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Sober L-convex spaces and L-join-semilattices

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Abstract

With a complete residuated lattice L as the truth value table, we extend the definition of sobriety of classical convex spaces to the framework of L-convex spaces. We provide a specific construction for the sobrification of an L-convex space and demonstrate that the full subcategory of sober L-convex spaces is reflective in the category of L-convex spaces with convexity-preserving mappings. Additionally, we introduce the concept of Scott L-convex structures on L-ordered sets. As an application of this type of sobriety, we obtain a characterization for the L-join-semilattice completion of an L-ordered set: an L-ordered set Q is an L-join-semilattice completion of an L-ordered set P if and only if the Scott L-convex space $(Q, \sigma^*(Q))$ is a sobrification of the Scott L-convex space $(P, \sigma^*(P))$.

Keywords: L-convex spaces, sobriety, Scott L-convex structure, L-ordered set, L-join-semilattice.

1 Introduction

A convex structure on a set is a family closed under arbitrary intersections and directed unions, which contains the empty set as a member. Convex structure can be seen as the axiomatization of the usual convex sets in Euclidean spaces. Monograph [31] provides an overview of the theory of convex structures in detail. Convex structure exists in many branches of mathematics, including lattices [5, 29], algebras [16, 19], metric spaces [18], graphs [3, 4, 8] and topological spaces [12, 30].

Analogous to topological structures, a convex structure is essentially a family of subsets of a background set. It is well-known that there are close and rich connections between topological structures and ordered structures. These connections are mainly reflected in domain theory [6] and locale theory [11]. In general, there are two primary methods to establish the connections between orders and topologies. One approach involves using intrinsic topologies on posets, such as the Scott topology, and the specialization orders induced by given topologies to link topological structures with ordered structures. Famous results include the categorical isomorphism between injective T_0 spaces and continuous lattices [6]. Another one takes a pointless approach, disregarding the underlying set. Specifically, we equip the family of open sets with set-inclusion order to obtain a complete lattice; conversely, we can define an appropriate spectral topology on a complete lattice. This method gives rise to category dualities between topological structures and ordered structures, such as the well-known duality between sober topological spaces and spatial frames [11].

Inspired by the above two approaches, scholars have studied the intersection of convex structures and ordered structures. In [27], Shen et al. studied the pointfree structure of convex spaces, demonstrating that sober convex spaces and algebraic lattices are categorically dual equivalent to each other. Recently, Xia [34] extended these findings, discovering additional categorical dualities between convex structures and ordered structures in pointfree convex geometry. The specialization order approach has also been employed to study the ordered properties of convex structure. In [10], Jankowski showed that the category of injective S_0-convex spaces and that of frames are isomorphic. Yao and Zhou in [40] established a categorical isomorphism between sober convex spaces and join-semilattices by means with specialization order. These works reveal that there are close relationships between ordered structures and convex structures.

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With the development of fuzzy mathematics, Shen and Shi in [26] highlighted that the fuzzy extensions of convex spaces are significant for both theoretical research and practical applications. In 1994, the notions of fuzzy convex spaces and hull operators were first proposed by Rosa [24], which are called I-convex ($I = [0, 1]$) structures nowadays. Later, Maruyama [17] extended Rosa's definition to completely distributive lattice-valued setting and obtained the notion of L-convex spaces. In recent years, Shi, Pang, and their team members have made substantial contributions to the study of L-convex structures. In [21], Pang and Shi introduced several kinds of L-convex spaces and discussed their categorical relationships. In [26], Shen and Shi introduced the notions of L-convex systems and Scott-hull spaces, establishing a categorical isomorphism between them. Similar to the bases and subbases of L-topology, Pang and Xiu in [23] studied the bases and subbases of L-convex structures. For further studies on fuzzy convex spaces, see [22, 28, 32].

Just as the the intersection of orders and topologies can be extended to the fuzzy setting (see [37, 38, 39, 46]), the intersection of ordered structures and convex structures can also be extended to the fuzzy setting. Yao and Zhou in [41] introduced sobriety of quantale-valued fuzzy convex spaces and algebraicness of fuzzy complete lattices and then they established a dually equivalent between the category of sober fuzzy convex spaces and that of algebraic fuzzy complete lattices, which is an extension of the work in [27]. Motivated by Yao's work [37], Xia in [33] extended the Jankowski's work [10] and established a categorical isomorphism between the category of balanced L-S_0-convex spaces and that of fuzzy frames. Following Xia' work, Liu, Yue and Wei in [14] studied frame-valued Scott open set monad and proved that the related Kleisli monoids are precisely the strong frame-valued convex spaces. Building on this, Pang in [20] further developed the monadic approach in the theory of fuzzy convex structures.

Recently, Liu and Yue [13] introduced algebraic irreducible convex sets (called compact convex sets in this paper) with the help of the fuzzy inclusion order between the convex sets in an L-convex space. This notion allowed them to extended the theory of sober convex spaces to the fuzzy setting. In the classical case, the notions of polytopes and compact convex sets are equivalent. It is thus natural to ask *whether one can postulate polytopes in an L-convex space and then introduce another type of sobriety of L-convex spaces with the help of polytopes instead of compact convex sets*. Fortunately, this paper provides a positive answer. Moreover, unlike Liu and Yue's sobriety, *our sobriety aims to foster deeper connections between fuzzy ordered structures and fuzzy convex structures*, inspired by the role of topological sobriety played in order theory/domain theory [35].

In this paper, we use a complete residuated lattice L as the truth value table. This paper is organized as follows: In Section 2, we recall basic concepts and results about lattices, L-orders and L-convex spaces. In Section 3, we propose a new type of sobriety of L-convex spaces and provide a specific construction for the sobrification of any given L-convex space. In Section 4, we introduce the notion of L-join-semilattices and obtain a construction for the L-join-semilattice completion of any given L-ordered set via sobrification.

2 Preliminaries

We refer to [2, 7, 25] for contents on residuated lattices. We refer to [9] for notions of fuzzy sets, and to [36, 37, 39, 44] for contents of fuzzy posets. For contents on category theory, we refer the reader to [1].

We say a subset D of a poset P is *directed* provided it is non-empty and every two elements of D has an upper bound in D. For $A \subseteq P$, write $\bigvee A$ for the least upper bound of A and $\bigwedge A$ for the greatest lower bound of A. In particular, we use the convenient notation $x = \bigvee^{\uparrow} D$ to claim that the set D is directed and x is its least upper bound.

Let L be a complete lattice with a bottom element 0 and a top element 1 and let $\otimes$ be a binary operation on L such that $(L, \otimes, 1)$ is a commutative monoid. The pair $(L, \otimes)$ is called a *complete residuated lattice*, or a *commutative integral quantale*, if the operation $\otimes$ is distributive over joins; that is,

$$a \otimes (\bigvee S) = \bigvee_{s \in S} (a \otimes s).$$

For a complete residuated lattice $(L, \otimes)$, the operation $\otimes$ gives rises to a right adjoint $\to$: $L \times L \longrightarrow L$ via the adjoint property

$$a \otimes b \leq c \iff a \leq b \to c \ (\forall a, b, c \in L).$$

Lemma 2.1. [2, 7] *Suppose that $(L, \otimes)$ is a complete residuated lattice. Then for all $a, b, c \in L$, $\{a_i |\ i \in I\}$, $\{b_j |\ j \in J\} \subseteq L$,*

(1) $1 = a \to b \Longleftrightarrow a \leq b$;
(2) $1 \to a = a$;
(3) $a \otimes (a \to b) \leq b$;
(4) $a \to (b \to c) = (a \otimes b) \to c$

(5) $(\bigvee_{i\in I} a_i) \to b = \bigwedge_{i\in I}(a_i \to b)$;
(6) $a \to (\bigwedge_{j\in J} b_j) = \bigwedge_{j\in J}(a \to b_j)$.

Every mapping $A : X \longrightarrow L$ is called an *L-subset* of X and we use L^X to denote the collection of L-subsets of X. Customarily, the crisp order $\leq$ on L^X is defined pointwisely; that is $A \leq B \Leftrightarrow A(x) \leq B(x)$ $(\forall x \in X)$. An L-subset A is said to be *nonempty* if $\bigvee_{x\in X} A(x) = 1$. Let $Y \subseteq X$ and $A \in L^X$, define $A|_Y \in L^Y$ by $A|_Y(y) = A(y)$ $(\forall y \in Y)$. For an element $a \in L$, the notation a_X denotes the constant L-subset of X with the value a, i.e., $a_X(x) = a$ $(\forall x \in X)$. For all $a \in L$ and $A \in L^X$, write $a \otimes A$, for the L-subset given by $(a \otimes A)(x) = a \otimes A(x)$.

