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Towards efficient solutions of space-time fractional fuzzy diffusion equations: A methodological approach

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Abstract

This paper aims to introduce a groundbreaking methodology for deriving analytical solutions to the space-time fractional fuzzy diffusion equation. Our approach uniquely incorporates a Caputo generalized Hukuhara fractional derivative (of order $\beta \in (0, 2]$) for the second-order spatial derivative, alongside a fuzzy Caputo-Katugampola generalized Hukuhara time-fractional derivative (of order $\alpha \in (0, 1)$) for the first-order temporal derivative. The primary objective is to develop explicit and fundamental solutions for both the space-time fractional fuzzy diffusion equation and the time fractional fuzzy diffusion equation, encompassing various forms of fuzzy Caputo-Katugampola generalized Hukuhara time-fractional differentiability. We initiate our study by thoroughly analyzing the fuzzy Fourier and fuzzy $\wp-$Laplace transforms of the equation. To demonstrate the practical utility and effectiveness of our proposed method, we apply it to two specific models: a fuzzy groundwater flow model for computing pressure head, and a fuzzy model for determining the concentration of tumor cells. The results obtained highlight the method's efficiency and precision in addressing the complexities of both the space-time fractional fuzzy diffusion equation and the time fractional fuzzy diffusion equation.

Keywords: The fuzzy Caputo-Katugampola generalized Hukuhara time-fractional derivative, the fuzzy space-time fractional diffusion equation, the fuzzy $\wp-$Laplace transform, the fuzzy Fourier transform, the Caputo generalized Hukuhara fractional derivative.

1 Introduction

In the field of mathematical modeling, traditional diffusion equations have been crucial for understanding how various quantities like heat, mass, or information spread and change across different spaces. However, when applied in real-world contexts within continuous space, space-time fractional diffusion equations often encounter difficulties in accurately capturing the complexities and uncertainties inherent to practical scenarios. The inherent disparities between controlled laboratory conditions and real-world dynamics pose challenges in achieving precise estimations. This brings us to the space-time fractional fuzzy diffusion equation, a formulation that combines fractional calculus principles. This integration allows us to account for non-local effects in both space and time, using fuzzy logic to handle uncertainties in the system.

Fuzzy fractional calculus has witnessed extensive utilization across various disciplines over the past few decades, leading to the development of numerous nonlocal fuzzy fractional differential operators. These operators, available in both singular and non-singular forms, find versatile applicability across a wide spectrum of scientific domains. Notable examples include the fuzzy Riemann-Liouville derivative, fuzzy generalized Hukuhara Caputo fractional derivative, fuzzy Caputo-Fabrizio fractional derivative, fuzzy Atangana-Baleanu fractional derivative, fuzzy Riemann-Liouville-Katugampola generalized Hukuhara fractional derivative, and fuzzy Caputo-Katugampola generalized Hukuhara fractional derivative. These operators, with their multifaceted functionalities, serve as indispensable tools for modeling and analyzing intricate systems encountered in diverse scientific inquiries.

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Given the crucial role time-fractional differential equations play in various industries and engineering fields, it becomes imperative to explore mathematical approaches that yield closed-form solutions for such equations. Both numerical and analytical methods employ mathematical techniques to derive closed-form or exact solutions for fuzzy time-fractional differential equations. Viet Long and colleagues [25] introduced a novel approach to handle fuzzy fractional partial differential equations by employing the Caputo generalized Hukuhara partial differentiability for fuzzy multi-variable functions. Following this work, Ghaffari and co-authors [12] further advanced the field by providing explicit solutions for the time-fractional fuzzy equation using fuzzy Laplace and Fourier transforms. In a related study, Keshavarz et al. [18] derived the fuzzy fundamental solution of the fractional diffusion equation, emphasizing generalized Hukuhara differentiability, and applied it to model the release of an anti-cancer drug within a tumor. Mohapatra et al. [21] employed the Elzaki transformed homotopy perturbation method and the fractional reduced differential transformation method to solve the $n+1$-dimensional fractional order heat equation with an external source term under uncertain conditions.

In this study, we propose enhancements to the existing fuzzy Caputo-Katugampola derivative for two-variable fuzzy functions, aimed at expanding its applicability in solving space-time fuzzy fractional differential equations. Initially, Hoa et al. [15] utilized the generalized Lipschitz condition to establish the existence of solutions for initial value fractional differential equations, employing the fuzzy Caputo-Katugampola derivative. Concurrently, Sajedi et al. [22] investigated the existence, uniqueness, and stability of solutions for an impulsive coupled system of Caputo-Katugampola fuzzy fractional differential equations. Thereafter, Alshbeel et al. [4] considered a fuzzy fractional heat equation and presented its numerical solutions using the optimal homotopy asymptotic method.

This research introduces a novel method for analyzing the space-time fractional diffusion equation within the framework of fuzzy space, leveraging the Caputo-Katugampola generalized Hukuhara time-fractional derivative. This equation is derived by replacing the second-order space derivative with a Caputo generalized Hukuhara fractional derivative of order $\beta \in (0, 2]$ and the first-order time derivative with a fuzzy Caputo-Katugampola generalized Hukuhara time-fractional derivative of order $\alpha \in (0, 1)$. We advocate for the adoption of this derivative type as a robust tool for qualitatively assessing such equations, thereby enhancing our understanding of complex systems governed by fractional dynamics. Furthermore, we utilize the capabilities of the fuzzy $\wp$-Laplace transform [10] and fuzzy Fourier transform [13] to derive explicit and fundamental solutions for the space-time fractional fuzzy diffusion equation.

Our focus extends to addressing a significant challenge in groundwater flow modeling, particularly in accurately determining the pressure head and a fuzzy model for determining the concentration of tumor cells. Through concrete examples and meticulous analysis, we demonstrate the efficacy and practical relevance of our approach. By tackling real-world challenges, our method not only validates its effectiveness but also underscores its potential utility in various practical applications, paving the way for new insights and advancements in the field.

The subsequent sections are carefully arranged to offer a thorough understanding. Section 2 provides necessary background information on fuzzy fractional derivatives. Section 3 introduces fuzzy Caputo-Katugampola time fractional derivatives using generalized Hukuhara differentiability and establishes important properties of this type of differentiation. Section 4 explores the fuzzy $\wp$-Laplace transform and the fuzzy Fourier transform along with their essential properties. Moving forward, Section 5 presents a fuzzy space-time fractional diffusion equation and derives fuzzy explicit and fundamental solutions for it. To demonstrate the practical applicability of our proposed method in addressing real-world issues, we examine the fuzzy saturated groundwater flow model and a fuzzy model for calculating the concentration of tumor cells in Section 6. Finally, Section 7 summarizes the paper with conclusive remarks.

2 Fuzzy fractional derivatives

In this section, we revisit the fundamental concepts essential for our exploration of fuzzy fractional partial derivatives. We begin by explaining important definitions and basic properties relevant to our study. Additionally, we introduce key notations and preliminary facts that will form the basis for the discussions in this paper. For those seeking a more in-depth understanding of the fuzzy theory underpinning our analysis, we recommend consulting pertinent articles and books [1, 3, 6, 8, 19].

To initiate our discussion, we define the set of fuzzy numbers as $\mathbb{R}_F$, which consists of normal, fuzzy convex, upper semi-continuous, and compactly supported fuzzy sets defined over the real line. A fuzzy function extends the classical notion of a function by accommodating uncertain or imprecise mappings between elements of two sets. A specific type of fuzzy function that we consider is the triangular fuzzy-valued function, denoted as $\breve{\vartheta} : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}_F$. It is represented as $\breve{\vartheta}(\mathfrak{x}, \mathfrak{t}) = (\vartheta_1(\mathfrak{x}, \mathfrak{t}), \vartheta_2(\mathfrak{x}, \mathfrak{t}), \vartheta_3(\mathfrak{x}, \mathfrak{t}))$.

Corollary 2.1. [18] *Let λ_1 and λ_2 be two real constants such that $\lambda_1, \lambda_2 \geq 0$ (or $\lambda_1, \lambda_2 \leq 0$). Then, the following*

relation holds

$$\lambda_1 \breve{\vartheta}(\mathfrak{x},\mathfrak{t}) \ominus_{gH} \lambda_2 \breve{\vartheta}(\mathfrak{x},\mathfrak{t}) = (\lambda_1-\lambda_2)\breve{\vartheta}(\mathfrak{x},\mathfrak{t}).$$

Definition 2.2. [11] *A fuzzy-valued function $\breve{\vartheta}:\Omega\to\mathbb{R}_F$ is said to be continuous at $\mathfrak{t}_0\in\Omega$ if for each $\epsilon>0$ there is $\delta>0$ such that $\mathbb{H}\left(\breve{\vartheta}(\mathfrak{t}),\breve{\vartheta}(\mathfrak{t}_0)\right)<\epsilon$, whenever $\mathfrak{t}\in\Omega$ and $|\mathfrak{t}-\mathfrak{t}_0|<\delta$. We say that $\breve{\vartheta}$ is fuzzy continuous on Ω if $\breve{\vartheta}$ is continuous at each $\mathfrak{t}_0\in\Omega$. Here, $\mathbb{H}$ is the Hausdorff distance* [19].

Definition 2.3. [23] *A fuzzy function $\breve{\vartheta}$ is piecewise continuous on the interval $[0,\infty)$ if*

1. $\lim\limits_{\mathfrak{t}\to 0^+}\breve{\vartheta}(\mathfrak{t})=\breve{\vartheta}(0^+)$.
2. *$\breve{\vartheta}$ is continuous on every finite interval $(0,b)$ expect possibly at a finite number of points $\mathfrak{t}_1,\mathfrak{t}_2,\cdots,\mathfrak{t}_n$ in $(0,b)$ at which $\breve{\vartheta}$ has jump discontinuity.*

Theorem 2.4. [27] *Let $\breve{\vartheta}(\mathfrak{t})$ be a fuzzy-valued function on $(-\infty,\infty)$ and it is represented by $\breve{\vartheta}(\mathfrak{t};\mathfrak{r})=[\vartheta^-(x;\mathfrak{r}),\vartheta^+(\mathfrak{t};\mathfrak{r})]$ for any fixed $\mathfrak{r}\in[0,1]$. Assume that $|\vartheta^-(\mathfrak{t};\mathfrak{r})|$ and $|\vartheta^+(\mathfrak{t};\mathfrak{r})|$ are Riemann integrable on $(-\infty,\infty)$ for all $\mathfrak{r}\in[0,1]$. Then $\breve{\vartheta}(\mathfrak{t})$ is improper fuzzy Riemann-integrable on $(-\infty,\infty)$ and the improper fuzzy Riemann integral is a fuzzy number. Furthermore, we have*

$$\int_{-\infty}^{\infty}\breve{\vartheta}(\mathfrak{t})\mathrm{d}\mathfrak{t}=\left[\int_{-\infty}^{\infty}\vartheta^-(\mathfrak{t};\mathfrak{r})\mathrm{d}\mathfrak{t}\quad,\quad\int_{-\infty}^{\infty}\vartheta^+(\mathfrak{t};\mathfrak{r})\mathrm{d}\mathfrak{t}\right].$$

Definition 2.5. [13] *Let $\breve{\vartheta}:\mathbb{R}\to\mathbb{R}_{\mathcal{F}}$. We say that $\breve{\vartheta}(\mathfrak{t})$ is fuzzy absolutely integrable if $\int_{-\infty}^{\infty}\mathbb{H}\left(\breve{\vartheta}(\mathfrak{t}),0\right)<\infty$.*

Definition 2.6. [8] *The fuzzy generalized Hukuhara partial derivative with respect to $\mathfrak{t}$ for the mapping $\breve{\vartheta}:\Omega\to\mathbb{R}_F$, denoted as $\frac{\partial\breve{\vartheta}(\mathfrak{x},\mathfrak{t})}{\partial\mathfrak{t}}$ (abbreviated as gH-partial derivative), is formally defined as:*

$$\frac{\partial\breve{\vartheta}(\mathfrak{x},\mathfrak{t})}{\partial\mathfrak{t}}\Big|_{(\mathfrak{x}_0,\mathfrak{t}_0)}=\lim_{\hbar\to 0}\frac{\breve{\vartheta}(\mathfrak{x}_0,\mathfrak{t}_0+\hbar)\ominus_{gH}\breve{\vartheta}(\mathfrak{x}_0,\mathfrak{t}_0)}{\hbar}.$$

Where $(\mathfrak{x}_0,\mathfrak{t}_0)\in\Omega$. It is assumed that $\frac{\partial\breve{\vartheta}(\mathfrak{x},\mathfrak{t})}{\partial\mathfrak{t}}\in\mathbb{R}_F$.

