin{cases} y + (1-x)\frac{\omega_\psi(\psi^{-1}(y)) - y}{1 - \psi^{-1}(y)}, & y \geq \psi(x), \\ \frac{\omega_\psi(x)}{\psi(x)} y, & y \leq \psi(x). \end{cases} \tag{21}$$

Next, we provide the necessary and sufficient conditions for the function $C^l_{\omega_\psi}$ of type (21) to be a copula in $\mathbb{C}_{\omega_\psi}$.

Proposition 6.1. *Let $\psi \in \Psi$ and ω_ψ be an opposite ψ-function. The function $C^l_{\omega_\psi}$ of type* (21) *is a copula if and only if:*

- (a) *the functions λ, κ defined by $\lambda(x) = \frac{\omega_\psi(x)}{\psi(x)}$ and $\kappa(x) = \frac{\omega_\psi(x) - \psi(x)}{1-x}$ are non-decreasing on* $[0.1]$*;*
- (b) *the function ζ defined by $\zeta(x) = \frac{\omega_\psi(x) - x\psi(x)}{(1-x)\psi(x)}$ is non-increasing on* $[0,1]$.

Proof. Clearly, the function $C^l_{\omega_\psi}$ defined by (21) satisfies boundary conditions (C1). Firstly, We verify the necessity. Assuming that $C^l_{\omega_\psi}$ is a copula, then we can get that

$$\frac{\partial C^l_{\omega_\psi}(x,y)}{\partial x \partial y} \geq 0,$$

for any $(x,y) \in D_{U,\psi} \cup D_{L,\psi}$. Thus, it holds that λ and κ are non-decreasing on $[0,1]$. Then condition (a) immediately holds. Moreover, for any rectangle $H = [x_1, x_2] \times [\psi(x_2), \psi(x_1)]$, if $V_{C^l_{\omega_\psi}}(H) \geq 0$ holds for any $0 \leq x_1 \leq x_2 \leq 1$, we can get that

$$V_{C^l_{\omega_\psi}}(H) = C^l_{\omega_\psi}(x_2, \psi(x_1)) + C^l_{\omega_\psi}(x_1, \psi(x_2)) - \omega_{C^l_{\omega_\psi},\psi}(x_1) - \omega_{C^l_{\omega_\psi},\psi}(x_2) \geq 0.$$

Dividing both sides of the inequality by $(x_2 - x_1)$ and taking the limit $x_1 \to x_2$ yields that condition (b) holds.

Now, we verify the sufficiency. Suppose that the conditions (a) and (b) hold on $[0,1]$. We will now demonstrate that $C^l_{\omega_\psi}$ satisfies the 2-increasing property. In this case, we focus on three special rectangles, that is, $H_1 \subset D_{L,\psi}$, $H_2 \subset D_{U,\psi}$ and $H_3 = [x_1, x_2] \times [\psi(x_2), \psi(x_1)] \subset [0,1]^2$. For any $H_1 = [x_1, x_2] \times [y_1, y_2] \subset D_{L,\psi}$, the $C^l_{\omega_\psi}$-*volume* is given by

$$V_{C^l_{\omega_\psi}}(H_1) = (\lambda(x_2) - \lambda(x_1))(y_2 - y_1),$$

which is non-negative since $\lambda(x_2) \geq \lambda(x_1)$. Similarly, For any $H_2 = [x_1, x_2] \times [y_1, y_2] \subset D_{U,\psi}$, the $C^l_{\omega_\psi}$-*volume* is given by

$$V_{C^l_{\omega_\psi}}(H_2) = (\kappa(\psi^{-1}(y_1)) - \kappa(\psi^{-1}(y_2)))(x_2 - x_1),$$

which is non-negative since $\kappa(\psi^{-1}(y_1)) \geq \kappa(\psi^{-1}(y_2))$. Let $H = [x_1, x_2] \times [\psi(x_2), \psi(x_1)]$ be a rectangle, then its volume is given by

$$V_{C^l_{\omega_\psi}}(H) = C^l_{\omega_\psi}(x_1, \psi(x_2)) + C^l_{\omega_\psi}(x_2, \psi(x_1)) - \omega_{C^l_{\omega_\psi},\psi}(x_1) - \omega_{C^l_{\omega_\psi},\psi}(x_2).$$