For each $a \in X$ and $Z \subseteq X \subseteq Y$. Define characteristic functions 1_a , $\chi_Z \in L^X$ by

$$1_a(x) = \begin{cases} 1, & x = a; \\ 0, & x \neq a, \end{cases} \qquad \chi_Z(x) = \begin{cases} 1, & x \in Z; \\ 0, & x \notin Z. \end{cases}$$

It is worth noting that in this paper, we also use the symbols 1_a , $\chi_Z \in L^Y$ to denote the characteristic functions, with the only difference in the domains of the mappings. Readers should determine the domain of each characteristic function from the context to avoid any confusion.

Definition 2.2. [36, 44] A mapping $e : P \times P \longrightarrow L$ is called an *L-order* if
(E1) $\forall x \in P$, $e(x, x) = 1$;
(E2) $\forall x, y, z \in P$, $e(x, y) \otimes e(y, z) \leq e(x, z)$;
(E3) $\forall x, y \in P$, if $e(x, y) \wedge e(y, x) = 1$, then $x = y$.
The pair (P, e) is called an *L-ordered set*. It is customary to write P for the pair (P, e).

To avoid confusion, we sometimes use e_p to denote the L-order on P. A mapping $f : P \longrightarrow Q$ between two L-ordered sets is said to be *L-order-preserving* if for all $x, y \in P$, $e_P(x, y) \leq e_Q(f(x), f(y))$; f is said to be *L-order-isomorphic* if f is a bijection and for all $x, y \in P$, $e_P(x, y) = e_Q(f(x), f(y))$.

Example 2.3. [37]
(1) Define $e_L : L \times L \longrightarrow L$ by $e_L(x, y) = x \to y$ $(\forall x, y \in L)$. Then e_L is an L-order on L.
(2) Define $\mathrm{sub}_X : L^X \times L^X \longrightarrow L$ by

$$\mathrm{sub}_X(A, B) = \bigwedge_{x\in X} A(x) \to B(x) \ (\forall A, B \in L^X).$$

Then sub_X is an L-order on L^X, which is called the *inclusion L-order* on L^X. If the background set is clear, then we always drop the subscript X to be sub.

Let $f : X \longrightarrow Y$ be a mapping between two sets. The *Zadeh extensions* $f^{\rightarrow} : L^X \longrightarrow L^Y$ and $f^{\leftarrow} : L^Y \longrightarrow L^X$ are respectively given by

$$f^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x) \ (\forall A \in L^X), \qquad f^{\leftarrow}(B) = B \circ f \ (\forall B \in L^Y).$$

Proposition 2.4. [43] *For each mapping $f : X \longrightarrow Y$,*
(1) *$f^{\rightarrow} : (L^X, \mathrm{sub}_X) \longrightarrow (L^Y, \mathrm{sub}_Y)$ is L-order-preserving;*
(2) *$f^{\leftarrow} : (L^Y, \mathrm{sub}_Y) \longrightarrow (L^X, \mathrm{sub}_X)$ is L-order-preserving;*
(3) *$f^{\rightarrow}$ is left adjoint to $f^{\leftarrow}$, written $f^{\rightarrow} \dashv f^{\leftarrow}$, that is*

$$\mathrm{sub}_Y(f^{\rightarrow}(A), B) = \mathrm{sub}_X(A, f^{\leftarrow}(B)) \ (\forall A \in L^X, B \in L^Y).$$

Define $\uparrow x$ and $\downarrow x$ respectively by $\uparrow x(y) = e(x, y)$, $\downarrow x(y) = e(y, x)$ $(\forall x, y \in P)$. An L-subset $S \in L^P$ is called a *lower set* (resp., an *upper set*) if $S(x) \otimes e(y, x) \leq S(y)$ (resp., $S(x) \otimes e(x, y) \leq S(y)$) for all $x, y \in P$. Clearly, $\downarrow x$ (resp., $\uparrow x$) is a lower (resp., an upper) set for every $x \in P$.

Definition 2.5. [36, 44] Let P be an L-ordered set. An element $x \in P$ is called a *supremum* of $A \in L^P$, denoted by $x = \sqcup A$, if

$$e(x, y) = \mathrm{sub}(A, \downarrow y) \ (\forall y \in P).$$

Dually, an element x is called an *infimum* of $A \in L^P$, denoted by $x = \sqcap A$, if

$$e(y, x) = \mathrm{sub}(A, \uparrow y) \ (\forall y \in P).$$

It is easy to check that if the supremum (resp., infimum) of an L-subset in an L-ordered set exists, then it must be unique.

For basic contents of L-convex spaces, we refer to [21, 26].

Definition 2.6. [13] Let X be a set and $\mathcal{C} \subseteq L^X$. The family $\mathcal{C}$ is called an *L-convex structure* on X if it satisfies the following conditions:

(C1) $0_X, 1_X \in \mathcal{C}$;

(C2) $\bigvee^{\uparrow}_{i\in I} C_i \in \mathcal{C}$ for every directed subset $\{C_i \mid i \in I\}$ of $\mathcal{C}$;

(C3) $\bigwedge_{j\in J} C_j \in \mathcal{C}$ for every subset $\{C_j \mid j \in J\}$ of $\mathcal{C}$;

(C4) $a \to C \in \mathcal{C}$ for all $a \in L$ and $C \in \mathcal{C}$.

The pair $(X, \mathcal{C})$ is called an *L-convex space*; every element of $\mathcal{C}$ is called a *convex set* of $(X, \mathcal{C})$.

The standard name of the L-convex space defined above is *stratified L-convex space* in the sense of [13]. While in this paper, every L-convex space is always assumed to be stratified, so we omit the word "stratified". When no confusion can arise, we often write X instead of $(X, \mathcal{C})$ for an L-convex space and write $\mathcal{C}(X)$ for the L-convex structure of X.

Definition 2.7. [26] Let X be an L-convex space. Define a mapping $co_X : L^X \longrightarrow L^X$ by

$$co_X(A) = \bigwedge\{B \in \mathcal{C}(X) \mid A \leq B\} \ (\forall A \in L^X),$$

called the *hull operator* of $(X, \mathcal{C}(X))$. For simplicity of notation, we always write co instead of co_X when no confusion can arise.

Proposition 2.8. [13, Proposition 2.10] *Let X be an L-convex space. Then*

(1) $a \otimes co(A) \leq co(a \otimes A)$ $(\forall a \in L, A \in L^X)$*;*

(2) *the hull operator* $co : (L^X, \mathrm{sub}) \longrightarrow (L^X, \mathrm{sub})$ *is L-order-preserving;*

(3) $\mathrm{sub}(A, B) = \mathrm{sub}(co(A), B)$ $(\forall A \in L^X, B \in \mathcal{C}(X))$.

Definition 2.9. [21] Let $f : X \longrightarrow Y$ be a mapping between two L-convex spaces. Then f is called

(1) *convexity-preserving* if for every $B \in \mathcal{C}(Y)$, $f^{\leftarrow}(B) \in \mathcal{C}(X)$;

(2) *convex-to-convex* if for every $A \in \mathcal{C}(X)$, $f^{\rightarrow}(A) \in \mathcal{C}(Y)$;

(3) *convex-homeomorphic* if it is bijective, convexity-preserving and convex-to-convex.

We say that X is *convex-homeomorphic* to Y if there exists *a convex-homeomorphism* between X and Y.