Moreover, the generalized Hukuhara partial differentiability concerning $\mathfrak{t}$ at a specific point $(\mathfrak{x},\mathfrak{t})\in\Omega$ can be classified into two types: $[i-p]$-differentiability and $[ii-p]$-differentiability.

When we state that $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ is $[i-p]$-differentiable, it signifies that the first $gH-$partial derivative with respect to t of this function can be expressed as:

$$\frac{\partial_i\breve{\vartheta}(\mathfrak{x},\mathfrak{t})}{\partial\mathfrak{t}}=\left(\frac{\partial\vartheta_1(\mathfrak{x},\mathfrak{t})}{\partial\mathfrak{t}},\frac{\partial\vartheta_2(\mathfrak{x},\mathfrak{t})}{\partial\mathfrak{t}},\frac{\partial\vartheta_3(\mathfrak{x},\mathfrak{t})}{\partial\mathfrak{t}}\right).$$

Similarly, $[ii-p]$-differentiability is attributed to $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ when the first $gH-$partial derivative with respect to t can be represented as:

$$\frac{\partial_{ii}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})}{\partial\mathfrak{t}}=\left(\frac{\partial\vartheta_3(\mathfrak{x},\mathfrak{t})}{\partial\mathfrak{t}},\frac{\partial\vartheta_2(\mathfrak{x},\mathfrak{t})}{\partial\mathfrak{t}},\frac{\partial\vartheta_1(\mathfrak{x},\mathfrak{t})}{\partial\mathfrak{t}}\right).$$

Throughout this paper, we make the following assumptions: Suppose that the space of all continuous fuzzy functions on a domain Ω is denoted by $C_F(\Omega,\mathbb{R}_F)$, and $AC(\Omega,\mathbb{R}_F)$ is the set of all absolutely continuous fuzzy functions $\breve{\vartheta}$ on Ω with values in $\mathbb{R}_F$ that are gH-partial differentiable and have no switching points in their domain.

Definition 2.7. [2, 25] *The map $\breve{\vartheta}:\Omega\to\mathbb{R}_F$ is called a Caputo generalized Hukuhara fractional differentiable function of order β with respect to $\mathfrak{x}$ (Caputo gH-space fractional differentiable), if ${}_g^C\mathcal{D}^{\beta}_{\mathfrak{x}_0,\mathfrak{x}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ is a fuzzy function. This derivative is defined as follows:*

$${}_g^C\mathcal{D}^{\beta}_{\mathfrak{x}_0,\mathfrak{x}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})=\frac{1}{\Gamma(1-\beta)}\int_{\mathfrak{x}_0}^{\mathfrak{x}}\frac{1}{(\mathfrak{x}-\xi)^{\beta}}\frac{\partial\breve{\vartheta}(\xi,\mathfrak{t})}{\partial\xi}\mathrm{d}\xi,$$

where $0 < \beta \leq 1$ *and* $\mathfrak{x} > \mathfrak{x}_0$.

Furthermore, the Caputo generalized Hukuhara space fractional differentiability with respect to $\mathfrak{x}$ *at a specific point can be classified into two distinct types:* ${}^{C}_{1.g}\mathcal{D}^{\beta}_{\mathfrak{x}_0,\mathfrak{x}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ *and* ${}^{C}_{2.g}\mathcal{D}^{\beta}_{\mathfrak{x}_0,\mathfrak{x}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$.

The first type, denoted as (1)-Caputo gH-space fractional differentiability (which is denoted as ${}^{C}_{1.g}\mathcal{D}^{\beta}_{\mathfrak{x}_0,\mathfrak{x}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$*), manifests when the Caputo gH-derivative takes the form:*

$$ {}^{C}_{1.g}\mathcal{D}^{\beta}_{\mathfrak{x}_0,\mathfrak{x}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) = \Big({}^{C}D^{\beta}_{\mathfrak{x}_0,\mathfrak{x}}\vartheta_1(\mathfrak{x},\mathfrak{t}),\ {}^{C}D^{\beta}_{\mathfrak{x}_0,\mathfrak{x}}\vartheta_2(\mathfrak{x},\mathfrak{t}),\ {}^{C}D^{\beta}_{\mathfrak{x}_0,\mathfrak{x}}\vartheta_3(\mathfrak{x},\mathfrak{t})\Big). $$

Similarly, the second type, termed (2)-Caputo gH-space fractional differentiability (which is denoted as ${}^{C}_{2.g}\mathcal{D}^{\beta}_{\mathfrak{x}_0,\mathfrak{x}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$*), emerges when the Caputo gH-derivative is articulated as:*

$$ {}^{C}_{2.g}\mathcal{D}^{\beta}_{\mathfrak{x}_0,\mathfrak{x}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) = \Big({}^{C}D^{\beta}_{\mathfrak{x}_0,\mathfrak{x}}\vartheta_3(\mathfrak{x},\mathfrak{t}),\ {}^{C}D^{\beta}_{\mathfrak{x}_0,\mathfrak{x}}\vartheta_2(\mathfrak{x},\mathfrak{t}),\ {}^{C}D^{\beta}_{\mathfrak{x}_0,\mathfrak{x}}\vartheta_1(\mathfrak{x},\mathfrak{t})\Big). $$

Here, the symbol ${}^{C}D^{\beta}_{\mathfrak{x}_0,\mathfrak{x}}$ *denotes a fractional Caputo derivative of a real-valued function.*

Definition 2.8. [10] *A function* $\breve{\vartheta} : \Omega \to \mathbb{R}_F$ *is termed a partially generalized Hukuhara conformable fractional differentiable fuzzy function of order* $0 < \wp < 1$ *at* $(\mathfrak{x},\mathfrak{t}) \in \Omega$*, such that for any sufficiently small* $\varepsilon > 0$*, the generalized Hukuhara difference* $\breve{\vartheta}\left(\mathfrak{x},\mathfrak{t}+\varepsilon(\mathfrak{t}-a)^{1-\wp}\right) \ominus_{gH} \breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ *is well-defined and*

$$ {}_{gH}\mathbb{T}^{\wp}_{a,\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) = \lim_{\varepsilon\to 0}\frac{\breve{\vartheta}\left(\mathfrak{x},\mathfrak{t}+\varepsilon(\mathfrak{t}-a)^{1-\wp}\right) \ominus_{gH} \breve{\vartheta}(\mathfrak{x},\mathfrak{t})}{\varepsilon}, $$

provided that ${}_{gH}\mathbb{T}^{\wp}_{a,\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) \in \mathbb{R}_F$ *exists. In simpler terms, we say that* $\breve{\vartheta}$ *is a* $[\wp - p]_{gH}$*-differentiable fuzzy function.*

Let $\breve{\vartheta}$ *be a* $[\wp - p]_{gH}$*-differentiable fuzzy function with* $0 < \wp < 1$*. We say that* $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ *is* $[\wp - p]_{i.gH}$*-differentiable over* Ω *(denoted as* ${}_{i.gH}\mathbb{T}^{\wp}_{a,\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$*) if the following holds*

$$ {}_{i.gH}\mathbb{T}^{\wp}_{a,\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) = \left(T^{\wp}_{a,\mathfrak{t}}\vartheta_1(\mathfrak{x},\mathfrak{t}), T^{\wp}_{a,\mathfrak{t}}\vartheta_2(\mathfrak{x},\mathfrak{t}), T^{\wp}_{a,\mathfrak{t}}\vartheta_3(\mathfrak{x},\mathfrak{t})\right). $$

Similarly, $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ *is* $[\wp - p]_{ii.gH}$*-differentiable over* Ω *(denoted as* ${}_{ii.gH}\mathbf{T}^{\wp}_{a,\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$*) if the following holds*

$$ {}_{ii.gH}\mathbb{T}^{\wp}_{a,\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) = \left(T^{\wp}_{a,\mathfrak{t}}\vartheta_3(\mathfrak{x},\mathfrak{t}), T^{\wp}_{a,\mathfrak{t}}\vartheta_2(\mathfrak{x},\mathfrak{t}), T^{\wp}_{a,\mathfrak{t}}\vartheta_1(\mathfrak{x},\mathfrak{t})\right). $$

Here, the symbol $T^{\wp}_{a,\mathfrak{t}}$ *denotes a fractional Conformable derivative of a real-valued function.*

Theorem 2.9. [10] *Let* $\breve{\vartheta} : \Omega \to \mathbb{R}_F$ *be a* $[\wp - p]_{gH}$*-differentiable fuzzy function, such that the type of* $[\wp - p]_{gH}$*-differentiability remains consistent throughout* Ω*. Then, for* $\wp \in (0,1)$*, we have*

$$ {}_{gH}\mathbb{T}^{\wp}_{0,\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) = \mathfrak{t}^{1-\wp}\frac{\partial\breve{\vartheta}(\mathfrak{x},\mathfrak{t})}{\partial\mathfrak{t}}. $$

3 The fuzzy Caputo-Katugampola generalized Hukuhara time fractional derivative and its key properties

This section presents a new definition of a fuzzy fractional derivative, the fuzzy Caputo-Katugampola generalized Hukuhara time-fractional derivative. Within this framework, we aim to elucidate and establish various crucial properties associated with this particular type of fuzzy fractional derivative through systematic presentation and rigorous proofs.

Definition 3.1. *Let* $0 < \alpha < 1$ *and* $\wp > 0$ *be two fixed real parameters. The fuzzy Caputo-Katugampola generalized Hukuhara time-fractional derivative of order* α *for a given fuzzy function* $\breve{\vartheta} : \Omega \to \mathbb{R}_F$*, with respect to* $\mathfrak{t}$ *at* $(\mathfrak{x},\mathfrak{t}) \in \Omega$*, is formally defined as follows:*

$$ {}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) = \frac{\wp^{\alpha}}{\Gamma(1-\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}^{\wp}-\xi^{\wp})^{-\alpha}\quad {}_{gH}\mathbb{T}^{\wp}_{0,\xi}\breve{\vartheta}(\mathfrak{x},\xi)\frac{\mathrm{d}\xi}{\xi^{1-\wp}}. $$

Provided that ${}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) \in \mathbb{R}_F$*. In simpler terms, we say that* $\breve{\vartheta}$ *is a* $[CK]_g$*-time differentiable fuzzy function.*

Using Theorem 2.9, we can rewrite the generalized Hukuhara Caputo-Katugampola time-fractional derivative as follows:

$$ {}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) = \frac{\wp^{\alpha}}{\Gamma(1-\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}^{\wp}-\xi^{\wp})^{-\alpha}\frac{\partial\breve{\vartheta}(\mathfrak{x},\xi)}{\partial\xi}\mathrm{d}\xi. $$

Definition 3.2. *Consider a function $\breve{\vartheta} : \Omega \to \mathbb{R}_F$ that is $[CK]_g$-time differentiable with an order of α. We refer to this as a $[CK]_{i.g}$-time differentiable function at $(\mathfrak{x}, \mathfrak{t}) \in \Omega$, symbolized by ${}^{*}_{i.g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x}, \mathfrak{t})$, if the following condition holds:*

$$\left[{}^{*}_{i.g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x}, \mathfrak{t}) \right]^{\mathfrak{r}} = \left[D^{\alpha,\wp}_{\mathfrak{t}}\vartheta^{-}_{\mathfrak{r}}(\mathfrak{x}, \mathfrak{t}), D^{\alpha,\wp}_{\mathfrak{t}}\vartheta^{+}_{\mathfrak{r}}(\mathfrak{x}, \mathfrak{t}) \right], \quad \forall \mathfrak{r} \in [0, 1]. \tag{1}$$

Similarly, we refer to this function as a $[CK]_{ii.g}$-time differentiable function at $(\mathfrak{x}, \mathfrak{t}) \in \Omega$, symbolized by ${}^{*}_{ii.g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x}, \mathfrak{t})$, if the following condition holds:

$$\left[{}^{*}_{ii.g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x}, \mathfrak{t}) \right]^{\mathfrak{r}} = \left[D^{\alpha,\wp}_{\mathfrak{t}}\vartheta^{+}_{\mathfrak{r}}(\mathfrak{x}, \mathfrak{t}), D^{\alpha,\wp}_{\mathfrak{t}}\vartheta^{-}_{\mathfrak{r}}(\mathfrak{x}, \mathfrak{t}) \right], \quad \forall \mathfrak{r} \in [0, 1]. \tag{2}$$

Here, the symbol $D^{\alpha,\wp}_{\mathfrak{t}}$ denotes the Caputo-Katugampola fractional derivative of a real-valued function [24].