From condition (a), it follows that

$$\frac{\partial C^l_{\omega_\psi}(x,y)}{\partial x \partial y} \geq 0,$$

for any $(x,y) \in D_{U,\psi} \cup D_{L,\psi}$, which implies that

$$V_1 = \int_{x_1}^{x_2} \mathrm{d}s \int_{\psi(x_2)}^{\psi(s)} \frac{\partial C^l_{\omega_\psi}(s,t)}{\partial s \partial t} \mathrm{d}t \geq 0,$$

and

$$V_2 = \int_{x_1}^{x_2} \mathrm{d}s \int_{\psi(s)}^{\psi(x_1)} \frac{\partial C^l_{\omega_\psi}(s,t)}{\partial s \partial t} \mathrm{d}t \geq 0.$$

After some elementary manipulations, it yields that

$$V_1 = C^l_{\omega_\psi}(x_1, \psi(x_2)) - \omega_{C^l_{\omega_\psi},\psi}(x_2) + \int_{x_1}^{x_2} \frac{\psi(s)\omega'_\psi(s) - \psi'(s)\omega_\psi(s)}{\psi(s)} \mathrm{d}s,$$

and

$$V_2 = C^l_{\omega_\psi}(x_2, \psi(x_1)) - \omega_{C^l_{\omega_\psi},\psi}(x_1) + \int_{x_1}^{x_2} \frac{\omega_\psi(s) - \psi(s)}{1 - s} \mathrm{d}s.$$

From condition (b), we can get that

$$-\psi(x)\omega_\psi(x) + \psi^2(x) - (1-x)[\omega'_\psi(x)\psi(x) - \psi'(x)\omega_\psi(x)] \geq 0,$$

for any $x \in [0,1]$. Then we have that

$$\frac{-\psi(x)\omega_\psi(x) + \psi^2(x) - (1-x)[\omega'_\psi(x)\psi(x) - \psi'(x)\omega_\psi(x)]}{(1-x)\psi(x)} \geq 0,$$

for any $x \in [0,1]$, which implies that

$$V_3 = \int_{x_1}^{x_2} \frac{-\psi(s)\omega_\psi(s) + \psi^2(s) - (1-s)[\omega'_\psi(s)\psi(s) - \psi'(s)\omega_\psi(s)]}{(1-s)\psi(s)} \mathrm{d}s \geq 0.$$

Put all things together, it follows that

$$V_1 + V_2 + V_3 = C^l_{\omega_\psi}(x_1, \psi(x_2)) + C^l_{\omega_\psi}(x_2, \psi(x_1)) - \omega_{C^l_{\omega_\psi},\psi}(x_1) - \omega_{C^l_{\omega_\psi},\psi}(x_2) = V_{C^l_{\omega_\psi}}(H) \geq 0,$$

then the 2-increasing property of $C^l_{\omega_\psi}$ holds. □

Example 6.2. *Consider the decreasing homeomorphism $\psi(x) = 1-x^2$ and let $\omega_\psi(x)$ be the opposite track section of W, that is, $\omega_\psi(x) = x(1-x)$. Clearly, conditions (a) and (b) of Proposition 6.1 hold in this case. Then the corresponding upper-lower semilinear function $C^l_{\omega_\psi}$ is*

$$C^l_{\omega_\psi}(x,y) = \begin{cases} x+y-1, & y \geq 1-x^2, \\ \frac{x}{1+x}y, & y \leq 1-x^2, \end{cases}$$

which is a copula in $\mathbb{C}_{\omega_\psi}$.

In the following, we construct the lower-upper semilinear copulas with a given opposite track section. Let $\psi \in \Psi$ and ω_ψ be an *opposite ψ-function*. We call the function $C^u_{\omega_\psi}$ defined on $[0,1]^2$ by (22) the lower-upper semilinear function with *opposite ψ-function* ω_ψ (with the convention $0/0 =: 0$).

$$C^u_{\omega_\psi}(x,y) = \begin{cases} x - \frac{x-\omega_\psi(x)}{1-\psi(x)}(1-y), & y \geq \psi(x), \\ \frac{\omega_\psi(\psi^{-1}(y))}{\psi^{-1}(y)}x, & y \leq \psi(x). \end{cases} \tag{22}$$

Next, we provide the necessary and sufficient conditions without proof for the function $C^u_{\omega_\psi}$ to be a copula in $\mathbb{C}_{\omega_\psi}$.