Proposition 2.10. [21] *Let $f : X \longrightarrow Y$ be a mapping between two L-convex spaces. Then f is convexity-preserving if and only if $f^{\rightarrow}(co_X(A)) \subseteq co_Y(f^{\rightarrow}(A))$ for every $A \in L^X$.*

3 Sober L-convex spaces

Let $(X, \mathcal{C})$ be a classical convex space. A subset E is called a *polytope* if there exists a nonempty finite subset $F \subseteq X$ such that $E = co(F)$. The space X is called *sober* if, for every polytope E, there is a unique element x such that $F = co(x)$. In order to extend the theory of sober convex spaces to the fuzzy setting, the first step is to postulate polytopes of L-convex spaces. Note that a subset F of X is finite if and only if, for every directed family $\{A_i \mid i \in I\}$, $F \subseteq \bigcup^{\uparrow}_{i\in I} A_i$ implies that there exists $i \in I$ such that $F \subseteq A_i$. Fortunately, making use of the fuzzy inclusion order between L-subsets, we will naturally define polytopes in L-convex spaces and establish a theory of sober L-convex spaces.

Definition 3.1. (1) An L-subset $F \in L^X$ is said to be *finite* if

$$\mathrm{sub}(F, \bigvee^{\uparrow}_{i\in I} A_i) = \bigvee^{\uparrow}_{i\in I} \mathrm{sub}(F, A_i),$$

for every directed family $\{A_i \mid i \in I\} \subseteq L^X$. A convex set C of X is called a *polytope* if it is the hull of a nonempty finite L-subset.

(2) An L-convex space is said to be *sober* if for every nonempty finite set $F \in L^X$, there exists a unique $x \in X$ such that $co(F) = co(1_x)$.

Remark 3.2. When L is a frame, we assert that for each crisp finite subset F of X, the characteristic function χ_F is a finite L-subset of X. In fact, for every directed family $\{A_i \mid i \in I\} \subseteq L^X$, due to the finiteness of F, we have

$$\mathrm{sub}(\chi_F, \bigvee\nolimits^{\uparrow}_{i\in I} A_i) = \bigwedge_{x\in F} \bigvee\nolimits^{\uparrow}_{i\in I} A_i(x) = \bigvee\nolimits^{\uparrow}_{i\in I} \bigwedge_{x\in F} A_i(x) = \bigvee\nolimits^{\uparrow}_{i\in I} \mathrm{sub}(\chi_F, A_i).$$

But conversely, a finite L-subset may not necessarily be a characteristic function of a crisp finite subset. For example, when both the background set X and the truth value table L are crisp finite sets, every L-subset of X is a finite L-subset of X.

Definition 3.3. [13] Let X be an L-convex space. A convex set K is said to be *compact* (called *algebraic irreducible* in [13]) if $\bigvee_{x\in X} K(x) = 1$ and

$$\mathrm{sub}(K, \bigvee\nolimits^{\uparrow}_{i\in I} C_i) = \bigvee\nolimits^{\uparrow}_{i\in I} \mathrm{sub}(K, C_i),$$

for every directed family $\{C_i \mid i \in I\} \subseteq \mathcal{C}(X)$.

In classical setting, monograph [31] shows that compact convex sets are equivalent to polytopes. In the fuzzy setting, their relationship is much more complex. By Proposition 2.8(3), polytopes are clearly compact convex sets. However, it remains unclear whether the reverse holds; that is, whether all compact convex sets are polytopes. We leave it as an open question.

Example 3.4. Let $L = ([0,1], \otimes)$ be a complete residuated lattice with $\otimes$ being $\wedge$. Now, L is a frame. Define a stratified L-convex structure

$$\mathcal{C} = \{a \wedge \phi \mid a \in [0,1], \phi : [0,1] \longrightarrow [0,1] \text{ is increasing}, \phi \geq id\},$$

on $[0,1]$. Specifically, a function $\mu : [0,1] \to [0,1]$ is a member of $\mathcal{C}$ if and only if μ is an increasing function; and there exists some $a \in [0,1]$ such that $\mu(x) \geq x$ if $x \in [0,a)$ and $\mu(x) = a$ if $x \in [a,1]$.

We next show that A is nonempty finite L-subset of $[0,1]$ if and only if there exists a nonempty finite subset $F \subseteq_{fin} [0,1]$ such that $A = \chi_F$. By Remark 3.2, it remains to prove the "only if" part. We divided this proof into two steps.

Step 1. We prove that $\{x \in [0,1] \mid A(x) \neq 0\}$ is a nonempty finite set. Since $\bigvee_{x\in X} A(x) = 1$, there exists $x \in [0,1]$ such that $A(x) \neq 0$. Assume that $\{x \in [0,1] \mid A(x) \neq 0\}$ is an infinite set. Then for every $F \subseteq_{fin} X$, there exists x such that $A(x) \neq 0$, but $x \notin F$. Noticing that $\otimes$ is $\wedge$, we have $\bigvee_{F\subseteq_{fin}[0,1]} \mathrm{sub}(A, A \wedge \chi_F) = 0$. Since A is finite, it holds that

$$1 = \mathrm{sub}(A, \bigvee_{F\subseteq_{fin}[0,1]} A \wedge \chi_F) = \bigvee_{F\subseteq_{fin}[0,1]} \mathrm{sub}(A, A \wedge \chi_F),$$

a contradiction. Thus, $\{x \in [0,1] \mid A(x) \neq 0\}$ is a nonempty finite set.

Step 2. We prove that there is no x_0 such that $A(x_0) \in (0,1)$.

Assume that there exists x_0 such that $A(x_0) \in (0,1)$. Write $A(x_0) = y_0$. It is clear that there exists $t_0 > 1$, such that $y_0 - \frac{1}{t_0} > 0$. Define $A_n : [0,1] \longrightarrow [0,1]$ by $A_n(x_0) = y_0 - \frac{1}{t_0+n}$; when $x \neq x_0$, $A_n(x) = A(x)$. We obtain a directed family $\{A_n \mid n \in \mathbb{N}\}$. Obviously, $\bigvee_{n\in\mathbb{N}} A_n = A$; that is to say $\mathrm{sub}(A, \bigvee_{n\in\mathbb{N}} A_n) = 1$. For every n, it holds that

$$\mathrm{sub}(A, A_n) = A(x_0) \to A_n(x_0) = y_0 - \frac{1}{t_0 + n}.$$

Therefore, $\bigvee_{n\in\mathbb{N}} \mathrm{sub}(A, A_n) = y_0 < 1$, a contradiction.

Thus, there exists a nonempty finite subset $F \subseteq_{fin} [0,1]$ such that $A = \chi_F$.

Write $\min(F) = b_0$. It is routine to check that the smallest convex set containing A is

$$\phi(x) = \begin{cases} 1, & x \geq b_0; \\ x, & x < b_0. \end{cases}$$

It is clear that $co(A) = co(1_{b_0})$. Thus $([0,1], \mathcal{C})$ is a sober L-convex space. By the way, it is mechanical to check that in $([0,1], \mathcal{C})$, every compact convex set is exactly a polytope; we leave this verification to the readers. □

In this paper, we use $\mathrm{cp}(\mathcal{C}(X))$ denote the set of all compact convex sets of X. For each $A \in \mathcal{C}(X)$, define

$$\phi(A) : \mathrm{cp}(\mathcal{C}(X)) \longrightarrow L,$$

by $\phi(A)(K) = \mathrm{sub}(K, A)$.

Lemma 3.5. *Let X be an L-convex space.*
(1) $\phi(a_X)(K) = a$ *for* $a = 0, 1$ *and for every* $K \in \mathrm{cp}(\mathcal{C}(X))$;
(2) $\phi(\bigvee^{\uparrow}_{i\in I} C_i) = \bigvee^{\uparrow}_{i\in I} \phi(C_i)$ *for every directed family* $\{C_i \mid i \in I\} \subseteq \mathcal{C}(X)$;
(3) $\phi(\bigwedge_{j\in J} C_j) = \bigwedge_{j\in J} \phi(C_j)$ *for every family* $\{C_j \mid j \in J\} \subseteq \mathcal{C}(X)$;
(4) $\phi(a \to C) = a \to \phi(C)$ *for* $a \in L$ *and* $C \in \mathcal{C}(X)$;
(5) $\mathrm{sub}_X(A, B) = \mathrm{sub}_{\mathrm{cp}(\mathcal{C}(X))}(\phi(A), \phi(B))$ *for all* $A, B \in \mathcal{C}(X)$.