Remark 3.3. *Consider a function $\breve{\vartheta} : \Omega \to \mathbb{R}_F$ that is $[CK]_g$-time differentiable with an order of α. Furthermore, $\breve{\vartheta}$ is a triangular fuzzy function, i.e., $\breve{\vartheta}(\mathfrak{x}, \mathfrak{t}) = \Big(\vartheta_1(\mathfrak{x}, \mathfrak{t}), \vartheta_2(\mathfrak{x}, \mathfrak{t}), \vartheta_3(\mathfrak{x}, \mathfrak{t})\Big)$.*

i. *$\breve{\vartheta}$ is considered $[CK]_{i.g}$-time differentiable at a point $(\mathfrak{x}, \mathfrak{t})$ within the set Ω if the following equation holds:*

$${}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x}, \mathfrak{t}) = \left(D^{\alpha,\wp}_{\mathfrak{t}}\vartheta_1(\mathfrak{x}, \mathfrak{t}), D^{\alpha,\wp}_{\mathfrak{t}}\vartheta_2(\mathfrak{x}, \mathfrak{t}), D^{\alpha,\wp}_{\mathfrak{t}}\vartheta_3(\mathfrak{x}, \mathfrak{t})\right).$$

ii. *$\breve{\vartheta}$ is considered $[CK]_{ii.g}$-time differentiable at a point $(\mathfrak{x}, \mathfrak{t})$ within the set Ω if the following equation holds:*

$${}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x}, \mathfrak{t}) = \left(D^{\alpha,\wp}_{\mathfrak{t}}\vartheta_3(\mathfrak{x}, \mathfrak{t}), D^{\alpha,\wp}_{\mathfrak{t}}\vartheta_2(\mathfrak{x}, \mathfrak{t}), D^{\alpha,\wp}_{\mathfrak{t}}\vartheta_1(\mathfrak{x}, \mathfrak{t})\right).$$

Example 3.4. *Consider the fuzzy function $\breve{\vartheta} \in AC([0, 2] \times [0, \frac{\pi}{2}], \mathbb{R}_F)$ defined as follows:*

$$\breve{\vartheta}(\mathfrak{x}, \mathfrak{t}) = \left(2\mathfrak{x}\sin(\mathfrak{t}), 5\mathfrak{x}\sin(\mathfrak{t}), 12\mathfrak{x}\sin(\mathfrak{t})\right).$$

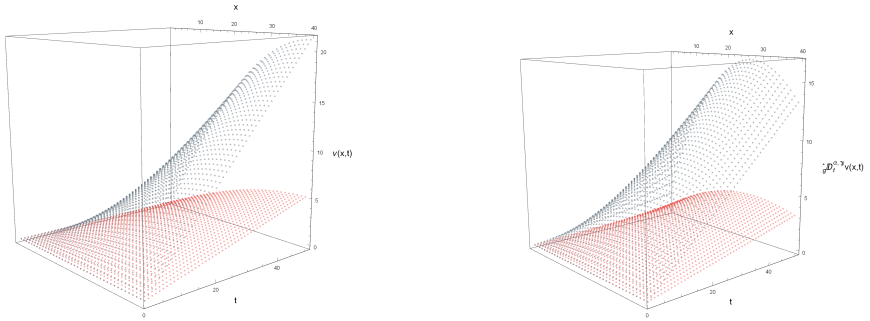
Next, we explore the fuzzy $[CK]_g$-time derivative using Definition 3.1, with $\wp = \alpha = \frac{1}{2}$. This derivative is represented as:

$${}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x}, \mathfrak{t}) = \left(\frac{8}{3}\sqrt{\frac{2}{\pi}}\mathfrak{t}^{\frac{3}{4}}\mathfrak{x}, \frac{20}{3}\sqrt{\frac{2}{\pi}}\mathfrak{t}^{\frac{3}{4}}\mathfrak{x}, 16\sqrt{\frac{2}{\pi}}\mathfrak{t}^{\frac{3}{4}}\mathfrak{x}\right) \; {}_pF_q\left(\left\{\frac{3}{4}, 1, \frac{5}{4}\right\}; \left\{\frac{5}{8}, \frac{7}{8}, \frac{9}{8}, \frac{11}{8}\right\}; \frac{-\mathfrak{t}^2}{4}\right).$$

In this equation, ${}_pF_q$ signifies the generalized hypergeometric function. Consequently, the fuzzy function $\breve{\vartheta}(\mathfrak{x}, \mathfrak{t})$ is deemed $[CK]_{i.g}$-time differentiable within the specified domain of $[0, 2] \times [0, \frac{\pi}{2}]$.

To visually comprehend the concept of $[CK]_{i.g}$-time differentiability for this solution, we have generated a plot showcasing the behavior of both $\breve{\vartheta}(\mathfrak{x}, \mathfrak{t})$ and ${}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x}, \mathfrak{t})$ with $\wp = \alpha = \frac{1}{2}$. This plot is depicted in Figure 1.

The lower cut (in red) and the upper cut (in blue) for both $\breve{\vartheta}(\mathfrak{x}, \mathfrak{t})$ and ${}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x}, \mathfrak{t})$ coincide, emphasizing that $\breve{\vartheta}(\mathfrak{x}, \mathfrak{t})$ is a $[CK]_{i.g}$-time differentiable function.



(A) $\breve{\vartheta}(\mathfrak{x}, \mathfrak{t})$ (B) ${}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x}, \mathfrak{t})$

Figure 1: Illustrating the behavior of $\breve{\vartheta}(\mathfrak{x}, \mathfrak{t})$ and its $[CK]_g$-time derivative ${}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x}, \mathfrak{t})$ for specific parameters in Example 3.4: $\wp = \alpha = \frac{1}{2}$, $\mathfrak{r} = 0.2$

Example 3.5. *Consider the function $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$, defined as follows*

$$\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) = (\mathfrak{x}\exp(-\mathfrak{t}), 12\mathfrak{x}\exp(-\mathfrak{t}), 24\mathfrak{x}\exp(-\mathfrak{t}))\,.$$

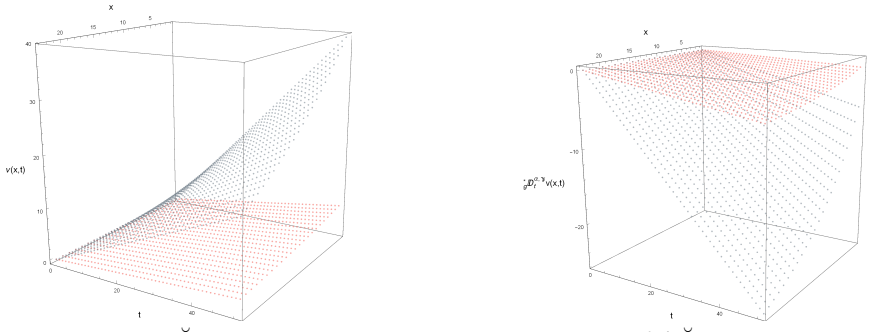
Now, we delve into the concept of a fuzzy $[CK]_g$-time derivative with an order of $\alpha = \frac{1}{3}$ and $\wp = \frac{1}{2}$. It can be represented as

$${}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) = \left(-\frac{108\times 2^{\frac{2}{3}}\mathfrak{t}^{\frac{5}{6}}\mathfrak{x}}{5\Gamma\left(\frac{2}{3}\right)}, -\frac{54\times 2^{\frac{2}{3}}\mathfrak{t}^{\frac{5}{6}}\mathfrak{x}}{5\Gamma\left(\frac{2}{3}\right)}, -\frac{9\mathfrak{t}^{\frac{5}{6}}\mathfrak{x}}{5\times 2^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)}\right)\ {}_pF_q\left(\left\{1,\frac{3}{2}\right\};\left\{\frac{4}{3},\frac{11}{6}\right\};-\mathfrak{t}\right).$$

Consequently, the fuzzy function $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ is considered $[CK]_{ii.g}$-time differentiable.

To visually grasp the idea of $[CK]_g$-time differentiability for this solution, we've created a plot illustrating the behavior of both $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ and ${}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ with $\alpha = \frac{1}{3}$ and $\wp = \frac{1}{2}$. This plot is depicted in Figure 2.

The positions of the lower cut (in red) and the upper cut (in blue) for both $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ and ${}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ coincide, indicating that $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ is a $[CK]_{ii.g}$-time differentiable function.



(A) $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ (B) ${}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$

Figure 2: Visualizing the dynamics of $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ and its derivative, ${}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$, under specific parameters outlined in Example 3.5: $\alpha = \frac{1}{3}$, $\wp = \frac{1}{2}$ and $\mathfrak{r} = 0.2$.