Proposition 6.3. *Let $\psi \in \Psi$ and ω_ψ be an opposite ψ-function. The function $C^u_{\omega_\psi}$ of type* (22) *is a copula if and only if:*

- *(c) the functions η, ϑ defined by $\eta(x) = -\frac{\omega_\psi(x)}{x}$ and $\vartheta(x) = \frac{x-\omega_\psi(x)}{1-\psi(x)}$ are non-decreasing on $[0,1]$;*
- *(d) the function ϱ defined by $\varrho(x) = \frac{x-\omega_\psi(x)}{x(1-\psi(x))}$ is non-increasing on $[0,1]$.*

Example 6.4. *As in Example 6.2, let $\psi(x) = 1-x^2$ and $\omega_\psi(x) = x(1-x)$. Clearly, the conditions (c) and (d) of Proposition 6.3 hold in this case. Then the corresponding lower-upper semilinear function $C^u_{\omega_\psi}$ is*

$$C^u_{\omega_\psi}(x,y) = \begin{cases} x+y-1, & y \geq 1-x^2, \\ x(1-\sqrt{1-y}), & y \leq 1-x^2, \end{cases}$$

which is a copula in $\mathbb{C}_{\omega_\psi}$.

Example 6.5. *Consider the decreasing homeomorphism $\psi(x) = 1-x^s$ with $s \geq 1$, and let $\omega_\psi(x) = x(1-x^{t-1})$ with $s \leq t \leq s+1$. Clearly, the conditions (c) and (d) of Proposition 6.3 hold. Then the corresponding lower-upper semilinear function $C^u_{\omega_\psi}$ is*

$$C^u_{\omega_\psi}(x,y) = \begin{cases} x(1-x^{t-s-1}(1-y)), & y \geq 1-x^s, \\ x(1-(1-y)^{(t-1)/s}), & y \leq 1-x^s, \end{cases}$$

which is a copula in $\mathbb{C}_{\omega_\psi}$.

It is worth noting that x-flipping of these semilinear copulas yields members in the Marshall-Olkin family of copulas[28]. When letting $s = 2$ and $t = 2$, it is easy to obtain the lower-upper semilinear copula $C^u_{\omega_\psi}$ in Example 6.4.

In the following, we consider the curved splicing of a lower-upper (respectively, an upper-lower) semilinear copula and an upper-lower (respectively, a lower-upper) semilinear copula that share the common opposite track section.

Proposition 6.6. *Let $\psi \in \Psi$. Assume that ω_ψ is an opposite ψ-function such that the lower semilinear function $C^l_{\omega_\psi}$ defined by* (21) *and the upper semilinear function $C^u_{\omega_\psi}$ defined by* (22) *are both copulas. Then ψ-splice $C^u_{\omega_\psi}\square_\psi C^l_{\omega_\psi}$ (or $C^l_{\omega_\psi}\square_\psi C^u_{\omega_\psi}$) is a copula if and only if the inequality $\omega_\psi(x) \leq x\psi(x)$ holds on $[0,1]$.*

Proof. From Theorem 4.3, we can get that the ψ-splice $C^u_{\omega_\psi}\square_\psi C^l_{\omega_\psi}$ is a copula if and only if (9) holds for all $0 \leq x_1 \leq x_2 \leq 1$, and which yields that

$$(\psi(x_1) - \psi(x_2))\left(\frac{x_2 - \omega_\psi(x_2)}{1-\psi(x_2)} - \frac{\omega_\psi(x_1)}{\psi(x_1)}\right) \geq 0.$$

By simplifying the above inequalities, we get that the above inequality holds if and only if $\omega_\psi(x) \leq x\psi(x)$ holds on $[0,1]$. Thus, the ψ-splice $C^u_{\omega_\psi}\square_\psi C^l_{\omega_\psi}$ is a copula if and only if the inequality $\omega_\psi(x) \leq x\psi(x)$ holds on $[0,1]$. Similarly, we can get that the ψ-splice $C^l_{\omega_\psi}\square_\psi C^u_{\omega_\psi}$ is a copula if and only if the inequality $\omega_\psi(x) \leq x\psi(x)$ holds on $[0,1]$. □