Proof. The verification is straightforward by Lemma 2.1. □

Given an L-convex space X, by Lemma 3.5, it is straightforward to check that $\{\phi(A) \mid A \in \mathcal{C}(X)\}$ is an L-convex space on $\mathrm{cp}(\mathcal{C}(X))$ and denoted by $\mathcal{C}(\mathrm{cp}(\mathcal{C}(X)))$. We write $\mathrm{Cp}(\mathcal{C}(X))$, rather than $(\mathrm{cp}(\mathcal{C}(X)), \mathcal{C}(\mathrm{cp}(\mathcal{C}(X))))$, for the resulting L-convex space. By Lemma 3.5(2)(5), it is easy to check that A is a compact convex set of X if and only if $\phi(A)$ is a compact convex set of $\mathrm{Cp}(\mathcal{C}(X))$.

Proposition 3.6. *Let X be an L-convex space. Then* $\mathrm{Cp}(\mathcal{C}(X))$ *is a sober L-convex space.*

Proof. We first show that $\phi(K) = co(1_K)$ for every $K \in \mathrm{cp}(\mathcal{C}(X))$. Since $\phi(K)(K) = 1$, we have $1_K \le \phi(K)$. Thus $co(1_K) \le \phi(K)$. For each convex set A, if $1_K \le \phi(A)$, then $\phi(A)(K) = \mathrm{sub}(K, A) = 1$. For every $G \in \mathrm{cp}(\mathcal{C}(X))$,

$$\phi(K)(G) = \mathrm{sub}(G, K) = \mathrm{sub}(G, K) \wedge \mathrm{sub}(K, A) \le \mathrm{sub}(G, A) = \phi(A)(G).$$

Thus $\phi(K) \le \phi(A)$. This shows that $\phi(K) = co(1_K)$.

Let $\phi(A) \in \mathcal{C}(\mathrm{cp}(\mathcal{C}(X)))$ be a polytope. Then $\phi(A)$ is a compact convex set of $\mathrm{Cp}(\mathcal{C}(X))$ and A is a compact convex set of X. Therefore, $\phi(A) = co(1_A)$. The uniqueness of A can be obtained directly from Lemma 3.5(5). Thus $\mathrm{Cp}(\mathcal{C}(X))$ is a sober L-convex space. □

In the following second part of this section, we will provide a construction of the sobrification of an L-convex space. We first need a definition as follows.

Definition 3.7. Let X be an L-convex space and $A \subseteq X$. Then A is called an *F-closed set* if for every nonempty finite L-subset with $F \le \chi_A$ and $co_X(F) = co_X(1_x)$ implies $x \in A$.

In this paper, let $\mathfrak{F}(X)$ denote the family of F-closed sets of X. The collection $\mathfrak{F}(X)$ is a closure system on X, that is to say $\mathfrak{F}(X)$ is closed under arbitrary intersections (including empty intersection). The collection $\mathfrak{F}(X)$ gives rise to a closure operator $\mathrm{cl}_{\mathfrak{F}} : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ defined by

$$\mathrm{cl}_{\mathfrak{F}}(B) = \bigcap \{A \in \mathfrak{F}(X) \mid B \subseteq A\}.$$

We call $\mathrm{cl}_{\mathfrak{F}}(B)$ the *F-closure* of B.

Let $f : X \longrightarrow Y$ be a mapping between two L-convex spaces. Then f is said to be *F-continuous* if for every $A \in \mathfrak{F}(Y)$, one has $f^{-1}(A) \in \mathfrak{F}(X)$. Recall that an L-convex space X is said to be S_0 if for all $x, y \in X$, $co(1_x) = co(1_y)$ implies $x = y$.

Proposition 3.8. *Let X and Y be two L-convex spaces and let $f : X \longrightarrow Y$ be convexity-preserving. The following statements hold:*
(1) *for every nonempty L-subset $A \in L^X$, $co_Y(f^{\to}(A)) = co_Y(1_{f(x)})$, where $co_X(A) = co_X(1_x)$;*
(2) *f is F-continuous;*
(3) *if Y is S_0, $g : X \longrightarrow Y$ is convexity-preserving and $Z \subseteq X$ with $g|_Z = f|_Z$, then $g|_{\mathrm{cl}_{\mathfrak{F}}(Z)} = f|_{\mathrm{cl}_{\mathfrak{F}}(Z)}$.*

Proof. (1) Since $f : X \longrightarrow Y$ be convexity-preserving, we have

$$\begin{aligned} 1 =& \mathrm{sub}(f^{\to}(A), f^{\to}(co_X(A))) \\ =& \mathrm{sub}(f^{\to}(A), f^{\to}(co_X(1_x))) \\ \le& \mathrm{sub}(f^{\to}(A), co_Y(f^{\to}(1_x))) \\ =& \mathrm{sub}(co_Y(f^{\to}(A)), co_Y(f^{\to}(1_x))) \\ =& \mathrm{sub}(co_Y(f^{\to}(A)), co_Y(1_{f(x)}). \end{aligned}$$

Thus $co_Y(f^{\to}(A)) \le co_Y(1_{f(x)})$.

On the other hand, since

$$1 = f^{\rightarrow}(co_X(1_x))(f(x)) = f^{\rightarrow}(co_X(A))(f(x)) \leq co_Y(f^{\rightarrow}(A))(f(x)),$$

we have $co_Y(1_{f(x)}) \leq co_Y(f^{\rightarrow}(A))$. Thus $co_Y(f^{\rightarrow}(A)) = co_Y(1_{f(x)})$.

(2) Let B be an F-closed set of Y. We will show that $f^{-1}(B)$ is an F-closed set of X. For nonempty set F with $F \leq \chi_{f^{-1}(B)}$ and $co_X(F) = co_X(1_x)$, we have

$$F \leq \chi_{f^{-1}(B)} = \chi_B \circ f = f^{\leftarrow}(\chi_B).$$

By $f^{\rightarrow} \dashv f^{\leftarrow}$, we have $f^{\rightarrow}(F) \leq \chi_B$ and $f^{\rightarrow}(F)$ is a nonempty finite L-subset of Y. By Part (1), we have $co_Y(f^{\rightarrow}(F)) = co_Y(1_{f(x)})$. Since B is an F-closed set, we have $f(x) \in B$. Therefore, $x \in f^{-1}(B)$. This shows that $f^{-1}(B)$ is an F-closed set of X. Thus f is F-continuous.

(3) Write $M = \{x \in X \mid g(x) = f(x)\}$. Clearly, $Z \subseteq M$. We will show that M is an F-closed set of X. Let F be a nonempty L-subset with $F \leq \chi_M$ and $co_X(F) = co_X(1_{x_0})$. By Part (1), we have

$$co_Y(1_{f(x_0)}) = co_Y(f^{\rightarrow}(F)) = co_Y(g^{\rightarrow}(F)) = co_Y(1_{g(x_0)}).$$

Since Y is S_0, it follows that $f(x_0) = g(x_0)$, i.e., $x_0 \in M$. This shows that M is an F-closed set and $\mathrm{cl}_{\mathfrak{F}}(Z) \subseteq M$. Thus $g|_{\mathrm{cl}_{\mathfrak{F}}(Z)} = f|_{\mathrm{cl}_{\mathfrak{F}}(Z)}$. □

Given an L-convex space X, define

$$\Theta(X) = \{co(1_x) \mid x \in X\}.$$

We use the symbol X^F to denote the F-closure of $\Theta(X)$ in $\mathrm{Cp}(\mathcal{C}(X))$, i.e., $X^F := \mathrm{cl}_{\mathfrak{F}}(\Theta(X))$. For $A \in \mathcal{C}(X)$, define $\varphi(A) : X^F \longrightarrow L$ by $\varphi(A)(C) = \mathrm{sub}(C, A)$, i.e., $\varphi(A) = \phi(A)|_{X^F}$. It is easy to see that $\{\varphi(A) \mid A \in \mathcal{C}(X)\}$ is an L-convex structure on X^F, denoted by $\mathcal{C}(X^F)$. When we see X^F as an L-convex space, the related L-convex structure is always assumed to be the resulting L-convex structure. It is easy to see that X^F is a subspace of $\mathrm{Cp}(\mathcal{C}(X))$. Similar to Lemma 3.5(5), we also have $\mathrm{sub}_X(A, B) = \mathrm{sub}_{X^F}(\varphi(A), \varphi(B))$ for all $A, B \in \mathcal{C}(X)$, which will be useful.