Definition 3.6. *Let $\tau_0 \in \mathbb{R}^+$ is fixed and there exist $(\mathfrak{x},\tau_1)$ and $(\mathfrak{x},\tau_2)$ in any neighborhood of $(\mathfrak{x},\tau_0)$ such that $(\mathfrak{x},\tau_1) < (\mathfrak{x},\tau_0) < (\mathfrak{x},\tau_2)$, then*

i. *The points $(\mathfrak{x},\tau_0)$, are switching points **Type I** for the $[CK]_g-$time differentiability of $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ if at $(\mathfrak{x},\tau_1)$ (1) holds while (2) does not hold and at $(\mathfrak{x},\tau_2)$ (2) holds and (1) does not hold.*

ii. *The points $(\mathfrak{x},\tau_0)$, are switching points **Type II** for the $[CK]_g-$time differentiability of $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ if at $(\mathfrak{x},\tau_1)$ (2) holds while (1) does not hold and at $(\mathfrak{x},\tau_2)$ (1) holds and (2) does not hold.*

Example 3.7. *Let $\breve{\vartheta} \in AC([0,2]\times[0,6],\mathbb{R}_F)$ which is defined by*

$$\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) = \left(\mathfrak{x}^2(6\mathfrak{t}-\mathfrak{t}^2), 8\mathfrak{x}^2(6\mathfrak{t}-\mathfrak{t}^2), 20\mathfrak{x}^2(6\mathfrak{t}-\mathfrak{t}^2)\right).$$

To calculate the $[CK]_g-$time derivative of $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$, we use Definition 3.1. In this case, for $(\mathfrak{x},\mathfrak{t}) \in [0,2]\times[0,\frac{35}{8})$ and $\alpha = \wp = \frac{1}{2}$ we have

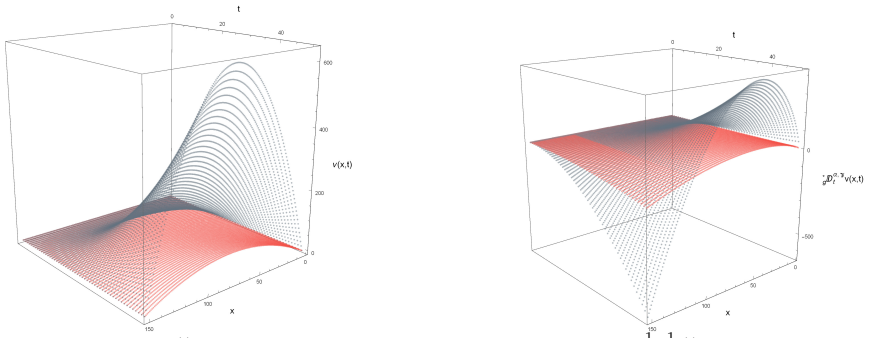
$${}^{*}_{g}\mathbb{D}^{\frac{1}{2},\frac{1}{2}}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) = \left(\frac{-8}{35}\sqrt{\frac{2}{\pi}}\mathfrak{t}^{\frac{3}{4}}(8\mathfrak{t}-35)\mathfrak{x}^2, \frac{-64}{35}\sqrt{\frac{2}{\pi}}\mathfrak{t}^{\frac{3}{4}}(8\mathfrak{t}-35)\mathfrak{x}^2, \frac{-32}{7}\sqrt{\frac{2}{\pi}}\mathfrak{t}^{\frac{3}{4}}(8\mathfrak{t}-35)\mathfrak{x}^2\right),$$

and for $(\mathfrak{x},\mathfrak{t}) \in [0,2]\times[\frac{35}{8},6]$, we have

$${}^{*}_{g}\mathbb{D}^{\frac{1}{2},\frac{1}{2}}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) = \left(\frac{-32}{7}\sqrt{\frac{2}{\pi}}\mathfrak{t}^{\frac{3}{4}}(8\mathfrak{t}-35)\mathfrak{x}^2, \frac{-64}{35}\sqrt{\frac{2}{\pi}}\mathfrak{t}^{\frac{3}{4}}(8\mathfrak{t}-35)\mathfrak{x}^2, \frac{-8}{35}\sqrt{\frac{2}{\pi}}\mathfrak{t}^{\frac{3}{4}}(8\mathfrak{t}-35)\mathfrak{x}^2\right).$$

Clearly the fuzzy-valued function $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ *is* $[CK]_{i.g}-$*time differentiable on* $(\mathfrak{x},\mathfrak{t})\in[0,2]\times[0,\frac{35}{8})$ *and it is a* $[CK]_{ii.g}-$*time differentiable for all* $(\mathfrak{x},\mathfrak{t})\in[0,2]\times[\frac{35}{8},6]$ *. Hence, the points* $(\mathfrak{x},\frac{35}{8})$*, for all* $\mathfrak{x}\in[0,2]$*, are switching points* ***Type I*** *for the* $[CK]_g-$*time differentiability of* $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$*.*

The fuzzy functions $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ *and* ${}^{*}_{g}\mathbb{D}^{\frac{1}{2},\frac{1}{2}}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ *are shown in Figures 3(a) and (b), respectively. As you can see, in Figure 3(b), the position of lower cut(blue color) and upper cut(red color) of the fuzzy function* ${}^{*}_{g}\mathbb{D}^{\frac{1}{2},\frac{1}{2}}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ *for* $\mathfrak{t}>\frac{35}{8}$ *is changed. It shows that, the points* $(\mathfrak{x},\frac{35}{8})$*, for all* $\mathfrak{x}\in[0,2]$*, are switching points* ***Type I****.*



(A) $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ (B) ${}^{*}_{g}\mathbb{D}^{\frac{1}{2},\frac{1}{2}}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$

Figure 3: Plots of $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ and ${}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ for Example 3.7 when $\wp=\alpha=\frac{1}{2}$, and r = 0.1.

Theorem 3.8. *Let* $\breve{\vartheta}\in AC(\Omega,\mathbb{R}_F)$ *be a fuzzy function and is* $[CK]_g$*-time differentiable.*

i. $\breve{\vartheta}$ *is* $[\wp-p]_{i.gH}-$*differentiable for all* $(\mathfrak{x},\mathfrak{t})\in\Omega$*, if and only if it is* $[CK]_{i.g}-$*time differentiable fuzzy function.*

ii. $\breve{\vartheta}$ *is* $[\wp-p]_{ii.gH}-$*differentiable for all* $(\mathfrak{x},\mathfrak{t})\in\Omega$ *if and only if it is* $[CK]_{ii.g}-$*time differentiable fuzzy function.*

Proof. Let $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ satisfy the assumptions of the theorem, making it $[CK]_g$-time differentiable. By applying Definition 3.1, we obtain the following expression

$$ {}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) \;=\; \frac{\wp^{\alpha}}{\Gamma(1-\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}^{\wp}-\xi^{\wp})^{-\alpha}_{gH}\,\mathbb{T}^{\wp}_{0,\xi}\breve{\vartheta}(\mathfrak{x},\xi)\frac{\mathrm{d}\xi}{\xi^{1-\wp}}. $$

Now, assuming that $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ is $[\wp-p]_{i.gH}$-differentiable for all $(\mathfrak{x},\mathfrak{t})\in\Omega$, we express ${}_{i.gH}\mathbb{T}^{\wp}_{0,\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$, as

$$ \left(T^{\wp}_{0,\mathfrak{t}}\vartheta_1(\mathfrak{x},\mathfrak{t}),T^{\wp}_{0,\mathfrak{t}}\vartheta_2(\mathfrak{x},\mathfrak{t}),T^{\wp}_{0,\mathfrak{t}}\vartheta_3(\mathfrak{x},\mathfrak{t})\right). $$

This allows us to rewrite the expression for ${}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ as:

$$\begin{aligned}
{}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) &= \frac{\wp^{\alpha}}{\Gamma(1-\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}^{\wp}-\xi^{\wp})^{-\alpha}\left({}_{i.gH}\mathbb{T}^{\wp}_{0,\xi}\breve{\vartheta}(\mathfrak{x},\xi)\right)\frac{\mathrm{d}\xi}{\xi^{1-\wp}}\\
&= \frac{\wp^{\alpha}}{\Gamma(1-\alpha)}\Bigg(\int_0^{\mathfrak{t}}(\mathfrak{t}^{\wp}-\xi^{\wp})^{-\alpha}T^{\wp}_{0,\xi}\vartheta_1(\mathfrak{x},\xi)\frac{\mathrm{d}\xi}{\xi^{1-\wp}},\int_0^{t}(\mathfrak{t}^{\wp}-\xi^{\wp})^{-\alpha}T^{\wp}_{0,\xi}\vartheta_2(\mathfrak{x},\xi)\frac{\mathrm{d}\xi}{\xi^{1-\wp}}\\
&\qquad ,\int_0^{\mathfrak{t}}(\mathfrak{t}^{\wp}-\xi^{\wp})^{-\alpha}T^{\wp}_{0,\xi}\vartheta_3(\mathfrak{x},\xi)\frac{\mathrm{d}\xi}{\xi^{1-\wp}}\Bigg)\\
&= (D^{\alpha,\wp}_{\mathfrak{t}}\vartheta_1(\mathfrak{x},\mathfrak{t}),D^{\alpha,\wp}_{\mathfrak{t}}\vartheta_2(\mathfrak{x},\mathfrak{t}),D^{\alpha,\wp}_{\mathfrak{t}}\vartheta_3(\mathfrak{x},\mathfrak{t}))\\
&= {}^{*}_{i.g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}).
\end{aligned}$$

Therefore, the desired result is achieved. A similar argument can be applied to establish the result for the other case. □

Theorem 3.9. *Let* $\breve{\vartheta}$ *and* $\breve{\mathfrak{u}}$ *be two fuzzy functions in* $AC(\Omega,\mathbb{R}_F)$*, each* $[CK]_g$*-time differentiable, with no switching points in the domain* Ω*. Additionally, if* λ_1 *and* λ_2 *are both positive real numbers, then the following statements hold:*

i. ${}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\left(\lambda_1\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\oplus\lambda_2\breve{\mathfrak{u}}(\mathfrak{x},\mathfrak{t})\right)=\lambda_1\,{}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\oplus\lambda_2\,{}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\mathfrak{u}}(\mathfrak{x},\mathfrak{t})$.

ii. ${}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\left(\lambda_1\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\ominus_{gH}\lambda_2\breve{\mathfrak{u}}(\mathfrak{x},\mathfrak{t})\right)=\lambda_1\ {}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\ominus_{gH}\lambda_2\ {}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\mathfrak{u}}(\mathfrak{x},\mathfrak{t})$.

Proof. Using Definition 3.1 and properties of fuzzy integrals, it can be concluded that

$$\begin{aligned}
{}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\left(\lambda_1\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\oplus\lambda_2\breve{\mathfrak{u}}(\mathfrak{x},\mathfrak{t})\right) &= \frac{\wp^{\alpha}}{\Gamma(1-\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}^{\wp}-\xi^{\wp})^{-\alpha}\frac{\partial}{\partial\xi}\left(\lambda_1\breve{\vartheta}(\mathfrak{x},\xi)\oplus\lambda_2\breve{\mathfrak{u}}(\mathfrak{x},\xi)\right)\mathrm{d}\xi\\
&= \lambda_1\frac{\wp^{\alpha}}{\Gamma(1-\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}^{\wp}-\xi^{\wp})^{-\alpha}\frac{\partial\breve{\vartheta}(\mathfrak{x},\xi)}{\partial\xi}\mathrm{d}\xi\\
&\oplus\ \lambda_2\frac{\wp^{\alpha}}{\Gamma(1-\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}^{\wp}-\xi^{\wp})^{-\alpha}\frac{\partial\breve{\mathfrak{u}}(\mathfrak{x},\xi)}{\partial\xi}\mathrm{d}\xi\\
&= \lambda_1\ {}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\oplus\lambda_2\ {}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\mathfrak{u}}(\mathfrak{x},\mathfrak{t}).
\end{aligned}$$

Similarly, the second statement can be proven. □

4 Fuzzy integral transforms: A comprehensive overview

In this section, we embark on a concise exploration of fuzzy integral transforms, specifically focusing on the fuzzy $\wp$-Laplace transform and the fuzzy Fourier transform. Through rigorous analysis, novel properties of these fuzzy transforms will be established, contributing to an enhanced understanding of their mathematical characteristics and potential applications.

4.1 The generalized fuzzy Laplace transform

Consider the set $\mathfrak{K}$ defined as follows

$$\mathfrak{K}=\left\{\breve{\vartheta}(\mathfrak{x},\mathfrak{t}):\exists M,\wp>0,\mathbb{H}\Big(\breve{\vartheta}(\mathfrak{x},\mathfrak{t}),0\Big)\leq M\mathrm{e}^{\mathrm{c}\frac{\mathfrak{t}^{\wp}}{\wp}}\right\},$$

where $\mathbb{H}$ is the Hausdorff distance, and $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\in\mathfrak{K}$ is a piecewise continuous fuzzy function on $[0,\infty)$.

The fuzzy $\wp-$Laplace transform for this fuzzy function $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ with respect to $\mathfrak{t}$ is defined as

$$\mathcal{L}_{\wp,\mathfrak{t}}\left\{\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s})=\int_0^{\infty}\mathfrak{t}^{\wp-1}\mathrm{e}^{-\mathfrak{s}\frac{\mathfrak{t}^{\wp}}{\wp}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\mathrm{d}\mathfrak{t},\qquad\wp>0,\tag{3}$$

subject to the condition that the limit exists as a finite fuzzy function.