Note that if ψ-splices $C^u_{\omega_\psi} \square_\psi C^l_{\omega_\psi}$ and $C^l_{\omega_\psi} \square_\psi C^u_{\omega_\psi}$ are both copulas, then we call them the vertical and horizontal semilinear copulas with opposite track section ω_ψ, and denote by $C^v_{\omega_\psi}$ and $C^h_{\omega_\psi}$ them, respectively. In practice, given $\psi \in \Psi$ and an *opposite ψ-function*, we may like to ensure that all four of the above semilinear copulas exist.

Example 6.7. *Let ψ be a decreasing homeomorphism given by*

$$\psi(x) = \begin{cases} 1 - 2x^2, & x \le 1/2, \\ 2(1-x)^2, & x \ge 1/2, \end{cases}$$

and opposite ψ-function ω_ψ be the opposite track section $\omega_{C,\psi}$ of the FGM copula given by $C(x,y) = xy - x(1-x)y(1-y)$ (see [3]), that is, $\omega_\psi = x\psi(x)(1-(1-x)(1-\psi(x)))$.

It is clear that the conditions (a) and (b) of Proposition 6.1 and conditions (c) and (d) of Proposition 6.3 hold, then the upper-lower semilinear function of type (21) *and the lower-upper semilinear function of type* (22) *with opposite ψ-function ω_ψ are both copulas in $\mathbb{C}_{\omega_\psi}$. Moreover, since $\omega_\psi(x) \le x\psi(x)$ on $[0,1]$, the corresponding vertical and horizontal semilinear functions are*

$$C^v_{\omega_\psi}(x,y) = \begin{cases} xy - x\psi(x)(1-x)(1-y), & y \ge \psi(x), \\ xy(1-(1-x)(1-\psi(x))), & y < \psi(x), \end{cases}$$

and

$$C^h_{\omega_\psi}(x,y) = \begin{cases} xy - y\psi^{-1}(y)(1-x)(1-y), & y \ge \psi(x), \\ xy(1-(1-y)(1-\psi^{-1}(y))), & y < \psi(x), \end{cases}$$

respectively, which are both copulas in $\mathbb{C}_{\omega_\psi}$.

7 Conclusions

In this paper, we have generalized the splicing construction of copulas with a given opposite diagonal section by proposing the curved splicing method to construct copulas with a given opposite track section. We have established a correspondence between copulas with a given opposite track section and those with a given track section, and based on this, we have discussed the best-possible bounds for all those copulas with a given opposite track section. Moreover, we also have generalized the semilinear copulas with a given opposite diagonal section to semilinear copulas with a given opposite track section using linear interpolation.

We have demonstrated that the best-possible lower bound for all those quasi-copulas sharing a common opposite track section is the quasi-copula A_{ω_ψ}, while the best-possible upper bound is the copula B_{ω_ψ}. Furthermore, it is proved that the curved splicing of any two quasi-copulas with a common opposite track section must be a quasi-copula. We also identify a necessary and sufficient condition and several sufficient conditions under which the curved splicing of two copulas with a common opposite track section results in a copula, which we call a *curved splice copula*.

We have demonstrated that a similar characterization to that of the opposite diagonal splicing is achievable in the case of curved splicing by generalizing the general notion of opposite symmetry to that of opposite symmetry w.r.t a curve in a subdomain of the unit square. Furthermore, we have introduced the hairpin splice, which is an extension of the curved splicing of two opposite symmetric copulas, and is also opposite symmetric on the unit square.

The necessary and sufficient condition for the absolute continuity of a *curved splice copula* is established and the density function of a *curved splice copula* is provided. For the copula that is non-absolutely continuous, the singular component is determined. The tail dependence coefficients of the *curved splice copula* are also provided. Drawing inspiration from the construction methods of semilinear copulas with a given opposite diagonal section, two new ways are proposed to construct semilinear copulas with a given opposite track section, which exhibit linearity on horizontal or vertical segments that connect points on the opposite track to those on the boundaries of the unit square. Furthermore, by applying the curved splicing operation, two other types of semilinear copulas are derived.

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