Proposition 3.9. *X^F is a sober L-convex space.*

Proof. Let $\mathcal{K}$ be a nonempty finite L-subset of X^F. Define $\mathcal{K}' : \mathrm{cp}(\mathcal{C}(X)) \longrightarrow L$ by

$$\mathcal{K}'(A) = \begin{cases} \mathcal{K}(A), & A \in X^F; \\ 0, & A \in \mathrm{cp}(\mathcal{C}(X)) - X^F. \end{cases}$$

We will show that $\mathcal{K}'$ is a nonempty finite L-subset of $\mathrm{cp}(\mathcal{C}(X))$. It is obvious that $\mathcal{K}'$ is nonempty. For every directed family $\{\mathcal{A}_i \mid i \in I\} \subseteq L^{\mathrm{cp}(\mathcal{C}(X))}$, write $\mathcal{A}_i|_{X^F} = \mathcal{A}_i^*$. Then $\{\mathcal{A}_i^* \mid i \in I\}$ is a directed family of L^{X^F}. Then we have

$$\begin{aligned}
\mathrm{sub}_{\mathrm{cp}(\mathcal{C}(X))}(\mathcal{K}', \bigvee_{i\in I} \mathcal{A}_i) &= \bigwedge_{A\in \mathrm{cp}(\mathcal{C}(X))} \mathcal{K}'(A) \to \bigvee_{i\in I} \mathcal{A}_i(A) \\
&= \bigwedge_{A\in X^F} \mathcal{K}(A) \to \bigvee_{i\in I} \mathcal{A}_i^*(A) \\
&= \mathrm{sub}_{X^F}(\mathcal{K}, \bigvee_{i\in I} \mathcal{A}_i^*) \\
&= \bigvee_{i\in I} \mathrm{sub}_{X^F}(\mathcal{K}, \mathcal{A}_i^*) \\
&= \bigvee_{i\in I} \bigwedge_{A\in X^F} \mathcal{K}(A) \to \mathcal{A}_i^*(A) \\
&= \bigvee_{i\in I} \bigwedge_{A\in \mathrm{cp}(\mathcal{C}(X))} \mathcal{K}'(A) \to \mathcal{A}_i(A) \\
&= \bigvee_{i\in I} \mathrm{sub}_{\mathrm{cp}(\mathcal{C}(X))}(\mathcal{K}', \mathcal{A}_i).
\end{aligned}$$

This shows that $\mathcal{K}'$ is a nonempty finite L-subset of $\mathrm{cp}(\mathcal{C}(X))$. Therefore, $co_{\mathrm{cp}(\mathcal{C}(X))}(\mathcal{K}')$ is a polytope in $\mathrm{Cp}(\mathcal{C}(X))$. It follows from the sobriety of $\mathrm{Cp}(\mathcal{C}(X))$ that there exists a unique $A_0 \in \mathrm{cp}(\mathcal{C}(X))$ such that

$$co_{\mathrm{cp}(\mathcal{C}(X))}(\mathcal{K}') = co_{\mathrm{cp}(\mathcal{C}(X))}(1_{A_0}) (= \phi(A_0)).$$

Since X^F is an F-closed set in $\mathrm{Cp}(\mathcal{C}(X))$ and $\mathcal{K}' \leq \chi_{X^F} \in L^{\mathrm{cp}(\mathcal{C}(X))}$, we have $A_0 \in X^F$. We claim that $co_{X^F}(\mathcal{K}) = co_{\mathrm{cp}(\mathcal{C}(X))}(\mathcal{K}')|_{X^F}$. In fact,

$$\begin{aligned} co_{X^F}(\mathcal{K}) &= \bigwedge\{\varphi(C) \mid C \in \mathcal{C}, \mathcal{K} \leq \varphi(C)\} \\ &= \bigwedge\{\phi(C)|_{X^F} \mid C \in \mathcal{C}, \mathcal{K}' \leq \phi(C)\} \\ &= (\bigwedge\{\phi(C) \mid C \in \mathcal{C}, \mathcal{K}' \leq \phi(C)\})|_{X^F} \\ &= co_{\mathrm{cp}(\mathcal{C}(X))}(\mathcal{K}')|_{X^F}. \end{aligned}$$

By this fact and $A_0 \in X^F$, it follows that

$$co_{X^F}(\mathcal{K}) = co_{\mathrm{cp}(\mathcal{C}(X))}(\mathcal{K}')|_{X^F} = \phi(A_0)|_{X^F} = \varphi(A_0).$$

Hence $co_{X^F}(\mathcal{K}) = co_{X^F}(1_{A_0})$, where the uniqueness of A_0 can be derived from Lemma 3.5(5). Thus, X^F is a sober L-convex space. □

Proposition 3.10. *Let X be an L-convex space. Define $\xi_X : X \longrightarrow X^F$ by $\xi_X(x) = co(1_x)$. Then*
(1) *ξ_X is a convexity-preserving mapping;*
(2) *X is sober if and only if ξ_X is a convex-homeomorphism.*

Proof. (1) Let $A \in \mathcal{C}(X)$ and $x \in X$. Then

$$\xi_X^{\leftarrow}(\varphi(A))(x) = \varphi(A)(co(1_x)) = \mathrm{sub}(co(1_x), A) = \mathrm{sub}(1_x, A) = A(x).$$

Therefore, $\xi_X^{\leftarrow}(\varphi(A)) = A$. Thus ξ_X is a convexity-preserving mapping.

(2) Suppose that ξ_X is a convex-homeomorphism. Since X^F is sober, it is clear that X is sober. Conversely, let X be a sober space. Then ξ_X is an injection. We claim that $\Theta(X)$ is F-closed in $\mathrm{Cp}(\mathcal{C}(X))$, i.e., $\mathrm{cl}_{\mathfrak{F}}(\Theta(X)) = \Theta(X)(= X^F)$. Let $\mathcal{K}$ be a nonempty finite L-subset of $\mathrm{cp}(\mathcal{C}(X))$ and $\mathcal{K} \leq \chi_{\Theta(X)}$. Define $K \in L^X$ by $K(x) = \mathcal{K}(co_X(1_x))$. It is routine to check that K is a nonempty finite L-subset of X and $\xi_X^{\rightarrow}(K) = \mathcal{K}$. Since X is sober, it follows that there exists a unique $x_0 \in X$ such that $co_X(K) = co_X(1_{x_0})$. It follows from Part (1) and Proposition 3.8(1) that

$$co_{\mathrm{cp}(\mathcal{C}(X))}(\mathcal{K}) = co_{\mathrm{cp}(\mathcal{C}(X))}(1_{co_X(1_{x_0})})(= \phi(co_X(1_{x_0}))).$$

Notice that $co_X(1_{x_0}) \in \Theta(X)$. Thus $\Theta(X)$ is F-closed in $\mathrm{Cp}(\mathcal{C}(X))$. Hence $\mathrm{cl}_{\mathfrak{F}}(\Theta(X)) = \Theta(X)(= X^F)$. It follows that ξ_X is a bijection. It is routine to check that $\xi_X^{\rightarrow}(A)(co_X(1_x)) = A(x) = \varphi(A)(co_X(1_x))$ for all $A \in \mathcal{C}(X)$. Hence $\xi_X^{\rightarrow}(A) = \varphi(A)$. Thus ξ_X is a convex-homeomorphism. □

Lemma 3.11. *If $Z \subseteq X^F$ is an F-closed set of X^F, then Z is an F-closed set of* $\mathrm{Cp}(\mathcal{C}(X))$.

Proof. Let K be a nonempty finite L-subset of $\mathrm{cp}(\mathcal{C}(X))$ and $K \leq \chi_Z \in L^{\mathrm{cp}(\mathcal{C}(X))}$. Since $\mathrm{Cp}(\mathcal{C}(X))$ is sober, there exists a unique $A_0 \in \mathrm{cp}(\mathcal{C}(X))$ such that

$$co_{\mathrm{cp}(\mathcal{C}(X))}(K) = \phi(A_0)(= co_{\mathrm{cp}(\mathcal{C}(X))}(1_{A_0})).$$

Since X^F is an F-closed set of $\mathrm{Cp}(\mathcal{C}(X))$ and $K \leq \chi_{X^F} \in L^{\mathrm{cp}(\mathcal{C}(X))}$, we have $A_0 \in X^F$. Define $K^* : X^F \longrightarrow L$ by $K^*(A) = K(A)$, i.e., $K^* = K|_{X^F}$. It is routine to check that K^* is a nonempty finite L-subset of X^F. Since X^F is a subspace of $\mathrm{Cp}(\mathcal{C}(X))$, it is routine to obtain that

$$co_{X^F}(K^*) = \varphi(A_0)(= co_{X^F}(1_{A_0})).$$

Since Z is an F-closed set of X^F and $K^* \leq \chi_Z \in L^{X^F}$, it follows that $A_0 \in Z$. This shows that Z is an F-closed set of $\mathrm{Cp}(\mathcal{C}(X))$. □

We give a standard definition of sobrification as follows.