Lemma 4.1. [10] *Consider $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ and $\breve{\mathfrak{u}}(\mathfrak{x},\mathfrak{t})$ are two fuzzy functions. Consider real constants λ_1, λ_2 both either positive or negative. If the fuzzy $\wp-$Laplace transform $\breve{\vartheta}$ and $\breve{\mathfrak{u}}$ exist hence the fuzzy $\wp-$Laplace transform of $\lambda_1\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\oplus\lambda_2\breve{\mathfrak{u}}(\mathfrak{x},\mathfrak{t})$ and $\lambda_1\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\ominus_{gH}\lambda_2\breve{\mathfrak{u}}(\mathfrak{x},\mathfrak{t})$ also exist. Additionally*

1. $\lambda_1\ \mathcal{L}_{\wp,\mathfrak{t}}\left\{\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s})\ominus_{gH}\lambda_2\ \mathcal{L}_{\wp,\mathfrak{t}}\left\{\breve{\mathfrak{u}}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s})=\mathcal{L}_{\wp,\mathfrak{t}}\left\{\lambda_1\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\ominus_{gH}\lambda_2\breve{\mathfrak{u}}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s})$.
2. $\lambda_1\ \mathcal{L}_{\wp,\mathfrak{t}}\left\{\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s})\oplus\lambda_2\ \mathcal{L}_{\wp,\mathfrak{t}}\left\{\breve{\mathfrak{u}}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s})=\mathcal{L}_{\wp,\mathfrak{t}}\left\{\lambda_1\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\oplus\lambda_2\breve{\mathfrak{u}}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s})$.

Theorem 4.2. [10] *Let $\breve{\vartheta}:\mathbb{R}\times[0,\infty)\to\mathbb{R}_F$ be a fuzzy piecewise continuous function that exhibits $\wp$-exponential order. Assume that the type of $[\wp-p]_{gH}$-differentiability for $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ does not change in the interval $\mathbb{R}\times[0,\infty)$, and that ${}_{gH}\mathbb{T}^{\wp}_{0,\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ is piecewise fuzzy continuous. Then, the fuzzy $\wp$-Laplace transform of ${}_{gH}\mathbb{T}^{\wp}_{0,\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ exists for $0<\vartheta<1$, $\mathfrak{s}>\mathrm{c}$, and*

i. *If $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ is $[\wp-p]_{i.gH}-$differentiable with respect to $\mathfrak{t}$, then we have*

$$\mathcal{L}_{\wp,\mathfrak{t}}\left\{\ {}_{i.gH}\mathbb{T}^{\wp}_{0,\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s})\ =\ \mathfrak{s}\mathcal{L}_{\wp,\mathfrak{t}}\left\{\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s})\ominus\breve{\vartheta}(\mathfrak{x},0).$$

ii. *If $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ is $[\wp-p]_{ii.gH}-$differentiable with respect to $\mathfrak{t}$, then we have*

$$\mathcal{L}_{\wp,\mathfrak{t}}\left\{{}_{ii.gH}\mathbf{T}^{\vartheta}_{0,\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s})\ =\ (-1)\breve{\vartheta}(\mathfrak{x},0)\ominus(-1)\mathfrak{s}\mathcal{L}_{\wp,\mathfrak{t}}\left\{\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s}).$$

Definition 4.3. *Let $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ be a piecewise fuzzy-valued continuous function on $\mathbb{R}\times[0,\infty)$, exhibiting $\wp$-exponential order. Also, let $\hbar(\mathfrak{x},\mathfrak{t})$ be a real-valued piecewise continuous function on $\mathbb{R}\times[0,\infty)$, showing $\wp$-exponential order. The $\wp-$convolution of two functions $\hbar(\mathfrak{x},\mathfrak{t})$ and $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ for $\mathfrak{t}>0$ is defined as*

$$(\hbar *_{\wp} \breve{\vartheta})(\mathfrak{x},\mathfrak{t}) = \int_0^t \frac{1}{\xi^{1-\wp}} \hbar\left((\mathfrak{t}^{\wp}-\xi^{\wp})^{\frac{1}{\wp}}\right) \breve{\vartheta}(\mathfrak{x},\xi)\mathrm{d}\xi.$$

Theorem 4.4. *($\wp-$Convolution Theorem) Let $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ be a piecewise fuzzy-valued continuous function on $\mathbb{R}\times[0,\infty)$, exhibiting $\wp$-exponential order, and $\hbar(\mathfrak{x},\mathfrak{t})$ be a real-valued piecewise continuous function on $\mathbb{R}\times[0,\infty)$, showing $\wp$-exponential order. Then, for $Re(\mathfrak{s})>\mathrm{c}$, the $\wp$-Laplace transform of the $\wp$-convolution $(\hbar *_{\wp} \breve{\vartheta})(\mathfrak{x},\mathfrak{t})$ is given by*

$$\mathcal{L}_{\wp,\mathfrak{t}}\left\{(\hbar *_{\wp} \breve{\vartheta})(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s}) = \mathcal{L}_{\wp,\mathfrak{t}}\left\{\hbar(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s}) \times \mathcal{L}_{\wp,\mathfrak{t}}\left\{\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s}).$$

Proof. Starting with the right-hand side of the equation

$$\begin{aligned}\mathcal{L}_{\wp,\mathfrak{t}}\{\hbar(\mathfrak{x},\mathfrak{t})\}(\mathfrak{s}) \times \mathcal{L}_{\wp,\mathfrak{t}}\{\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\}(\mathfrak{s}) &= \left(\int_0^{\infty} \mathrm{e}^{-\mathfrak{s}\frac{\mathfrak{t}^{\wp}}{\wp}} \hbar(\mathfrak{x},\mathfrak{t})\frac{\mathrm{d}\mathfrak{t}}{\mathfrak{t}^{1-\wp}}\right)\left(\int_0^{\infty} \mathrm{e}^{-\mathfrak{s}\frac{\xi^{\wp}}{\wp}} \breve{\vartheta}(\mathfrak{x},\xi)\frac{\mathrm{d}\xi}{\xi^{1-\wp}}\right)\\ &= \int_0^{\infty}\int_0^{\infty} \mathrm{e}^{-\mathfrak{s}\frac{\mathfrak{t}^{\wp}+\xi^{\wp}}{\wp}} \hbar(\mathfrak{x},\mathfrak{t})\breve{\vartheta}(\mathfrak{x},\xi)\frac{\mathrm{d}\mathfrak{t}}{\mathfrak{t}^{1-\wp}}\frac{\mathrm{d}\xi}{\xi^{1-\wp}}.\end{aligned}$$

By holding $\mathfrak{t}$ fixed in the interior integral and substituting $\tau^{\wp}=\mathfrak{t}^{\wp}+\xi^{\wp}$, we obtain

$$\begin{aligned}&\mathcal{L}_{\wp,\mathfrak{t}}\{\hbar(\mathfrak{x},\mathfrak{t})\}(\mathfrak{s}) \times \mathcal{L}_{\wp,\mathfrak{t}}\{\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\}(\mathfrak{s}) = \int_0^{\infty}\int_{\xi}^{\infty} \mathrm{e}^{-\mathfrak{s}\frac{\tau^{\wp}}{\wp}} \hbar\left((\tau^{\wp}-\xi^{\wp})^{\frac{1}{\wp}}\right) \breve{\vartheta}(\mathfrak{x},\xi)\frac{\mathrm{d}\tau}{\tau^{1-\wp}}\frac{\mathrm{d}\xi}{\xi^{1-\wp}}\\ &\left(\int_0^{\infty}\int_{\xi}^{\infty} \mathrm{e}^{-\mathfrak{s}\frac{\tau^{\wp}}{\wp}} \hbar\left((\tau^{\wp}-\xi^{\wp})^{\frac{1}{\wp}}\right) \vartheta_1(\mathfrak{x},\xi)\frac{\mathrm{d}\tau}{\tau^{1-\wp}}\frac{\mathrm{d}\xi}{\xi^{1-\wp}}, \int_0^{\infty}\int_{\xi}^{\infty} \mathrm{e}^{-\mathfrak{s}\frac{\tau^{\wp}}{\wp}} \hbar\left((\tau^{\wp}-\xi^{\wp})^{\frac{1}{\wp}}\right) \vartheta_2(\mathfrak{x},\xi)\frac{\mathrm{d}\tau}{\tau^{1-\wp}}\frac{\mathrm{d}\xi}{\xi^{1-\wp}},\right.\\ &\left.\int_0^{\infty}\int_{\xi}^{\infty} \mathrm{e}^{-\mathfrak{s}\frac{\tau^{\wp}}{\wp}} \hbar\left((\tau^{\wp}-\xi^{\wp})^{\frac{1}{\wp}}\right) \vartheta_3(\mathfrak{x},\xi)\frac{\mathrm{d}\tau}{\tau^{1-\wp}}\frac{\mathrm{d}\xi}{\xi^{1-\wp}}\right).\end{aligned}$$

Since $\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) = \Big(\vartheta_1(\mathfrak{x},\mathfrak{t}),\vartheta_2(\mathfrak{x},\mathfrak{t}),\vartheta_3(\mathfrak{x},\mathfrak{t})\Big)$, and using Theorem 3.2 in [13], we can reverse the order of integration:

$$\begin{aligned}\mathcal{L}_{\wp,\mathfrak{t}}\{\hbar(\mathfrak{x},\mathfrak{t})\}(\mathfrak{s}) \times \mathcal{L}_{\wp,\mathfrak{t}}\{\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\}(\mathfrak{s}) &= \left(\int_0^{\infty} \mathrm{e}^{-\mathfrak{s}\frac{\tau^{\wp}}{\wp}}\Big[\int_0^{\tau} \hbar\left((\tau^{\wp}-\xi^{\wp})^{\frac{1}{\wp}}\right) \vartheta_1(\mathfrak{x},\xi)\frac{\mathrm{d}\xi}{\xi^{1-\wp}}\Big]\frac{\mathrm{d}\tau}{\tau^{1-\wp}},\right.\\ &\quad \int_0^{\infty} \mathrm{e}^{-\mathfrak{s}\frac{\tau^{\wp}}{\wp}}\Big[\int_0^{\tau} \hbar\left((\tau^{\wp}-\xi^{\wp})^{\frac{1}{\wp}}\right) \vartheta_2(\mathfrak{x},\xi)\frac{\mathrm{d}\xi}{\xi^{1-\wp}}\Big]\frac{\mathrm{d}\tau}{\tau^{1-\wp}},\\ &\quad \left.\int_0^{\infty} \mathrm{e}^{-\mathfrak{s}\frac{\tau^{\wp}}{\wp}}\Big[\int_0^{\tau} \hbar\left((\tau^{\wp}-\xi^{\wp})^{\frac{1}{\wp}}\right) \vartheta_3(\mathfrak{x},\xi)\frac{\mathrm{d}\xi}{\xi^{1-\wp}}\Big]\frac{\mathrm{d}\tau}{\tau^{1-\wp}}\right).\end{aligned}$$

Finally

$$\begin{aligned}\mathcal{L}_{\wp,\mathfrak{t}}\{\hbar(\mathfrak{x},\mathfrak{t})\}(\mathfrak{s}) \times \mathcal{L}_{\wp,\mathfrak{t}}\{\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\}(\mathfrak{s}) &= \int_0^{\infty} \mathrm{e}^{-\mathfrak{s}\frac{\tau^{\wp}}{\wp}}\Big[\int_0^{\tau} \hbar\left((\tau^{\wp}-\xi^{\wp})^{\frac{1}{\wp}}\right) \breve{\vartheta}(\mathfrak{x},\xi)\frac{\mathrm{d}\xi}{\xi^{1-\wp}}\Big]\frac{\mathrm{d}\tau}{\tau^{1-\wp}}\\ &= \mathcal{L}_{\wp,\mathfrak{t}}\{\hbar(\mathfrak{x},\mathfrak{t}) *_{\wp} \breve{\vartheta}(\mathfrak{x},\mathfrak{t})\}(\mathfrak{s}).\end{aligned}$$

The desired result obtained. □

Theorem 4.5. *Let $\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) \in AC(\mathbb{R}\times[0,\infty),\mathbb{R}_F)$ is a fuzzy function and $0<\alpha<1$. Moreover, the function $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ be of $\wp-$exponential order such that ${}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ is fuzzy piecewise continuous on $\mathbb{R}\times[0,\infty)$. Then*

i. *If $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ be a $[CK]_{i.g}-$time differentiable on $\mathbb{R}\times[0,\infty)$, then*

$$\mathcal{L}_{\wp,\mathfrak{t}}\left\{{}^{*}_{i.g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s}) = \mathfrak{s}^{\alpha}\mathcal{L}_{\wp,\mathfrak{t}}\left\{\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s}) \ominus \mathfrak{s}^{\alpha-1}\breve{\vartheta}(\mathfrak{x},0).$$

ii. *If $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ be a $[CK]_{ii.g}-$time differentiable on $\mathbb{R}\times[0,\infty)$, then*

$$\mathcal{L}_{\wp,\mathfrak{t}}\left\{{}^{*}_{ii.g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s}) = (-1)\mathfrak{s}^{\alpha-1}\breve{\vartheta}(\mathfrak{x},0) \ominus (-1)\mathfrak{s}^{\alpha}\mathcal{L}_{\wp,\mathfrak{t}}\left\{\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s}).$$

Proof. According to the definition of the fuzzy Caputo-Katugampola generalized Hukuhara time-fractional derivative (Definition 3.1) and the definition of the fuzzy $\wp$-Laplace transform (Equation (3)), we derive

$$\begin{aligned}\mathcal{L}_{\wp,\mathfrak{t}}\left\{ {}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s}) &= \int_0^{\infty}\mathfrak{t}^{\wp-1}\mathrm{e}^{-\mathfrak{s}\frac{\mathfrak{t}^{\wp}}{\wp}}\left(\frac{\wp^{\alpha}}{\Gamma(1-\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}^{\wp}-\xi^{\wp})^{-\alpha}\left({}_{gH}\mathbb{T}^{\wp}_{0,\xi}\breve{\vartheta}(\mathfrak{x},\xi)\right)\frac{\mathrm{d}\xi}{\xi^{1-\wp}}\right)\mathrm{d}\mathfrak{t}\\ &= \frac{\wp^{\alpha}}{\Gamma(1-\alpha)}\int_0^{\infty}\mathfrak{t}^{\wp-1}\mathrm{e}^{-\mathfrak{s}\frac{\mathfrak{t}^{\wp}}{\wp}}\left(\int_0^{\mathfrak{t}}\xi^{\wp-1}\left((\mathfrak{t}^{\wp}-\xi^{\wp})^{\frac{1}{\wp}}\right)^{-\wp\alpha}\left({}_{gH}\mathbb{T}^{\wp}_{0,\xi}\breve{\vartheta}(\mathfrak{x},\xi)\right)\right)\mathrm{d}\xi\mathrm{d}\mathfrak{t}.\end{aligned}$$

Next, by comparing this expression with the definition of the $\wp$-convolution of two functions (Definition 4.3) and utilizing Theorem 4.4, we obtain

$$\begin{aligned}\mathcal{L}_{\wp,\mathfrak{t}}\left\{ {}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s}) &= \frac{\wp^{\alpha}}{\Gamma(1-\alpha)}\int_0^{\infty}\mathfrak{t}^{\wp-1}\mathrm{e}^{-\mathfrak{s}\frac{\mathfrak{t}^{\wp}}{\wp}}\left(\mathfrak{t}^{-\wp\alpha}*_{\wp}\ {}_{gH}\mathbb{T}^{\wp}_{0,\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right)\mathrm{d}\mathfrak{t}\\ &= \frac{\wp^{\alpha}}{\Gamma(1-\alpha)}\mathcal{L}_{\wp,\mathfrak{t}}\left\{\mathfrak{t}^{-\wp\alpha}*_{\wp}\ {}_{gH}\mathbb{T}^{\wp}_{0,\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s})\\ &= \frac{\wp^{\alpha}}{\Gamma(1-\alpha)}\mathcal{L}_{\wp,\mathfrak{t}}\{\mathfrak{t}^{-\wp\alpha}\}(\mathfrak{s})\times\mathcal{L}_{\wp,\mathfrak{t}}\left\{{}_{gH}\mathbb{T}^{\wp}_{0,\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s}).\end{aligned}\tag{4}$$

Consider the function $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ to be a fuzzy-valued function that is $[CK]_{i.g}$-differentiable. According to Theorem 3.8, it is also $[\wp-p]_{i.gH}$-differentiable. Utilizing Case 1 of Theorem 4.2, we can deduce that

$$\mathcal{L}_{\wp,\mathfrak{t}}\left\{ {}_{i.gH}\mathbb{T}^{\wp}_{0,\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s}) = \mathfrak{s}\mathcal{L}_{\wp,\mathfrak{t}}\left\{\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s})\ominus\breve{\vartheta}(\mathfrak{x},0).$$

Additionally, the $\wp$-Laplace transform of $\mathfrak{t}^{-\wp\alpha}$ is given by [17]

$$\mathcal{L}_{\wp}\left\{\mathfrak{t}^{-\wp\alpha}\right\}(\mathfrak{s}) = \wp^{-\alpha}\frac{\Gamma(1-\alpha)}{\mathfrak{s}^{1-\alpha}}.$$

Using this result, together with the previous result and Equation 4, we can express the $\wp$-Laplace transform of ${}^{*}_{i.g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}$ of $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ as follows:

$$\begin{aligned}\mathcal{L}_{\wp,\mathfrak{t}}\left\{ {}^{*}_{i.g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s}) &= \frac{\wp^{\alpha}}{\Gamma(1-\alpha)}\left(\wp^{-\alpha}\frac{\Gamma(1-\alpha)}{\mathfrak{s}^{1-\alpha}}\times\left(\mathfrak{s}\mathcal{L}_{\wp,\mathfrak{t}}\left\{\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s})\ominus\breve{\vartheta}(\mathfrak{x},0)\right)\right)\\ &= \mathfrak{s}^{\alpha}\mathcal{L}_{\wp,\mathfrak{t}}\left\{\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s})\ominus\mathfrak{s}^{\alpha-1}\breve{\vartheta}(\mathfrak{x},0).\end{aligned}$$

A similar approach can be extended to $[CK]_{ii.g}$-time differentiable functions.

□

4.2 The fuzzy Fourier transform

Let $\breve{\vartheta}:\mathbb{R}\times[0,\infty)\to\mathbb{R}_F$. The fuzzy Fourier transform of $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ with respect to $\mathfrak{x}$ is given by the integral [13]

$$\mathcal{F}_{\mathfrak{x}}\left\{\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(w) = \int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}w\mathfrak{x}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})d\mathfrak{x} = \mathcal{V}(w,\mathfrak{t}).\tag{5}$$

The result is a fuzzy function in the frequency domain, denoted as $\mathcal{V}(w,\mathfrak{t})$.

Likewise, the fuzzy inverse Fourier transform of $\mathcal{V}(w,\mathfrak{t})$ is given by

$$\mathcal{F}^{-1}_{w}\left\{\mathcal{V}(w,\mathfrak{t})\right\}(\mathfrak{x}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}w\mathfrak{x}}\mathcal{V}(w,\mathfrak{t})dw = \breve{\vartheta}(\mathfrak{x},\mathfrak{t}).$$

Theorem 4.6. *Let $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ be fuzzy continuous, fuzzy absolutely integrable. Furthermore, let ${}^{C}_{g}\mathcal{D}^{\beta}_{0,\mathfrak{x}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ is fuzzy absolutely integrable on every $(-\infty,\infty)$. Then*

$$\mathcal{F}_{\mathfrak{x}}\left\{ {}^{C}_{g}\mathcal{D}^{\beta}_{0,\mathfrak{x}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(w) = (\mathrm{i}w)^{\beta}\mathcal{V}(w,\mathfrak{t}).$$

Where $\mathcal{V}(w,\mathfrak{t}) = \mathcal{F}_{\mathfrak{x}}\left\{\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(w)$.

Proof. Using Definition of fuzzy Fourier transform and its inverse, we get

$$\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}w\mathfrak{x}}\mathcal{V}(w,\mathfrak{t})\mathrm{d}w. \tag{6}$$

Applying the Caputo generalized Hukuhara fractional differential operator of order β to both sides of (6), we obtain

$$\begin{aligned} {}_{g}^{C}\mathcal{D}_{0,\mathfrak{x}}^{\beta}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) &= {}_{g}^{C}\mathcal{D}_{0,\mathfrak{x}}^{\beta}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{V}(w,\mathfrak{t})\mathrm{e}^{\mathrm{i}w\mathfrak{x}}\,\mathrm{d}w\right) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{V}(w,\mathfrak{t})\left({}^{C}D_{0,\mathfrak{x}}^{\beta}\mathrm{e}^{\mathrm{i}w\mathfrak{x}}\right)\mathrm{d}w. \end{aligned} \tag{7}$$

Since ${}^{C}D_{0,z}^{\beta}\mathrm{e}^{az} = a^{\beta}\mathrm{e}^{az}$ holds, take $a = \mathrm{i}w$ and $z = \mathfrak{x}$, then

$${}^{C}D_{0,\mathfrak{x}}^{\beta}\mathrm{e}^{\mathrm{i}w\mathfrak{x}} = (\mathrm{i}w)^{\beta}\mathrm{e}^{\mathrm{i}w\mathfrak{x}}. \tag{8}$$

Substituting (8) into (7), we obtain

$${}_{g}^{C}\mathcal{D}_{0,\mathfrak{x}}^{\beta}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(\mathrm{i}w)^{\beta}\mathcal{V}(w,\mathfrak{t})\mathrm{e}^{\mathrm{i}w\mathfrak{x}}\mathrm{d}w.$$

Thus, the fuzzy Fourier transform of the Caputo generalized Hukuhara fractional derivative of order β, i.e. ${}_{g}^{C}\mathcal{D}_{0,\mathfrak{x}}^{\alpha}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$, is given by $(\mathrm{i}w)^{\beta}\mathcal{V}(w,\mathfrak{t})$. □

5 Space-time fractional fuzzy diffusion equation

The space-time fractional fuzzy diffusion equation is an advanced mathematical model that describes how a substance, like heat, mass, or concentration, behaves as it spreads through a medium. This equation helps us understand the complex movement of substances from areas of high concentration to low concentration, influenced by the random motions of molecules.

In this section, our focus is on investigating fuzzy solutions for a specific case of the space-time fractional diffusion equation, represented by

$${}_{g}^{*}\mathbb{D}_{\mathfrak{t}}^{\alpha,\wp}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) = \kappa\,{}_{g}^{C}\mathcal{D}_{0,\mathfrak{x}}^{\beta}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}), \quad (\mathfrak{x},\mathfrak{t})\in\mathbb{R}\times[0,\infty), \tag{9}$$

where $0 < \alpha < 1$, $0 < \beta \leq 2$ and κ is a positive real constant, representing the diffusion constant. The terms ${}_{g}^{*}\mathbb{D}_{\mathfrak{t}}^{\alpha,\wp}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ and ${}_{g}^{C}\mathcal{D}_{0,\mathfrak{x}}^{\beta}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ are the generalized Hukuhara Caputo-Katugampola time-fractional derivative of order α and the Caputo generalized Hukuhara space-fractional derivative of order β, respectively.