Definition 3.12. Let X be an L-convex space, let Y be a sober L-convex space and let $j : X \longrightarrow Y$ be a convexity-preserving mapping. Then (Y, j), or Y, is called a *sobrification* of X if for every sober L-convex space Z and every convexity-preserving mapping $f : X \longrightarrow Z$, there exists a unique convexity-preserving mapping $\overline{f} : Y \longrightarrow Z$ such that $f = \overline{f} \circ j$.

By the universal property of sobrifications, it is easy to see that the sobrification of an L-convex space is unique up to convex-homeomorphism. Next, we present the main result of this section as follows.

Theorem 3.13. *X^F is a sobrification of X with the mapping ξ_X.*

Proof. Let Y be a sober L-convex space and let $f : X \longrightarrow Y$ be a convexity-preserving mapping. Define $g : \mathrm{cp}(\mathcal{C}(X)) \longrightarrow \mathrm{cp}(\mathcal{C}(Y))$ by $g(K) = co_Y(f^{\rightarrow}(K))$. For every $x \in X$,

$$g(\xi_X(x)) = g(co_X(1_x)) = co_Y(f^{\rightarrow}(co_X(1_x))) = co_Y(1_{f(x)}),$$

Thus $g(co_X(1_x)) = co_Y(1_{f(x)})$, which shows that $g(\Theta(X)) \subseteq \Theta(Y)$. For every $A \in \mathrm{cp}(\mathcal{C}(X))$ and $C \in \mathcal{C}(Y)$,

$$\begin{aligned} g^{\leftarrow}(\phi(C))(A) &= \phi(C)(co_Y(f^{\rightarrow}(A)) \\ &= \mathrm{sub}(co_Y(f^{\rightarrow}(A)), C) \\ &= \mathrm{sub}(f^{\rightarrow}(A), C) \\ &= \mathrm{sub}(A, f^{\leftarrow}(C)) \\ &= \phi(f^{\leftarrow}(C))(A). \end{aligned}$$

Thus $g^{\leftarrow}(\phi(C)) = \phi(f^{\leftarrow}(C))$. This shows that g is convexity-preserving. It follows from Proposition 3.10(2) that $Y^F = \Theta(Y)$ which is an F-closed set of $\mathrm{cp}(\mathcal{C}(Y))$. By Proposition 3.8(2), $g^{-1}(\Theta(Y))$ is an F-closed set of $\mathrm{cp}(\mathcal{C}(X))$. Thus

$$X^F = \mathrm{cl}_{\mathfrak{F}}(\Theta(X)) \subseteq g^{-1}(\Theta(Y)).$$

Therefore $g(X^F) \subseteq \Theta(Y)$. Thus for every $K \in X^F$, there exists a unique $y \in Y$ such that $g(K) = co_Y(1_y)$. Define $\overline{f}(K)$ to be this y. For every convex set B of Y and every $K \in X^F$, we have

$$\begin{aligned} \overline{f}^{\leftarrow}(B)(K) = B(\overline{f}(K)) &= \mathrm{sub}(co_Y(1_{\overline{f}(K)}), B) \\ &= \mathrm{sub}(g(K), B) \\ &= \mathrm{sub}(co_Y(f^{\rightarrow}(K)), B) \\ &= \mathrm{sub}(f^{\rightarrow}(K), B) \\ &= \mathrm{sub}(K, f^{\leftarrow}(B)) \\ &= \varphi(f^{\leftarrow}(B))(K). \end{aligned}$$

Thus, $\overline{f}^{\leftarrow}(B) = \varphi(f^{\leftarrow}(B)) \in \mathcal{C}(X^F)$. This shows that $\overline{f} : X^F \longrightarrow Y$ is convexity-preserving. Notice that for every $x \in X$, $g(co_X(1_x)) = co_Y(1_{f(x)})$. Thus $\overline{f} \circ \xi_X = f$.

In order to show the uniqueness of $\overline{f}$, let $h : X^F \longrightarrow Y$ such that $f = h \circ \xi_X$. Then $\overline{f}|_{\Theta(X)} = h|_{\Theta(X)}$. By Lemma 3.11, we know that the F-closure of $\Theta(X)$ in X^F coincides with that in $\mathrm{cp}(\mathcal{C}(X))$; that is X^F. Then by Proposition 3.8(3), we know that $\overline{f} = h$. □

We denote the category of L-convex spaces with convexity-preserving mappings as morphisms by L-**CS**. Theorem 3.13 shows that the full subcategory of sober L-convex spaces is reflective in L-**CS**.

4 L-join-semilattice completion via sobrification

In this section, we introduce the notion of L-join-semilattices and Scott L-convex structures on L-ordered sets. By means with specialization L-order and Scott L-convex structure, we will discuss the relationship between sober L-convex spaces and L-join-semilattices. Finally, we also present a construction for the L-join-semilattice completion of any given L-ordered set via sobrifications. These results have not been reflected in the framework of Liu and Yue's sobreity [13].

Given an S_0 L-convex space X, define $e_{\mathcal{C}(X)} : X \times X \longrightarrow L$ by

$$e_{\mathcal{C}(X)}(x, y) = \bigwedge_{A \in \mathcal{C}(X)} A(y) \to A(x).$$

It is easy to see that $e_{\mathcal{C}(X)}$ is an L-order on X, called the *specialization L-order* of the space X. Accordingly, we will write $\Omega(X)$ for the L-ordered set obtained by equipping X with its specialization L-order. In the following, unless otherwise specified, the L-order of a given L-convex space refers to its specialization L-order.

Proposition 4.1. *Let X be an L-convex space. Then $e_{\mathcal{C}(X)}(x,y) = co(1_y)(x)$ for all $x, y \in X$.*

Proof. On one hand,

$$e_{\mathcal{C}(X)}(x,y) \leq co(1_y)(y) \to co(1_y)(x) = 1 \to co(1_y)(x) = co(1_y)(x).$$

On the other hand, we have

$$\begin{aligned} co(1_y)(x) \otimes A(y) &= co(1_y)(x) \otimes \mathrm{sub}(1_y, A) \\ &= co(1_y)(x) \otimes \mathrm{sub}(co(1_y), A) \leq A(x). \end{aligned}$$

It follows that

$$co(1_y)(x) \leq \bigwedge_{A \in \mathcal{C}(X)} A(y) \to A(x) = e_{\mathcal{C}(X)}(x,y).$$

The proof is finished. □

Proposition 4.2. *Let X be an L-convex space. Then for every $A, B \in \mathrm{cp}(\mathcal{C}(X))$, $e_{\mathrm{cp}(\mathcal{C}(X))}(A,B) = \mathrm{sub}(A,B)$, and for every $A, B \in X^F$, $e_{X^F}(A,B) = \mathrm{sub}(A,B)$.*

Proof. By the proof of Proposition 3.6 and Proposition 4.1.

$$e_{\mathrm{cp}(\mathcal{C}(X))}(A,B) = co_{\mathrm{cp}(\mathcal{C}(X))}(1_B)(A) = \phi(B)(A) = \mathrm{sub}(A,B).$$

Similarly, for every $A, B \in X^F$, $e_{X^F}(A,B) = \mathrm{sub}(A,B)$. □

Yue, Yao, and Ho introduced the notion of Scott convex structures on a join-semilattice (see [42, Example 5.1]), which precisely corresponds to the collection of all ideals of the join-semilattice. This notion of Scott convex structures on a join-semilattice can be straightforwardly extended to a more general poset. We now define the Scott L-convex structure on an L-ordered set.