The solution for Equation (9) is obtained through the fuzzy $\wp$-Laplace transform of a fuzzy function$\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ with respect to $\mathfrak{t}$ for any fixed $\mathfrak{x}\in\mathbb{R}$ and the fuzzy Fourier transform with respect to $\mathfrak{x}$ for any fixed $\mathfrak{t} > 0$.

Let LF represent the space of all fuzzy functions $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ for which the fuzzy $\wp$-Laplace transform and the fuzzy Fourier transform exist. We introduce the notation:

$$\tilde{\mathcal{V}}(w,\mathfrak{s}) := \mathcal{F}_{\mathfrak{x}}\mathcal{L}_{\wp,\mathfrak{t}}\left\{\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\} = \int_{-\infty}^{\infty}\int_{0}^{\infty}\mathfrak{t}^{\wp-1}\mathrm{e}^{-\mathfrak{s}\frac{\mathfrak{t}^{\wp}}{\wp}}\mathrm{e}^{-\mathrm{i}w\mathfrak{x}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\mathrm{d}\mathfrak{t}\mathrm{d}\mathfrak{x}.$$

5.1 Fuzzy explicit solution

In this subsection, our goal is to determine the explicit fuzzy solution denoted as $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ for the space-time fractional fuzzy diffusion equation (9), with the following fuzzy initial-boundary conditions

$$\breve{\vartheta}(\mathfrak{x},0) = \breve{\varphi}(\mathfrak{x}), \quad \mathfrak{x}\in\mathbb{R}, \quad \mathfrak{t} > 0. \tag{10}$$

Here, $\breve{\varphi}(\mathfrak{x})$ is a given continuous fuzzy function.

Theorem 5.1. *Consider a fuzzy function denoted by $\breve{\varphi}(\mathfrak{x})$, for which a fuzzy Fourier transform $\breve{\Phi}(w)$ exists, defined as $\breve{\Phi}(w) = \mathcal{F}_{\mathfrak{x}}\{\breve{\varphi}(\mathfrak{x})\}(w)$. The solution $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\in LF$ to the fuzzy problem (9) with fuzzy initial-boundary values (10) is expressed as follows*

$$\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\breve{\Phi}(w)\mathbb{E}_{\alpha}\left(\kappa(\mathrm{i}w)^{\beta}\left(\frac{\mathfrak{t}^{\wp}}{\wp}\right)^{\alpha}\right)\mathrm{e}^{\mathrm{i}w\mathfrak{x}}\mathrm{d}w,$$

assuming that the integral on the right-hand side of this equation exists. In this expression, $\mathbb{E}_\alpha(z)$ denotes the Mittag-Leffler function.

Proof. Consider the function $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$, which possesses $[CK]_{i.g}$-time differentiability up to order α. We apply the fuzzy $\wp$-Laplace transform with respect to $\mathfrak{t}$ to Equation (9)

$$\mathcal{L}_{\wp,\mathfrak{t}}\left\{ {}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s}) \quad = \quad \kappa\mathcal{L}_{\wp,\mathfrak{t}}\left\{ {}^{C}_{g}\mathcal{D}^{\beta}_{0,\mathfrak{x}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s}).$$

Using Theorem 4.5 and the fuzzy initial conditions (10), we obtain:

$$\mathfrak{s}^{\alpha}\mathcal{L}_{\wp,\mathfrak{t}}\left\{\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s})\ominus\mathfrak{s}^{\alpha-1}\breve{\varphi}(\mathfrak{x}) = \kappa\ {}^{C}_{g}\mathcal{D}^{\beta}_{0,\mathfrak{x}}\mathcal{L}_{\wp,\mathfrak{t}}\left\{\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\}(\mathfrak{s}).$$

Next, we apply the fuzzy Fourier transform (5) to the above equation, utilizing Theorem 4.6, which leads to the following expression

$$\mathfrak{s}^{\alpha}\breve{\mathcal{V}}(w,\mathfrak{s})\ominus\mathfrak{s}^{\alpha-1}\breve{\Phi}(w) \quad = \quad \kappa(\mathrm{i}w)^{\beta}\tilde{\mathcal{V}}(w,\mathfrak{s}),$$

then

$$\mathfrak{s}^{\alpha}\tilde{\mathcal{V}}(w,\mathfrak{s})\ominus\kappa(\mathrm{i}w)^{\beta}\tilde{\mathcal{V}}(w,\mathfrak{s}) = \breve{\Phi}(w)\mathfrak{s}^{\alpha-1}.$$

Subsequently, by employing Corollary 2.1, we can conclude that

$$\tilde{\mathcal{V}}(w,\mathfrak{s}) = \breve{\Phi}(w)\Big(\frac{\mathfrak{s}^{\alpha-1}}{\mathfrak{s}^{\alpha}-\kappa(\mathrm{i}w)^{\beta}}\Big). \tag{11}$$

Use the fuzzy inverse $\wp-$Laplace transform in Equation (11), we recall the $\wp-$Laplace transform of the Mittag-Leffler function, $\mathbb{E}_\alpha\Big(\lambda\Big(\frac{\mathfrak{t}^{\wp}}{\wp}\Big)^{\alpha}\Big)$ [17] as follows

$$\mathcal{L}_{\wp,\mathfrak{t}}\left\{\mathbb{E}_\alpha\left(\lambda\Big(\frac{\mathfrak{t}^{\wp}}{\wp}\Big)^{\alpha}\right)\right\}(\mathfrak{s}) = \frac{\mathfrak{s}^{\alpha-1}}{\mathfrak{s}^{\alpha}-\lambda}, \qquad (|\frac{\lambda}{\mathfrak{s}^{-\alpha}}|<1).$$

where $\mathbb{E}_\alpha(\lambda\mathfrak{t}^{\alpha})$ denotes the entire transcendental function, known as the Mittag-Leffler function of order α, defined in the complex plane by the power series $\mathbb{E}_\alpha(z)=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\alpha j+1)}$ [17]. Going back to the time domain, by assuming $\lambda=\kappa(\mathrm{i}w)^{\beta}$, we can conclude that

$$\mathcal{F}_{\mathfrak{x}}\left\{\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\right\} = \breve{\Phi}(w)\mathbb{E}_\alpha\left(\kappa(\mathrm{i}w)^{\beta}\left(\frac{\mathfrak{t}^{\wp}}{\wp}\right)^{\alpha}\right), \tag{12}$$

and applying the inverse fuzzy Fourier transform to Equation (12)

$$\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty}\breve{\Phi}(w)\mathbb{E}_\alpha\left(\kappa(\mathrm{i}w)^{\beta}\left(\frac{\mathfrak{t}^{\wp}}{\wp}\right)^{\alpha}\right)\mathrm{e}^{\mathrm{i}w\mathfrak{x}}\mathrm{d}w.$$

The solution for other cases of $[CK]_{ii.g}-$time differentiability can be obtained in a similar way. □

5.2 Fuzzy fundamental solution

In this section, we explore a specific situation involving the space-time fractional diffusion equation, which was previously discussed in Section 5. Our focus is on analyzing the unique features and properties that arise in this particular instance. To start, we consider the initial value as $\breve{\varphi}(\mathfrak{x})=\breve{v}_0\delta(\mathfrak{x})$. Here, $\breve{v}_0$ is a fuzzy number, a mathematical concept that accounts for imprecision or uncertainty, and $\delta(\mathfrak{x})$ is the Dirac delta function, a mathematical tool used to model concentrated point-like sources.

By applying this fuzzy initial condition to Equation (9), we derive a solution that demonstrates fuzzy characteristics. This solution is commonly known as the fuzzy fundamental solution. Analyzing this specific solution provides valuable insights into the behavior and dynamics of the space-time fractional diffusion equation under fuzzy conditions.

Theorem 5.2. *Consider the space-time fractional fuzzy diffusion equation represented by Equation (9), wherein fuzzy initial values are defined as follows*

$$\breve{\vartheta}(\mathfrak{x},0)=\breve{\upsilon}_0\delta(\mathfrak{x}),\qquad \mathfrak{x}\in\mathbb{R}.$$

The solution to this equation, denoted as $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})\in LF$*, can be expressed through the following integral representation*

$$\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) \;=\; \frac{\breve{\upsilon}_0}{2\pi}\int_{-\infty}^{\infty}\mathbb{E}_\alpha\left(\kappa(\mathrm{i}w)^\beta\left(\frac{\mathfrak{t}^\wp}{\wp}\right)^\alpha\right)\mathrm{e}^{\mathrm{i}w\mathfrak{x}}\mathrm{d}w.$$

Provided that the integral on the right-hand side of this equation converges.

Proof. Consider the function $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$, assumed to be $[CK]_g$-time differentiable. Utilizing Theorem 5.1 and invoking Lemma 3.1 from reference [18], we achieve the desired outcomes for this theorem by defining $\breve{\Phi}(\mathfrak{x})=\breve{\upsilon}_0\delta(\mathfrak{x})$. □

Remark 5.3. *By setting* $\beta=1$ *in Equation (9), we derive the following time fractional diffusion equation, utilizing the generalized Hukuhara Caputo-Katugampola time-fractional derivative of order* α*:*

$$\begin{cases} {}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t})=\kappa\frac{\partial\breve{\vartheta}(\mathfrak{x},\mathfrak{t})}{\partial\mathfrak{x}}, & (\mathfrak{x},\mathfrak{t})\in\mathbb{R}\times[0,\infty),\\ \breve{\vartheta}(\mathfrak{x},0)=\breve{\varphi}(\mathfrak{x}). \end{cases} \tag{13}$$

Here, $0<\alpha<1$*,* $\wp>0$*, and* κ *is a positive constant representing the diffusion coefficient.*

To determine the fuzzy explicit solution for Equation (13), we employ the methodology outlined in Theorem 5.1 and the result from Theorem 3.4 in [18]*, which state that* $\mathcal{F}_{\mathfrak{x}}\left\{\frac{\partial\breve{\vartheta}(\mathfrak{x},\mathfrak{t})}{\partial\mathfrak{x}}\right\}(w)=\mathrm{i}w\mathcal{V}(w,\mathfrak{t})$*. This leads us to the following fuzzy explicit solution*

$$\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty}\breve{\Phi}(w)\mathbb{E}_\alpha\left(\kappa\mathrm{i}w\left(\frac{\mathfrak{t}^\wp}{\wp}\right)^\alpha\right)\mathrm{e}^{\mathrm{i}w\mathfrak{x}}\mathrm{d}w,$$

where the convergence of the integral on the right-hand side is assumed. Here, $\breve{\Phi}(w)=\mathcal{F}_{\mathfrak{x}}\left\{\breve{\varphi}(\mathfrak{x})\right\}(w)$*.*

Similarly, given the initial condition $\breve{\varphi}(\mathfrak{x})=\breve{\upsilon}_0\delta(\mathfrak{x})$*, the fuzzy fundamental solution for Equation (13) can be expressed as follows*

$$\breve{\vartheta}(\mathfrak{x},\mathfrak{t})=\frac{\breve{\upsilon}_0}{2\pi}\int_{-\infty}^{\infty}\mathbb{E}_\alpha\left(\kappa\mathrm{i}w\left(\frac{\mathfrak{t}^\wp}{\wp}\right)^\alpha\right)\mathrm{e}^{\mathrm{i}w\mathfrak{x}}\mathrm{d}w,$$

under the condition that the integral on the right-hand side converges.

6 Application

6.1 Fuzzy Groundwater modeling

Groundwater modeling serves as an indispensable tool for hydrogeologists, enabling them to predict fluctuations in water levels and flow velocities under various conditions. By using mathematical expressions to describe groundwater dynamics in specific areas, hydrogeologists gain deeper insights into how water moves through underground aquifers. One crucial aspect of groundwater modeling is the concept of pressure head, which refers to the force exerted by water due to gravity within an aquifer. Pressure head is influenced by factors such as the depth of the water table, characteristics of the surrounding soil or rock formations, and extraction rates. Understanding pressure head is essential for developing accurate groundwater models and effectively managing groundwater reservoirs [5, 14, 16, 20].

However, assessing pressure head often poses challenges in terms of precision and complexity, leading to uncertainties about the initial conditions of the problem. To address this challenge, employing a fuzzy groundwater model offers a promising solution. By incorporating uncertainties, a fuzzy groundwater model provides a more comprehensive and realistic representation of the various factors affecting groundwater dynamics.

In this study, we enhance an existing groundwater model [7] by integrating fuzzy functions, aiming to improve its representation of an aquifer with continuous vertical flow over time. This model focuses on an aquifer with unbounded vertical flow over time. The initial distribution of fuzzy pressure head in this aquifer is represented by the fuzzy function $\breve{\varphi}(\mathfrak{x})$.

To determine the fuzzy pressure head within the aquifer, we utilize a fuzzy saturated groundwater flow model expressed as follows:

$$ {}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) = \frac{\mathcal{K}}{\mathcal{N}}\ {}^{C}_{g}\mathcal{D}^{\beta}_{0,\mathfrak{x}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}). \tag{14} $$

Here, $\mathfrak{x}$ represents vertical flow, and $\mathfrak{t}$ indicates time.

In this equation, $\mathcal{K}$ stands for saturated hydraulic conductivity, reflecting the rate at which water can traverse a porous medium under a given hydraulic gradient. Hydraulic conductivity is impacted by several factors including the physical attributes of the porous medium (such as porosity and permeability), the nature of the fluid, and the direction and slope of water movement. Additionally, $\mathcal{N}$ denotes specific yield, which quantifies the volume of water that a saturated aquifer or soil can release under the influence of gravity, typically expressed as a fraction or percentage.

These factors significantly influence an aquifer's or soil's ability to transmit water, consequently affecting the rate and direction of groundwater flow. Table 1 provides insights into the typical ranges of specific yield and hydraulic conductivity for various earth materials, collating measurements obtained through diverse laboratory and field techniques [9, 26].

Material	**Range of Hydraulic Conductivity $\mathcal{K}$**	**Range of Specific Yield $\mathcal{N}$**
Clay	10^{-13}-10^{-9}	1-18
Silt	10^{-8}-10^{-5}	1-40
Sand	10^{-5}-10^{-2}	32-47
Gravel	10^{-3}-1	13-40

Table 1: Range of hydraulic conductivity and specific yield of soil materials

Example 6.1. *Imagine a shallow aquifer with layers of saturated silt, nourished by rainwater from a recharge zone above. Water flows downward through these layers, driven by gravity, until it reaches a discharge zone, where it exits the aquifer into a nearby stream. This process can be modeled mathematically using a fuzzy groundwater flow equation:*

$$ {}^{*}_{g}\mathbb{D}^{\frac{1}{2},\frac{1}{2}}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) = \frac{\mathcal{K}}{\mathcal{N}}\ {}^{C}_{g}\mathcal{D}^{\frac{1}{2}}_{0,\mathfrak{x}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}), \quad -\infty < \mathfrak{x} < \infty, \mathfrak{t} \geq 0. $$

Suppose we set $\mathcal{K} = 10^{-5}$ and $\mathcal{N} = 2$, as indicated in Table 1. Initially, the aquifer has a certain level of pressure, which we can describe using a mathematical fuzzy function denoted as $\breve{\varphi}(\mathfrak{x}) = \left(\mathfrak{x}^2, 7\mathfrak{x}^2, 15\mathfrak{x}^2\right)$. By employing a particular method outlined in Theorem 5.1, we derive the following fuzzy Explicit solution for this model

$$ \begin{aligned} \breve{\vartheta}(\mathfrak{x},\mathfrak{t}) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\breve{\Phi}(w)\mathbb{E}_{\frac{1}{2}}\left(\frac{10^{-5}\sqrt{2}}{2}(\mathrm{i}w)^{\frac{1}{2}}\mathfrak{t}^{\frac{1}{4}}\right)\mathrm{e}^{\mathrm{i}w\mathfrak{x}}\mathrm{d}w, \\ &= \left(\frac{\left(\sqrt{\mathfrak{t}}-10^{10}\mathfrak{x}\right)^2}{10^{20}}, \frac{7\left(\sqrt{\mathfrak{t}}-10^{10}\mathfrak{x}\right)^2}{10^{20}}, \frac{3\left(\sqrt{\mathfrak{t}}-10^{10}\mathfrak{x}\right)^2}{2\times 10^{19}}\right). \end{aligned} $$

Example 6.2. *Consider a saturated sand aquifer with multiple layers, typical in sandy regions, river valleys, and coastal areas. To characterize this system, we define a fuzzy initial condition, $\breve{\varphi}(\mathfrak{x}) = \left(3\mathfrak{x}^4, 5\mathfrak{x}^4, 9\mathfrak{x}^4\right)$, representing the initial water pressure distribution. Two critical parameters, $\mathcal{K} = 10^{-2}$ and $\mathcal{N} = 35$, are set based on Table 1.*

The dynamics of the aquifer and pressure distribution can be mathematically modeled using a fuzzy groundwater flow equation

$$ {}^{*}_{g}\mathbb{D}^{\frac{1}{2},3}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) = \frac{10^{-2}}{35}\ {}^{C}_{g}\mathcal{D}^{\frac{3}{2}}_{0,\mathfrak{x}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}), \quad -\infty < \mathfrak{x} < \infty, \mathfrak{t} \geq 0. $$

Utilizing Theorem 5.1, we derive the following explicit solution to understand the aquifer's dynamics and pressure distribution

$$ \begin{aligned} \breve{\vartheta}(\mathfrak{x},\mathfrak{t}) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\breve{\Phi}(w)\mathbb{E}_{\frac{1}{2}}\left(\frac{10^{-2}}{35}(\mathrm{i}w)^{\frac{3}{2}}\left(\frac{\mathfrak{t}^2}{2}\right)^{\frac{1}{2}}\right)\mathrm{e}^{\mathrm{i}w\mathfrak{x}}\mathrm{d}w \\ &= \left(3\left(\frac{-\mathfrak{t}^3\mathfrak{x}}{1531250}+\mathfrak{x}^4\right), \frac{-\mathfrak{t}^3\mathfrak{x}}{306250}+5\mathfrak{x}^4, 9\left(\frac{-\mathfrak{t}^3\mathfrak{x}}{1531250}+\mathfrak{x}^4\right)\right). \end{aligned} $$

6.2 Fuzzy modeling of tumor cell concentration

The concentration of tumor cells, indicating the quantity or density of cancer cells within a specific volume of tissue, fluid, or area, is crucial for diagnosing cancer, determining its stage and severity, devising treatment plans, and evaluating therapeutic efficacy.

Accurately estimating tumor cell concentration in real-world scenarios is challenging due to the differences between controlled laboratory conditions and the complex environment of the human body. These variations introduce uncertainty, complicating measurement accuracy. To address this, a fuzzy model incorporating uncertainty and imprecision can more accurately represent real-world conditions.

Let $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ denote the concentration of tumor cells at location $\mathfrak{x}$ and time $\mathfrak{t}$, and let $\breve{\vartheta}(\mathfrak{x},\mathfrak{t}_0)$ represent the initial concentration of tumor cells at time $\mathfrak{t}_0$. The model examines the dynamics of tumor cell populations through proliferation and diffusion, which can be mathematically expressed as:

$$ {}^{*}_{g}\mathbb{D}^{\alpha,\wp}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) = \kappa\frac{\partial\breve{\vartheta}(\mathfrak{x},\mathfrak{t})}{\partial\mathfrak{x}}, \quad -\infty<\mathfrak{x}<\infty, \mathfrak{t}\geq 0. \tag{15}$$

In this equation, κ represents the diffusion coefficient.

Example 6.3. *Consider a patient diagnosed with brain cancer. At the time of diagnosis, the initial distribution of tumor cells* $\breve{\vartheta}(\mathfrak{x},\mathfrak{t}_0) = (6,7,8)$ *nanograms per milliliter (ng/mL) characterizes the spatial arrangement of cancerous cells.*

In addition, the diffusion process governing the spread of tumor cells is quantified by the diffusion coefficient $\kappa = 0.01$*. To mathematically model this scenario, we employ a fuzzy time-fractional diffusion equation:*

$$ {}^{*}_{g}\mathbb{D}^{\frac{1}{2},2}_{\mathfrak{t}}\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) = 0.01\frac{\partial\breve{\vartheta}(\mathfrak{x},\mathfrak{t})}{\partial\mathfrak{x}},$$

where ${}^{*}_{g}\mathbb{D}^{\frac{1}{2},2}_{\mathfrak{t}}$ *denotes the fuzzy generalized Hukuhara Caputo-Katugampola time-fractional derivative operator, reflecting the non-local temporal dynamics in the evolution of tumor cell concentrations. This equation encapsulates how the concentration* $\breve{\vartheta}(\mathfrak{x},\mathfrak{t})$ *evolves over time* $\mathfrak{t}$ *due to diffusion governed by* κ*.*

To obtain a fundamental solution for this mathematical model, we will apply the methodology discussed in Remark 5.3, ensuring a rigorous approach to understanding the spatial and temporal characteristics of tumor cell spread within brain tissue.

$$\begin{aligned}
\breve{\vartheta}(\mathfrak{x},\mathfrak{t}) &= \frac{(6,7,8)}{2\pi}\int_{-\infty}^{\infty}\mathbb{E}_{\frac{1}{2}}\left(0.01\mathrm{i}w\left(\frac{\mathfrak{t}^2}{2}\right)^{\frac{1}{2}}\right)\mathrm{e}^{\mathrm{i}w\mathfrak{x}}\mathrm{d}w\\
&= \left(e^{-\frac{5000\mathfrak{x}^2}{\mathfrak{t}^2}}\left(\frac{95.493\mathfrak{t}^2+7619.24\mathfrak{t}\mathfrak{x}}{\mathfrak{t}^3}\right), e^{-\frac{5000\mathfrak{x}^2}{\mathfrak{t}^2}}\left(\frac{111.408\mathfrak{t}^2+8889.11\mathfrak{t}\mathfrak{x}}{\mathfrak{t}^3}\right),\right.\\
&\qquad \left. e^{-\frac{5000\mathfrak{x}^2}{\mathfrak{t}^2}}\left(\frac{127.324\mathfrak{t}^2+10159\mathfrak{t}\mathfrak{x}}{\mathfrak{t}^3}\right)\right).
\end{aligned}$$

Which is the exact fuzzy fundamental solution.

7 Conclusion

In summary, this research developed a novel methodology to derive analytical solutions for the space-time fractional fuzzy diffusion equation, incorporating both Caputo generalized Hukuhara fractional derivatives and fuzzy Caputo-Katugampola generalized Hukuhara time-fractional derivatives. Utilizing the fuzzy Fourier and fuzzy $\wp$-Laplace transforms, the study successfully obtained explicit and fundamental solutions, accounting for various degrees of fuzzy time-fractional differentiability.

The effectiveness of the proposed method was demonstrated through applications to a fuzzy groundwater flow model and a fuzzy tumor cell concentration model, accurately determining the pressure head and cell concentration, respectively. This highlights the method's efficiency and accuracy in solving complex space-time and time fractional fuzzy diffusion equations.

However, the research has certain limitations. The complexity of the fractional derivatives and fuzzy transforms may pose computational challenges, especially for large-scale problems. Additionally, the method's applicability to non-linear or higher-dimensional systems remains unexplored.

Future research should focus on extending this methodology to handle non-linear and higher-dimensional fractional fuzzy diffusion equations. Further development of numerical techniques could also enhance the practicality and efficiency of the proposed method for real-world applications.

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