Definition 4.3. Let P be an L-ordered set. An L-subset $A \in L^P$ is called a *Scott convex set* if A is a lower set and for every nonempty finite L-subset F with supremum exists, it holds that

$$\mathrm{sub}(F, A) \leq A(\sqcup F).$$

It is to observe that if we replace the nonempty finite L-subsets with directed L-subsets in the above definition, we precisely obtain the definition of Scott closed sets (see [45, Definition 5.1]) in L-cotopological spaces. Let $\sigma^*(P)$ denote the set of all Scott L-convex sets of P. It is straightforward to check that $\sigma^*(P)$ is an actual S_0 L-convex structure and we call $\sigma^*(P)$ the *Scott L-convex structure* on P. It is easy to check that for an L-ordered set (P, e), $\Omega(P, \sigma^*(P)) = (P, e)$.

Definition 4.4. An L-order-preserving mapping $f : P \longrightarrow Q$ is said to be *finite-join-preserving* if for every nonempty finite L-subset F with a supremum, the supremum of $f^{\rightarrow}(F)$ exists and $f(\sqcup F) = \sqcup f^{\rightarrow}(F)$.

The following shows that finite-join-preserving mappings consistent with the convexity-preserving mappings between the related Scott L-convex spaces.

Proposition 4.5. *$f : P \longrightarrow Q$ is finite-join-preserving if and only if $f : (P, \sigma^*(P)) \longrightarrow (Q, \sigma^*(Q))$ is convexity-preserving.*

Proof. To check the necessity, let $B \in \sigma^*(Q)$. Since f is L-order-preserving, we have that $f^{\leftarrow}(B)$ is a lower set. For every nonempty finite L-subset F of P, $f^{\rightarrow}(F)$ is a nonempty finite L-subset of Q since $f^{\rightarrow} \dashv f^{\leftarrow}$. If F has a supremum, we have

$$\begin{aligned} \mathrm{sub}(F, f^{\leftarrow}(B)) &= \mathrm{sub}(f^{\rightarrow}(F), B) \leq B(\sqcup f^{\rightarrow}(F)) \\ &= B(f(\sqcup F)) \\ &= f^{\leftarrow}(B)(\sqcup F). \end{aligned}$$

Thus, $f^{\leftarrow}(B) \in \sigma^*(P)$. This shows that $f : (P, \sigma^*(P)) \longrightarrow (Q, \sigma^*(Q))$ is convexity-preserving.

In what follows, we prove the sufficiency. For all $a, b \in P$, since $\downarrow f(b) \in \sigma^*(Q)$, we have $f^{\leftarrow}(\downarrow f(b)) \in \sigma^*(P)$. Hence, $f^{\leftarrow}(\downarrow f(b))$ is a lower set. Thus, we have

$$e(a,b) = e(a,b) \otimes f^{\leftarrow}(\downarrow f(b))(b) \leq f^{\leftarrow}(\downarrow f(b))(a) = e(f(a), f(b)).$$

This show that f is L-order-preserving. Let F be a nonempty finite L-subset with a supremum. We claim that $f(\sqcup F) = \sqcup f^{\rightarrow}(F)$. In fact, for very $y \in Q$,

$$\mathrm{sub}(f^{\rightarrow}(F), \downarrow y) = \mathrm{sub}(F, f^{\leftarrow}(\downarrow y)) = f^{\leftarrow}(\downarrow y)(\sqcup F) = e(f(\sqcup F), y).$$

Thus, $f(\sqcup F) = \sqcup f^{\rightarrow}(F)$. This shows that $f : P \longrightarrow Q$ is finite-join-preserving. □

In [15], Mao and Xu provided a topological representation of directed completions of continuous posets via sobrification. Recently, Zhang et al. [45] studied the relationship between D-completions of L-cotopological space and fuzzy directed completions of L-ordered sets. In [45], fuzzy directed sets are crucial in defining monotone convergence L-cotopological spaces, also called fuzzy d-spaces. While in this paper, the definition of sobriety of L-convex spaces is introduced using fuzzy nonempty finite sets. These motivate us to investigate the relationship between sober L-convex spaces and a certain type of fuzzy finite complete L-ordered sets.

Definition 4.6. An L-ordered set P is called an *L-join-semilattice* if every nonempty finite L-subset F of P has a supremum.

Proposition 4.7. *Let $(X, \mathcal{C})$ be a sober L-convex space. Then ΩX is an L-join-semilattice.*

Proof. Let $F \in L^X$ be a nonempty finite L-subset. Since X is sober, there exists a unique $a \in X$ such that $co(F) = co(1_a)$. We claim that a is the supremum of F in ΩX. In fact, by Proposition 4.1

$$\begin{aligned} e_{\mathcal{C}(X)}(a, x) &= co(1_x)(a) = \mathrm{sub}(1_a, co(1_x)) \\ &= \mathrm{sub}(co(1_a), co(1_x)) \\ &= \mathrm{sub}(co(F), co(1_x)) \\ &= \mathrm{sub}(F, co(1_x)) \\ &= \mathrm{sub}(F, \downarrow x). \end{aligned}$$

This shows that a is the supremum of F. Thus ΩX is an L-join-semilattice. □

Proposition 4.8. *An S_0 L-convex space $(X, \mathcal{C})$ is sober iff ΩX is an L-join-semilattice and $\mathcal{C} \subseteq \sigma^*(\Omega X)$.*

Proof. **Necessity**. By Proposition 4.7, ΩX is an L-join-semilattice. For every $A \in \mathcal{C}$, let F be a nonempty finite L-subset of X , we have

$$\mathrm{sub}(F, A) = \mathrm{sub}(co(F), A) = \mathrm{sub}(co(1_{\sqcup F}), A) = A(\sqcup F).$$

This shows that $A \in \sigma^*(\Omega X)$. Thus $\mathcal{C} \subseteq \sigma^*(\Omega X)$.

Sufficiency. Let F be a nonempty finite L-subset of X. Since $(X, e_{\mathcal{C}})$ is an L-join-semilattice, the supremum of F exists. We claim that $co(F) = co(1_{\sqcup F})$. In fact, since $co(F) \in \mathcal{C} \subseteq \sigma^*(\Omega X)$ and $F \leq co(F)$, we have $co(F)(\sqcup F) = 1$. That is to say, $1_{\sqcup F} \leq co(F)$. Thus $co(1_{\sqcup F}) \leq co(F)$. On the other hand, since

$$1 = e_{\mathcal{C}}(\sqcup F, \sqcup F) = \bigwedge_{x \in X} F(x) \longrightarrow \downarrow \sqcup F(x) = \bigwedge_{x \in X} F(x) \to co(1_{\sqcup F})(x),$$

we have $F \leq co(1_{\sqcup F})$. Thus $co(F) = co(1_{\sqcup F})$, which shows that $(X, \mathcal{C})$ is a sober L-convex space. □

Note the difference between the theorem above and [40, Theorem 3.13]. [40, Theorem 3.13] shows that for every classical sober convex space $(X, \mathcal{C})$, $\mathcal{C} = \sigma^*(\Omega X)$. But in the fuzzy setting, $\sigma^*(\Omega X) \subseteq \mathcal{C}$ cannot be obtained. Moreover, by the proof of Proposition 4.8, we know that the Scott L-convex structure on an L-join-semilattice is indeed sober.

Unless otherwise stated, whenever an order-theoretic concept is mentioned in the context of an S_0 L-convex space X, it is to be interpreted with respect to the specialization L-order on X. For example, we use order-theoretic concept to restate Definition 3.7 in the case of sober L-convex spaces: Let X be a sober L-convex space and $A \subseteq X$. Then A is called an *F-closed set*, if for every nonempty finite L-subset with $F \leq \chi_A$ implies $\sqcup F \in A$.

Moreover, we use order-theoretic concept to recall Proposition 3.8(1), we deduce that: Let X be a sober L-convex space and Y be an S_0 L-convex spaces. If $f : X \longrightarrow Y$ be a convexity-preserving mapping, then $f : \Omega X \longrightarrow \Omega Y$ is finite-join-preserving.

We use $(P^F, \mathcal{C}(P^F))$ with the mapping ξ_P to denote the sobrification of $(P, \sigma^*(P))$.

Lemma 4.9. *The mapping $\xi_P : (P, e_P) \longrightarrow (P^F, \mathrm{sub}_P)$ is finite-join-preserving.*

Proof. If K is a nonempty finite L-subset of P and the supremum $\sqcup K$ exists, then $\sqcup\xi_P^{\rightarrow}(K) = co(1_{\sqcup K})(= \xi_P(\sqcup K))$. In fact, for every $B \in P^F$,

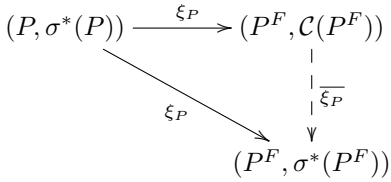
$$\begin{aligned}\mathrm{sub}(\xi_P^{\rightarrow}(K), \mathrm{sub}(-, B)) &= \bigwedge_{x\in X} \xi_P^{\rightarrow}(K)(co(1_x)) \to \mathrm{sub}(co(1_x), B)\\ &= \bigwedge_{x\in X} K(x) \to B(x)\\ &= \mathrm{sub}(K, B)\\ &= B(\sqcup K)\\ &= \mathrm{sub}(co(1_{\sqcup K}), B).\end{aligned}$$

This shows that ξ_P is finite-join-preserving. □

By proposition 4.2, when see P^F as an L-ordered set, the related L-order is always assumed to be the specialization L-order of $(P^F, \mathcal{C}(P^F))$; that is inclusion L-order sub_P. The following proposition shows that the original L-convex structure of the sobrification P^F coincides with the Scott L-convex structure of (P^F, sub_P).

Proposition 4.10. *Let P be an L-ordered set. Then $\mathcal{C}(P^F) = \sigma^*(P^F)$.*

Proof. By Proposition 4.8, we only need to show $\sigma^*(P^F) \subseteq \mathcal{C}(P^F)$. It follows from 4.8 that (P^F, sub_P) is an L-join-semilattice and $(P, \sigma^*(P^F))$ is a sober L-convex space. By Lemma 4.9, $\xi_P : (P, \sigma^*(P)) \longrightarrow (P^F, \sigma^*(P^F))$ is convexity-preserving. Since $(P^F, \mathcal{C}(P^F))$ with mapping ξ_P is a sobrification of $(P, \sigma^*(P))$, there exists a unique convexity-preserving mapping $\overline{\xi_P} : (P^F, \mathcal{C}(P^F)) \longrightarrow (P^F, \sigma^*(P^F))$ such that the following diagram commutes.



Write $\mathcal{A} = \{A \in P^F \mid \overline{\xi}_P(A) = A\}$. It is clear that $\Theta(P) \subseteq \mathcal{A}$. We next prove that $\mathcal{A}$ is an F-closed set of $(P^F, \mathcal{C}(P^F))$. Here, we need to note that the specialization L-orders of $(P^F, \mathcal{C}(P^F))$ and $(P^F, \sigma^*(P^F))$ are consistent; that is sub_P. Let $\mathcal{K}$ be a nonempty finite L-subset of P^F and $\mathcal{K} \le \chi_{\mathcal{A}}$. By Proposition 3.8(1), we have $\sqcup\overline{\xi_P}^{\rightarrow}(\mathcal{K}) = \overline{\xi}_P(\sqcup\mathcal{K})$. Since $\mathcal{K} \le \chi_A$, we have $\overline{\xi_P}^{\rightarrow}(\mathcal{K}) = \mathcal{K}$. Thus, $\sqcup\overline{\xi_P}^{\rightarrow}(\mathcal{K}) = \sqcup\mathcal{K}$. Hence, $\overline{\xi}_P(\sqcup\mathcal{K}) = \sqcup\mathcal{K}$. This shows that $\sqcup\mathcal{K} \in \mathcal{A}$. Thus $\mathcal{A}$ is an F-closed set of $(P^F, \mathcal{C}(P^F))$. By Lemma 3.11, $\mathcal{A}$ is also an F-closed set of $\mathrm{Cp}(\sigma^*(P))$. Noticing that $\Theta(P) \subseteq \mathcal{A} \subseteq P^F$, we have $\mathcal{A} = P^F$. Thus $\overline{\xi_p} = id_{P^F}$. Since $\overline{\xi_p}$ is convexity-preserving, we have $\sigma^*(P^F) \subseteq \mathcal{C}(P^F)$, as desired. □

Definition 4.11. Let P and Q be two L-ordered sets. If Q is an L-join-semilattice and $j : P \longrightarrow Q$ is finite-join-preserving, then (Q, j), or Q, is called an *L-join-semilattice completion* of P if for every L-join-semilattice M and finite-join-preserving mapping $f : P \longrightarrow M$, there exists a unique finite-join-preserving mapping $\overline{f} : Q \to M$ such that $\overline{f} \circ j = f$.

By the universal property of L-join-semilattice completions, the L-join-semilattice completion of an L-ordered set is unique up to L-order-isomorphism.

Theorem 4.12. *(P^F, sub_P) is an L-join-semilattice completion of (P, e_p) with the mapping ξ_P .*

Proof. Let (Q, e_Q) be an L-join-semilattice and Let $f : P \longrightarrow Q$ be Scott convexity-preserving. Then by Proposition 4.8, $(Q, \sigma^*(Q))$ is a sober L-convex space and $f : (P, \sigma^*(P)) \longrightarrow (Q, \sigma^*(Q))$ is convexity-preserving. By Theorem 3.13 and Proposition 4.10, we have $(P^F, \sigma^*(P^F))$ is a sobrification of $(P, \sigma^*(P))$ with ξ_P. Thus, there exists a unique convexity-preserving mapping $\overline{f} : (P^F, \sigma^*(P^F)) \longrightarrow (Q, \sigma^*(Q))$ such that $\overline{f} \circ \xi_P = f$. By Proposition 4.5, we have $\xi_P : (P, e_p) \longrightarrow (P^F, \mathrm{sub}_P)$ and $\overline{f} : (P^F, \mathrm{sub}_P) \longrightarrow (Q, e_Q)$ are finite-join-preserving mappings. Thus, (P^F, sub_P) with ξ_P is an L-join-semilattice completion of (P, e_p). □

We now provide a characterization for the L-join-semilattice completion of an L-ordered set via sobrification. The proof is straightforward and thus omitted. This result highlights the significance of our notion of sobriety in bridging fuzzy ordered structures and fuzzy convex structures.

Theorem 4.13. *An L-ordered set Q is an L-join-semilattice completion of an L-ordered set P if and only if $(Q, \sigma^*(Q))$ is a sobrification of $(P, \sigma^*(P))$.*

We denote the category of L-ordered sets with finite-join-preserving mappings as morphisms by L-**Ord**. Theorem 4.12 shows that the full subcategory of L-join-semilattices is reflective in L-**Ord**.

5 Conclusions

This paper explore the connection between sober L-convex spaces and L-join-semilattices for a complete residuated lattice L as the truth value table. We construct a sobrification for any L-convex space, demonstrating that the category of sober L-convex spaces is a reflective subcategory of that of L-convex spaces (cf. Theorem 3.13). We also show that Q is an L-join-semilattice completion of P if and only if $(Q, \sigma^*(Q))$ is a sobrification of $(P, \sigma^*(P))$ (cf. Theorems 4.12, 4.13). These work promote a closer connection between L-convex structures and L-ordered structures. Consequently, both ordered and categorical approaches can be effectively combined in future studies of fuzzy convex structure theory.

We offer two future research directions as follows:

- In the classical case, the notions of polytopes and compact convex sets are equivalent. However, in the fuzzy setting, this equivalence is not obvious. This is indeed a question to which we possess no answer. Liu and Yue in [13] have defined a type of sobriety for L-convex spaces based on compact convex sets. In this paper, we introduce another type of sobriety based on polytopes. Future research can focus on the relationship between these two types of sobriety. The first step in bridging this relationship is to either prove the equivalence between compact convex sets and polytopes or provide a counterexample.
- In the theory of topology, studying generalizations of sobriety is an interesting topic. Zhao and Fan in [47] introduced a weaker form of sobriety, called bounded sobriety. Zhang and Wang [45] further extended bounded sobriety to the framework of $\mathcal{Q}$-cotopological spaces. Following these work, one can introduce the notion of bounded sobriety in the framework of L-convex structure and subsequently investigate the relationship between bounded sober L-convex spaces and conditional L-join-semilattices.

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